Switch Distribution

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1 Formulate the problem

Our distribution should start as a linear function f until time t_0 and continue as an exponential function g until the end of time. The total budget shall be B. Then the functions

$$\begin{aligned} f(x) &= ax + 1 \\ g(x) &= b\exp(cx) \end{aligned}$$

should satisfy:

$$f(t_0) = g(t_0) \tag{1}$$
$$f'(t_0) = g'(t_0) \tag{2}$$
$$B = \int_0^{t_0} f(x)dx + \int_{t_0}^{\infty} g(x)dx \tag{3}$$

2 Solve for a,b and c

Equation (1) implies:

$$at_0 + 1 = b\exp(ct_0) \tag{4}$$

Equation (2) implies:

$$\begin{aligned} & a = bc\exp(ct_0) \\ \implies & \frac{a}{c} = b\exp(ct_0) \overset{(4)}{=} at_0 + 1 \qquad (5) \\ \implies & a = cat_0 + c \\ \implies & a - act_0 = c \\ \implies & a(1 - ct_0) = c \\ \implies & a = \frac{c}{1 - ct_0} \qquad (6) \end{aligned}$$

Note that $1 - ct_0 \geq 1 > 0$, since $c < 0$ will be a necessary assumption for the integral to be finite (c.f. (8)), and $t_0 \geq 0$ simply means that the switch happens in the future.
(6) in (5):

$$\begin{aligned}
b \exp(ct_0) &\overset{(5)}{=} \frac{a}{c} \overset{(6)}{=} \frac{c}{c(1-ct_0)} \\
\implies b &= \frac{\exp(-ct_0)}{1-ct_0} \qquad (7)
\end{aligned}$$

So we only have one unknown variable left, which is c. Equation (3) implies:

$$\begin{aligned}
B &= \int_0^{t_0} ax + 1 dx + \int_{t_0}^{\infty} b \exp(cx) dx \\
&= \left[\frac{a}{2}x^2 + x\right]_0^{t_0} + \left[\frac{b}{c}\exp(cx)\right]_{t_0}^{\infty} \\
&= \frac{a}{2}{t_0}^2 + t_0 - \frac{b}{c}\exp(ct_0) \qquad (8) \\
&\overset{(7)}{=} \frac{a}{2}{t_0}^2 + t_0 - \frac{1}{c(1-ct_0)} \\
&\overset{(6)}{=} \frac{c}{2(1-ct_0)}{t_0}^2 + t_0 - \frac{1}{c(1-ct_0)}
\end{aligned}$$

This implies:

$$\begin{aligned}
c(1-ct_0)B &= c^2\frac{{t_0}^2}{2} + c(1-ct_0)t_0 - 1 \\
\implies cB - c^2 t_0 B &= c^2\frac{{t_0}^2}{2} + ct_0 - c^2 t_0^2 - 1 \\
\implies c^2 \underbrace{\left(\frac{{t_0}^2}{2} - t_0^2 + t_0 B\right)}_{=t_0 B - \frac{{t_0}^2}{2}} &+ c\,(t_0 - B) - 1 = 0
\end{aligned}$$

Which has solutions:

$$c_{1/2} = \frac{B - t_0 \pm \sqrt{(B-t_0)^2 + 4\left(t_0 B - \frac{{t_0}^2}{2}\right)}}{2\left(t_0 B - \frac{{t_0}^2}{2}\right)} \qquad (9)$$

As we assumed $c < 0$ in (8), we need to ensure this condition is satisfied: