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1

(a) *Find the* $\mathbb{E}[\bar{X}(N)]$ *analytically. Does it really estimate the mean of* X*?*

We have that

$$\bar{X}(N) = \frac{1}{N}\sum_{i=1}^{N} X_i, \quad X_i \sim \text{Uniform}(0,1)\ .$$

We first compute the expectation of X_i. We have

$$\begin{aligned}
\mathbb{E}\left[X_i\right] &= \int_{-\infty}^{\infty} x \cdot p(X = x)\ \ \mathrm{d}x \\
&= \int_0^1 x \cdot p(X = x)\ \mathrm{d}x \\
&= \int_0^1 x \cdot \frac{1}{1-0}\ \mathrm{d}x \\
&= \int_0^1 x\ \mathrm{d}x \\
&= \left[\frac{1}{2}x^2\right]_0^1 \\
&= \frac{1}{2}(1^2 - 0^2) \\
&= \frac{1}{2} \qquad (1)
\end{aligned}$$

Therefore, we can compute the expectation of $\bar{X}(N)$ as follows:

$$\begin{aligned}
\mathbb{E}[\bar{X}(N)] &= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[X_i\right] \\
&\overset{(1)}{=} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{2} \\
&= \frac{1}{n} \cdot n \cdot \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

Therefore, the expected value of $\mathbb{E}[\bar{X}(N)]$ estimates the true mean of X.

(b) *Find the* $Var(\bar{X}(N))$ *analytically. Hint: find the variance of a single* X_i *first.*

Let us first compute the exectation of $\mathbb{E}[X_i^2]$:

$$\begin{aligned}
\mathbb{E}[X_i^2] &= \int_0^1 x^2 \cdot p(X = x) \;\; \mathrm{d}x \\
&= \int_0^1 x^2 \cdot \frac{1}{1-0} \;\; \mathrm{d}x \\
&= \int_0^1 x^2 \;\; \mathrm{d}x \\
&= \left[\frac{1}{3} \cdot x^3\right]_0^1 \\
&= \frac{1}{3}(1-0) \\
&= \frac{1}{3}
\end{aligned} \tag{2}$$

We can now compute the variance of a X_i as follows:

$$\begin{aligned}
\mathrm{Var}(X_i) &= \mathbb{E}\left[(X_i - \mathbb{E}[X_i])^2\right] \\
&= \mathbb{E}\left[(X_i - \frac{1}{2})^2\right] \\
&= \mathbb{E}\left[X_i^2 + \frac{1^2}{2^2} - 2 \cdot X_i \cdot \frac{1}{2}\right] \\
&= \mathbb{E}\left[X_i^2 + \frac{1^2}{2^2} - X_i\right] \\
&= \mathbb{E}\left[X_i^2\right] + \mathbb{E}\left[\frac{1}{4}\right] - \mathbb{E}\left[X_i\right] \\
&= \mathbb{E}\left[X_i^2\right] + \frac{1}{4} - \frac{1}{2} \\
&\overset{(2)}{=} \frac{1}{3} + \frac{1}{4} - \frac{1}{2} \\
&= \frac{4}{12} + \frac{3}{12} - \frac{6}{12} \\
&= \frac{1}{12}\;.
\end{aligned}$$

We can now compute the variance of $\bar{X}(N)$ as follows:

$$\begin{aligned}
\operatorname{Var}(\bar{X}(N)) &= \operatorname{Var}\left(\frac{1}{N}\sum_{i=1}^{N} X_i\right) \\
&= \frac{1}{N^2}\cdot \operatorname{Var}\left(\sum_{i=1}^{N} X_i\right) \\
&= \frac{1}{N^2}\cdot \sum_{i=1}^{N} \operatorname{Var}(X_i) \qquad (3) \\
&= \frac{1}{N^2}\cdot \sum_{i=1}^{N} \frac{1}{12} \\
&= \frac{1}{N^2}\cdot N \cdot \frac{1}{12} \\
&= \frac{1}{N}\cdot \frac{1}{12}
\end{aligned}$$

where Equation 3 holds as each drawing X_i is independent of each other drawing X_j ($\Pr(X_i \mid X_j) = \Pr(X_i)$), and we can therefore use the property of the variance for i.i.d. variables $\operatorname{Var}(X+Y) = \operatorname{Var}(X) + \operatorname{Var}(Y)$.

Therefore, in the limit, we have

$$\lim_{n\to\infty} \operatorname{Var}(\bar{(}X)(N)) = \lim_{n\to\infty} \frac{1}{N}\cdot\frac{1}{12} = 0$$

with a linear convergence rate.

(c) The probability density function of $\hat{X}(N)$ is given as

$$f_{\hat{X}(N)}(x) = \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot x^{\frac{N-1}{2}} \cdot (1-x)^{\frac{N-1}{2}} \ .$$

Let us now compute the expectation of $\hat{X}(N)$:

$$
\begin{aligned}
\mathbb{E}[\hat{X}(N)] &= \int_0^1 x \cdot f_{\hat{X}(N)}(x) \, \mathrm{d}x \\
&= \int_0^1 x \cdot \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot x^{\frac{N-1}{2}} \cdot (1-x)^{\frac{N-1}{2}} \, \mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \int_0^1 x \cdot x^{\frac{N-1}{2}} \cdot (1-x)^{\frac{N-1}{2}} \, \mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \int_0^1 x^{\frac{N-1}{2}+1} \cdot (1-x)^{\frac{N-1}{2}} \, \mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \int_0^1 x^{\frac{N+1}{2}} \cdot (1-x)^{\frac{N-1}{2}} \, \mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+1}{2}!\,\frac{N-1}{2}!}{\left(\frac{N+1}{2} + \frac{N-1}{2} + 1\right)!} \qquad (4) \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+1}{2}!\,\frac{N-1}{2}!}{\left(\frac{N-1}{2} + \frac{N-1}{2} + 1 + 1\right)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+1}{2}!\,\frac{N-1}{2}!}{\left(2 \cdot \frac{N-1}{2} + 2\right)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+1}{2}!\,\frac{N-1}{2}!}{(N-1+2)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+1}{2}!\,\frac{N-1}{2}!}{(N+1)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{1}{2} \cdot \frac{\left(\frac{N+1}{2}\right)!\,\frac{N-1}{2}!}{N!} \\
&= \frac{N!}{\left(\frac{N-1}{2}\right)!\left(N - \frac{N-1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\left(\frac{N+1}{2}\right)!\,\frac{N-1}{2}!}{N!} \\
&= \frac{1}{\left(\frac{N-1}{2}\right)!\left(N - \frac{N-1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\left(\frac{N+1}{2}\right)!\,\frac{N-1}{2}!}{1} \\
&= \frac{1}{\left(N - \frac{N-1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\left(\frac{N+1}{2}\right)!}{1} \\
&= \frac{1}{\left(\frac{N+1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\left(\frac{N+1}{2}\right)!}{1} \qquad (5) \\
&= \frac{1}{2}
\end{aligned}
$$

where we used the given known integral rule to obtain Equation 4, and Equation 5 results from

$$
\begin{aligned}
N - \frac{N-1}{2} &= \frac{2N}{2} - \frac{N-1}{2} \\
&= \frac{N + N - (N-1)}{2} \\
&= \frac{N+1}{2} \; .
\end{aligned}
$$

Therefore, the exectation of the of $\hat{X}(N)$ is $\frac{1}{2}$, i.e., the true mean of X.

(d) *Find the Var*$(\hat{X}(N))$ *analytically. Which distribution has the better variance* $\bar{X}(N)$ *or* $\hat{X}(N)$*?*

We compute the expectation of $\hat{X}(N)^2$:

$$
\begin{aligned}
\mathbb{E}[\hat{X}(N)^2] &= \int_0^1 x^2 \cdot f_{\hat{X}(N)}(x) \, \mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \int_0^1 x^2 \cdot x^{\frac{N-1}{2}} \cdot (1-x)^{\frac{N-1}{2}} \, \mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \int_0^1 x^{\frac{N-1}{2}+2} \cdot (1-x)^{\frac{N-1}{2}} \, \mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \int_0^1 x^{\frac{N+3}{2}} \cdot (1-x)^{\frac{N-1}{2}} \, \mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+3}{2}!\,\frac{N-1}{2}!}{\left(\frac{N+3}{2}+\frac{N-1}{2}+1\right)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+3}{2}!\,\frac{N-1}{2}!}{\left(\frac{N-1}{2}+\frac{N-1}{2}+1+2\right)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+3}{2}!\,\frac{N-1}{2}!}{\left(2\cdot\frac{N-1}{2}+3\right)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+3}{2}!\,\frac{N-1}{2}!}{(N-1+3)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+3}{2}!\,\frac{N-1}{2}!}{(N+2)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+3}{2}!\,\frac{N-1}{2}!}{(N+2)\cdot(N+1)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{1}{2} \cdot \frac{\frac{N+3}{2}!\,\frac{N-1}{2}!}{(N+2)\cdot N!} \\
&= \frac{N!}{\left(\frac{N-1}{2}\right)!\left(N-\frac{N-1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\frac{N+3}{2}!\,\frac{N-1}{2}!}{(N+2)\cdot N!} \\
&= \frac{1}{\left(\frac{N-1}{2}\right)!\left(N-\frac{N-1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\frac{N+3}{2}!\,\frac{N-1}{2}!}{(N+2)} \\
&= \frac{1}{\left(N-\frac{N-1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\frac{N+3}{2}!}{(N+2)} \\
&= \frac{1}{\frac{N+1}{2}!} \cdot \frac{1}{2} \cdot \frac{\frac{N+3}{2}!}{(N+2)} \\
&= \frac{1}{\frac{N+1}{2}!} \cdot \frac{1}{2} \cdot \frac{\frac{N+3}{2}\cdot\left(\frac{N+1}{2}\right)!}{(N+2)} \\
&= \frac{1}{2} \cdot \frac{\frac{N+3}{2}}{(N+2)} \\
&= \frac{1}{2} \cdot \frac{\frac{N+3}{2}\cdot 2}{(N+2)\cdot 2} \\
&= \frac{1}{2} \cdot \frac{N+3}{(N+2)\cdot 2} \\
&= \frac{1}{2} \cdot \frac{N+2+1}{(N+2)\cdot 2} \\
&= \frac{1}{2} \cdot \left(\frac{N+2}{(N+2)\cdot 2} + \frac{1}{(N+2)\cdot 2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2} \cdot \left(\frac{1}{2} + \frac{1}{2N+4} \right) \\
&= \frac{1}{4} + \frac{1}{4N+8}
\end{aligned}
$$

We can finally compute the variance of $\hat{X}(N)$ as follows:

$$
\begin{aligned}
\mathrm{Var}(\hat{X}(N)) &= \mathbb{E}[\hat{X}(N)^2] - \mathbb{E}[\hat{X}(N)]^2 \\
&= \mathbb{E}[\hat{X}(N)^2] - \left(\frac{1}{2}\right)^2 \\
&= \mathbb{E}[\hat{X}(N)^2] - \frac{1}{4} \\
&= \frac{1}{4} + \frac{1}{4N+8} - \frac{1}{4} \\
&= \frac{1}{4N+8}
\end{aligned}
$$

which, in the limit, converges to zero:

$$
\lim_{N\to\infty} \mathrm{Var}(\hat{X}(N)) = \lim_{N\to\infty} \frac{1}{4N+8} = 0
$$

As the convergence rate of $\mathrm{Var}(\hat{X}(N))$ is faster than $\mathrm{Var}(\bar{X}(N))$, and that both have the same (correct) expectation, we can conclude that $\hat{X}(N)$ has the better variance.