Switch Distribution

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1 Formulate the problem

Our distribution should start as a linear function f until time t_0 and continue as an exponential function g until the end of time. The total budget shall be B. Then the functions

$$\begin{aligned} f(x) &= ax + 1 \\ g(x) &= b\exp(cx) \end{aligned}$$

should satisfy:

$$f(t_0) = g(t_0) \tag{1}$$

$$f'(t_0) = g'(t_0) \tag{2}$$

$$B = \int_0^{t_0} f(x)dx + \int_{t_0}^{\infty} g(x)dx \tag{3}$$

2 Solve for a,b and c

Equation (1) implies:

$$at_0 + 1 = b\exp(ct_0) \tag{4}$$

Equation (2) implies:

$$\begin{aligned} & a = bc\exp(ct_0) \\ \implies & \frac{a}{c} = b\exp(ct_0) \overset{(4)}{=} at_0 + 1 && (5) \\ \implies & a = cat_0 + c \\ \implies & a - act_0 = c \\ \implies & a(1 - ct_0) = c \\ \implies & a = \frac{c}{1 - ct_0} && (6) \end{aligned}$$

Note that $1 - ct_0 \geq 1 > 0$, since $c < 0$ will be a necessary assumption for the integral to be finite (c.f. (8)), and $t_0 \geq 0$ simply means that the switch happens in the future.
(6) in (5):

$$b \exp(ct_0) \overset{(5)}{=} \frac{a}{c} \overset{(6)}{=} \frac{c}{c(1 - ct_0)}$$
$$\implies b = \frac{\exp(-ct_0)}{1 - ct_0} \tag{7}$$

So we only have one unknown variable left, which is c. Equation (3) implies:

$$\begin{aligned} B &= \int_0^{t_0} ax + 1 dx + \int_{t_0}^{\infty} b \exp(cx) dx \\ &= \left[\frac{a}{2}x^2 + x\right]_0^{t_0} + \left[\frac{b}{c}\exp(cx)\right]_{t_0}^{\infty} \\ &= \frac{a}{2}{t_0}^2 + t_0 - \frac{b}{c}\exp(ct_0) \\ &\overset{(7)}{=} \frac{a}{2}{t_0}^2 + t_0 - \frac{1}{c(1 - ct_0)} \\ &\overset{(6)}{=} \frac{c}{2(1 - ct_0)}{t_0}^2 + t_0 - \frac{1}{c(1 - ct_0)} \end{aligned} \tag{8}$$

This implies:

$$\begin{aligned} c(1 - ct_0)B &= c^2\frac{{t_0}^2}{2} + c(1 - ct_0)t_0 - 1 \\ \implies cB - c^2 t_0 B &= c^2\frac{{t_0}^2}{2} + ct_0 - c^2 t_0^2 - 1 \\ \implies c^2 &\underbrace{\left(\frac{{t_0}^2}{2} - t_0^2 + t_0 B\right)}_{= t_0 B - \frac{{t_0}^2}{2}} + c\left(t_0 - B\right) - 1 = 0 \end{aligned}$$

Which has solutions:

$$c_{1/2} = \frac{B - t_0 \pm \sqrt{(B - t_0)^2 + 4\left(t_0 B - \frac{{t_0}^2}{2}\right)}}{2\left(t_0 B - \frac{{t_0}^2}{2}\right)}$$

As we assumed $c < 0$ in (8), we need to ensure this condition is satisfied. After a case by case analysis of the cases $t_0 \lesseqgtr 2B$, it becomes clear that this is only satisfied for $t_0 \leq 2B$ and

$$c = \frac{B - t_0 - \sqrt{(B - t_0)^2 + 4\left(t_0 B - \frac{{t_0}^2}{2}\right)}}{2\left(t_0 B - \frac{{t_0}^2}{2}\right)} \tag{9}$$

3 Numerical Stability

While the current parameters are correct, their formulas appear to be numerically unstable. For this reason simplify (9) to

$$c = \frac{B - t_0 - \sqrt{(B-t_0)^2 + 2t_0(2B-t_0)}}{t_0(2B-t_0)} \tag{10}$$

and notice that the denominator approaches zero for $t_0 \to 2B$. This implies c approaches infinity. For this reason a (see (6)) is unstable, and we rewrite it as:

$$\begin{aligned}
a &= \frac{c}{1 - ct_0} \\
&= \frac{1}{\frac{1}{c} - t_0} \\
&\overset{(9)}{=} \frac{1}{\frac{t_0(2B-t_0)}{B-t_0-\sqrt{(B-t_0)^2+2t_0(2B-t_0)}} - t_0} \\
&= \frac{B - t_0 - \sqrt{(B-t_0)^2 + 2t_0(2B-t_0)}}{t_0\left((2B-t_0) - \left(B - t_0 - \sqrt{(B-t_0)^2 + 2t_0(2B-t_0)}\right)\right)} \\
&= \frac{B - t_0 - \sqrt{(B-t_0)^2 + 2t_0(2B-t_0)}}{t_0\left(B + \sqrt{(B-t_0)^2 + 2t_0(2B-t_0)}\right)} \\
&= \frac{B}{t_0(B + \text{rootTempVar})} - \frac{1}{B + \text{rootTempVar}} - \frac{\text{rootTempVar}}{t_0(B + \text{rootTempVar})}
\end{aligned}$$

With

$$\text{rootTempVar} := \sqrt{(B-t_0)^2 + 2t_0(2B-t_0)}$$

As we need to calculate

$$f(t_0) = at_0 + 1 =: \text{switchValue}$$

The calculation of b can be avoided entirely, since:

$$g(x) = b\exp(cx) = \underbrace{b\exp(ct_0)}_{=g(t_0)=f(t_0)} \exp(c(x - t_0))$$

And c can be calculated as (cf. (5)):

$$c = \frac{a}{b\exp(ct_0)} = \frac{a}{f(t_0)}$$

4 Algorithm

Algorithm 1 calcSwitchParams(B, t_0)

rootTempVar $\leftarrow sqrt(B - t_0)^2 + 2t_0(2B - t_0)$
denominator $\leftarrow t_0(B + \text{rootTempVar})$

$a = \text{linFactor} \leftarrow \frac{B}{\text{denominator}} - \frac{1}{B+\text{rootTempVar}} - \frac{\text{rootTempVar}}{\text{denominator}}$
$f(t_0) = \text{switchValue} \leftarrow \text{linFactor} \times t_0 + 1$
$c = \text{logBase} \leftarrow \frac{a}{\text{linFactor}}$

return data.frame(
linFactor = linFactor, logBase = logBase, switchValue = switchValue
)