

## 第六讲：不定积分 > 换元法 > 倒代换

当被积函数是分式时，并且当分母次数明显高于分子次数时，我们可以考虑倒代换  
倒代换可以起到对分母降次的作用，但是倒代换后分子次数一般会增加  
那么我们怎么对分子进行降次呢？

1. 多项式的除法
2. 因式分解
3. 二项式展开

这样我们就对分子和分母都进行了降次，从而达到简化分式的目的

# 第六讲：不定积分 > 换元法 > 倒代换

$$\int \frac{dx}{x^8(1+x^2)}$$

$$\text{令 } x = \frac{1}{t}$$

$$\int \frac{dx}{x^8(1+x^2)} = \int \frac{-\frac{1}{t^2} dt}{\frac{1}{t^8} \left(1 + \frac{1}{t^2}\right)} = \int \frac{-t^8 dt}{1+t^2} = \int \left(1 - t^2 + t^4 - t^6 - \frac{1}{1+t^2}\right) dt$$

$$= t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} - \arctan t + C$$

$$= \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} - \arctan \frac{1}{x} + C$$

$$\begin{aligned} -t^8 &= 1 - t^8 - 1 = (1 - t^4)(1 + t^4) - 1 = (1 + t^2)(1 - t^2)(1 + t^4) - 1 \\ &= (1 + t^2)(1 - t^2 + t^4 - t^6) - 1 \end{aligned}$$

$$\begin{array}{r} -t^6 + t^4 - t^2 + 1 \\ t^2 + 1 \overline{) -t^8} \\ \underline{-t^8 - t^6} \phantom{+ 1} \\ t^6 \\ \underline{t^6 + t^4} \phantom{+ 1} \\ -t^4 \\ \underline{-t^4 - t^2} \phantom{+ 1} \\ t^2 \\ \underline{t^2 + 1} \\ -1 \end{array}$$

$$-t^8 = (1+t^2)(1-t^2+t^4-t^6) - 1$$

# 第六讲：不定积分 > 换元法 > 倒代换

$$\int \frac{dx}{(x^2 - 1)^n} \quad \text{令 } x^2 - 1 = \frac{1}{t} \Rightarrow x = \pm \sqrt{1 + \frac{1}{t}}$$

$$\int \frac{dx}{(x^2 - 1)^n} = \int t^n \cdot \frac{\pm 1}{2\sqrt{1 + \frac{1}{t}}} \cdot \left(-\frac{1}{t^2}\right) dt = \mp \int \frac{t^{n-2}}{2\sqrt{1 + \frac{1}{t}}} dt$$

$$\text{令 } x = \frac{1}{t}$$

$$\int \frac{dx}{(x^2 - 1)^n} = \int \frac{-\frac{1}{t^2} dt}{\left(\frac{1}{t^2} - 1\right)^n} = \int \frac{-t^{2n-2} dt}{(1 - t^2)^n} \quad \int \frac{dx}{(x-1)^n (x+1)^n} = \int \frac{-\frac{1}{t^2} dt}{\left(\frac{1}{t} - 1\right)^n \left(\frac{1}{t} + 1\right)^n} = \int \frac{-t^{2n-2} dt}{(1-t)^n (1+t)^n}$$

$$\text{令 } x+1 = \frac{1}{t}$$

$$\int \frac{dx}{(x^2 - 1)^n} = \int \frac{dx}{(x-1)^n (x+1)^n} = \int \frac{-\frac{1}{t^2} dt}{\left(\frac{1}{t} - 2\right)^n \left(\frac{1}{t}\right)^n} = \int \frac{-t^{2n-2} dt}{(1-2t)^n}$$

# 第六讲：不定积分 > 换元法 > 倒代换

$$\begin{aligned}
 \int \frac{-t^{2n-2} dt}{(1-2t)^n} &= \int \frac{-\left(\frac{1-s}{2}\right)^{2n-2} \left(-\frac{1}{2}\right) ds}{s^n} = \left(\frac{1}{2}\right)^{2n-1} \int \frac{(1-s)^{2n-2} ds}{s^n} \\
 &= \left(\frac{1}{2}\right)^{2n-1} \int \frac{\sum_{k=0}^{2n-2} (-1)^k C_{2n-2}^k s^k ds}{s^n} = \left(\frac{1}{2}\right)^{2n-1} \int \sum_{k=0}^{2n-2} (-1)^k C_{2n-2}^k s^{k-n} ds \\
 &= \left(\frac{1}{2}\right)^{2n-1} \left[ \int \sum_{\substack{k=0 \\ k \neq n-1}}^{2n-2} (-1)^k C_{2n-2}^k s^{k-n} ds + \int (-1)^{n-1} C_{2n-2}^{n-1} s^{-1} ds \right] \\
 &= \left(\frac{1}{2}\right)^{2n-1} \left[ \sum_{\substack{k=0 \\ k \neq n-1}}^{2n-2} (-1)^k C_{2n-2}^k \frac{s^{k-n+1}}{k-n+1} + (-1)^{n-1} C_{2n-2}^{n-1} \ln|s| \right] + C \\
 &= \left(\frac{1}{2}\right)^{2n-1} \left[ \sum_{\substack{k=0 \\ k \neq n-1}}^{2n-2} (-1)^k \frac{C_{2n-2}^k}{k-n+1} \left(\frac{x-1}{x+1}\right)^{k-n+1} + (-1)^{n-1} C_{2n-2}^{n-1} \ln \left| \frac{x-1}{x+1} \right| \right] + C
 \end{aligned}$$

$$\text{令 } 1-2t=s \Rightarrow t=\frac{1-s}{2}$$

变量代换简化分母

$$1-2t=s \text{ 且 } x+1=\frac{1}{t} \Rightarrow s=\frac{x-1}{x+1}$$

## 第六讲：不定积分 > 换元法 > 倒代换

有时候我们作倒代换后可以凑微分，不用对分子进行降次

## 第六讲：不定积分 > 换元法 > 倒代换

$$a \neq 0, \int \frac{dx}{x(x^n + a)} \quad \text{令 } x = \frac{1}{t}$$

$$\begin{aligned} \int \frac{dx}{x(x^n + a)} &= \int \frac{-\frac{1}{t^2} dt}{\frac{1}{t} \left( \frac{1}{t^n} + a \right)} = \int \frac{-t^{n-1} dt}{1 + at^n} = \int \frac{-\frac{1}{an} d(1 + at^n)}{1 + at^n} \\ &= -\frac{1}{an} \ln |1 + at^n| + C \\ &= -\frac{1}{an} \ln \left| \frac{x^n + a}{x^n} \right| + C \end{aligned}$$

# 第六讲：不定积分 > 换元法 > 倒代换

$$\int \frac{1}{(1+x^4)^{\frac{5}{4}} \sqrt[4]{1+x^4}} dx \quad \text{令 } x = \frac{1}{t}$$

$$\begin{aligned} \text{当 } x > 0 \text{ 时 } \int \frac{1}{(1+x^4)^{\frac{5}{4}} \sqrt[4]{1+x^4}} dx &= \int \frac{-\frac{1}{t^2}}{\left(1+\frac{1}{t^4}\right)^{\frac{5}{4}} \sqrt[4]{1+\frac{1}{t^4}}} dt = \int \frac{-t^3}{t(1+t^4)^{\frac{5}{4}} \sqrt[4]{1+\frac{1}{t^4}}} dt = \int \frac{-t^3}{(1+t^4)^{\frac{5}{4}} \sqrt[4]{t^4+1}} dt \\ &= \int \frac{-t^3}{(1+t^4)^{\frac{5}{4}}} dt = \int \frac{-\frac{1}{4} d(1+t^4)}{(1+t^4)^{\frac{5}{4}}} = (1+t^4)^{-\frac{1}{4}} + C = \frac{x}{\sqrt[4]{1+x^4}} + C \end{aligned}$$

$$\text{当 } x < 0 \text{ 时 } \int \frac{1}{(1+x^4)^{\frac{5}{4}} \sqrt[4]{1+x^4}} dx = \frac{x}{\sqrt[4]{1+x^4}} + C$$

# 第六讲：不定积分 > 换元法 > 倒代换

$$\int \frac{1}{(1+x^4)^{\sqrt[4]{1+x^4}}} dx \quad \text{令 } x = \frac{1}{t} \quad t = \frac{t}{|t|} \cdot |t|$$

$$\begin{aligned} \int \frac{1}{(1+x^4)^{\sqrt[4]{1+x^4}}} dx &= \int \frac{-t^3}{t(1+t^4)^{\sqrt[4]{1+\frac{1}{t^4}}}} dt = \int \frac{-t^3}{\frac{t}{|t|} \cdot |t|(1+t^4)^{\sqrt[4]{1+\frac{1}{t^4}}}} dt = \int \frac{|t|}{t} \cdot \frac{-t^3}{(1+t^4)^{\sqrt[4]{1+\frac{1}{t^4}}}} dt \\ &= \int \frac{|t|}{t} \cdot \frac{-\frac{1}{4}d(1+t^4)}{(1+t^4)^{\frac{5}{4}}} = \frac{|t|}{t} \cdot (1+t^4)^{-\frac{1}{4}} + C = \frac{x}{|x|} \cdot \frac{1}{\sqrt[4]{1+\frac{1}{x^4}}} + C = \frac{x}{\sqrt[4]{1+x^4}} + C \end{aligned}$$



# 第六讲：不定积分 > 换元法 > 倒代换

$$\int \frac{dx}{x \sqrt{x^2 - 1}} \quad \text{令 } x = \frac{1}{t}$$

$$\text{当 } x > 0 \text{ 时} \quad \int \frac{dx}{x \sqrt{x^2 - 1}} = \int \frac{-\frac{1}{t^2} dt}{\frac{1}{t} \sqrt{\frac{1}{t^2} - 1}} = \int \frac{-dt}{\sqrt{1 - t^2}} = -\arcsin t + C = -\arcsin \frac{1}{x} + C$$

$$\text{当 } x < 0 \text{ 时} \quad \int \frac{dx}{x \sqrt{x^2 - 1}} = \int \frac{-\frac{1}{t^2} dt}{\frac{1}{t} \sqrt{\frac{1}{t^2} - 1}} = \int \frac{dt}{\sqrt{1 - t^2}} = \arcsin t + C = \arcsin \frac{1}{x} + C$$

$$-\frac{|x|}{x} \arcsin \frac{1}{x} + C$$

## 第六讲：不定积分 > 换元法 > 倒代换

$$\int \frac{dx}{(x+1)^3 \sqrt{x^2 + 2x}} \quad \text{令 } x+1=t \quad \text{平移}$$

$$\int \frac{dx}{(x+1)^3 \sqrt{x^2 + 2x}} = \int \frac{dx}{(x+1)^3 \sqrt{(x+1)^2 - 1}} = \int \frac{dt}{t^3 \sqrt{t^2 - 1}}$$

## 第六讲：不定积分 > 换元法 > 三角代换 > 消去三角

当被积函数只含有三角函数时

我们一般通过凑微分  $-\sin x dx = d \cos x$   $\cos x dx = d \sin x$   $\sec^2 x dx = d \tan x$

把积分变换成  $\int f(\sin x) d \sin x$   $\int f(\cos x) d \cos x$   $\int f(\tan x) d \tan x$  这样的形式

把被积函数化成关于  $\sin x$  或  $\cos x$  或  $\tan x$  的函数

积分变量对应地化成  $\sin x$  或  $\cos x$  或  $\tan x$

这样我们就可以通过换元 **消去三角**

## 第六讲：不定积分 > 换元法 > 三角代换 > 消去三角

$\int R(\sin x, \cos x) dx$  被积函数是关于  $\sin x, \cos x$  的三角有理式

$R(\sin x, -\cos x) = -R(\sin x, \cos x)$ , 即  $R(\sin x, \cos x)$  是关于  $\cos x$  的奇函数, 则令  $t = \sin x$  或  $\csc x$

$$\int R(\sin x, \cos x) dx = \int f(\sin x) d\sin x = \int f(t) dt$$

$R(-\sin x, \cos x) = -R(\sin x, \cos x)$ , 即  $R(\sin x, \cos x)$  是关于  $\sin x$  的奇函数, 则令  $t = \cos x$  或  $\sec x$

$$\int R(\sin x, \cos x) dx = \int f(\cos x) d\cos x = \int f(t) dt$$

$R(-\sin x, -\cos x) = R(\sin x, \cos x)$ , 则令  $t = \tan x$  或  $\cot x$

$$\int R(\sin x, \cos x) dx = \int f(\tan x) d\tan x = \int f(t) dt$$

$\int R(\sin x, \cos x) dx$  转化成了有理函数  $f(t)$  的不定积分

# 第六讲：不定积分 > 换元法 > 三角代换 > 消去三角

$$\int \frac{dx}{\sin x \cos^4 x} \quad \frac{1}{\sin x \cos^4 x} \text{关于} \sin x \text{的奇函数}$$

故可化  $\int f(\cos x) d\cos x$

$$\int \frac{dx}{\sin x \cos^4 x} = \int \frac{\sin x dx}{\sin^2 x \cos^4 x} = \int \frac{-d\cos x}{\sin^2 x \cos^4 x} = \int \frac{-d\cos x}{(1 - \cos^2 x) \cos^4 x} = \int \frac{-dt}{(1 - t^2) t^4} \quad \text{令 } \cos x = t$$

$$\int \frac{dx}{\sin x \cos^4 x} = \int \frac{\cos x dx}{\sin x \cos^5 x} = \int \frac{d\sin x}{\sin x \cos^5 x} = \int \frac{d\sin x}{\pm \sin x (1 - \sin^2 x)^{\frac{5}{2}}} = \int \frac{dt}{\pm t (1 - t^2)^{\frac{5}{2}}} \quad \text{令 } \sin x = t$$

$$\cos x = \pm (1 - \sin^2 x)^{\frac{1}{2}}$$

$$\cos^5 x = \pm (1 - \sin^2 x)^{\frac{5}{2}}$$

# 第六讲：不定积分 > 换元法 > 三角代换 > 消去三角

## 倒代换对分母进行降次

$$\int \frac{-dt}{(1-t^2)t^4} = \int \frac{\frac{1}{s^2} ds}{\left(1 - \frac{1}{s^2}\right) \frac{1}{s^4}} = \int \frac{s^4 ds}{s^2 - 1} \quad \text{令 } t = \frac{1}{s}$$

$$= \int \left[ s^2 + 1 + \frac{1}{2} \left( \frac{1}{s-1} - \frac{1}{s+1} \right) \right] ds$$

$$= \frac{1}{3} s^3 + s + \frac{1}{2} (\ln|s-1| - \ln|s+1|) + C = \frac{1}{3} s^3 + s + \frac{1}{2} \ln \left| \frac{s-1}{s+1} \right| + C$$

$$= \frac{1}{3 \cos^3 x} + \frac{1}{\cos x} + \frac{1}{2} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| + C$$

$$\frac{s^4}{s^2 - 1} = \frac{(s^4 - 1) + 1}{s^2 - 1} = s^2 + 1 + \frac{1}{s^2 - 1} = s^2 + 1 + \frac{1}{2} \left( \frac{1}{s-1} - \frac{1}{s+1} \right)$$

$$\begin{array}{r} s^2 + 1 \\ s^2 - 1 \overline{) s^4} \\ \underline{s^4 - s^2} \phantom{00} \\ s^2 \phantom{00} \\ \underline{s^2 - 1} \phantom{00} \\ 1 \end{array}$$

## 第六讲：不定积分 > 换元法 > 三角代换 > 消去三角

直接对分式进行分解达到对分母进行降次的目的

$$\frac{-1}{(1-t^2)t^4} = \frac{(t^2-1)-t^2}{(1-t^2)t^4} = -\frac{1}{t^4} + \frac{1}{(t^2-1)t^2} = -\frac{1}{t^4} + \frac{1}{t^2-1} - \frac{1}{t^2} = -\frac{1}{t^4} + \frac{1}{2} \left( \frac{1}{t-1} - \frac{1}{t+1} \right) - \frac{1}{t^2}$$

$$\frac{-1}{(1-t^2)t^4} = \frac{at^3 + bt^2 + ct + d}{t^4} + \frac{e}{t-1} + \frac{f}{t+1}$$

待定系数法

分式分解定理

# 第六讲：不定积分 > 换元法 > 三角代换 > 消去三角

$$\int \frac{dx}{\sin^3 x + 3 \sin x} = \int \frac{\sin x dx}{\sin^4 x + 3 \sin^2 x} = \int \frac{-d \cos x}{(1 - \cos^2 x)(4 - \cos^2 x)}$$

$\frac{1}{\sin^3 x + 3 \sin x}$  关于  $\sin x$  的奇函数  
故可化  $\int f(\cos x) d \cos x$

$$= \int \frac{-dt}{(1 - t^2)(4 - t^2)} \quad \text{令 } \cos x = t$$

$$= \int \frac{1}{3} \left( \frac{1}{t^2 - 1} - \frac{1}{t^2 - 4} \right) dt$$

$$= \frac{1}{6} \ln \left| \frac{1-t}{1+t} \right| - \frac{1}{12} \ln \left| \frac{2-t}{2+t} \right| + C = \frac{1}{6} \ln \left| \frac{1-\cos x}{1+\cos x} \right| - \frac{1}{12} \ln \left| \frac{2-\cos x}{2+\cos x} \right| + C$$

$$\int \frac{dx}{x^2 - a^2} = \int \frac{1}{2a} \left( \frac{1}{x-a} - \frac{1}{x+a} \right) dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$



# 第六讲：不定积分 > 换元法 > 三角代换 > 消去三角

$$\sec^2 x = \tan^2 x + 1 \quad \sec^2 x dx = d \tan x$$

$$\int \sec^6 x dx = \int \sec^4 x \cdot \sec^2 x dx = \int \sec^4 x d \tan x = \int (1 + \tan^2 x)^2 d \tan x$$

$$= \int (1 + t^2)^2 dt \quad \text{令 } \tan x = t$$

$$= \int (1 + 2t^2 + t^4) dt$$

$$= t + \frac{2t^3}{3} + \frac{t^5}{5} + C$$

$$= \tan x + \frac{2 \tan^3 x}{3} + \frac{\tan^5 x}{5} + C$$

$$\sec^6 x = \frac{1}{\cos^6 x} = R(\sin x, \cos x)$$

$$R(\sin x, \cos x) = R(-\sin x, -\cos x)$$

$$\text{故可化 } \int f(\tan x) d \tan x$$

# 第六讲：不定积分 > 换元法 > 三角代换 > 消去三角

$$\int \frac{1}{5+8\sin\theta\cos\theta} d\theta = \int \frac{\sec^2\theta d\theta}{5\sec^2\theta+8\tan\theta}$$

$$= \int \frac{\sec^2\theta d\theta}{5\tan^2\theta+5+8\tan\theta}$$

$$= \int \frac{d\tan\theta}{5\tan^2\theta+5+8\tan\theta}$$

令  $\tan\theta = t$

$$= \int \frac{dt}{5t^2+5+8t}$$

$$= \int \frac{dt}{5\left(t+\frac{4}{5}\right)^2+\frac{9}{5}} = \int \frac{\frac{5}{9}dt}{\frac{25}{9}\left(t+\frac{4}{5}\right)^2+1} = \int \frac{\frac{1}{3}d\left(\frac{5}{3}t+\frac{4}{3}\right)}{\left(\frac{5}{3}t+\frac{4}{3}\right)^2+1} = \frac{1}{3}\arctan\left(\frac{5}{3}t+\frac{4}{3}\right)+C$$

$$= \frac{1}{3}\arctan\left(\frac{5}{3}\tan\theta+\frac{4}{3}\right)+C$$

$$\frac{1}{5+8\sin\theta\cos\theta} = R(\sin\theta, \cos\theta)$$

$$R(\sin\theta, \cos\theta) = R(-\sin\theta, -\cos\theta)$$

故可化  $\int f(\tan\theta) d\tan\theta$

# 第六讲：不定积分 > 换元法 > 三角代换 > 消去三角

$$\begin{aligned}
 ab \neq 0 \int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} &= \int \frac{\sec^2 x dx}{a^2 \tan^2 x + b^2} = \int \frac{d \tan x}{a^2 \tan^2 x + b^2} & \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} &= R(\sin x, \cos x) \\
 &= \int \frac{dt}{a^2 t^2 + b^2} & \text{令 } \tan x &= t & R(\sin x, \cos x) &= R(-\sin x, -\cos x) \\
 & & & & \text{故可化 } \int f(\tan x) d \tan x & \\
 &= \int \frac{\frac{1}{b^2} dt}{\left(\frac{a}{b}\right)^2 t^2 + 1} = \int \frac{\frac{1}{ab} d\left(\frac{a}{b} t\right)}{\left(\frac{a}{b} t\right)^2 + 1} = \frac{1}{ab} \arctan\left(\frac{a}{b} t\right) + C \\
 &= \frac{1}{ab} \arctan\left(\frac{a}{b} \tan x\right) + C
 \end{aligned}$$

$$\begin{aligned}
 ab \neq 0 \int \frac{dx}{a^2 \sin^2 x - b^2 \cos^2 x} &= \frac{1}{2ab} \ln \left| \frac{a \tan x - b}{a \tan x + b} \right| + C & ab \neq 0 \int \frac{dx}{a \sin^2 x + b} & \int \frac{dx}{a \cos^2 x + b}
 \end{aligned}$$

# 第六讲：不定积分 > 换元法 > 三角代换 > 消去三角

前面我们是通过**凑微分**的方法来消去三角，我们还可以利用**万能代换**来消去三角

$$\text{令 } u = \tan \frac{x}{2} \Rightarrow \cos x = \frac{1-u^2}{1+u^2} \quad \sin x = \frac{2u}{1+u^2} \quad \tan x = \frac{2u}{1-u^2} \quad \cot x = \frac{1-u^2}{2u} \quad dx = \frac{2}{1+u^2} du$$

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \left( \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right) / \left( \cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} \right) = \left( 1 - \tan^2 \frac{x}{2} \right) / \left( 1 + \tan^2 \frac{x}{2} \right) = \frac{1-u^2}{1+u^2}$$

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = 2 \sin \frac{x}{2} \cos \frac{x}{2} / \left( \cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} \right) = 2 \tan \frac{x}{2} / \left( 1 + \tan^2 \frac{x}{2} \right) = \frac{2u}{1+u^2}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{2u}{1-u^2} \quad \cot x = \frac{\sin x}{\cos x} = \frac{1-u^2}{2u}$$

$$du = d \tan \frac{x}{2} = \frac{1}{2} \sec^2 \frac{x}{2} dx = \frac{1}{2} \left( 1 + \tan^2 \frac{x}{2} \right) dx = \frac{1}{2} (1+u^2) dx$$

# 第六讲：不定积分 > 换元法 > 三角代换 > 消去三角

万能代换实际上是第二类换元

凑微分法是第一类换元

$$\frac{x}{2} = \arctan u \Rightarrow \tan \frac{x}{2} = \tan(\arctan u) = u \text{ 但 } \tan \frac{x}{2} = u \not\Rightarrow \frac{x}{2} = \arctan u$$

$$\text{故 } u = \tan \frac{x}{2} \not\Leftrightarrow \frac{x}{2} = \arctan u$$

$$u = \tan \frac{x}{2} \Leftrightarrow \tan(\arctan u) = \tan \frac{x}{2}$$

$\arctan u$  与  $\frac{x}{2}$  不一定相等

$\arctan u$  与  $\frac{x}{2}$  相差  $\pi$  的整数倍

$$\text{当 } x \in ((2k-1)\pi, (2k+1)\pi) \text{ 时 } \Rightarrow \frac{x}{2} \in (k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}) \Rightarrow \frac{x}{2} - k\pi \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

$$\text{此时 } \frac{x}{2} - k\pi = \arctan u \Leftrightarrow \tan\left(\frac{x}{2} - k\pi\right) = \tan(\arctan u) \Leftrightarrow \tan \frac{x}{2} = u$$

故当  $x \in ((2k-1)\pi, (2k+1)\pi)$  时，令  $\tan \frac{x}{2} = u$ ，实际上是令  $x = 2 \arctan u + 2k\pi$

## 第六讲：不定积分 > 换元法 > 三角代换 > 消去三角

$$\int \frac{1}{\cos x + \sin x + 2} dx = \int \frac{\frac{2}{1+u^2}}{\frac{1-u^2}{1+u^2} + \frac{2u}{1+u^2} + 2} du = \int \frac{2}{u^2 + 2u + 3} du \quad \text{令 } u = \tan \frac{x}{2}$$

$$= \int \frac{2}{(u+1)^2 + 2} du$$

$$= \int \frac{du}{\left(\frac{u+1}{\sqrt{2}}\right)^2 + 1}$$

$$= \int \frac{\sqrt{2} d\left(\frac{u+1}{\sqrt{2}}\right)}{\left(\frac{u+1}{\sqrt{2}}\right)^2 + 1} = \sqrt{2} \arctan \frac{u+1}{\sqrt{2}} + C = \sqrt{2} \arctan \frac{\tan \frac{x}{2} + 1}{\sqrt{2}} + C$$

# 第六讲：不定积分 > 换元法 > 三角代换 > 消去三角

$$\int_{\frac{\pi}{2}}^{2\pi} \frac{1}{\cos x + \sin x + 2} dx = \int_1^0 \frac{2}{u^2 + 2u + 3} du \quad ?? \quad \text{令 } u = \tan \frac{x}{2}$$

$$\begin{array}{lcl} x & \frac{\pi}{2} \rightarrow \pi & \pi \rightarrow 2\pi \\ u & 1 \rightarrow +\infty & -\infty \rightarrow 0 \end{array}$$

无穷间断点

$$\int_{\frac{\pi}{2}}^{2\pi} \frac{1}{\cos x + \sin x + 2} dx = \int_1^{+\infty} \frac{2}{u^2 + 2u + 3} du + \int_{-\infty}^0 \frac{2}{u^2 + 2u + 3} du$$

# 第六讲：不定积分 > 换元法 > 三角代换 > 消去三角

$$\int_{\frac{\pi}{2}}^{2\pi} \frac{1}{\cos x + \sin x + 2} dx = \int_{\frac{\pi}{2}}^{\pi} \frac{1}{\cos x + \sin x + 2} dx + \int_{\pi}^{2\pi} \frac{1}{\cos x + \sin x + 2} dx$$

$$x = 2 \arctan u$$

$$\int_{\frac{\pi}{2}}^{\pi} \frac{1}{\cos x + \sin x + 2} dx = \int_1^{+\infty} \frac{2}{u^2 + 2u + 3} du$$

$$\tan \frac{x}{2} = u$$

$$x = 2 \arctan u + 2\pi$$

$$\int_{\pi}^{2\pi} \frac{1}{\cos x + \sin x + 2} dx = \int_{-\infty}^0 \frac{2}{u^2 + 2u + 3} du$$



# 第六讲：不定积分 > 换元法 > 三角代换 > 消去三角

$$\int \frac{1}{\sin x + 2 \cos x} dx = \int \frac{\frac{2}{1+u^2}}{\frac{2u}{1+u^2} + 2 \cdot \frac{1-u^2}{1+u^2}} du \quad \text{令 } u = \tan \frac{x}{2}$$

$$= \int \frac{1}{-u^2 + u + 1} du$$

$$= \int \frac{1}{\left(\frac{1+\sqrt{5}}{2} - u\right)\left(u - \frac{1-\sqrt{5}}{2}\right)} du = \int \frac{1}{\sqrt{5}} \left( \frac{1}{\frac{1+\sqrt{5}}{2} - u} + \frac{1}{u - \frac{1-\sqrt{5}}{2}} \right) du$$

$$= \frac{1}{\sqrt{5}} \left( -\ln \left| \frac{1+\sqrt{5}}{2} - u \right| + \ln \left| u - \frac{1-\sqrt{5}}{2} \right| \right) + C = \frac{1}{\sqrt{5}} \ln \left| \frac{\tan \frac{x}{2} - \frac{1-\sqrt{5}}{2}}{\frac{1+\sqrt{5}}{2} - \tan \frac{x}{2}} \right| + C$$

# 第六讲：不定积分 > 换元法 > 三角代换 > 消去三角

$$\int \frac{1}{\sin x + 2 \cos x} dx$$

$$\sin x + 2 \cos x = \sqrt{1^2 + 2^2} \left( \frac{1}{\sqrt{1^2 + 2^2}} \sin x + \frac{2}{\sqrt{1^2 + 2^2}} \cos x \right)$$

$$\text{设 } \frac{2}{\sqrt{1^2 + 2^2}} = \sin \varphi \quad \varphi \in (0, \frac{\pi}{2}) \Rightarrow \frac{1}{\sqrt{1^2 + 2^2}} = \cos \varphi$$

$$\sin x + 2 \cos x = \sqrt{5} (\cos \varphi \sin x + \sin \varphi \cos x) = \sqrt{5} \sin (x + \varphi)$$

$$= \int \frac{1}{\sqrt{5} \sin (x + \varphi)} dx = \int \frac{\sin (x + \varphi) dx}{\sqrt{5} \sin^2 (x + \varphi)} = \frac{1}{\sqrt{5}} \int \frac{-d \cos (x + \varphi)}{1 - \cos^2 (x + \varphi)} \quad \text{令 } \cos (x + \varphi) = t$$

$$= \frac{1}{\sqrt{5}} \int \frac{-dt}{1 - t^2} = \frac{1}{2\sqrt{5}} \ln \left| \frac{t-1}{t+1} \right| + C = \frac{1}{2\sqrt{5}} \ln \left| \frac{\cos (x + \varphi) - 1}{\cos (x + \varphi) + 1} \right| + C = \frac{1}{2\sqrt{5}} \ln \left| \frac{\cos (x + \arcsin 2/\sqrt{5}) - 1}{\cos (x + \arcsin 2/\sqrt{5}) + 1} \right| + C$$

# 第六讲：不定积分 > 换元法 > 三角代换 > 产生三角

通过三角代换来产生三角目的是为了消去根号

$$\sqrt{a^2 - x^2} \Rightarrow -a \leq x \leq a \Rightarrow \text{令 } x = a \sin t \text{ 或 } a \cos t$$

$$\text{令 } x = a \sin t, t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \sqrt{a^2 - x^2} = a \cos t$$

$$\text{令 } x = a \cos t, t \in [0, \pi] \Rightarrow \sqrt{a^2 - x^2} = a \sin t$$

$$\sqrt{a^2 + x^2} \Rightarrow \text{对 } x \text{ 无限制} \Rightarrow \text{令 } x = a \tan t \text{ 或 } a \cot t$$

$$\text{令 } x = a \tan t, t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \sqrt{a^2 + x^2} = a \sec t$$

$$\text{令 } x = a \cot t, t \in (0, \pi) \Rightarrow \sqrt{a^2 + x^2} = a \csc t$$

$$\sqrt{x^2 - a^2} \Rightarrow a \leq x \text{ 或 } x \leq -a \Rightarrow \text{令 } x = a \sec t \text{ 或 } a \csc t$$

$$\text{令 } x = a \sec t, t \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right) \Rightarrow \sqrt{x^2 - a^2} = a \tan t$$

$$\text{令 } x = a \csc t, t \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right) \Rightarrow \sqrt{x^2 - a^2} = a \cot t$$

$$x = a \sin t, t \in \left[-\frac{\pi}{2}, 0\right] \cup \left[\frac{\pi}{2}, \pi\right]$$

$$\Rightarrow \sqrt{a^2 - x^2} = \pm a \cos t$$

$$\text{令 } x = a \tan t, t \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$

$$\Rightarrow \sqrt{a^2 + x^2} = \pm a \sec t$$

# 第六讲：不定积分 > 换元法 > 三角代换 > 产生三角

$$\int (3-x^2)^{\frac{3}{2}} dx$$

降次

$$|x| \leq \sqrt{3} \Rightarrow \text{令 } x = \sqrt{3} \sin t \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$= \int (3-3\sin^2 t)^{\frac{3}{2}} \cdot \sqrt{3} \cos t dt = \int 9 \cos^4 t dt$$

$$= \int 9 \cdot \frac{4 \cos 2t + 3 + \cos 4t}{8} dt = \frac{9}{4} \sin 2t + \frac{27}{8} t + \frac{9}{32} \sin 4t + C$$

$$t = \arcsin \frac{x}{\sqrt{3}}$$

$$\sin 2t = 2 \sin t \cos t = 2 \sin t \sqrt{1 - \sin^2 t} = 2 \cdot \frac{x}{\sqrt{3}} \sqrt{1 - \left(\frac{x}{\sqrt{3}}\right)^2} = \frac{2x}{3} \sqrt{3 - x^2}$$

$$\sin 4t = 2 \sin 2t \cos 2t = 2 \sin 2t (1 - 2 \sin^2 t) = 2 \cdot \frac{2x}{3} \sqrt{3 - x^2} \cdot \left(1 - 2 \left(\frac{x}{\sqrt{3}}\right)^2\right) = \frac{4}{9} (3x - 2x^3) \sqrt{3 - x^2}$$

$$\cos^4 t = (\cos^2 t)^2 = \left(\frac{1 + \cos 2t}{2}\right)^2 = \frac{1 + 2 \cos 2t + \cos^2 2t}{4}$$

$$= \frac{1 + 2 \cos 2t + \frac{1 + \cos 4t}{2}}{4} = \frac{4 \cos 2t + 3 + \cos 4t}{8}$$

# 第六讲：不定积分 > 换元法 > 三角代换 > 产生三角

$$\int \frac{1}{x^2 \sqrt{x^2 + 3}} dx \quad \text{令 } x = \sqrt{3} \tan t \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

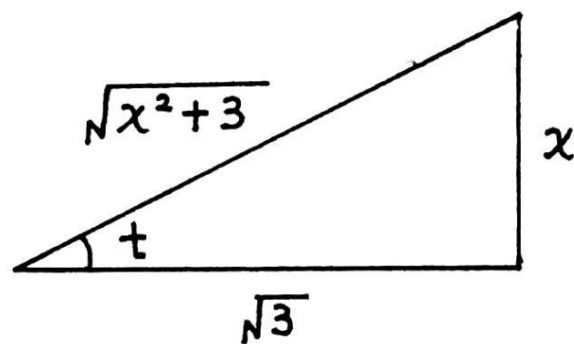
$$\text{若令 } x = \sqrt{3} \tan t \quad t \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$$

$$\sqrt{x^2 + 3} = \sqrt{3 \tan^2 t + 3} = \sqrt{3 \sec^2 t} = \pm \sqrt{3} \sec t$$

需讨论麻烦

$$= \int \frac{\sqrt{3} \sec^2 t}{3 \tan^2 t \sqrt{3 \tan^2 t + 3}} dt = \int \frac{\sec t}{3 \tan^2 t} dt$$

$$= \int \frac{\cos t dt}{3 \sin^2 t} = \int \frac{d \sin t}{3 \sin^2 t} = -\frac{1}{3 \sin t} + C = -\frac{\sqrt{x^2 + 3}}{3x} + C$$



$$|\sin t| = \left| \frac{x}{\sqrt{x^2 + 3}} \right|$$

$$x > 0 \Rightarrow t \in \left(0, \frac{\pi}{2}\right) \Rightarrow \sin t > 0 \Rightarrow \sin t = \frac{x}{\sqrt{x^2 + 3}}$$

$$x < 0 \Rightarrow t \in \left(-\frac{\pi}{2}, 0\right) \Rightarrow \sin t < 0 \Rightarrow \sin t = \frac{x}{\sqrt{x^2 + 3}}$$

# 第六讲：不定积分 > 换元法 > 三角代换 > 产生三角

$$\int \frac{x^2}{(x^2-3)^{\frac{3}{2}}} dx \quad |x| > \sqrt{3} \quad \text{令 } x = \sqrt{3} \sec t \quad t \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$$

$$= \int \frac{3 \sec^2 t \cdot \sqrt{3} \sec t \tan t dt}{(3 \tan^2 t)^{\frac{3}{2}}} = \int \frac{\sec^3 t dt}{\tan^2 t} \quad (x^2-3)^{\frac{3}{2}} = (3 \tan^2 t)^{\frac{3}{2}} = 3^{\frac{3}{2}} \tan^3 t$$

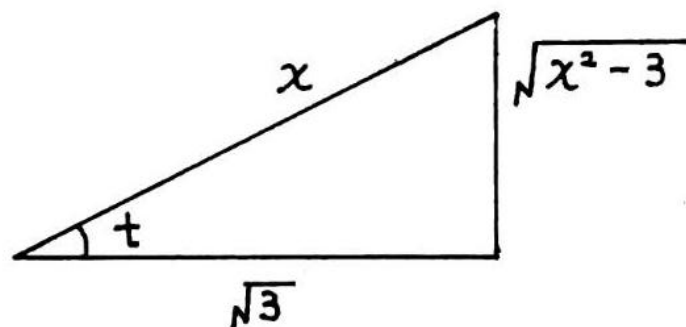
$$= \int \frac{dt}{\sin^2 t \cos t} = \int \frac{\cos t dt}{\sin^2 t \cos^2 t} = \int \frac{d \sin t}{\sin^2 t (1 - \sin^2 t)} \quad \text{令 } \sin t = s$$

$$= \int \frac{ds}{s^2 (1-s^2)} = \int \left( \frac{1}{s^2} + \frac{1}{1-s^2} \right) ds = -\frac{1}{s} + \frac{1}{2} \ln \left| \frac{s+1}{s-1} \right| + C = -\frac{1}{\sin t} + \frac{1}{2} \ln \left| \frac{\sin t + 1}{\sin t - 1} \right| + C$$

$$|\sin t| = \left| \frac{\sqrt{x^2-3}}{x} \right|$$

$$x > 0 \Rightarrow t \in (0, \frac{\pi}{2}) \Rightarrow \sin t > 0 \Rightarrow \sin t = \frac{\sqrt{x^2-3}}{x}$$

$$x < 0 \Rightarrow t \in (\pi, \frac{3\pi}{2}) \Rightarrow \sin t < 0 \Rightarrow \sin t = \frac{\sqrt{x^2-3}}{x}$$



## 第六讲：不定积分 > 换元法 > 三角代换 > 产生三角

$$\sqrt{Ax^2 + Bx + C} \xrightarrow{\text{配方}} \sqrt{a^2 - x^2} \text{ 或 } \sqrt{a^2 + x^2} \text{ 或 } \sqrt{x^2 - a^2}$$

再作三角代换即可消去根号

# 第六讲：不定积分 > 换元法 > 三角代换 > 产生三角

$$\int \frac{dx}{\sqrt{1-x+x^2}} = \int \frac{dx}{\sqrt{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}}} = \int \frac{\frac{4}{3}dx}{\sqrt{\left(\frac{2x-1}{\sqrt{3}}\right)^2 + 1}} = \int \frac{\frac{2}{\sqrt{3}}d\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{\left(\frac{2x-1}{\sqrt{3}}\right)^2 + 1}} \quad \text{令 } t = \frac{2x-1}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}} \int \frac{dt}{\sqrt{t^2 + 1}} \quad \text{令 } t = \tan \theta \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$= \frac{2}{\sqrt{3}} \int \frac{\sec^2 \theta d\theta}{\sqrt{\tan^2 \theta + 1}} = \frac{2}{\sqrt{3}} \int \sec \theta d\theta$$

$$= \frac{2}{\sqrt{3}} \int \frac{d\theta}{\cos \theta} = \frac{2}{\sqrt{3}} \int \frac{\cos \theta d\theta}{\cos^2 \theta}$$

$$= \frac{2}{\sqrt{3}} \int \frac{d \sin \theta}{1 - \sin^2 \theta} = \frac{1}{\sqrt{3}} \ln \left| \frac{1 + \sin \theta}{1 - \sin \theta} \right| + C$$

欧拉替换法



# 第六讲：不定积分 > 换元法 > 欧拉替换法

$$\int R(x, \sqrt{ax^2 + bx + c}) dx$$

其中  $a \neq 0$  且  $\Delta = b^2 - 4ac \neq 0$   $R(x, y)$  表示以  $x, y$  的为变量的有理函数

对于这样一种形式的不定积分我们用欧拉代换都可以将被积函数有理化

欧拉第一代换

$$a > 0 \quad \text{令 } \sqrt{ax^2 + bx + c} = t + \sqrt{a}x \text{ 或 } t - \sqrt{a}x$$

欧拉第二代换

$$c > 0 \quad \text{令 } \sqrt{ax^2 + bx + c} = xt + \sqrt{c} \text{ 或 } xt - \sqrt{c}$$

欧拉第三代换

$$\Delta > 0 \quad \text{令 } \sqrt{ax^2 + bx + c} = t(x - \alpha) \text{ 或 } t(x - \beta)$$

其中  $\alpha, \beta$  是  $ax^2 + bx + c = 0$  的两互异实根

# 第六讲：不定积分 > 换元法 > 欧拉替换法

欧拉第一代换

$$a > 0 \quad \text{令 } \sqrt{ax^2 + bx + c} = t + \sqrt{ax} \text{ 或 } t - \sqrt{ax}$$

$$\sqrt{ax^2 + bx + c} = t + \sqrt{ax} \Rightarrow ax^2 + bx + c = t^2 + 2\sqrt{a}xt + ax^2$$

$$\Rightarrow x = \frac{c - t^2}{2\sqrt{at} - b} \quad \sqrt{ax^2 + bx + c} = \frac{\sqrt{a}t^2 - bt + c\sqrt{a}}{2\sqrt{at} - b} \quad dx = \frac{-2\sqrt{a}t^2 + 2bt - 2c\sqrt{a}}{(2\sqrt{at} - b)^2} dt$$

$$\int R(x, \sqrt{ax^2 + bx + c}) dx = \int R\left(\frac{c - t^2}{2\sqrt{at} - b}, \frac{\sqrt{a}t^2 - bt + c\sqrt{a}}{2\sqrt{at} - b}\right) \frac{-2\sqrt{a}t^2 + 2bt - 2c\sqrt{a}}{(2\sqrt{at} - b)^2} dt$$

# 第六讲：不定积分 > 换元法 > 欧拉替换法

欧拉第二代换

$$c > 0 \quad \text{令 } \sqrt{ax^2 + bx + c} = xt + \sqrt{c} \text{ 或 } xt - \sqrt{c}$$

$$\sqrt{ax^2 + bx + c} = xt + \sqrt{c} \Rightarrow ax^2 + bx + c = x^2 t^2 + 2\sqrt{c}xt + c$$

$$\Rightarrow x = \frac{2\sqrt{c}t - b}{a - t^2} \quad \sqrt{ax^2 + bx + c} = \frac{\sqrt{c}t^2 - bt + a\sqrt{c}}{a - t^2} \quad dx = \frac{2\sqrt{c}t^2 - 2bt + 2a\sqrt{c}}{(a - t^2)^2} dt$$

$$\int R(x, \sqrt{ax^2 + bx + c}) dx = \int R\left(\frac{2\sqrt{c}t - b}{a - t^2}, \frac{\sqrt{c}t^2 - bt + a\sqrt{c}}{a - t^2}\right) \frac{2\sqrt{c}t^2 - 2bt + 2a\sqrt{c}}{(a - t^2)^2} dt$$

# 第六讲：不定积分 > 换元法 > 欧拉替换法

欧拉第三代换

$$\Delta > 0 \quad \text{令 } \sqrt{ax^2 + bx + c} = t(x - \alpha) \text{ 或 } t(x - \beta)$$

其中  $\alpha, \beta$  是  $ax^2 + bx + c = 0$  的两互异实

$$\text{令 } t = \sqrt{\frac{a(x - \beta)}{x - \alpha}} \text{ 或 } \sqrt{\frac{a(x - \alpha)}{x - \beta}}$$

$$\sqrt{ax^2 + bx + c} = t(x - \alpha) \Rightarrow ax^2 + bx + c = t^2(x - \alpha)^2$$

$$\Rightarrow a(x - \beta)(x - \alpha) = t^2(x - \alpha)^2 \Rightarrow a(x - \beta) = t^2(x - \alpha)$$

$$\Rightarrow x = \frac{\alpha t^2 - a\beta}{t^2 - a} \quad \sqrt{ax^2 + bx + c} = \frac{a(\alpha - \beta)t}{t^2 - a} \quad dx = \frac{2a(\beta - \alpha)t}{(t^2 - a)^2} dt$$

$$\int R(x, \sqrt{ax^2 + bx + c}) dx = \int R\left(\frac{\alpha t^2 - a\beta}{t^2 - a}, \frac{a(\alpha - \beta)t}{t^2 - a}\right) \frac{2a(\beta - \alpha)t}{(t^2 - a)^2} dt$$

# 第六讲：不定积分 > 换元法 > 欧拉替换法

$$\int \frac{dx}{\sqrt{1-x+x^2}}$$

$$\text{令 } \sqrt{1-x+x^2} = t+x \Rightarrow 1-x+x^2 = t^2 + 2xt + x^2$$

$$\Rightarrow x = \frac{1-t^2}{2t+1} \quad \sqrt{1-x+x^2} = \frac{t^2+t+1}{2t+1} \quad dx = \frac{-2t^2-2t-2}{(2t+1)^2} dt$$

$$\int \frac{dx}{\sqrt{1-x+x^2}} = \int \frac{\frac{-2(t^2+t+1)}{(2t+1)^2} dt}{\frac{t^2+t+1}{2t+1}} = \int \frac{-2 dt}{2t+1} = -\ln|2t+1| + C = -\ln\left|2\sqrt{1-x+x^2} - 2x + 1\right| + C$$

$$a > 0 \quad b^2 - 4ac \neq 0$$

$$\int \frac{1}{\sqrt{ax^2+bx+c}} dx = \frac{1}{\sqrt{a}} \ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right| + C$$

## 第六讲：不定积分 > 换元法 > 欧拉替换法

$$\begin{aligned} -\ln\left|2\sqrt{1-x+x^2}-2x+1\right| &= -\ln\left|\frac{(2\sqrt{1-x+x^2}-2x+1)(2\sqrt{1-x+x^2}+2x-1)}{2\sqrt{1-x+x^2}+2x-1}\right| \\ &= -\ln\left|\frac{(2\sqrt{1-x+x^2})^2-(2x-1)^2}{2\sqrt{1-x+x^2}+2x-1}\right| = -\ln\left|\frac{3}{2\sqrt{1-x+x^2}+2x-1}\right| = \ln\left|2\sqrt{1-x+x^2}+2x-1\right| - \ln 3 \end{aligned}$$

# 第六讲：不定积分 > 换元法 > 欧拉替换法

$$\int \frac{dx}{\sqrt{1-x+x^2}}$$

$$\text{令 } \sqrt{1-x+x^2} = xt+1 \Rightarrow x^2 - x + 1 = x^2 t^2 + 2xt + 1$$

$$\Rightarrow x = \frac{2t+1}{1-t^2} \quad \sqrt{1-x+x^2} = \frac{t^2+t+1}{1-t^2} \quad dx = \frac{2(t^2+t+1)}{(1-t^2)^2} dt$$

$$\int \frac{dx}{\sqrt{1-x+x^2}} = \int \frac{\frac{2(t^2+t+1)}{(1-t^2)^2} dt}{\frac{t^2+t+1}{1-t^2}} = \int \frac{2dt}{1-t^2} = \ln \left| \frac{t+1}{t-1} \right| + C = \ln \left| \frac{tx+x}{tx-x} \right| + C = \ln \left| \frac{\sqrt{1-x+x^2}-1+x}{\sqrt{1-x+x^2}-1-x} \right| + C$$

## 第六讲：不定积分 > 换元法 > 欧拉替换法

$$\begin{aligned} \ln \left| \frac{\sqrt{1-x+x^2}-1+x}{\sqrt{1-x+x^2}-1-x} \right| &= \ln \left| \frac{(\sqrt{1-x+x^2}-1+x)(\sqrt{1-x+x^2}+1+x)}{(\sqrt{1-x+x^2}-1-x)(\sqrt{1-x+x^2}+1+x)} \right| \\ &= \ln \left| \frac{(\sqrt{1-x+x^2}+x)^2-1}{(\sqrt{1-x+x^2})^2-(1+x)^2} \right| = \ln \left| \frac{2x\sqrt{1-x+x^2}+2x^2-x}{-3x} \right| = \ln |2\sqrt{1-x+x^2}+2x-1| - \ln 3 \end{aligned}$$



# 第六讲：不定积分 > 换元法 > 欧拉替换法

$$n > 2 \quad \int \frac{dx}{x \sqrt{-x^2 + nx - 1}} \quad -x^2 + nx - 1 = 0 \quad \alpha = \frac{n - \sqrt{n^2 - 4}}{2} \quad \beta = \frac{n + \sqrt{n^2 - 4}}{2}$$

$$\text{令 } \sqrt{-x^2 + nx - 1} = t(x - \alpha) \Rightarrow -x^2 + nx - 1 = t^2(x - \alpha)^2 \quad \alpha < x < \beta \quad t = \frac{\sqrt{-x^2 + nx - 1}}{x - \alpha}$$

$$\Rightarrow -(x - \beta)(x - \alpha) = t^2(x - \alpha)^2 \Rightarrow -(x - \beta) = t^2(x - \alpha) \quad = \sqrt{\frac{-(x - \beta)(x - \alpha)}{(x - \alpha)^2}}$$

$$\Rightarrow x = \frac{\alpha t^2 + \beta}{t^2 + 1} \quad \sqrt{-x^2 + nx - 1} = \frac{-(\alpha - \beta)t}{t^2 + 1} \quad dx = \frac{-2(\beta - \alpha)t}{(t^2 + 1)^2} dt$$

$$= \int \frac{\frac{-2(\beta - \alpha)t}{(t^2 + 1)^2} dt}{\frac{\alpha t^2 + \beta}{t^2 + 1} \cdot \frac{-(\alpha - \beta)t}{t^2 + 1}} = \int \frac{-2 dt}{\alpha t^2 + \beta} = \int \frac{-\frac{2}{\beta} dt}{\frac{\alpha}{\beta} t^2 + 1} = \int \frac{-2 \sqrt{\frac{1}{\alpha \beta}} d\left(\sqrt{\frac{\alpha}{\beta}} t\right)}{\left(\sqrt{\frac{\alpha}{\beta}} t\right)^2 + 1}$$

$$= -2 \arctan \sqrt{\frac{\alpha}{\beta}} t + C = -2 \arctan \sqrt{\frac{2 - (n - \sqrt{n^2 - 4})x}{(n + \sqrt{n^2 - 4})x - 2}} + C$$

$$= \sqrt{\frac{\beta - x}{x - \alpha}}$$

$$\sqrt{\frac{\alpha}{\beta}} t = \sqrt{\frac{\alpha \beta - \alpha x}{\beta x - \alpha \beta}}$$

$$= \sqrt{\frac{1 - \alpha x}{\beta x - 1}}$$

$$= \sqrt{\frac{2 - (n - \sqrt{n^2 - 4})x}{(n + \sqrt{n^2 - 4})x - 2}}$$

# 第六讲：不定积分 > 待定系数法 > 有理分式

分式分解定理

理论依据

设  $\frac{P(x)}{Q(x)}$  为有理真分式，其中  $Q(x) = Q_1(x)Q_2(x)$  且  $Q_1(x), Q_2(x)$  互素

则存在唯一一组多项式  $P_1(x), P_2(x)$  使得  $\frac{P(x)}{Q(x)} = \frac{P_1(x)}{Q_1(x)} + \frac{P_2(x)}{Q_2(x)}$

其中  $\frac{P_1(x)}{Q_1(x)}, \frac{P_2(x)}{Q_2(x)}$  为真分式

分式分解定理的推论

设  $\frac{P(x)}{Q(x)}$  为有理真分式，其中  $Q(x) = Q_1(x)Q_2(x)\cdots Q_m(x)$  且  $Q_1(x), Q_2(x), \dots, Q_m(x)$  互素

则存在唯一一组多项式  $P_1(x), P_2(x), \dots, P_m(x)$  使得  $\frac{P(x)}{Q(x)} = \frac{P_1(x)}{Q_1(x)} + \frac{P_2(x)}{Q_2(x)} + \cdots + \frac{P_m(x)}{Q_m(x)}$

其中  $\frac{P_1(x)}{Q_1(x)}, \frac{P_2(x)}{Q_2(x)}, \dots, \frac{P_m(x)}{Q_m(x)}$  为真分式

# 第六讲：不定积分 > 待定系数法 > 有理分式

$$a > 0 \quad b^2 - 4ac \neq 0$$

$$\int \frac{1}{ax^2 + bx + c} dx = \begin{cases} \frac{2}{\sqrt{4ac - b^2}} \arctan \frac{2ax + b}{\sqrt{4ac - b^2}} + C & b^2 < 4ac \\ \frac{1}{\sqrt{b^2 - 4ac}} \ln \left| \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right| + C & b^2 > 4ac \end{cases}$$

$$\int \frac{1}{t^2 + 1} dt$$

$$\int \frac{1}{t^2 - 1} dt$$

# 第六讲：不定积分 > 待定系数法 > 有理分式

$$\int \frac{1}{(x^2+1)(x^2+x+1)} dx$$

分式分解定理

省去一些中间过程

$$\frac{1}{(x^2+1)(x^2+x+1)} = \frac{M+Nx}{x^2+1} + \frac{P+Qx}{x^2+x+1} = A \frac{1}{x^2+1} + B \frac{2x}{x^2+1} + C \frac{1}{x^2+x+1} + D \frac{2x+1}{x^2+x+1}$$

$$\begin{cases} M = A \\ N = 2B \\ P = C + D \\ Q = 2D \end{cases}$$

$$= A(\arctan x)' + B(\ln(x^2+1))' + C\left(\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}\right)' + D(\ln(x^2+x+1))'$$

$$\frac{1}{(x^2+1)(x^2+x+1)} = A \frac{1}{x^2+1} + B \frac{2x}{x^2+1} + C \frac{1}{x^2+x+1} + D \frac{2x+1}{x^2+x+1} = \frac{(A+2Bx)(x^2+x+1) + (C+2Dx+D)(x^2+1)}{(x^2+1)(x^2+x+1)}$$

$$x = 0, -1, i$$

$$\begin{cases} A + C + D = 1 \\ (A - 2B) + 2(C - D) = 1 \\ (A + 2Bi)i = Ai - 2B = 1 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = -1/2 \\ C = D = 1/2 \end{cases}$$

赋值法

$$\int \frac{1}{(x^2+1)(x^2+x+1)} dx = -\frac{1}{2}\ln(x^2+1) + \frac{1}{2} \cdot \frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + \frac{1}{2}\ln(x^2+x+1) + C$$

## 第六讲：不定积分 > 待定系数法 > 有理分式

$$\int \frac{1}{x^3 - 1} dx$$

$$x^3 - 1 = (x - 1)(1 + x + x^2)$$

$$\frac{1}{x^3 - 1} = \frac{A}{x - 1} + \frac{P + Qx}{1 + x + x^2} = A \frac{1}{x - 1} + B \frac{1}{1 + x + x^2} + C \frac{2x + 1}{1 + x + x^2} \quad \begin{cases} P = B + C \\ Q = 2C \end{cases}$$

$$= A(\ln|x - 1|)' + B \left( \frac{2}{\sqrt{3}} \arctan \frac{2x + 1}{\sqrt{3}} \right)' + C(\ln(1 + x^2))'$$

# 第六讲：不定积分 > 待定系数法 > 有理分式

$$\int \frac{1}{x^5 + 1} dx$$

实系数多项式的虚根成对存在

$$(x - \alpha)(x - \bar{\alpha}) = x^2 - (\alpha + \bar{\alpha})x + \alpha\bar{\alpha}$$

$$(\alpha + \bar{\alpha})^2 - 4\alpha\bar{\alpha} = (\alpha - \bar{\alpha})^2 < 0$$

复数域上多项式因式分解定理

每个次数  $\geq 1$  的复系数多项式在复数域上都可以唯一地分解成一次因式的乘积

其标准分解式为：  $f(x) = a_n (x - \alpha_1)^{k_1} (x - \alpha_2)^{k_2} \cdots (x - \alpha_s)^{k_s}$

其中  $a_n$  为  $f(x)$  的首项系数，  $\alpha_1, \alpha_2, \dots, \alpha_s$  是互异的复数

$k_1, k_2, \dots, k_s$  是正整数，  $k_1 + k_2 + \cdots + k_s = n = \partial(f(x))$

实数域上多项式因式分解定理

每个次数  $\geq 1$  的实系数多项式在实数域上都可以唯一地分解成一次因式和二次不可约因式的乘积

其标准分解式为：  $f(x) = a_n (x - c_1)^{k_1} \cdots (x - c_s)^{k_s} (x^2 + p_1x + q_1)^{j_1} \cdots (x^2 + p_rx + q_r)^{j_r}$

其中  $a_n$  为  $f(x)$  的首项系数，  $c_1, \dots, c_s, p_1, \dots, p_r, q_1, \dots, q_r$  是实数且  $p_i^2 - 4q_i < 0 \quad i = 1, \dots, r$

$k_1, \dots, k_s, j_1, \dots, j_r$  是正整数，  $k_1 + \cdots + k_s + 2(j_1 + \cdots + j_r) = n = \partial(f(x))$

# 第六讲：不定积分 > 待定系数法 > 有理分式

$$x^n - a \quad a \neq 0$$

**n次复系数多项式在复数域恰有n个根**

$$x = \begin{cases} \sqrt[n]{a} e^{\frac{2k}{n}\pi i} & k = 0, \dots, n-1 \quad a > 0 \\ \sqrt[n]{-a} e^{\frac{2k+1}{n}\pi i} & k = 0, \dots, n-1 \quad a < 0 \end{cases}$$

$$\left( \sqrt[n]{a} e^{\frac{2k}{n}\pi i} \right)^n = a e^{2k\pi i} = a (\cos 2k\pi + i \sin 2k\pi) = a$$

$$\left( \sqrt[n]{-a} e^{\frac{2k+1}{n}\pi i} \right)^n = -a e^{(2k+1)\pi i} = -a [\cos(2k+1)\pi + i \sin(2k+1)\pi] = a$$

$e^{\frac{2k}{n}\pi i}$  ( $k = 0, \dots, n-1$ ) 是  $x^n - 1$  的所有复根       $e^{\frac{2k+1}{n}\pi i}$  ( $k = 0, \dots, n-1$ ) 是  $x^n + 1$  的所有复根

$$x^n - 1 = \prod_{k=0}^{n-1} \left( x - e^{\frac{2k}{n}\pi i} \right)$$

$$x^n + 1 = \prod_{k=0}^{n-1} \left( x - e^{\frac{2k+1}{n}\pi i} \right)$$

# 第六讲：不定积分 > 待定系数法 > 有理分式

$$\begin{aligned}
 x^5 + 1 &= \left(x - e^{\frac{1}{5}\pi i}\right) \left(x - e^{\frac{3}{5}\pi i}\right) \left(x - e^{\frac{5}{5}\pi i}\right) \left(x - e^{\frac{7}{5}\pi i}\right) \left(x - e^{\frac{9}{5}\pi i}\right) \\
 &= \left(x - e^{\frac{1}{5}\pi i}\right) \left(x - e^{\frac{3}{5}\pi i}\right) \left(x - e^{\frac{5}{5}\pi i}\right) \left(x - e^{-\frac{3}{5}\pi i}\right) \left(x - e^{-\frac{1}{5}\pi i}\right) \\
 &= \left[x^2 - \left(e^{\frac{1}{5}\pi i} + e^{-\frac{1}{5}\pi i}\right)x + e^{\frac{1}{5}\pi i} e^{-\frac{1}{5}\pi i}\right] \left[x^2 - \left(e^{\frac{3}{5}\pi i} + e^{-\frac{3}{5}\pi i}\right)x + e^{\frac{3}{5}\pi i} e^{-\frac{3}{5}\pi i}\right] \left(x - e^{\frac{5}{5}\pi i}\right) \\
 &= \left(x^2 - 2\cos\frac{1}{5}\pi \cdot x + 1\right) \left(x^2 - 2\cos\frac{3}{5}\pi \cdot x + 1\right) (x + 1)
 \end{aligned}$$

赋值法

$$x = i \Rightarrow i + 1 = -4\cos\frac{1}{5}\pi \cos\frac{3}{5}\pi (i + 1) \Rightarrow \cos\frac{1}{5}\pi \cos\frac{3}{5}\pi = -\frac{1}{4}$$

$$x = 1 \Rightarrow 2 = \left(2 - 2\cos\frac{1}{5}\pi\right) \left(2 - 2\cos\frac{3}{5}\pi\right) 2 \Rightarrow 1 - \cos\frac{1}{5}\pi - \cos\frac{3}{5}\pi + \cos\frac{1}{5}\pi \cos\frac{3}{5}\pi = \frac{1}{4} \Rightarrow \cos\frac{1}{5}\pi + \cos\frac{3}{5}\pi = \frac{1}{2}$$

$$\cos\frac{1}{5}\pi, \cos\frac{3}{5}\pi \text{ 是方程 } t^2 - \frac{1}{2}t - \frac{1}{4} = 0 \text{ 的两根} \quad \cos\frac{1}{5}\pi = \frac{1+\sqrt{5}}{4} \quad \cos\frac{3}{5}\pi = \frac{1-\sqrt{5}}{4}$$

$$x^5 + 1 = \left(x^2 - \frac{1+\sqrt{5}}{2}x + 1\right) \left(x^2 - \frac{1-\sqrt{5}}{2}x + 1\right) (x + 1)$$



# 第六讲：不定积分 > 待定系数法 > 有理分式

$$x^5 + 1 = \left( x^2 - \frac{1+\sqrt{5}}{2}x + 1 \right) \left( x^2 - \frac{1-\sqrt{5}}{2}x + 1 \right) (x+1)$$

$$\frac{1}{x^5 + 1} = \frac{Px + Q}{x^2 - \frac{1+\sqrt{5}}{2}x + 1} + \frac{Mx + N}{x^2 - \frac{1-\sqrt{5}}{2}x + 1} + \frac{E}{x+1}$$

$$\begin{cases} P = 2A \\ Q = B - \frac{1+\sqrt{5}}{2}A \\ M = 2C \\ N = D - \frac{1-\sqrt{5}}{2}C \end{cases}$$

$$= A \frac{2x - \frac{1+\sqrt{5}}{2}}{x^2 - \frac{1+\sqrt{5}}{2}x + 1} + B \frac{1}{x^2 - \frac{1+\sqrt{5}}{2}x + 1} + C \frac{2x - \frac{1-\sqrt{5}}{2}}{x^2 - \frac{1-\sqrt{5}}{2}x + 1} + D \frac{1}{x^2 - \frac{1-\sqrt{5}}{2}x + 1} + E \frac{1}{x+1}$$

$$= A \left( \ln \left| x^2 - \frac{1+\sqrt{5}}{2}x + 1 \right| \right)' + B \frac{1}{x^2 - \frac{1+\sqrt{5}}{2}x + 1} + C \left( \ln \left| x^2 - \frac{1-\sqrt{5}}{2}x + 1 \right| \right)' + D \frac{1}{x^2 - \frac{1-\sqrt{5}}{2}x + 1} + E (\ln|x+1|)'$$

$$a > 0 \text{ 且 } b^2 - 4ac < 0$$

$$\int \frac{1}{ax^2 + bx + c} dx = \frac{2}{\sqrt{4ac - b^2}} \arctan \frac{2ax + b}{\sqrt{4ac - b^2}} + C$$

# 第六讲：不定积分 > 待定系数法 > 有理分式

$$\int \frac{1}{x^5 - 1} dx$$

$$\int \frac{1}{x^n + 1} dx \quad \int \frac{1}{x^n - 1} dx$$

$$\begin{aligned} x^5 - 1 &= \left( x - e^{\frac{0}{5}\pi i} \right) \left( x - e^{\frac{2}{5}\pi i} \right) \left( x - e^{\frac{4}{5}\pi i} \right) \left( x - e^{\frac{6}{5}\pi i} \right) \left( x - e^{\frac{8}{5}\pi i} \right) \\ &= \left( x - e^{\frac{0}{5}\pi i} \right) \left( x - e^{\frac{2}{5}\pi i} \right) \left( x - e^{\frac{4}{5}\pi i} \right) \left( x - e^{-\frac{4}{5}\pi i} \right) \left( x - e^{-\frac{2}{5}\pi i} \right) \\ &= \left( x - e^{\frac{0}{5}\pi i} \right) \left[ x^2 - \left( e^{\frac{2}{5}\pi i} + e^{-\frac{2}{5}\pi i} \right) x + e^{\frac{2}{5}\pi i} e^{-\frac{2}{5}\pi i} \right] \left[ x^2 - \left( e^{\frac{4}{5}\pi i} + e^{-\frac{4}{5}\pi i} \right) x + e^{\frac{4}{5}\pi i} e^{-\frac{4}{5}\pi i} \right] \\ &= (x - 1) \left( x^2 - 2 \cos \frac{2}{5} \pi \cdot x + 1 \right) \left( x^2 - 2 \cos \frac{4}{5} \pi \cdot x + 1 \right) \end{aligned}$$

$$x = i \quad x = -1 \Rightarrow \cos \frac{2}{5} \pi = \frac{\sqrt{5} - 1}{4} \quad \cos \frac{4}{5} \pi = \frac{-\sqrt{5} - 1}{4}$$

$$x^5 - 1 = (x - 1) \left( x^2 - \frac{\sqrt{5} - 1}{2} x + 1 \right) \left( x^2 + \frac{\sqrt{5} + 1}{2} x + 1 \right)$$

# 第六讲：不定积分 > 待定系数法 > 有理分式

$$\int \frac{-x^2 - 2}{(x^2 + x + 1)^2} dx \quad \frac{1}{x^2 + x + 1} \quad \frac{2x + 1}{x^2 + x + 1} \quad \left( \frac{Ax + B}{x^2 + x + 1} \right)'$$

$$\frac{-x^2 - 2}{(x^2 + x + 1)^2} = \left( \frac{Ax + B}{x^2 + x + 1} \right)' + C \frac{2x + 1}{x^2 + x + 1} + D \frac{1}{x^2 + x + 1}$$

猜测

$$= \left( \frac{Ax + B}{x^2 + x + 1} \right)' + C (\ln(x^2 + x + 1))' + D \left( \frac{2}{\sqrt{3}} \arctan \frac{2x + 1}{\sqrt{3}} \right)'$$

$$\frac{-x^2 - 2}{(x^2 + x + 1)^2} = \frac{A(x^2 + x + 1) - (Ax + B)(2x + 1) + (2Cx + C + D)(x^2 + x + 1)}{(x^2 + x + 1)^2}$$

$$\begin{cases} 0 = 2C \\ -1 = A - 2A + C + D + 2C \\ 0 = A - (2B + A) + C + D + 2C \\ -2 = A - B + C + D \end{cases} \quad \begin{cases} A = -1 \\ B = -1 \\ C = 0 \\ D = -2 \end{cases}$$

$$\int \frac{-x^2 - 2}{(x^2 + x + 1)^2} dx = \frac{-x - 1}{x^2 + x + 1} - \frac{4}{\sqrt{3}} \arctan \frac{2x + 1}{\sqrt{3}} + C$$

# 第六讲：不定积分 > 待定系数法 > 有理分式

$$\int \frac{1+x+x^2+x^3+x^4+x^5}{x^2(1+x^2)^2} dx$$

$$\frac{1+x+x^2+x^3+x^4+x^5}{x^2(1+x^2)^2} = \frac{P+Qx+Rx^2+Sx^3}{(1+x^2)^2} + \frac{M+Nx}{x^2}$$

分式分解定理

$$\frac{P+Qx+Rx^2+Sx^3}{(1+x^2)^2} = \left( \frac{A+Bx}{1+x^2} \right)' + C \frac{2x}{1+x^2} + D \frac{1}{1+x^2}$$

猜测

$$\frac{1+x+x^2+x^3+x^4+x^5}{x^2(1+x^2)^2} = \left( \frac{A+Bx}{1+x^2} \right)' + C \frac{2x}{1+x^2} + D \frac{1}{1+x^2} + M \frac{1}{x^2} + N \frac{1}{x}$$

## 第六讲：不定积分 > 待定系数法 > 有理分式

$$\begin{aligned}\frac{1+x+x^2+x^3+x^4+x^5}{x^2(1+x^2)^2} &= \left( \frac{A+Bx}{1+x^2} \right)' + C \frac{2x}{1+x^2} + D \frac{1}{1+x^2} + M \frac{1}{x^2} + N \frac{1}{x} \\ &= \left( \frac{A+Bx}{1+x^2} \right)' + C(\ln(1+x^2))' + D(\arctan x)' + M \left( -\frac{1}{x} \right)' + N(\ln x)' \\ &= \left( \frac{1-x}{2(1+x^2)} \right)' + 0(\ln(1+x^2))' - \frac{1}{2}(\arctan x)' + \left( -\frac{1}{x} \right)' + (\ln x)'\end{aligned}$$

$$1+x+x^2+x^3+x^4+x^5$$

$$= x^2 [B(1+x^2) - 2x(A+Bx)] + 2Cx^3(1+x^2) + Dx^2(1+x^2) + M(1+x^2)^2 + Nx(1+x^2)^2$$

$$\begin{cases} 1 = M \\ 1 = N \\ 1 = B + D + 2M \\ 1 = -2A + 2C + 2N \\ 1 = -B + D + M \\ 1 = 2C + N \end{cases} \quad \begin{cases} A = 1/2 \\ B = -1/2 \\ C = 0 \\ D = -1/2 \\ M = 1 \\ N = 1 \end{cases}$$

# 第六讲：不定积分 > 待定系数法 > 非有理分式

$$a > 0 \quad b^2 - 4ac \neq 0$$

$$\int \frac{1}{\sqrt{ax^2 + bx + c}} dx = \frac{1}{\sqrt{a}} \ln \left| 2ax + b + 2\sqrt{a} \sqrt{ax^2 + bx + c} \right| + C$$

$$b^2 + 4ac > 0$$

$$a > 0 \quad b^2 + 4ac \neq 0$$

$$\int \frac{1}{\sqrt{-ax^2 + bx + c}} dx = \frac{1}{\sqrt{a}} \arcsin \frac{2ax - b}{\sqrt{b^2 + 4ac}} + C$$

$$\int \frac{1}{\sqrt{1-t^2}} dt$$

# 第六讲：不定积分 > 待定系数法 > 非有理分式

$$\int \frac{x}{\sqrt{1-x+x^2}} dx \quad \frac{1}{\sqrt{1-x+x^2}} \quad \left( \sqrt{1-x+x^2} \right)' = \frac{x - \frac{1}{2}}{\sqrt{1-x+x^2}}$$

$$\frac{x}{\sqrt{1-x+x^2}} = A \frac{x - \frac{1}{2}}{\sqrt{1-x+x^2}} + B \frac{1}{\sqrt{1-x+x^2}}$$

猜测

$$\begin{cases} A = 1 \\ -\frac{1}{2}A + B = 0 \end{cases} \quad \begin{cases} A = 1 \\ B = \frac{1}{2} \end{cases}$$

$$\begin{aligned} \frac{x}{\sqrt{1-x+x^2}} &= \frac{x - \frac{1}{2}}{\sqrt{1-x+x^2}} + \frac{1}{2} \frac{1}{\sqrt{1-x+x^2}} \\ &= \left( \sqrt{1-x+x^2} \right)' + \frac{1}{2} \left( \ln |2x-1+2\sqrt{1-x+x^2}| \right)' \end{aligned}$$

# 第六讲：不定积分 > 待定系数法 > 非有理分式

$$\int \frac{x}{\sqrt{1-x-x^2}} dx \quad \frac{1}{\sqrt{1-x-x^2}} \quad \left( \sqrt{1-x-x^2} \right)' = \frac{-x-\frac{1}{2}}{\sqrt{1-x-x^2}}$$

$$\frac{x}{\sqrt{1-x-x^2}} = A \frac{-x-\frac{1}{2}}{\sqrt{1-x-x^2}} + B \frac{1}{\sqrt{1-x-x^2}}$$

猜测

$$\begin{cases} -A = 1 \\ -\frac{1}{2}A + B = 0 \end{cases} \quad \begin{cases} A = -1 \\ B = -\frac{1}{2} \end{cases}$$

$$\begin{aligned} \frac{x}{\sqrt{1-x-x^2}} &= -\frac{-x-\frac{1}{2}}{\sqrt{1-x-x^2}} - \frac{1}{2} \frac{1}{\sqrt{1-x-x^2}} \\ &= -\left( \sqrt{1-x-x^2} \right)' - \frac{1}{2} \left( \arcsin \frac{2x+1}{\sqrt{5}} \right)' \end{aligned}$$



# 第六讲：不定积分 > 待定系数法 > 非有理分式

$$a^2 + b^2 \neq 0 \quad \int \frac{c \sin x + d \cos x}{a \sin x + b \cos x} dx$$

待定  $m$   $n$

$$c \sin x + d \cos x = m(a \sin x + b \cos x) + n(a \sin x + b \cos x)'$$

$$c \sin x + d \cos x = m(a \sin x + b \cos x) + n(a \cos x - b \sin x)$$

$$\begin{cases} c = am - bn \\ d = bm + an \end{cases} \quad \begin{cases} m = \frac{ac + bd}{a^2 + b^2} \\ n = \frac{ad - bc}{a^2 + b^2} \end{cases}$$

$$c \sin x + d \cos x = \frac{ac + bd}{a^2 + b^2} (a \sin x + b \cos x) + \frac{ad - bc}{a^2 + b^2} (a \sin x + b \cos x)'$$

$$\frac{c \sin x + d \cos x}{a \sin x + b \cos x} = \frac{ac + bd}{a^2 + b^2} + \frac{ad - bc}{a^2 + b^2} \frac{(a \sin x + b \cos x)'}{a \sin x + b \cos x}$$

$$= \frac{ac + bd}{a^2 + b^2} (x)' + \frac{ad - bc}{a^2 + b^2} (\ln|a \sin x + b \cos x|)'$$

# 第六讲：不定积分 > 待定系数法 > 非有理分式

$$\int e^{\lambda x} (p \sin kx + q \cos kx) dx = e^{\lambda x} (m \sin kx + n \cos kx) + C$$

$$\int e^x (2 \sin 4x + \cos 4x) dx = \int 2e^x \sin 4x dx + \int e^x \cos 4x dx$$

待定  $m$   $n$

$$\int e^x (2 \sin 4x + \cos 4x) dx = e^x (m \sin 4x + n \cos 4x) + C$$

$$e^x (2 \sin 4x + \cos 4x) = e^x (m \sin 4x + n \cos 4x + 4m \cos 4x - 4n \sin 4x)$$

$$2e^x \sin 4x + e^x \cos 4x = (m - 4n)e^x \sin 4x + (n + 4m)e^x \cos 4x$$

$$\begin{cases} 2 = m - 4n \\ 1 = n + 4m \end{cases} \quad \begin{cases} m = \frac{6}{17} \\ n = -\frac{7}{17} \end{cases}$$

$$\int e^{ax} \sin bxdx \quad \int e^{ax} \cos bxdx$$

$$\int e^x (2 \sin 4x + \cos 4x) dx = e^x \left( \frac{6}{17} \sin 4x - \frac{7}{17} \cos 4x \right) + C$$

# 第六讲：不定积分 > 待定系数法 > 非有理分式

$$\int e^{\lambda x} [P_1(x) \cos kx + P_2(x) \sin kx] dx = e^{\lambda x} [Q_1(x) \cos kx + Q_2(x) \sin kx] + C$$

其中 $P_1(x)$ ,  $P_2(x)$ ,  $Q_1(x)$ ,  $Q_2(x)$ 是多项式且 $\partial(Q_1(x)), \partial(Q_2(x)) \leq \max\{\partial(P_1(x)), \partial(P_2(x))\}$

$$\int x e^x (\cos 4x + 2 \sin 4x) dx$$

待定  $a b c d$

$$\int x e^x (\cos 4x + 2 \sin 4x) dx = e^x [(ax + b) \cos 4x + (cx + d) \sin 4x] + C$$

$$x e^x (\cos 4x + 2 \sin 4x) = e^x [(ax + b) \cos 4x + (cx + d) \sin 4x + a \cos 4x - 4(ax + b) \sin 4x + c \sin 4x + 4(cx + d) \cos 4x]$$

$$x e^x \cos 4x + 2 x e^x \sin 4x = (ax + 4cx + b + a + 4d) e^x \cos 4x + (cx - 4ax + d + c - 4b) e^x \sin 4x$$

$$\begin{cases} 1 = a + 4c \\ 0 = b + a + 4d \\ 2 = c - 4a \\ 0 = d + c - 4b \end{cases} \quad \begin{cases} a = -7/17 \\ b = 31/17^2 \\ c = 6/17 \\ d = 22/17^2 \end{cases}$$

$$\int x e^x (\cos 4x + 2 \sin 4x) dx = e^x \left[ \left( -\frac{7}{17}x + \frac{31}{17^2} \right) \cos 4x + \left( \frac{6}{17}x + \frac{22}{17^2} \right) \sin 4x \right] + C$$

## 第六讲：不定积分 > 待定系数法

$$\int g(x) e^{f(x)} dx$$

$$\int g(x) e^{f(x)} dx = h(x) e^{f(x)} + C$$

待定  $h(x)$

$$g(x) e^{f(x)} = (h'(x) + f'(x) h(x)) e^{f(x)}$$

$$g(x) = h'(x) + f'(x) h(x)$$

猜测  $h(x)$

$$h(x) = e^{-f(x)} \left( \int g(x) e^{f(x)} dx + C \right)$$

## 第六讲：不定积分 > 待定系数法

$$\int \left(1 + x - \frac{1}{x}\right) e^{x + \frac{1}{x}} dx \quad (\text{第三届决赛})$$

$$\int \left(1 + x - \frac{1}{x}\right) e^{x + \frac{1}{x}} dx = h(x) e^{x + \frac{1}{x}} + C$$

待定  $h(x)$

$$\left(1 + x - \frac{1}{x}\right) e^{x + \frac{1}{x}} = \left[ \left(1 - \frac{1}{x^2}\right) h(x) + h'(x) \right] e^{x + \frac{1}{x}}$$

$$1 + x - \frac{1}{x} = \left(1 - \frac{1}{x^2}\right) h(x) + h'(x) \quad h(x) = x$$

$$\int \left(1 + x - \frac{1}{x}\right) e^{x + \frac{1}{x}} dx = \int \left[ x \left(1 - \frac{1}{x^2}\right) e^{x + \frac{1}{x}} + e^{x + \frac{1}{x}} \right] dx = \int \left[ x \left( e^{x + \frac{1}{x}} \right)' + (x)' e^{x + \frac{1}{x}} \right] dx = x e^{x + \frac{1}{x}} + C$$

## 第六讲：不定积分 > 待定系数法

$$\int \frac{e^{-\sin x} \sin 2x}{(1-\sin x)^2} dx \quad (\text{第九届初赛}) \quad \text{消去三角} \quad \text{令 } \sin x = t$$

$$= \int \frac{e^{-\sin x} 2 \sin x \cos x}{(1-\sin x)^2} dx = \int \frac{e^{-\sin x} 2 \sin x d \sin x}{(1-\sin x)^2} = \int \frac{e^{-t} 2 t dt}{(1-t)^2} = h(t) e^{-t} + C \quad \text{待定 } h(t)$$

$$\frac{e^{-t} 2 t}{(1-t)^2} = (-h(t) + h'(t)) e^{-t} \quad \frac{2 t}{(1-t)^2} = -h(t) + h'(t) \quad \text{猜测 } h(t) = \frac{a+bt}{1-t}$$

$$\Rightarrow -h(t) + h'(t) = -\frac{a+bt}{1-t} + \frac{a+b}{(1-t)^2} = \frac{-(a+bt)(1-t) + a+b}{(1-t)^2} \Rightarrow a=2 \quad b=0 \quad h(t) = \frac{2}{1-t}$$

$$\begin{aligned} \int \frac{e^{-t} 2 t dt}{(1-t)^2} &= \int \left[ \frac{2}{1-t} (-e^{-t}) + \frac{2}{(1-t)^2} e^{-t} \right] dt = \int \left[ \frac{2}{1-t} (e^{-t})' + \left( \frac{2}{1-t} \right)' e^{-t} \right] dt = \frac{2}{1-t} e^{-t} + C \\ &= \frac{2}{1-\sin x} e^{-\sin x} + C \end{aligned}$$

## 第六讲：不定积分 > 待定系数法

$$\int \frac{e^{\sin 2x} \sin^2 x}{e^{2x}} dx$$

$$\int \frac{e^{\sin 2x} \sin^2 x}{e^{2x}} dx = \int e^{\sin 2x - 2x} \sin^2 x dx = h(x) e^{\sin 2x - 2x} + C \quad \text{待定 } h(x)$$

$$e^{\sin 2x - 2x} \sin^2 x = e^{\sin 2x - 2x} [(2 \cos 2x - 2)h(x) + h'(x)]$$

$$\sin^2 x = (2 \cos 2x - 2)h(x) + h'(x)$$

$$\frac{1 - \cos 2x}{2} = (2 \cos 2x - 2)h(x) + h'(x)$$

$$h(x) = -\frac{1}{4}$$

## 第六讲：不定积分 > 待定系数法

$$\int \left( x^3 + x^2 + \frac{1}{3} \right) e^{x^3} dx$$

$$\int \left( x^3 + x^2 + \frac{1}{3} \right) e^{x^3} dx = h(x) e^{x^3} + C$$

待定  $h(x)$

$$\left( x^3 + x^2 + \frac{1}{3} \right) e^{x^3} = (3x^2 h(x) + h'(x)) e^{x^3} \quad x^3 + x^2 + \frac{1}{3} = 3x^2 h(x) + h'(x)$$

$$\text{猜测 } h(x) = ax + b \Rightarrow 3x^2 h(x) + h'(x) = 3ax^3 + 3bx^2 + a \Rightarrow a = b = \frac{1}{3} \quad h(x) = \frac{x+1}{3}$$



## 第六讲：不定积分 > 待定系数法

待定  $h(x)$

$$\int \frac{1 + \sin x}{1 + \cos x} e^x dx$$

$$\int \frac{1 + \sin x}{1 + \cos x} e^x dx = \int \frac{1 + 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} e^x dx = \int \left( \frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right) e^x dx = e^x h(x) + C$$

$$\left( \frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right) e^x = (h(x) + h'(x)) e^x$$

$$\frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} = h(x) + h'(x) \quad h(x) = \tan \frac{x}{2}$$

## 第六讲：不定积分 > 待定系数法

$$\int \left( te^{2t} - te^{-2t} + \frac{1}{2} \right) e^{(e^t - e^{-t})^2} dt = e^{(e^t - e^{-t})^2} h(t) + C \quad \text{待定 } h(t)$$

$$\left( te^{2t} - te^{-2t} + \frac{1}{2} \right) e^{(e^t - e^{-t})^2} = [2(e^t - e^{-t})(e^t + e^{-t})h(t) + h'(t)] e^{(e^t - e^{-t})^2}$$

$$te^{2t} - te^{-2t} + \frac{1}{2} = 2(e^{2t} - e^{-2t})h(t) + h'(t) \quad h(t) = \frac{t}{2}$$

## 第六讲：不定积分 > 待定系数法

$$\int \left[ \left( x - \frac{1}{x^3} \right) \ln x + \frac{1}{2x} \right] e^{\left( x - \frac{1}{x} \right)^2} dx = h(x) e^{\left( x - \frac{1}{x} \right)^2} + C \quad \text{待定 } h(x)$$

$$\left[ \left( x - \frac{1}{x^3} \right) \ln x + \frac{1}{2x} \right] e^{\left( x - \frac{1}{x} \right)^2} = \left[ 2 \left( x - \frac{1}{x} \right) \left( 1 + \frac{1}{x^2} \right) h(x) + h'(x) \right] e^{\left( x - \frac{1}{x} \right)^2}$$

$$\left( x - \frac{1}{x^3} \right) \ln x + \frac{1}{2x} = 2 \left( x - \frac{1}{x} \right) h(x) + h'(x) \quad h(x) = \frac{1}{2} \ln x$$

## 第六讲：不定积分 > 分部积分法 > 分部积分法的推广

$$\int u'v dx = -\int uv' dx + uv$$

$$u'v = -uv' + (uv)'$$

$$\int u^{(n)}v dx = (-1)^n \int uv^{(n)} dx + \sum_{k=1}^n (-1)^{n-k} v^{(n-k)} u^{(k-1)}$$

## 第六讲：不定积分 > 分部积分法 > 分部积分法的推广

$$\int u^{(n)} v dx = (-1)^n \int u v^{(n)} dx + \sum_{k=1}^n (-1)^{k-1} v^{(k-1)} u^{(n-k)}$$

当 $v$ 是一个 $n-1$ 次多项式函数时， $v^{(n)} = 0 \Rightarrow (-1)^n \int u v^{(n)} dx = C$

$$\int u^{(n)} v dx = C + \sum_{k=1}^n (-1)^{k-1} v^{(k-1)} u^{(n-k)}$$

# 第六讲：不定积分 > 分部积分法 > 分部积分法的推广

$$\int u^{(n)} v dx = (-1)^n \int u v^{(n)} dx + \sum_{k=1}^n (-1)^{n-k} v^{(n-k)} u^{(k-1)}$$

$$u^{(n)} v = (-1)^n u v^{(n)} + \left( \sum_{k=1}^n (-1)^{n-k} v^{(n-k)} u^{(k-1)} \right)'$$

$$\left( \sum_{k=1}^n (-1)^{n-k} v^{(n-k)} u^{(k-1)} \right)' = \sum_{k=1}^n \left( (-1)^{n-k} v^{(n-k)} u^{(k-1)} \right)' \quad \text{设 } (-1)^{n-k} v^{(n-k)} u^{(k)} = I_k$$

凑差分

消项

$$= \sum_{k=1}^n \left( (-1)^{n-k} v^{(n-k)} u^{(k)} + (-1)^{n-k} v^{(n-k+1)} u^{(k-1)} \right)$$

$$= \sum_{k=1}^n \left( (-1)^{n-k} v^{(n-k)} u^{(k)} - (-1)^{n-k+1} v^{(n-k+1)} u^{(k-1)} \right)$$

$$= \sum_{k=1}^n (I_k - I_{k-1}) = I_n - I_0 = (-1)^0 v^{(0)} u^{(n)} - (-1)^n v^{(n)} u^{(0)}$$

# 第六讲：不定积分 > 分部积分法 > 分部积分法的推广

$$m \in \mathbb{N}^+ \quad \int x^m e^x dx \quad \int u^{(n)} v dx = (-1)^n \int u v^{(n)} dx + \sum_{k=1}^n (-1)^{n-k} v^{(n-k)} u^{(k-1)}$$

$$= \int (e^x)^{(m+1)} x^m dx$$

$$= (-1)^{m+1} \int e^x (x^m)^{(m+1)} dx + \sum_{k=1}^{m+1} (-1)^{m+1-k} (x^m)^{(m+1-k)} (e^x)^{(k-1)}$$

$$= \sum_{k=1}^{m+1} (-1)^{m+1-k} \frac{m!}{(m - (m+1-k))!} x^{k-1} e^x + C$$

$$= \sum_{k=1}^{m+1} (-1)^{m+1-k} \frac{m!}{(k-1)!} x^{k-1} e^x + C$$

$$= \sum_{k=0}^m (-1)^{m-k} \frac{m!}{k!} x^k e^x + C$$

## 第六讲：不定积分 > 分部积分法 > 分部积分法的推广

$$m \in \mathbb{N}^+ \int (\ln x)^m dx$$

$$\text{令 } \ln x = t \Rightarrow x = e^t$$

$$\int (\ln x)^m dx = \int t^m e^t dt$$

$$m, n \in \mathbb{N}^+ \int x^n (\ln x)^m dx$$

$$\int x^n (\ln x)^m dx = \int (e^t)^n t^m de^t = \int e^{(n+1)t} t^m dt$$

$$= \int e^s \left( \frac{s}{n+1} \right)^m d\left( \frac{s}{n+1} \right) = \left( \frac{1}{n+1} \right)^{m+1} \int e^s s^m ds$$

$$\text{令 } \ln x = t \Rightarrow x = e^t \quad \text{令 } (n+1)t = s \Rightarrow t = \frac{s}{n+1}$$



# 第六讲：不定积分 > 分部积分法 > 分部积分法的推广

$$\begin{aligned}
 m \in \mathbb{N}^+ \quad \int x^m \sin x dx & \quad \int u^{(n)} v dx = (-1)^n \int u v^{(n)} dx + \sum_{k=1}^n (-1)^{n-k} v^{(n-k)} u^{(k-1)} \\
 & = \int \left( \sin \left( x - \frac{(m+1)\pi}{2} \right) \right)^{(m+1)} x^m dx \\
 & = (-1)^{m+1} \int \sin \left( x - \frac{(m+1)\pi}{2} \right) (x^m)^{(m+1)} dx + \sum_{k=1}^{m+1} (-1)^{m+1-k} (x^m)^{(m+1-k)} \left( \sin \left( x - \frac{(m+1)\pi}{2} \right) \right)^{(k-1)} \\
 & = \sum_{k=1}^{m+1} (-1)^{m+1-k} \frac{m!}{(m - (m+1-k))!} x^{k-1} \sin \left( x - \frac{(m+1)\pi}{2} + \frac{(k-1)\pi}{2} \right) + C \\
 & = \sum_{k=1}^{m+1} (-1)^{m+1-k} \frac{m!}{(k-1)!} x^{k-1} \sin \left( x + \frac{(k-m-2)\pi}{2} \right) + C \\
 & = \sum_{k=0}^m (-1)^{m-k} \frac{m!}{k!} x^k \sin \left( x + \frac{(k-m-1)\pi}{2} \right) + C
 \end{aligned}$$

# 第六讲：不定积分 > 分部积分法 > 分部积分法的推广

$$\int e^x \sin x dx$$

$$\int u^{(n)} v dx = (-1)^n \int u v^{(n)} dx + \sum_{k=1}^n (-1)^{n-k} v^{(n-k)} u^{(k-1)}$$

$$\begin{aligned} \int e^x \sin x dx &= \int (e^x)'' \sin x dx = (-1)^2 \int e^x (\sin x)'' dx + \sum_{k=1}^2 (-1)^{2-k} (\sin x)^{(2-k)} (e^x)^{(k-1)} \\ &= -\int e^x \sin x dx - e^x \cos x + e^x \sin x \end{aligned}$$

$$\int e^x \sin x dx = \frac{-e^x \cos x + e^x \sin x}{2} + C$$

$$\int \sin \ln x dx$$

$$\text{令 } \ln x = t \Rightarrow x = e^t \Rightarrow \int \sin \ln x dx = \int \sin t de^t = \int e^t \sin t dt$$

$$\int e^{\arccos x} dx$$

$$\text{令 } \arccos x = t \Rightarrow x = \cos t \Rightarrow \int e^{\arccos x} dx = \int e^t d \cos t = -\int e^t \sin t dt$$

## 第六讲：不定积分 > 组合法

当原不定积分不易直接求出时

我们设原不定积分为  $I$ ，我们找出原不定积分的一个对偶式  $J$

通过求出  $I, J$  两组不同的线性组合，从而求出  $I, J$

$$\begin{cases} AI + BJ = P \\ CI + DJ = Q \end{cases} \Rightarrow \begin{cases} I = \frac{PD - QB}{AD - BC} \\ J = \frac{QA - PC}{AD - BC} \end{cases} \quad \begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC \neq 0$$

所以我们可以将计算  $I$  的问题转化成计算  $AI+BJ$  和  $CI+DJ$  的问题

这样我们避开了对原不定积分的直接计算

## 第六讲：不定积分 > 组合法

$$ab \neq 0 \quad \int \frac{\sin x}{(a \sin x + b \cos x)^2} dx \quad \text{令 } u = \tan \frac{x}{2}$$

$$= \int \frac{\frac{2u}{1+u^2} \cdot \frac{2}{1+u^2}}{\left( a \frac{2u}{1+u^2} + b \frac{1-u^2}{1+u^2} \right)^2} du$$

$$= \int \frac{4u}{(2au + b - bu^2)^2} du$$

间接计算

## 第六讲：不定积分 > 组合法

$$ab \neq 0 \quad \int \frac{\sin x}{(a \sin x + b \cos x)^2} dx \quad \text{对偶式} \quad \int \frac{\cos x}{(a \sin x + b \cos x)^2} dx$$

$$\text{设 } I = \int \frac{\sin x}{(a \sin x + b \cos x)^2} dx \quad J = \int \frac{\cos x}{(a \sin x + b \cos x)^2} dx \quad \text{线性组合}$$

$$aI + bJ = \int \frac{a \sin x + b \cos x}{(a \sin x + b \cos x)^2} dx = \int \frac{1}{a \sin x + b \cos x} dx = \frac{1}{2\sqrt{a^2 + b^2}} \ln \left| \frac{a \cos x - b \sin x - \sqrt{a^2 + b^2}}{a \cos x - b \sin x + \sqrt{a^2 + b^2}} \right| + C$$

$$-bI + aJ = \int \frac{-b \sin x + a \cos x}{(a \sin x + b \cos x)^2} dx = \int \frac{d(a \sin x + b \cos x)}{(a \sin x + b \cos x)^2} = -\frac{1}{a \sin x + b \cos x} + C$$

$$d(a \sin x + b \cos x) = (-b \sin x + a \cos x) dx$$

## 第六讲：不定积分 > 组合法

$$ab \neq 0 \quad \int \frac{\sin x}{(a \sin x + b \cos x)^2} dx \quad \exists \varphi \in (0, 2\pi] \text{ 使得 } \frac{a}{\sqrt{a^2 + b^2}} = \cos \varphi \quad \frac{b}{\sqrt{a^2 + b^2}} = \sin \varphi$$

$$= \int \frac{\sin x}{(a^2 + b^2) \sin^2(x + \varphi)} dx \quad \text{令 } x + \varphi = t$$

简化分母

$$= \int \frac{\sin(t - \varphi)}{(a^2 + b^2) \sin^2 t} dt$$

平移

$$= \int \frac{\sin t \cos \varphi - \cos t \sin \varphi}{(a^2 + b^2) \sin^2 t} dt$$

$$= \int \left( \frac{\cos \varphi}{a^2 + b^2} \frac{1}{\sin t} - \frac{\sin \varphi}{a^2 + b^2} \frac{\cos t}{\sin^2 t} \right) dt$$

## 第六讲：不定积分 > 组合法

$$ab \neq 0 \quad \int \frac{\sin^2 x}{a \sin x + b \cos x} dx \quad \text{令 } u = \tan \frac{x}{2}$$

$$= \int \frac{\left( \frac{2u}{1+u^2} \right)^2 \frac{2}{1+u^2}}{a \frac{2u}{1+u^2} + b \frac{1-u^2}{1+u^2}} du$$

$$= \int \frac{8u^2}{(2au + b - bu^2)(1+u^2)^2} du$$

## 第六讲：不定积分 > 组合法

$$ab \neq 0 \quad \int \frac{\sin^2 x}{a \sin x + b \cos x} dx \quad \text{对偶式} \int \frac{\cos^2 x}{a \sin x + b \cos x} dx$$

$$\text{设 } I = \int \frac{\sin^2 x}{a \sin x + b \cos x} dx \quad J = \int \frac{\cos^2 x}{a \sin x + b \cos x} dx$$

$$I + J = \int \frac{\sin^2 x + \cos^2 x}{a \sin x + b \cos x} dx = \int \frac{1}{a \sin x + b \cos x} dx = \frac{1}{2\sqrt{a^2 + b^2}} \ln \left| \frac{a \cos x - b \sin x - \sqrt{a^2 + b^2}}{a \cos x - b \sin x + \sqrt{a^2 + b^2}} \right| + C$$

$$a^2 I - b^2 J = \int \frac{a^2 \sin^2 x - b^2 \cos^2 x}{a \sin x + b \cos x} dx = \int (a \sin x - b \cos x) dx = -a \cos x - b \sin x + C$$



# 第六讲：不定积分 > 组合法

$$ab \neq 0 \quad \int \frac{\sin^2 x}{a \sin x + b \cos x} dx \quad \exists \varphi \in (0, 2\pi] \text{ 使得 } \frac{a}{\sqrt{a^2 + b^2}} = \cos \varphi \quad \frac{b}{\sqrt{a^2 + b^2}} = \sin \varphi$$

$$= \int \frac{\sin^2 x}{\sqrt{a^2 + b^2} \sin(x + \varphi)} dx \quad \text{令 } x + \varphi = t$$

$$= \int \frac{\sin^2(t - \varphi)}{\sqrt{a^2 + b^2} \sin t} dt$$

$$= \int \frac{(\sin t \cos \varphi - \cos t \sin \varphi)^2}{\sqrt{a^2 + b^2} \sin t} dt$$

$$= \int \frac{\sin^2 t \cos^2 \varphi - 2 \sin t \cos t \sin \varphi \cos \varphi + \cos^2 t \sin^2 \varphi}{\sqrt{a^2 + b^2} \sin t} dt$$

$$= \int \left( \frac{\cos^2 \varphi}{\sqrt{a^2 + b^2}} \sin t - \frac{2 \sin \varphi \cos \varphi}{\sqrt{a^2 + b^2}} \cos t + \frac{\sin^2 \varphi}{\sqrt{a^2 + b^2}} \frac{\cos^2 t}{\sin t} \right) dt$$

## 第六讲：不定积分 > 组合法

$$\int \frac{\sin x}{1 + \sin x \cos x} dx \quad \text{对偶式} \int \frac{\cos x}{1 + \sin x \cos x} dx \quad \sin x \cos x = \frac{(\sin x + \cos x)^2 - 1}{2} = \frac{1 - (\sin x - \cos x)^2}{2}$$

$$\text{设 } I = \int \frac{\sin x}{1 + \sin x \cos x} dx \quad J = \int \frac{\cos x}{1 + \sin x \cos x} dx$$

$$\begin{aligned} I + J &= \int \frac{\sin x + \cos x}{1 + \sin x \cos x} dx = \int \frac{d(\sin x - \cos x)}{1 + \frac{1 - (\sin x - \cos x)^2}{2}} = \int \frac{2 d(\sin x - \cos x)}{3 - (\sin x - \cos x)^2} \\ &= \frac{1}{\sqrt{3}} \ln \left| \frac{\sin x - \cos x + \sqrt{3}}{\sin x - \cos x - \sqrt{3}} \right| + C \end{aligned}$$

$$\begin{aligned} I - J &= \int \frac{\sin x - \cos x}{1 + \sin x \cos x} dx = \int \frac{-d(\sin x + \cos x)}{1 + \frac{(\sin x + \cos x)^2 - 1}{2}} = \int \frac{-2 d(\sin x + \cos x)}{1 + (\sin x + \cos x)^2} \\ &= -2 \arctan(\sin x + \cos x) + C \end{aligned}$$

## 第六讲：不定积分 > 组合法

$$\int \frac{\sin^3 x}{1 + \sin x \cos x} dx \quad \text{对偶式} \int \frac{\cos^3 x}{1 + \sin x \cos x} dx \quad \sin x \cos x = \frac{(\sin x + \cos x)^2 - 1}{2} = \frac{1 - (\sin x - \cos x)^2}{2}$$

$$\text{设 } I = \int \frac{\sin^3 x}{1 + \sin x \cos x} dx \quad J = \int \frac{\cos^3 x}{1 + \sin x \cos x} dx$$

$$I + J = \int \frac{\sin^3 x + \cos^3 x}{1 + \sin x \cos x} dx = \int \frac{(\sin x + \cos x)(\sin^2 x - \sin x \cos x + \cos^2 x)}{1 + \sin x \cos x} dx = \int \frac{1 - \sin x \cos x}{1 + \sin x \cos x} d(\sin x - \cos x)$$

$$= \int \frac{1 - \frac{1 - (\sin x - \cos x)^2}{2}}{1 + \frac{1 - (\sin x - \cos x)^2}{2}} d(\sin x - \cos x) = \int \left( \frac{4}{3 - (\sin x - \cos x)^2} - 1 \right) d(\sin x - \cos x)$$

$$= \frac{4}{2\sqrt{3}} \ln \left| \frac{\sin x - \cos x + \sqrt{3}}{\sin x - \cos x - \sqrt{3}} \right| - (\sin x - \cos x) + C$$

$$I - J = \int \frac{\sin^3 x - \cos^3 x}{1 + \sin x \cos x} dx = \int \frac{(\sin x - \cos x)(\sin^2 x + \sin x \cos x + \cos^2 x)}{1 + \sin x \cos x} dx = \int (\sin x - \cos x) dx$$

$$= -\cos x - \sin x + C$$

## 第六讲：不定积分 > 组合法

$$\int \frac{\sin^2 x}{1 + \sin x \cos x} dx \quad \int \frac{\sin x}{1 + \sin x \cos x} dx \quad \int \frac{\sin^3 x}{1 + \sin x \cos x} dx$$

$$= \int \frac{\tan^2 x \sec^2 x}{(\sec^2 x + \tan x) \sec^2 x} dx$$

$$= \int \frac{\tan^2 x}{(\tan^2 x + 1 + \tan x)(\tan^2 x + 1)} d \tan x \quad \text{令 } \tan x = u$$

$$= \int \frac{u^2}{(u^2 + 1 + u)(u^2 + 1)} du$$

## 第六讲：不定积分 > 组合法

$$n > 1 \quad \int \frac{\sin^2 x}{1 + n \sin x} dx \quad \text{对偶式} \quad \int \frac{1}{1 + n \sin x} dx$$

$$\text{设 } I = \int \frac{\sin^2 x}{1 + n \sin x} dx \quad J = \int \frac{1}{1 + n \sin x} dx$$

$$J - n^2 I = \int \frac{1 - n^2 \sin^2 x}{1 + n \sin x} dx = \int (1 - n \sin x) dx = x + n \cos x + C$$

$$J = \int \frac{1}{1 + n \sin x} dx = \int \frac{\frac{2}{1+u^2}}{1+n \frac{2u}{1+u^2}} du = \int \frac{2}{1+u^2+2nu} du = \frac{1}{\sqrt{n^2-1}} \ln \left| \frac{u+n-\sqrt{n^2-1}}{u+n+\sqrt{n^2-1}} \right| + C$$

$$\text{令 } u = \tan \frac{x}{2} \quad = \frac{1}{\sqrt{n^2-1}} \ln \left| \frac{\tan \frac{x}{2} + n - \sqrt{n^2-1}}{\tan \frac{x}{2} + n + \sqrt{n^2-1}} \right| + C$$

# 第六讲：不定积分 > 组合法

$$a \neq 0 \quad \int \frac{dx}{x(x^n + a)} \quad \frac{1}{x} = \frac{x^n + a}{x(x^n + a)} \quad \text{对偶式} \int \frac{dx}{x}$$

$$\text{设 } I = \int \frac{dx}{x(x^n + a)} \quad J = \int \frac{dx}{x}$$

$$J - aI = \int \frac{x^n}{x(x^n + a)} dx = \int \frac{x^{n-1} dx}{x^n + a} = \frac{1}{n} \int \frac{d(x^n + a)}{x^n + a} = \frac{1}{n} \ln|x^n + a| + C$$

$$J = \ln|x| + C$$

## 第六讲：不定积分 > 组合法

$$\int \frac{x^2 \arctan x}{1+x^2} dx \quad \arctan x = \frac{(1+x^2)\arctan x}{1+x^2} \quad \text{对偶式} \int \arctan x dx$$

$$\text{设 } I = \int \frac{x^2 \arctan x}{1+x^2} dx \quad J = \int \arctan x dx$$

$$J - I = \int \frac{\arctan x}{1+x^2} dx = \int \arctan x d(\arctan x) = \frac{1}{2} \arctan^2 x + C$$

$$J = \int \arctan x dx = x \arctan x - \int \frac{x}{1+x^2} dx = x \arctan x - \frac{1}{2} \ln(1+x^2) + C$$

$$\text{对偶式} \int \frac{\arctan x}{1+x^2} dx$$

## 第六讲：不定积分 > 组合法

$$\int e^x (2 \sin 4x + \cos 4x) dx = I + 2J$$

$$\text{设 } I = \int e^x \cos 4x dx \quad J = \int e^x \sin 4x dx$$

$$(e^x \cos 4x)' = e^x \cos 4x - 4e^x \sin 4x \Rightarrow e^x \cos 4x + C = I - 4J$$

$$(e^x \sin 4x)' = e^x \sin 4x + 4e^x \cos 4x \Rightarrow e^x \sin 4x + C = J + 4I$$



## 第六讲：不定积分 > 组合法

$$\int x e^x (\cos 4x + 2 \sin 4x) dx = I + 2J$$

$$\text{设 } I = \int x e^x \cos 4x dx \quad J = \int x e^x \sin 4x dx$$

$$(x e^x \cos 4x)' = e^x \cos 4x + x e^x \cos 4x - 4 x e^x \sin 4x \quad \Rightarrow x e^x \cos 4x = \int e^x \cos 4x dx + I - 4J$$

$$(x e^x \sin 4x)' = e^x \sin 4x + x e^x \sin 4x + 4 x e^x \cos 4x \quad \Rightarrow x e^x \sin 4x = \int e^x \sin 4x dx + J + 4I$$