Felix Céard-Falkenberg
7174020

1 Question 2

(a) **Draw an example graph such that $k > 4$ and $n > 20$. Mark all the layers $V_1, \ldots$ with a different color.**

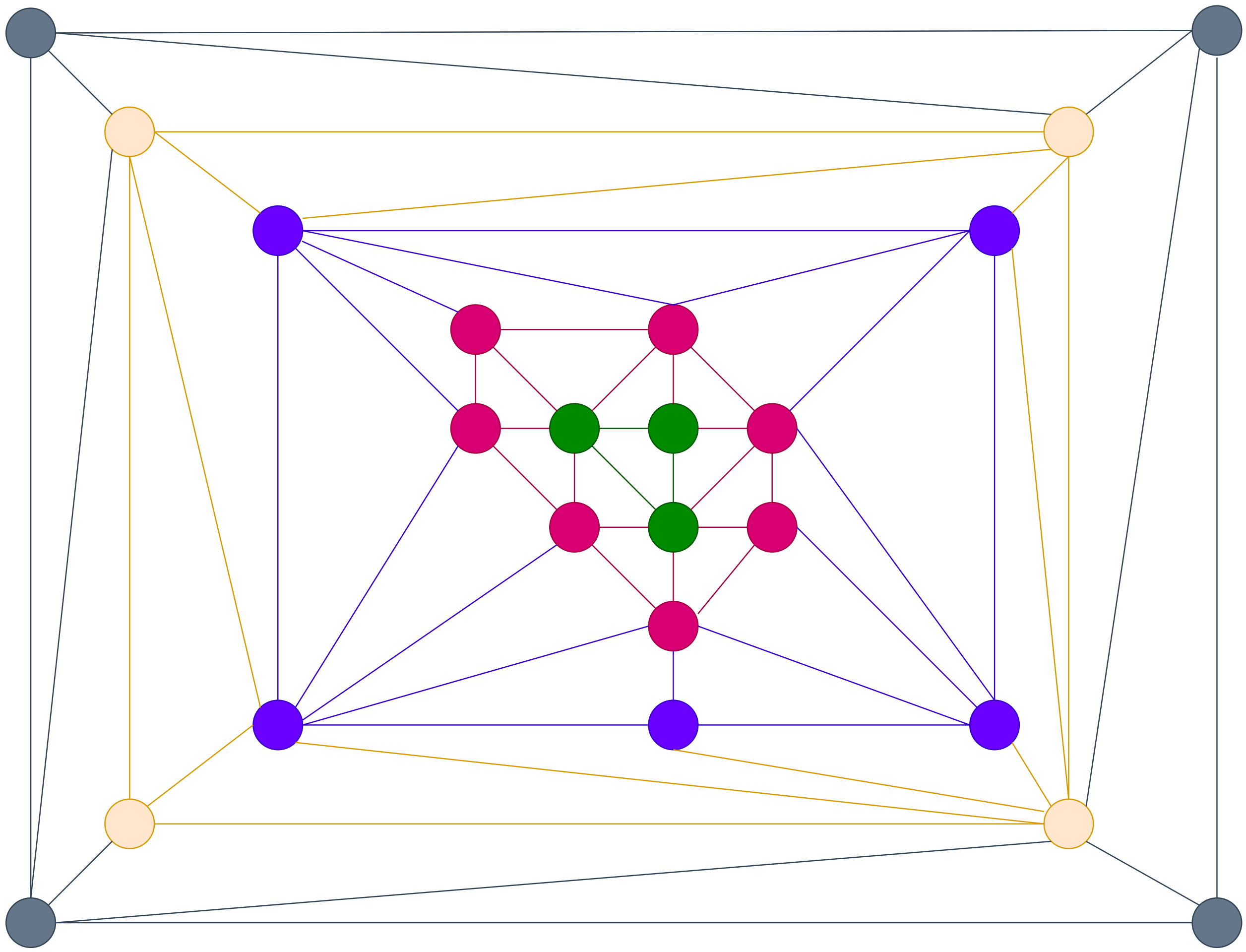

Figure 1: THE *graph* >:3.

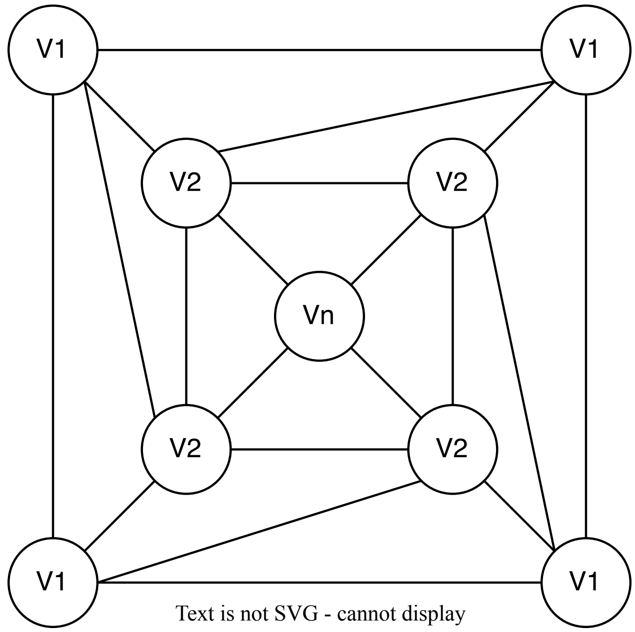

Figure 2: The minimal graph for V_1 and V_2, where V_n denotes the graph of the subsequent layers. Note that we assume that V_n is a triangulation too; we exclude the empty set.

(b) **Show that there are paths of length at most k from V_1 to V_k. If you take two of them together you have a set of size $2k$ that separates the graph into at least two components. Assuming $k = O(\sqrt{n})$ we have a small separator of the graph. It is unclear if this separator is balanced. Describe a graph together with two such paths that lead to an unbalanced separator.**

We start by proving that there exists at least one path of length at most k that connect V_1 to V_k. To this end, we will first prove a lemma to demonstrate that the graph in Figure 2 is the minimal graph for constructing V_1 and V_2.

Lemma 1. *The graph in Figure 2 contains the minimal number of nodes and edges for constructing V_1 and V_2, up to equivalence in their ordering.*

Proof. We can prove this lemma by using a proof by contradiction.

Edges. Suppose that there exists an edge that can be removed, such that the graph still is a triangulation, and the sets V_1 and V_2 remain the same. In that case, the graph would not be minimal, as we could remove that edge and still have a triangulation.

However, removing any edge between two nodes in V_1 (or two nodes in V_2) would result in at least one node of the subsequent layer being moved to a higher layer (e.g., a node $v \in V_2$ would be moved to the set V_1 if such an edge is removed). Hence, these edges are strictly necessary for the graph to be a triangulation.

Furthermore, removing any edge between V_1 and V_2 in this graph would result a inner face between V_1 and V_2 to not be a triangle. This would violate the definition of a triangulation graph, and thus, these edges are strictly necessary too.

We can reuse the same two arguments to show that the edges between V_2 and V_n are strictly necessary too. Note that while we represent V_n as a single node, in a real graph, V_n is an entire graph, and thus the exact connection between V_2 and V_n can differ based on the exact construction of V_n. Thus, all edges in this graph are stricly necessary for not violating the triangulation property, and therefore, the graph is the minimal graph for constructing V_1 and V_2 around a graph V_n.

Nodes. By removing any node from this graph, we would move the outer face to a sub-layer, and therefore violate the constraint that the sets remain the same. Thus, all nodes in this graph are strictly necessary for not violating the triangulation property.

Therefore, because every edge and node is strictly necessary for not violating the triangulation property, we can conclude that the graph in Figure 2 is the minimal graph for constructing V_1 and V_2 around a graph V_n. $\square$

Before proving Theorem 1, we will first introduce Lemma 2, which will be used to prove the final theorem.

Lemma 2. *Given a triangulation graph, we can add an infinite number of nodes to each set V_i while maintaining the triangulation property iff we constrain their placement to edges (u, v), where $u, v \in V_i$.*

Proof. Without loss of generality, we can assume that we have the minimal graph as shown in Figure 2. To prove this lemma, we will prove both directions of the implication:

$\Rightarrow$

We prove this lemma using a proof by induction over the number of nodes added to a set V_i.

Base case ($n = 1$). We can easily extend V_i by one node by adding it on any edge (u, v), where $u, v \in V_i$, as long as the position is not exactly u or v. This can be easily shown as the edge (u, v) is a straight line, and that the shape of the face is not affected by the addition of a single node on a straight line. Furthermore, as the new node is also incident to the same face as another node in V_i, the new node will be removed in G_{i+1}. Therefore, the new node does not affect the triangulation property of the graph.

If however, we would place the new node on an edge (u, w) where $u \in V_i$ and $w \notin V_i$ (either upper or lower layer), we would violate the triangulation property, as in the graph G_i, the new node would be "floating around", which would result in a non-triangular face, thus violating the triangulation property.

Induction step. We can repeat the same argument, except that we now include the previously added nodes to the set V_i, as they are also incident to the same face as the new node. Therefore, the new node does not affect the triangulation property of the graph.

$\Leftarrow$

We repeat the same argument as before. $\square$

Theorem 1. *There exists at least one path of length at most k from V_1 to V_k.*

Proof.

By combining Lemma 1 and Lemma 2, we know that it is possible to define the minimal structure of any triangulation graph for V_i with $k > 2$ by recursively creating new graphs based on Figure 2.

Therefore, following from Lemma 2, we know that there will always exist a direct connection from V_i to V_{i+1} for any $i \in \{1, \dots, k-1\}$, as adding nodes to these connecting edges violates the triangulation property. As each of these connecting edges between layers must lie on a direct path from V_1 to V_k, we know that there exists a path of length at most k from V_1 to V_k.

$\square$