Problem 1

(a) *Find the $\mathbb{E}[\bar{X}(N)]$ analytically. Does it really estimate the mean of X?*

We have that

$$\bar{X}(N) = \frac{1}{N} \sum_{i=1}^{N} X_i, \quad X_i \sim \text{Uniform}(0, 1) \ .$$

We first compute the expectation of X_i. We have

$$
\begin{aligned}
\mathbb{E}\left[X_i\right] &= \int_{-\infty}^{\infty} x \cdot p(X = x) \ \mathrm{d}x \\
&= \int_{0}^{1} x \cdot p(X = x) \ \mathrm{d}x \\
&= \int_{0}^{1} x \cdot \frac{1}{1-0} \ \mathrm{d}x \\
&= \int_{0}^{1} x \ \mathrm{d}x \\
&= \left[\frac{1}{2}x^2\right]_{0}^{1} \\
&= \frac{1}{2}(1^2 - 0^2) \\
&= \frac{1}{2}
\end{aligned}
\tag{1}
$$

Therefore, we can compute the expectation of $\bar{X}(N)$ as follows:

$$
\begin{aligned}
\mathbb{E}[\bar{X}(N)] &= \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} X_i\right] \\
&= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[X_i\right] \\
&\stackrel{(1)}{=} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \\
&= \frac{1}{n} \cdot n \cdot \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}
$$

Therefore, the expected value of $\mathbb{E}[\bar{X}(N)]$ estimates the true mean of X.

(b) *Find the $Var(\bar{X}(N))$ analytically. Hint: find the variance of a single X_i first.*

Let us first compute the exectation of $\mathbb{E}[X_i^2]$:

$$\mathbb{E}[X_i^2] = \int_{0}^{1} x^2 \cdot p(X = x) \ \mathrm{d}x$$

$$= \int_0^1 x^2 \cdot \frac{1}{1-0} \ \mathrm{d}x$$

$$= \int_0^1 x^2 \ \mathrm{d}x$$

$$= \left[\frac{1}{3} \cdot x^3 \right]_0^1$$

$$= \frac{1}{3}(1-0)$$

$$= \frac{1}{3} \tag{2}$$

We can now compute the variance of a X_i as follows:

$$\mathrm{Var}(X_i) = \mathbb{E}\left[(X_i - \mathbb{E}[X_i])^2\right]$$

$$= \mathbb{E}\left[(X_i - \frac{1}{2})^2\right]$$

$$= \mathbb{E}\left[X_i^2 + \frac{1^2}{2^2} - 2 \cdot X_i \cdot \frac{1}{2}\right]$$

$$= \mathbb{E}\left[X_i^2 + \frac{1^2}{2^2} - X_i\right]$$

$$= \mathbb{E}\left[X_i^2\right] + \mathbb{E}\left[\frac{1}{4}\right] - \mathbb{E}\left[X_i\right]$$

$$\overset{(2)}{=} \frac{1}{3} + \frac{1}{4} - \frac{1}{2}$$

$$= \frac{4}{12} + \frac{3}{12} - \frac{6}{12}$$

$$= \frac{1}{12} \ .$$

We can now compute the variance of $\bar{X}(N)$ as follows:

$$\mathrm{Var}(\bar{X}(N)) = \mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^N X_i\right)$$

$$= \frac{1}{N^2} \cdot \mathrm{Var}\left(\sum_{i=1}^N X_i\right)$$

$$= \frac{1}{N^2} \cdot \sum_{i=1}^N \mathrm{Var}(X_i) \tag{3}$$

$$= \frac{1}{N^2} \cdot \sum_{i=1}^N \frac{1}{12}$$

$$= \frac{1}{N^2} \cdot N \cdot \frac{1}{12}$$

$$= \frac{1}{N} \cdot \frac{1}{12}$$

where Equation 3 holds as each drawing X_i is independent of each other drawing X_j (i.e., for each $i \neq j$, $\mathrm{Pr}(X_i \mid X_j) = \mathrm{Pr}(X_i)$), therefore we can use the property of the variance for i.i.d. variables X, Y with

$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$.

Therefore, in the limit, we have

$$\lim_{n \to \infty} \text{Var}(\bar{X}(N)) = \lim_{n \to \infty} \frac{1}{N} \cdot \frac{1}{12} = 0$$

with a linear convergence rate.

(c) The probability density function of $\hat{X}(N)$ is given as

$$f_{\hat{X}(N)}(x) = \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot x^{\frac{N-1}{2}} \cdot (1-x)^{\frac{N-1}{2}}.$$

Let us now compute the expectation of $\hat{X}(N)$:

$$
\begin{aligned}
\mathbb{E}[\hat{X}(N)] &= \int_0^1 x \cdot f_{\hat{X}(N)}(x) \, \mathrm{d}x \\
&= \int_0^1 x \cdot \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot x^{\frac{N-1}{2}} \cdot (1-x)^{\frac{N-1}{2}} \, \mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \int_0^1 x \cdot x^{\frac{N-1}{2}} \cdot (1-x)^{\frac{N-1}{2}} \, \mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \int_0^1 x^{\frac{N-1}{2}+1} \cdot (1-x)^{\frac{N-1}{2}} \, \mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \int_0^1 x^{\frac{N+1}{2}} \cdot (1-x)^{\frac{N-1}{2}} \, \mathrm{d}x \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+1}{2}! \frac{N-1}{2}!}{\left(\frac{N+1}{2} + \frac{N-1}{2} + 1\right)!} \qquad (4) \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+1}{2}! \frac{N-1}{2}!}{\left(\frac{N-1}{2} + \frac{N-1}{2} + 1 + 1\right)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+1}{2}! \frac{N-1}{2}!}{\left(2 \cdot \frac{N-1}{2} + 2\right)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+1}{2}! \frac{N-1}{2}!}{(N - 1 + 2)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+1}{2}! \frac{N-1}{2}!}{(N + 1)!} \\
&= \binom{N}{\frac{N-1}{2}} \cdot \frac{1}{2} \cdot \frac{\left(\frac{N+1}{2}\right)! \frac{N-1}{2}!}{N!} \\
&= \frac{N!}{\left(\frac{N-1}{2}\right)! \left(N - \frac{N-1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\left(\frac{N+1}{2}\right)! \frac{N-1}{2}!}{N!} \\
&= \frac{1}{\left(\frac{N-1}{2}\right)! \left(N - \frac{N-1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\left(\frac{N+1}{2}\right)! \frac{N-1}{2}!}{1} \\
&= \frac{1}{\left(N - \frac{N-1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\left(\frac{N+1}{2}\right)!}{1}
\end{aligned}
$$

$$
= \frac{1}{\left(\frac{N+1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\left(\frac{N+1}{2}\right)!}{1} \tag{5}
$$

$$
= \frac{1}{2}
$$

where we used the given known integral rule to obtain Equation 4. Equation 5 results from the following:

$$
N - \frac{N-1}{2} = \frac{2N}{2} - \frac{N-1}{2}
$$
$$
= \frac{N + N - (N-1)}{2}
$$
$$
= \frac{N+1}{2} \ .
$$

Therefore, the exectation of the of $\hat{X}(N)$ is $\frac{1}{2}$, i.e., the true mean of X.

(d) *Find the $Var(\hat{X}(N))$ analytically. Which distribution has the better variance $\bar{X}(N)$ or $\hat{X}(N)$?*

We compute the expectation of $\hat{X}(N)^2$:

$$
\mathbb{E}[\hat{X}(N)^2] = \int_0^1 x^2 \cdot f_{\hat{X}(N)}(x) \, dx
$$
$$
= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \int_0^1 x^2 \cdot x^{\frac{N-1}{2}} \cdot (1-x)^{\frac{N-1}{2}} \, dx
$$
$$
= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \int_0^1 x^{\frac{N-1}{2}+2} \cdot (1-x)^{\frac{N-1}{2}} \, dx
$$
$$
= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \int_0^1 x^{\frac{N+3}{2}} \cdot (1-x)^{\frac{N-1}{2}} \, dx
$$
$$
= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+3}{2}! \frac{N-1}{2}!}{\left(\frac{N+3}{2} + \frac{N-1}{2} + 1\right)!}
$$
$$
= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+3}{2}! \frac{N-1}{2}!}{\left(\frac{N-1}{2} + \frac{N-1}{2} + 1 + 2\right)!}
$$
$$
= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+3}{2}! \frac{N-1}{2}!}{\left(2 \cdot \frac{N-1}{2} + 3\right)!}
$$
$$
= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+3}{2}! \frac{N-1}{2}!}{(N-1+3)!}
$$
$$
= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+3}{2}! \frac{N-1}{2}!}{(N+2)!}
$$
$$
= \binom{N}{\frac{N-1}{2}} \cdot \frac{N+1}{2} \cdot \frac{\frac{N+3}{2}! \frac{N-1}{2}!}{(N+2) \cdot (N+1)!}
$$
$$
= \binom{N}{\frac{N-1}{2}} \cdot \frac{1}{2} \cdot \frac{\frac{N+3}{2}! \frac{N-1}{2}!}{(N+2) \cdot N!}
$$
$$
= \frac{N!}{\left(\frac{N-1}{2}\right)! \left(N - \frac{N-1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\frac{N+3}{2}! \frac{N-1}{2}!}{(N+2) \cdot N!}
$$
$$
= \frac{1}{\left(\frac{N-1}{2}\right)! \left(N - \frac{N-1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\frac{N+3}{2}! \frac{N-1}{2}!}{(N+2)}
$$

$$= \frac{1}{\left(N - \frac{N-1}{2}\right)!} \cdot \frac{1}{2} \cdot \frac{\frac{N+3}{2}!}{(N+2)}$$

$$= \frac{1}{\frac{N+1}{2}!} \cdot \frac{1}{2} \cdot \frac{\frac{N+3}{2}!}{(N+2)}$$

$$= \frac{1}{\frac{N+1}{2}!} \cdot \frac{1}{2} \cdot \frac{\frac{N+3}{2} \cdot \left(\frac{N+1}{2}\right)!}{(N+2)}$$

$$= \frac{1}{2} \cdot \frac{\frac{N+3}{2}}{(N+2)}$$

$$= \frac{1}{2} \cdot \frac{\frac{N+3}{2} \cdot 2}{(N+2) \cdot 2}$$

$$= \frac{1}{2} \cdot \frac{N+3}{(N+2) \cdot 2}$$

$$= \frac{1}{2} \cdot \frac{N+2+1}{(N+2) \cdot 2}$$

$$= \frac{1}{2} \cdot \left(\frac{N+2}{(N+2) \cdot 2} + \frac{1}{(N+2) \cdot 2}\right)$$

$$= \frac{1}{2} \cdot \left(\frac{1}{2} + \frac{1}{2N+4}\right)$$

$$= \frac{1}{4} + \frac{1}{4N+8}$$

We can finally compute the variance of $\hat{X}(N)$ as follows:

$$\mathrm{Var}(\hat{X}(N)) = \mathbb{E}[\hat{X}(N)^2] - \mathbb{E}[\hat{X}(N)]^2$$

$$= \mathbb{E}[\hat{X}(N)^2] - \left(\frac{1}{2}\right)^2$$

$$= \mathbb{E}[\hat{X}(N)^2] - \frac{1}{4}$$

$$= \frac{1}{4} + \frac{1}{4N+8} - \frac{1}{4}$$

$$= \frac{1}{4N+8}$$

which, in the limit, converges to zero:

$$\lim_{N \to \infty} \mathrm{Var}(\hat{X}(N)) = \lim_{N \to \infty} \frac{1}{4N+8} = 0$$

As the convergence rate of $\mathrm{Var}(\hat{X}(N))$ is slower than $\mathrm{Var}(\bar{X}(N))$ (because $\mathrm{Var}(\hat{X}(N)) \propto \frac{1}{4N}$ whereas $\mathrm{Var}(\bar{X}(N)) \propto \frac{1}{12 \cdot N}$), and that both have the same (correct) expectation, we can conclude that $\bar{X}(N)$ has the better variance, i.e., it is better to estimate the mean of a uniform distribution by using the mean of the samples instead of the median.

(e) See provided code.

Problem 2

(a) *Show that if $E < 0$, then the game is losing in a long run and if $E > 0$, then the game is winning in a long run.*

Let π be the stationary distribution of the Markov chain, and

$$E = \pi_0 \cdot (B1_+ - B1_-) + (\pi_1 + \pi_2) \cdot (B2_+ - B2_-) \,.$$

We remark that E is composed of two sums, the first sum (which includes π_0) denotes the expected return when in state 0. As π is the stationary distribution, the state 0 is reached with π_0 probability at each time step in the long run. The $(B1_+ - B1_-)$ part is positive if probability of gaining one dollar is greater than losing one dollar when playing game B while in state 0.[1] Therefore, this term denotes the expected *gain* when playing game B while in state 0.

Because the second part of the equation follows the same structure (one could multiply out $(\pi_1 + \pi_2)$ to obtain the same argument for the two remaning states of the Markov chain), we can conclude that E denotes the expected *gain* (or *return*) when playing game B in the long run.

Therefore, if the expected gain is negative, the game is losing in the long run. On the other hand, a positive expected gain would imply that the game is winning in the long run.

(b) *Compute the stationary distribution π and the value of E to show that the game B is losing in a long run.*

We have three states in our Markov chain: 0, 1, and 2, which denote the number that results from the current capital of the player modulo 3. We now define the transition matrix P as follows:

$$P = \begin{pmatrix} 0.0 & 0.095 & 0.905 \\ 0.255 & 0.0 & 0.745 \\ 0.745 & 0.255 & 0.0 \end{pmatrix}$$

We remark that each play results in either a win or a loss. Therefore, there is a probability of 0.0 to stay in the same state after playing a round. Hence, the diagonal elements of P are all 0.0.

The stationary distribution of the Markov chain is denoted as

$$\pi := \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 \end{pmatrix} \,.$$

We now compute the stationary distribution from P by computing the stationary value for each π_i.

$$\begin{aligned} \pi_0 &= \pi_0 \cdot 0.0 + \pi_1 \cdot 0.255 + \pi_2 \cdot 0.745 \\ &= \pi_1 \cdot 0.255 + \pi_2 \cdot 0.745 \end{aligned} \tag{6}$$

---

[1] See the definition of the states in (b).

$$\pi_1 = \pi_0 \cdot 0.095 + \pi_1 \cdot 0.0 + \pi_2 \cdot 0.255$$

$$\pi_1 = \pi_0 \cdot 0.095 + \pi_2 \cdot 0.255$$

$$\pi_1 \stackrel{(6)}{=} (\pi_1 \cdot 0.255 + \pi_2 \cdot 0.745) \cdot 0.095 + \pi_2 \cdot 0.255$$

$$\pi_1 = \pi_1 \cdot 0.255 \cdot 0.095 + \pi_2 \cdot 0.745 \cdot 0.095 + \pi_2 \cdot 0.255 \qquad | - (\pi_1 \cdot 0.255 \cdot 0.095)$$

$$\pi_1(1 - 0.024225) = \pi_2 \cdot 0.745 \cdot 0.095 + \pi_2 \cdot 0.255 \qquad | \div (1 - 0.024225)$$

$$\pi_1 = \frac{\pi_2 \cdot (0.255 + 0.070775)}{(1 - 0.024225)}$$

$$\pi_1 = \pi_2 \cdot \frac{0.325775}{0.975775}$$

$$\pi_1 = \pi_2 \cdot 0.333862827 \tag{7}$$

Using the fact that $\pi_0 + \pi_1 + \pi_2 = 1$, we can now compute π_2:

$$1 = \pi_0 + \pi_1 + \pi_2$$

$$1 \stackrel{(6)}{=} (\pi_1 \cdot 0.255 + \pi_2 \cdot 0.745) + \pi_1 + \pi_2$$

$$1 = \pi_1 \cdot 1.255 + \pi_2 \cdot 1.745$$

$$1 \stackrel{(7)}{=} (\pi_2 \cdot 0.333862827) \cdot 1.255 + \pi_2 \cdot 1.745$$

$$1 = \pi_2 \cdot 0.4189978479 + \pi_2 \cdot 1.745$$

$$1 = \pi_2 \cdot 2.1639978479$$

$$\pi_2 = \frac{1}{2.1639978479}$$

$$\pi_2 = 0.4621076684$$

from which we can infer π_1 from Equation 7:

$$\pi_1 = \pi_2 \cdot 0.333862827$$

$$= 0.4621076684 \cdot 0.333862827$$

$$= 0.1542805726$$

and π_0 from Equation 6:

$$\pi_0 = \pi_1 \cdot 0.255 + \pi_2 \cdot 0.745$$

$$= 0.1542805726 \cdot 0.255 + 0.4621076684 \cdot 0.745$$

$$= 0.03934154601 + 0.344270213$$

$$= 0.383611759 \; .$$

Therefore, we have the stationary distribution

$$\pi = \begin{pmatrix} 0.383611759 & 0.1542805726 & 0.4621076684 \end{pmatrix}$$

with $\pi = \pi P$ and total sum of one $(0.383611759 + 0.1542805726 + 0.4621076684 = 1)$.

Therefore, the expected gain when playing game B in the long run is

$$
\begin{aligned}
E &= \pi_0 \cdot (B1_+ - B1_-) + (\pi_1 + \pi_2) \cdot (B2_+ - B2_-) \\
&= \pi_0 \cdot (0.095 - 0.905) + (\pi_1 + \pi_2) \cdot (0.745 - 0.255) \\
&= 0.383611759 \cdot (-0.81) + (0.1542805726 + 0.4621076684) \cdot 0.49 \\
&= 0.383611759 \cdot (-0.81) + 0.616388241 \cdot 0.49 \\
&= -0.0086952867 \, ,
\end{aligned}
$$

which indicates that the game is losing in the long run.

(c) *Use Monte Carlo simulation to numerically validate that the game B is losing in a long run.*

See provided code.

(d) *Construct the Markov chain for the random game. Compute its stationary distribution to see that it is winning in a long run.*

We first observe that we can keep the same three states of the Markov chain as introduced in (b). However, we need to update the transition matrix P. We now construct P to model the transition between the states given that the player *plays the random game with equal probabilities every time*, i.e., plays game A with probability 0.5 and game B with probability 0.5. Let us denote the transition matrix of game B as introduced in (b) as P_B. We now construct the transition matrix P_A for game A as follows:

$$
P_A = \begin{pmatrix}
0.0 & 0.495 & 0.505 \\
0.505 & 0.0 & 0.495 \\
0.495 & 0.505 & 0.0
\end{pmatrix}
$$

where in each state, the probability of winning one dollar is 0.495 and the probability of losing one dollar is 0.505. We can now construct the transition matrix for the combined Markov chain, i.e., when playing both games with equal probabilities, as follows:

$$
\begin{aligned}
P &= 0.5 \cdot P_A + 0.5 \cdot P_B \\
&= \begin{pmatrix}
0.0 & 0.295 & 0.705 \\
0.38 & 0.0 & 0.62 \\
0.62 & 0.38 & 0.0
\end{pmatrix} .
\end{aligned}
$$

We can now compute the stationary distribution of the new Markov chain as done in (b). In the following

derivation, we may skip some steps for brevity, as the exact same derivation process as in (b) is followed.

We start by computing π_0 as

$$\pi_0 = \pi_1 \cdot 0.38 + \pi_2 \cdot 0.62 \tag{8}$$

and

$$\pi_1 = \pi_0 \cdot 0.295 + \pi_2 \cdot 0.38$$
$$\pi_1 = (\pi_1 \cdot 0.38 + \pi_2 \cdot 0.62) \cdot 0.295 + \pi_2 \cdot 0.38$$
$$\pi_1 = \pi_1 \cdot 0.1121 + \pi_2 \cdot 0.1829 + \pi_2 \cdot 0.38$$
$$\pi_1 = \pi_2 \cdot 0.6339677892 \ .$$

Thus, we can compute π_2:

$$1 = \pi_0 + \pi_1 + \pi_2$$
$$1 = (\pi_1 \cdot 0.38 + \pi_2 \cdot 0.62) + \pi_1 + \pi_2$$
$$1 = \pi_1 \cdot 1.38 + \pi_2 \cdot 1.62$$
$$1 = \pi_2 \cdot 0.6339677892 \cdot 1.38 + \pi_2 \cdot 1.62$$
$$1 = \pi_2 \cdot 2.4948755491$$
$$\pi_2 = \frac{1}{2.4948755491}$$
$$\pi_2 = 0.4008215962$$

and infer π_0 and π_1:

$$\pi_1 = \pi_2 \cdot 0.6339677892$$
$$= 0.2541079812$$

and

$$\pi_0 = \pi_1 \cdot 0.38 + \pi_2 \cdot 0.62$$
$$= 0.2541079812 \cdot 0.38 + 0.4008215962 \cdot 0.62$$
$$= 0.3450704225$$

Therefore, we can now compute the expected gain when playing the new game by first extracting the probabilities of winning and losing in each state. Let us denote the state $0, 1, 2$ as A, B, C to ease the

notation. We have

$$A_+ = 0.295$$
$$A_- = 0.705$$
$$B_+ = 0.62$$
$$B_- = 0.38$$
$$C_+ = 0.62$$
$$C_- = 0.38$$

which we can directly read from the transition matrix P.[2] We can finally compute the expected gain of the new Markov chain as follows:[3]

$$
\begin{aligned}
E &= \pi_0 \cdot (A_+ - A_-) + \pi_1 \cdot (B_+ - B_-) + \pi_2 \cdot (C_+ - C_-) \\
&= \pi_0 \cdot (0.295 - 0.705) + \pi_1 \cdot (0.62 - 0.38) + \pi_2 \cdot (0.62 - 0.38) \\
&= 0.3450704225 \cdot (-0.41) + 0.2541079812 \cdot 0.24 + 0.4008215962 \cdot 0.24 \\
&= 0.01570422535
\end{aligned}
\tag{9}
$$

Therefore, the expected gain when playing the new game is positive, indicating that, on average, the player will win in the long run.

(e) *Confirm the result of (d) using Monte Carlo simulation.*

See provided code.

(f) *Show analytically that the provided periodic strategy is winning in a long run.*

We wish to investigate whether playing both games in a given sequence (ABBABBABB...) is also winning.

We observe that the sequence is a repeating pattern of playing game A followed by playing game B two times. We can therefore compute the transition matrix for one of such patterns as follows:

$$P_{ABB} = P_A \cdot P_B \cdot P_B = P_A \cdot P_B^2 \ .$$

Note that the complete transition matrix of the Markov chain $P = (P_{ABB})^\infty$, which is equivalent to playing the game in a repeating pattern of ABBABBABB... in the long run. To this end, we will compute the stationary distribution of P_{ABB} which will indicate whether playing the game in this pattern is winning.

---

[2]We could also compute the probabilities of winning and losing from each of the games separately. For example, $A_+ = 0.5 \cdot A_+^{(A)} + 0.5 \cdot A_+^{(B)}$ where $A_+^{(X)}$ denotes the probability of winning one dollar when playing game X while being in state 0.

[3]We could also summarize Equation 9 by clusting π_1 and π_2 into a single term as done in the assignment sheet. However, the formulation in Equation 9 makes the derivation more transparent.

We therefore use P_A and P_B as previously introduced in (b) and (d) and compute P_B^2:

$$P_B^2 = P_B \cdot P_B$$

$$= \begin{pmatrix} 0.69845 & 0.230775 & 0.070775 \\ 0.555025 & 0.2142 & 0.230775 \\ 0.065025 & 0.070775 & 0.8642 \end{pmatrix} .$$

Finally, we can compute P_{ABB}:

$$P_{ABB} = P_A \cdot P_B^2$$

$$= \begin{pmatrix} 0.307575 & 0.14177038 & 0.55065462 \\ 0.38490463 & 0.151575 & 0.46352037 \\ 0.62602038 & 0.22240463 & 0.151575 \end{pmatrix} .$$

We now compute the stationary distribution π of P_{ABB}. We, again, reuse the same derivation process as in (b) and may therefore skip some steps for brevity.

$$\pi_0 = \pi_0 \cdot 0.307575 + \pi_1 \cdot 0.38490463 + \pi_2 \cdot 0.62602038$$

$$\pi_0 \cdot (1 - 0.307575) = \pi_1 \cdot 0.38490463 + \pi_2 \cdot 0.62602038$$

$$\pi_0 = \pi_1 \cdot \frac{0.38490463}{0.692425} + \pi_2 \cdot \frac{0.62602038}{0.692425}$$

$$\pi_0 = \pi_1 \cdot 0.5558791638 + \pi_2 \cdot 0.9040984655 \tag{10}$$

and

$$\pi_1 = \pi_0 \cdot 0.14177038 + \pi_1 \cdot 0.151575 + \pi_2 \cdot 0.22240463$$

$$\pi_1 \cdot (1 - 0.151575) = \pi_0 \cdot 0.14177038 + \pi_2 \cdot 0.22240463$$

$$\pi_1 = \pi_0 \cdot 0.1670983057 + \pi_2 \cdot 0.2621382326$$

$$\pi_1 \overset{(10)}{=} (\pi_1 \cdot 0.5558791638 + \pi_2 \cdot 0.9040984655) \cdot 0.1670983057 + \pi_2 \cdot 0.2621382326$$

$$\pi_1 = \pi_2 \cdot 0.455523525 .$$

Therefore, we have

$$1 = \pi_0 + \pi_1 + \pi_2$$

$$1 = (\pi_1 \cdot 0.5558791638 + \pi_2 \cdot 0.9040984655) + \pi_1 + \pi_2$$

$$1 = \pi_1 \cdot 1.5558791638 + \pi_2 \cdot 1.9040984655$$

$$1 = \pi_2 \cdot 0.455523525 \cdot 1.5558791638 + \pi_2 \cdot 1.9040984655$$

$$1 = \pi_2 \cdot 2.6128380267$$

$$\pi_2 = \frac{1}{2.6128380267}$$

$$\pi_2 = 0.3827255994 \ .$$

Furthermore,

$$\pi_1 = \pi_2 \cdot 0.455523525$$

$$= 0.3827255994 \cdot 0.455523525$$

$$= 0.1743405141$$

and

$$\pi_0 = \pi_1 \cdot 0.5558791638 + \pi_2 \cdot 0.9040984655$$

$$= 0.1743405141 \cdot 0.5558791638 + 0.3827255994 \cdot 0.9040984655$$

$$= 0.4429338863 \ .$$

Lastly, we compute the expected gain when playing this repeating pattern in the long run. As P now represents three rounds instead of one, we need to compute the values for $A_-, A_+, \ldots$.

From this, we can compute the expected gain for a playing a round of game A in the long run. Note that we can read $A_+, A_-, \ldots$ directly from P_A.

$$E^{(A)} = \pi_0 \cdot (A_+ - A_-) + \pi_1 \cdot (B_+ - B_-) + \pi_2 \cdot (C_+ - C_-)$$

$$= \pi_0 \cdot (0.495 - 0.505) + \pi_1 \cdot (0.495 - 0.505) + \pi_2 \cdot (0.495 - 0.505)$$

$$= (0.495 - 0.505) \cdot (\pi_0 + \pi_1 + \pi_2)$$

$$= (-0.01) \cdot 1$$

$$= -0.01 \ .$$

For computing the average reward for playing the first round of game B after game A, we compute the probability of being in each state after having played game A:

$$\pi^{(A)} = \pi \cdot P_A$$

$$= \begin{pmatrix} 0.4429338863 & 0.1743405141 & 0.3827255994 \end{pmatrix} \cdot \begin{pmatrix} 0.0 & 0.495 & 0.505 \\ 0.505 & 0.0 & 0.495 \\ 0.495 & 0.505 & 0.0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.27749113 & 0.4125287 & 0.30998017 \end{pmatrix} \ .$$

The expected gain is

$$E^{(AB)} = \pi_0^{(A)} \cdot (A_+ - A_-) + \pi_1^{(A)} \cdot (B_+ - B_-) + \pi_2^{(A)} \cdot (C_+ - C_-)$$
$$= \pi_0^{(A)} \cdot (0.095 - 0.905) + \pi_1^{(A)} \cdot (0.745 - 0.255) + \pi_2^{(A)} \cdot (0.745 - 0.255)$$
$$= 0.27749113 \cdot (-0.81) + 0.4125287 \cdot 0.49 + 0.30998017 \cdot 0.49$$
$$= 0.129261531 \ .$$

Similarly, we can compute the expected gain for playing a round of game B after playing a round of game A.

$$\pi^{(AB)} = \pi^{(A)} \cdot P_B$$
$$= \begin{pmatrix} 0.33613004 & 0.1054066 & 0.55846336 \end{pmatrix}$$

and

$$E^{(ABB)} = \pi_0^{(AB)} \cdot (A_+ - A_-) + \pi_1^{(AB)} \cdot (B_+ - B_-) + \pi_2^{(AB)} \cdot (C_+ - C_-)$$
$$= 0.33613004 \cdot (-0.81) + 0.1054066 \cdot 0.49 + 0.55846336 \cdot 0.49$$
$$= 0.053030948 \ .$$

To total expected reward for playing all three rounds can be computed as follows:

$$E = E^{(A)} + E^{(AB)} + E^{(ABB)}$$
$$= (-0.01) + 0.129261531 + 0.053030948$$
$$= 0.172292479 \ .$$

this shows that playing all three rounds in this specific order yields a positive expected reward, and therefore is winning in the long run.

(Optional)
We can verify that $\pi^{(AB)}$ is valid by computing the probability of each state after the second round of game B:

$$\pi^{(ABB)} = \pi^{(AB)} \cdot P_B$$
$$= \begin{pmatrix} 0.44293388 & 0.17434051 & 0.38272561 \end{pmatrix} \ .$$

We remark that $\pi^{(ABB)} = \pi$, which is expected, as we just computed $\pi \cdot P$ step by step.

Problem 4

(a) *Use any algorithm of your choice to estimate the t_{start} and t_{end}. Plot the data and visually verify that your estimates are correct.*

I tested two different algorithms, one based on the shift of the phase of the Fourier transform, and one based on Baysian denoising, which is a popular general denoising algorithm.

The provided code contains both the details of the implementation and the results.

(b) *Find the MLE estimates for λ_{bg} and λ_{burst} analytically or numerically.*

We got given the following formula for the likelihood of observing counts with selected model parameters as

$$
\begin{aligned}
L(\lambda_{bg}, \lambda_{burst}) &= \prod_i P(c_i \mid \lambda_{bg}, \lambda_{burst}) \\
&= \prod_{i:t_i<t_{start} \text{ or } t_i>t_{end}} \frac{\exp(-\lambda_{bg}) \cdot \lambda_{bg}^{c_i}}{c_i!} \cdot \prod_{i:t_{start}\leq t_i\leq t_{end}} \frac{\exp(-\lambda_{burst}) \cdot \lambda_{burst}^{c_i}}{c_i!}
\end{aligned} \tag{11}
$$

where c_i is the number of counts in bin i starting at time t_i.

We now derive the maximum likelihood estimate for the model parameters λ_{bg} and λ_{burst}.

$$
\begin{aligned}
\log L(\lambda_{bg}, \lambda_{burst}) &= \sum_{i:t_i<t_{start} \text{ or } t_i>t_{end}} \log\left(\frac{\exp(-\lambda_{bg}) \cdot \lambda_{bg}^{c_i}}{c_i!}\right) + \sum_{i:t_{start}\leq t_i\leq t_{end}} \log\left(\frac{\exp(-\lambda_{burst}) \cdot \lambda_{burst}^{c_i}}{c_i!}\right) \\
&= \sum_{i:t_i<t_{start} \text{ or } t_i>t_{end}} \log\left(\exp(-\lambda_{bg}) \cdot \lambda_{bg}^{c_i}\right) - \log(c_i!) + \sum_{i:t_{start}\leq t_i\leq t_{end}} \log\left(\exp(-\lambda_{burst}) \cdot \lambda_{burst}^{c_i}\right) - \log(c_i!) \\
&= \sum_{i:t_i<t_{start} \text{ or } t_i>t_{end}} -\lambda_{bg} + c_i \cdot \log(\lambda_{bg}) - \log(c_i!) + \sum_{i:t_{start}\leq t_i\leq t_{end}} -\lambda_{burst} + c_i \cdot \log(\lambda_{burst}) - \log(c_i!)
\end{aligned}
$$

which we can now use to compute the gradients w.r.t. to λ_{bg} and λ_{burst}:

$$
\frac{\partial \log L(\lambda_{bg}, \lambda_{burst})}{\partial \lambda_{bg}} = \sum_{i:t_i<t_{start} \text{ or } t_i>t_{end}} -1 + c_i \cdot \frac{1}{\lambda_{bg}}
$$

$$
\frac{\partial \log L(\lambda_{bg}, \lambda_{burst})}{\partial \lambda_{burst}} = \sum_{i:t_{start}\leq t_i\leq t_{end}} -1 + c_i \cdot \frac{1}{\lambda_{burst}}
$$

thus

$$
\nabla \log L(\lambda_{bg}, \lambda_{burst}) = \begin{pmatrix} \sum_{i:t_i<t_{start} \text{ or } t_i>t_{end}} -1 + c_i \cdot \frac{1}{\lambda_{bg}} \\ \sum_{i:t_{start}\leq t_i\leq t_{end}} -1 + c_i \cdot \frac{1}{\lambda_{burst}} \end{pmatrix}.
$$

We can derive the stationary points by setting the gradients to zero.

$$\frac{\partial \log L(\lambda_{bg}, \lambda_{burst})}{\partial \lambda_{bg}} \stackrel{!}{=} 0$$

$$\sum_{i:t_i < t_{start} \text{ or } t_i > t_{end}} -1 + c_i \cdot \frac{1}{\lambda_{bg}} = 0$$

$$\frac{1}{\lambda_{bg}} \cdot \sum_{i:t_i < t_{start} \text{ or } t_i > t_{end}} c_i = \sum_{i:t_i < t_{start} \text{ or } t_i > t_{end}} 1$$

$$\lambda_{bg} = \frac{\sum_{i:t_i < t_{start} \text{ or } t_i > t_{end}} c_i}{\sum_{i:t_i < t_{start} \text{ or } t_i > t_{end}} 1}$$

and analogously for λ_{burst}:

$$\frac{\partial \log L(\lambda_{bg}, \lambda_{burst})}{\partial \lambda_{burst}} \stackrel{!}{=} 0$$

$$\lambda_{burst} = \frac{\sum_{i:t_{start} \leq t_i \leq t_{end}} c_i}{\sum_{i:t_{start} \leq t_i \leq t_{end}} 1} .$$

Therefore, the maximal likelihood estimator for both λ_{bg} and λ_{burst} are the mean of all the counts in the respective time intervals, as the denominator is the number of counts in the respective time interval, and the numerator is the sum of all the counts in the respective time interval.

We verify that the stationary points are maxima by checking the second derivatives. We therefore compute the hessian matrix:

$$H = \begin{pmatrix} \frac{\partial^2 \log L(\lambda_{bg}, \lambda_{burst})}{\partial \lambda_{bg}^2} & \frac{\partial^2 \log L(\lambda_{bg}, \lambda_{burst})}{\partial \lambda_{bg} \cdot \lambda_{burst}} \\ \frac{\partial^2 \log L(\lambda_{bg}, \lambda_{burst})}{\partial \lambda_{burst} \cdot \lambda_{bg}} & \frac{\partial^2 \log L(\lambda_{bg}, \lambda_{burst})}{\partial \lambda_{burst}^2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial^2 \log L(\lambda_{bg}, \lambda_{burst})}{\partial \lambda_{bg}^2} & 0 \\ 0 & \frac{\partial^2 \log L(\lambda_{bg}, \lambda_{burst})}{\partial \lambda_{burst}^2} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i:t_i < t_{start} \text{ or } t_i > t_{end}} -c_i \cdot \frac{1}{\lambda_{bg}^2} & 0 \\ 0 & \sum_{i:t_{start} \leq t_i \leq t_{end}} -c_i \cdot \frac{1}{\lambda_{burst}^2} \end{pmatrix} .$$

We omit substituting the equation we previously derived for λ_{bg} and λ_{burst} into the second derivatives, as the values for the lambdas are positive, due to the the counts being positive integers, and therefore at the minimum zero, but never negative. It is easy to observe that $\frac{1}{\lambda_{bg}^2}$ will remain positive when substituting the equation we derived, and that $-c_i$ will always be negative or zero. By multiplying and summing up all the terms, we know that the second derivatives are either negative or zero, depending on the observed values. Thus, the derived equations for λ_{bg} and λ_{burst} are indeed maxima.

Finally, the computed values for λ_{bg}, λ_{burst} using the closed form solution are

$$\lambda_{bg} = 1.955719557195572$$

$$\lambda_{burst} = 3.046511627906977 .$$

Note that gradient descent and Newton's method converge to the same stationary points.

(Optional) We further compute the variance of the MLE estimates, as we will use it later for the statistical testing. While it is not required for the assignment, I believe that it remains interesting to show.

After some research [4] the variance of the MLE estimates are the inverse of the negative Hessian matrix of the log-likelihood estimation. Hence, using H, we compute the variance of the MLE estimates as follows:

$$\mathrm{Var}(\lambda_{bg}) \geq -\frac{1}{\frac{\partial^2 \log L(\lambda_{bg}, \lambda_{burst})}{\partial \lambda_{bg}{}^2}}$$

$$= -\frac{1}{-\sum_{i:t_i < t_{start} \text{ or } t_i > t_{end}} c_i \cdot \frac{1}{\lambda_{bg}^2}}$$

$$= \frac{1}{\frac{1}{\lambda_{bg}^2} \cdot \sum_{i:t_i < t_{start} \text{ or } t_i > t_{end}} c_i}$$

$$= \frac{\lambda_{bg}^2}{\sum_{i:t_i < t_{start} \text{ or } t_i > t_{end}} c_i} \ .$$

Analogously, we can use the same steps to derive the variance of the estimation of λ_{burst}:

$$\mathrm{Var}(\lambda_{burst}) \geq \frac{\lambda_{burst}^2}{\sum_{i:t_{start} \leq t_i \leq t_{end}} c_i} \ .$$

(c) *Under the null hypothesis, find the MLE estimate for λ.*

We now investigate the null hypothesis that no burst happened, i.e., $\lambda_{bg} = \lambda_{burst} = \lambda$. To this end, we can simply replace λ_{bg} and λ_{burst} with λ in Equation 11. We now have

$$L(\lambda) = \prod_{i:t_i < t_{start} \text{ or } t_i > t_{end}} \frac{\exp(-\lambda) \cdot \lambda^{c_i}}{c_i!} \cdot \prod_{i:t_{start} \leq t_i \leq t_{end}} \frac{\exp(-\lambda) \cdot \lambda^{c_i}}{c_i!}$$

$$= \prod_{i=1}^{T} \frac{\exp(-\lambda) \cdot \lambda^{c_i}}{c_i!}$$

where T denotes the total number of timesteps observed.

We now derive the maximum likelihood estimator for λ.

$$\frac{\partial \log L(\lambda)}{\partial \lambda} = \frac{\partial}{\partial \lambda} \sum_{i=1}^{T} -\lambda + c_i \cdot \log(\lambda) - \log(c_i!)$$

$$= \sum_{i=1}^{T} -1 + c_i \cdot \frac{1}{\lambda} \ .$$

---

[4]

- https://math.arizona.edu/~jwatkins/n-mle.pdf
- https://stats.stackexchange.com/questions/60596/estimate-the-variance-of-mle
- https://en.wikipedia.org/wiki/Observed_information

Setting the gradient to zero, we get

$$\sum_{i=1}^{T} -1 + c_i \cdot \frac{1}{\lambda} \overset{!}{=} 0$$

$$\sum_{i=1}^{T} c_i \cdot \frac{1}{\lambda} = \sum_{i=1}^{T} 1$$

$$\frac{1}{\lambda} \cdot \sum_{i=1}^{T} c_i = T$$

$$\lambda = \frac{\sum_{i=1}^{T} c_i}{T} \; .$$

And we can verify that it is a maximum by checking the second derivative:

$$\frac{\partial^2 \log L(\lambda)}{\partial \lambda^2} = \frac{\partial}{\partial \lambda} \sum_{i=1}^{T} -1 + c_i \cdot \frac{1}{\lambda}$$

$$= \sum_{i=1}^{T} -c_i \cdot \frac{1}{\lambda^2}$$

$$= -\frac{1}{\lambda^2} \cdot \sum_{i=1}^{T} c_i$$

and by plugging the MLE into the second derivative, we get

$$-\frac{1}{\lambda^2} \cdot \sum_{i=1}^{T} c_i = -\frac{1}{\left(\frac{\sum_{i=1}^{T} c_i}{T}\right)^2} \cdot \sum_{i=1}^{T} c_i$$

$$= -\frac{1}{\frac{\left(\sum_{i=1}^{T} c_i\right)^2}{T^2}} \cdot \sum_{i=1}^{T} c_i$$

$$= -\frac{T^2}{\left(\sum_{i=1}^{T} c_i\right)^2} \cdot \sum_{i=1}^{T} c_i$$

$$= -\frac{T^2}{\sum_{i=1}^{T} c_i}$$

$$< 0 \; .$$

Therefore, the computed MLE is a maximum.

We use python to compute the MLE for λ for the given data and obtain a value of $\lambda = 2.3075$.

(d) *Compute the value of the test statistic T for the provided dataset.*

See provided code.

(e) *Generate 1000 synthetic datasets under the null hypothesis using the MLE estimate $\hat{\lambda}$ from (c). For each dataset compute the test statistic T. Plot the histogram of the resulting test statistics. Which conclusion can you make about the presence of a burst in the original dataset?*

See provided code.

Statement on the Useage of Gen. AI

Unless stated otherwise, all code, text, and math derivations have been entirely thought and written by me with no external help.

I used ChatGPT to help find literature on the properties of integration, differentiation.

Furthermore, when researching for denoising algorithms, I used ChatGPT to get a general overview of the available general denoising algorithms. After finding the Bayesian denoising algorithm, I used Claude 4.5 Sonnet to write the code for the Bayesian denoising algorithm (based on existing code for Bayesian denoising implemented for 2D images). As the exact algorithm is not at the core of the assignment or the course, and that the results of both denoising methods are very similar, I believe that this usage is acceptable.