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Problem 1

(a) *Find the $\mathbb{E}[\bar{X}(N)]$ analytically. Does it really estimate the mean of X?*

We have that

$$\bar{X}(N) = \frac{1}{N}\sum_{i=1}^{N} X_i, \quad X_i \sim \text{Uniform}(0,1)\ .$$

We first compute the expectation of X_i. We have

$$\begin{aligned}
\mathbb{E}\left[X_i\right] &= \int_{-\infty}^{\infty} x \cdot p(X=x)\ \ \mathrm{d}x \\
&= \int_0^1 x \cdot p(X=x)\ \mathrm{d}x \\
&= \int_0^1 x \cdot \frac{1}{1-0}\ \mathrm{d}x \\
&= \int_0^1 x\ \mathrm{d}x \\
&= \left[\frac{1}{2}x^2\right]_0^1 \\
&= \frac{1}{2}(1^2 - 0^2) \\
&= \frac{1}{2}
\end{aligned} \tag{1}$$

Therefore, we can compute the expectation of $\bar{X}(N)$ as follows:

$$\begin{aligned}
\mathbb{E}[\bar{X}(N)] &= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[X_i\right] \\
&\overset{(1)}{=} \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2} \\
&= \frac{1}{n}\cdot n \cdot \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

Therefore, the expected value of $\mathbb{E}[\bar{X}(N)]$ estimates the true mean of X.

(b) *Find the Var$(\bar{X}(N))$ analytically. Hint: find the variance of a single X_i first.*

Let us first compute the exectation of $\mathbb{E}[X_i^2]$:

$$\begin{aligned}
\mathbb{E}[X_i^2] &= \int_0^1 x^2 \cdot p(X = x) \ \ \mathrm{d}x \\
&= \int_0^1 x^2 \cdot \frac{1}{1-0} \ \ \mathrm{d}x \\
&= \int_0^1 x^2 \ \ \mathrm{d}x \\
&= \left[\frac{1}{3} \cdot x^3\right]_0^1 \\
&= \frac{1}{3}(1-0) \\
&= \frac{1}{3}
\end{aligned} \tag{2}$$

We can now compute the variance of a X_i as follows:

$$\begin{aligned}
\mathrm{Var}(X_i) &= \mathbb{E}\left[(X_i - \mathbb{E}[X_i])^2\right] \\
&= \mathbb{E}\left[(X_i - \frac{1}{2})^2\right] \\
&= \mathbb{E}\left[X_i^2 + \frac{1^2}{2^2} - 2 \cdot X_i \cdot \frac{1}{2}\right] \\
&= \mathbb{E}\left[X_i^2 + \frac{1^2}{2^2} - X_i\right] \\
&= \mathbb{E}\left[X_i^2\right] + \mathbb{E}\left[\frac{1}{4}\right] - \mathbb{E}\left[X_i\right] \\
&= \mathbb{E}\left[X_i^2\right] + \frac{1}{4} - \frac{1}{2} \\
&\overset{(2)}{=} \frac{1}{3} + \frac{1}{4} - \frac{1}{2} \\
&= \frac{4}{12} + \frac{3}{12} - \frac{6}{12} \\
&= \frac{1}{12} \ .
\end{aligned}$$

We can now compute the variance of $\bar{X}(N)$ as follows:

$$\begin{aligned}
\operatorname{Var}(\bar{X}(N)) &= \operatorname{Var}\left(\frac{1}{N}\sum_{i=1}^{N} X_i\right) \\
&= \frac{1}{N^2}\cdot \operatorname{Var}\left(\sum_{i=1}^{N} X_i\right) \\
&= \frac{1}{N^2}\cdot \sum_{i=1}^{N} \operatorname{Var}(X_i) \qquad (3) \\
&= \frac{1}{N^2}\cdot \sum_{i=1}^{N} \frac{1}{12} \\
&= \frac{1}{N^2}\cdot N \cdot \frac{1}{12} \\
&= \frac{1}{N}\cdot \frac{1}{12}
\end{aligned}$$

where Equation 3 holds as each drawing X_i is independent of each other drawing X_j ($\Pr(X_i \mid X_j) = \Pr(X_i)$), and we can therefore use the property of the variance for i.i.d. variables $\operatorname{Var}(X+Y) = \operatorname{Var}(X) + \operatorname{Var}(Y)$.

Therefore, in the limit, we have

$$\lim_{n\to\infty} \operatorname{Var}(\bar(X)(N)) = \lim_{n\to\infty} \frac{1}{N}\cdot\frac{1}{12} = 0$$

with a linear convergence rate.

(c) The probability density function of $\hat{X}(N)$ is given as

$$f_{\hat{X}(N)}(x) = \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot x^{\frac{N-1}{2}} \cdot (1-x)^{\frac{N-1}{2}} \ .$$

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Let us now compute the expectation of $\hat{X}(N)$:

$$\begin{aligned}
\mathbb{E}[\hat{X}(N)] &= \int_0^1 x \cdot f_{\hat{X}(N)}(x) \, \mathrm{d}x \\
&= \int_0^1 x \cdot \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot x^{\frac{N-1}{2}} \cdot (1-x)^{\frac{N-1}{2}} \, \mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \int_0^1 x \cdot x^{\frac{N-1}{2}} \cdot (1-x)^{\frac{N-1}{2}} \, \mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \int_0^1 x^{\frac{N-1}{2}+1} \cdot (1-x)^{\frac{N-1}{2}} \, \mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \int_0^1 x^{\frac{N+1}{2}} \cdot (1-x)^{\frac{N-1}{2}} \, \mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+1}{2}!\,\frac{N-1}{2}!}{\left(\frac{N+1}{2} + \frac{N-1}{2} + 1\right)!} && (4) \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+1}{2}!\,\frac{N-1}{2}!}{\left(\frac{N-1}{2} + \frac{N-1}{2} + 1 + 1\right)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+1}{2}!\,\frac{N-1}{2}!}{\left(2 \cdot \frac{N-1}{2} + 2\right)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+1}{2}!\,\frac{N-1}{2}!}{(N-1+2)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+1}{2}!\,\frac{N-1}{2}!}{(N+1)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{1}{2} \cdot \frac{\left(\frac{N+1}{2}\right)!\,\frac{N-1}{2}!}{N!} \\
&= \frac{N!}{\left(\frac{N-1}{2}\right)!\left(N - \frac{N-1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\left(\frac{N+1}{2}\right)!\,\frac{N-1}{2}!}{N!} \\
&= \frac{1}{\left(\frac{N-1}{2}\right)!\left(N - \frac{N-1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\left(\frac{N+1}{2}\right)!\,\frac{N-1}{2}!}{1} \\
&= \frac{1}{\left(N - \frac{N-1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\left(\frac{N+1}{2}\right)!}{1} \\
&= \frac{1}{\left(\frac{N+1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\left(\frac{N+1}{2}\right)!}{1} && (5) \\
&= \frac{1}{2}
\end{aligned}$$

where we used the given known integral rule to obtain Equation 4, and Equation 5 results from

$$\begin{aligned}
N - \frac{N-1}{2} &= \frac{2N}{2} - \frac{N-1}{2} \\
&= \frac{N + N - (N-1)}{2} \\
&= \frac{N+1}{2} \, .
\end{aligned}$$

Therefore, the exectation of the of $\hat{X}(N)$ is $\frac{1}{2}$, i.e., the true mean of X.

(d) *Find the Var($\hat{X}(N)$) analytically. Which distribution has the better variance $\bar{X}(N)$ or $\hat{X}(N)$?*

We compute the expectation of $\hat{X}(N)^2$:

$$
\begin{aligned}
\mathbb{E}[\hat{X}(N)^2] &= \int_0^1 x^2 \cdot f_{\hat{X}(N)}(x) \;\mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \int_0^1 x^2 \cdot x^{\frac{N-1}{2}} \cdot (1-x)^{\frac{N-1}{2}} \;\mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \int_0^1 x^{\frac{N-1}{2}+2} \cdot (1-x)^{\frac{N-1}{2}} \;\mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \int_0^1 x^{\frac{N+3}{2}} \cdot (1-x)^{\frac{N-1}{2}} \;\mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+3}{2}!\,\frac{N-1}{2}!}{\left(\frac{N+3}{2} + \frac{N-1}{2} + 1\right)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+3}{2}!\,\frac{N-1}{2}!}{\left(\frac{N-1}{2} + \frac{N-1}{2} + 1 + 2\right)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+3}{2}!\,\frac{N-1}{2}!}{\left(2 \cdot \frac{N-1}{2} + 3\right)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+3}{2}!\,\frac{N-1}{2}!}{(N-1+3)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+3}{2}!\,\frac{N-1}{2}!}{(N+2)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+3}{2}!\,\frac{N-1}{2}!}{(N+2) \cdot (N+1)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{1}{2} \cdot \frac{\frac{N+3}{2}!\,\frac{N-1}{2}!}{(N+2) \cdot N!} \\
&= \frac{N!}{\left(\frac{N-1}{2}\right)!\left(N - \frac{N-1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\frac{N+3}{2}!\,\frac{N-1}{2}!}{(N+2) \cdot N!} \\
&= \frac{1}{\left(\frac{N-1}{2}\right)!\left(N - \frac{N-1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\frac{N+3}{2}!\,\frac{N-1}{2}!}{(N+2)} \\
&= \frac{1}{\left(N - \frac{N-1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\frac{N+3}{2}!}{(N+2)} \\
&= \frac{1}{\frac{N+1}{2}!} \cdot \frac{1}{2} \cdot \frac{\frac{N+3}{2}!}{(N+2)} \\
&= \frac{1}{\frac{N+1}{2}!} \cdot \frac{1}{2} \cdot \frac{\frac{N+3}{2} \cdot \left(\frac{N+1}{2}\right)!}{(N+2)} \\
&= \frac{1}{2} \cdot \frac{\frac{N+3}{2}}{(N+2)} \\
&= \frac{1}{2} \cdot \frac{\frac{N+3}{2} \cdot 2}{(N+2) \cdot 2} \\
&= \frac{1}{2} \cdot \frac{N+3}{(N+2) \cdot 2} \\
&= \frac{1}{2} \cdot \frac{N+2+1}{(N+2) \cdot 2} \\
&= \frac{1}{2} \cdot \left(\frac{N+2}{(N+2) \cdot 2} + \frac{1}{(N+2) \cdot 2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2} \cdot \left(\frac{1}{2} + \frac{1}{2N+4} \right) \\
&= \frac{1}{4} + \frac{1}{4N+8}
\end{aligned}
$$

We can finally compute the variance of $\hat{X}(N)$ as follows:

$$
\begin{aligned}
\mathrm{Var}(\hat{X}(N)) &= \mathbb{E}[\hat{X}(N)^2] - \mathbb{E}[\hat{X}(N)]^2 \\
&= \mathbb{E}[\hat{X}(N)^2] - \left(\frac{1}{2}\right)^2 \\
&= \mathbb{E}[\hat{X}(N)^2] - \frac{1}{4} \\
&= \frac{1}{4} + \frac{1}{4N+8} - \frac{1}{4} \\
&= \frac{1}{4N+8}
\end{aligned}
$$

which, in the limit, converges to zero:

$$
\lim_{N\to\infty} \mathrm{Var}(\hat{X}(N)) = \lim_{N\to\infty} \frac{1}{4N+8} = 0
$$

As the convergence rate of $\mathrm{Var}(\hat{X}(N))$ is slower than $\mathrm{Var}(\bar{X}(N))$, and that both have the same (correct) expectation, we can conclude that $\bar{X}(N)$ has the better variance, i.e., it is better to estimate the mean of a uniform distribution by using the mean of the samples instead of the median.

(e) See provided code.

Problem 2

(a) *Show that if $E < 0$, then the game is losing in a long run and if $E > 0$, then the game is winning in a long run.*

Let π be the stationary distribution of the Markov chain, and

$$E = \pi_0 \cdot (B1_+ - B1_-) + (\pi_1 + \pi_2) \cdot (B2_+ - B2_-) \,.$$

We remark that E is composed of two sums, the first sum (which includes π_0) denotes the expected return when in state 0. As π is the stationary distribution, the state 0 is reached with π_0 probability at each time step in the long run. The $(B1_+ - B1_-)$ part is positive if probability of gaining one dollar is greater than losing one dollar when playing game B while in state 0.[1]

Because the second part of the equation follows the same structure (one could multiply out $(\pi_1 + \pi_2)$ to obtain the same argument for the two remaning states of the Markov chain), we can conclude that E denotes the expected *gain* (or *return*) when playing game B in the long run.

Therefore, if the expected gain is negative, the game is losing in the long run. On the other hand, a positive expected gain would imply that the game is winning.

(b) We three states in our Markov chain: 0, 1, and 2, which denotes the number that results from the current capital modulo 3. We now define the transition matrix P as follows:

$$P = \begin{pmatrix} 0.0 & 0.095 & 0.905 \\ 0.255 & 0.0 & 0.745 \\ 0.745 & 0.255 & 0.0 \end{pmatrix}$$

We remark that each play results in a win or a loss, and therefore a probability of 0.0 to stay in the same state after playing a round, hence, the diagonal elements of P are all 0.0.

The stationary distribution of the Markov chain is denoted as

$$\pi := \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 \end{pmatrix} \,.$$

We now compute the stationary distribution from P by computing the stationary value for each π_i.

$$\begin{aligned} \pi_0 &= \pi_0 \cdot 0.0 + \pi_1 \cdot 0.255 + \pi_2 \cdot 0.745 \\ &= \pi_1 \cdot 0.255 + \pi_2 \cdot 0.745 \end{aligned} \tag{6}$$

[1] See the definition of the states in (b).

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$$
\begin{aligned}
\pi_1 &= \pi_0 \cdot 0.095 + \pi_1 \cdot 0.0 + \pi_2 \cdot 0.255 \\
\pi_1 &= \pi_0 \cdot 0.095 + \pi_2 \cdot 0.255 \\
\pi_1 &\overset{(6)}{=} (\pi_1 \cdot 0.255 + \pi_2 \cdot 0.745) \cdot 0.095 + \pi_2 \cdot 0.255 \\
\pi_1 &= \pi_1 \cdot 0.255 \cdot 0.095 + \pi_2 \cdot 0.745 \cdot 0.095 + \pi_2 \cdot 0.255 && \mid -(\pi_1 \cdot 0.255 \cdot 0.095) \\
\pi_1(1 - 0.024225) &= \pi_2 \cdot 0.745 \cdot 0.095 + \pi_2 \cdot 0.255 && \mid \div(1 - 0.024225) \\
\pi_1 &= \frac{\pi_2 \cdot (0.255 + 0.070775)}{(1 - 0.024225)} \\
\pi_1 &= \pi_2 \cdot \frac{0.325775}{0.975775} \\
\pi_1 &= \pi_2 \cdot 0.333862827 \qquad (7)
\end{aligned}
$$

Using the fact that $\pi_0 + \pi_1 + \pi_2 = 1$, we can now compute π_2:

$$
\begin{aligned}
1 &= \pi_0 + \pi_1 + \pi_2 \\
1 &\overset{(6)}{=} (\pi_1 \cdot 0.255 + \pi_2 \cdot 0.745) + \pi_1 + \pi_2 \\
1 &= \pi_1 \cdot 1.255 + \pi_2 \cdot 1.745 \\
1 &\overset{(7)}{=} (\pi_2 \cdot 0.333862827) \cdot 1.255 + \pi_2 \cdot 1.745 \\
1 &= \pi_2 \cdot 0.4189978479 + \pi_2 \cdot 1.745 \\
1 &= \pi_2 \cdot 2.1639978479 \\
\pi_2 &= \frac{1}{2.1639978479} \\
\pi_2 &= 0.4621076684
\end{aligned}
$$

from which we can infer

$$
\begin{aligned}
\pi_1 &= \pi_2 \cdot 0.333862827 \\
&= 0.4621076684 \cdot 0.333862827 \\
&= 0.1542805726
\end{aligned}
$$

and

$$
\begin{aligned}
\pi_0 &= \pi_1 \cdot 0.255 + \pi_2 \cdot 0.745 \\
&= 0.1542805726 \cdot 0.255 + 0.4621076684 \cdot 0.745 \\
&= 0.03934154601 + 0.344270213 \\
&= 0.383611759\,.
\end{aligned}
$$

Therefore, we have the stationary distribution

$$\pi = \begin{pmatrix} 0.383611759 & 0.1542805726 & 0.4621076684 \end{pmatrix}$$

with $\pi = \pi P$ and total sum of one ($0.383611759 + 0.1542805726 + 0.4621076684 = 1$).

Therefore, the expected gain when playing game B in the long run is

$$\begin{aligned}
E &= \pi_0 \cdot (B1_+ - B1_-) + (\pi_1 + \pi_2) \cdot (B2_+ - B2_-) \\
&= \pi_0 \cdot (0.095 - 0.905) + (\pi_1 + \pi_2) \cdot (0.745 - 0.255) \\
&= 0.383611759 \cdot (-0.81) + (0.1542805726 + 0.4621076684) \cdot 0.49 \\
&= 0.383611759 \cdot (-0.81) + 0.616388241 \cdot 0.49 \\
&= -0.0086952867
\end{aligned}$$