例1 （2011）： 是否存在区间 $[0,2]$ 上的连续可微函数 $f(x)$

满足：$f(0)=f(2)=1,\ |f'(x)|\le 1;\ \left|\int_0^2 f(x)dx\right|\le 1$

说明理由。 中值定理 → 常经理

0 x 2

假设 $f(x)$ 存在的. $f\in C[0,2]$，可微. $f(0)=f(2)=1$

$|f'(x)|\le 1$ $\left|\int_0^2 f(x)dx\right|\le 1$

$x\in(0,1]$ $f(x)-f(0)=f'(\xi_1)\cdot x$ $\xi_1\in(0,x)$ ①

$x\in[1,2)$ $f(2)-f(x)=f'(\xi_2)(2-x)$ $\xi_2\in(x,2)$ ②

由①，及 $f(0)=f(2)=1$ $\Rightarrow$ $f(x)=1+f'(\xi_1)x$ $\xi_1\in(0,x)$

由② $\Rightarrow$ $f(x)=1+f'(\xi_2)(2-x)$ $\xi_2\in(x,2)$

由 $|f'(x)|\le 1$

$f(x)\ge 1-x$ $x\in(0,1]$

$f(x)\ge 1-(2-x)=x-1$ $x\in[1,2)$

$\left|\int_0^2 f(x)dx\right|\le 1$

$\int_0^2 f(x)dx=\int_0^1 f(x)dx+\int_1^2 f(x)dx$

$\boxed{>}\int_0^1(1-x)dx+\int_1^2(x-1)dx$

$=-\frac{1}{2}(1-x)^2\Big|_0^1+\frac{1}{2}(x-1)^2\Big|_1^2$

$=\frac{1}{2}+\frac{1}{2}=1$ $\int_0^2 f(x)dx>1$

矛盾 假设不成立 从而不存在这样的函数

例2： 设 $f\in C^3[0,1],\ f(0)=0,\ f(1)=\frac{1}{2},\ f'(\frac{1}{2})=0.$

0 $\frac{1}{2}$ 1

证 $\exists\xi\in(0,1)$，使得 $\boxed{f'''(\xi)=12.}$

$f'(\xi)=\frac{f(b)-f(a)}{b-a}$

$f(x)=f(x_0)+f'(x_0)(x-x_0)+\cdots$

$f(x)=f(x_0)+f'(x_0)(x-x_0)+\cdots$

$f(0)=f(\frac{1}{2})+\cancel{f'(\frac{1}{2})(0-\frac{1}{2})}+\frac{1}{2}f''(\frac{1}{2})(0-\frac{1}{2})^2+\frac{1}{3!}f'''(\xi_1)(0-\frac{1}{2})^3$ $\quad \xi_1\in(0,\frac{1}{2})$

$f(1)=f(\frac{1}{2})+\cancel{f'(\frac{1}{2})(1-\frac{1}{2})}+\frac{1}{2}f''(\frac{1}{2})(1-\frac{1}{2})^2+\frac{1}{3!}f'''(\xi_2)(1-\frac{1}{2})^3$ $\quad \xi_2\in(\frac{1}{2},1)$

$0=f(\frac{1}{2})+\frac{1}{8}f''(\frac{1}{2})-\frac{1}{48}f'''(\xi_1)$ ①

$\frac{1}{2}=f(\frac{1}{2})+\frac{1}{8}f''(\frac{1}{2})+\frac{1}{48}f'''(\xi_2)$ ②

xy huang 于 2021/10/10 9:44 修改

② − ① $\Rightarrow$ $\frac{1}{2}=\frac{1}{48}[f'''(\xi_2)+f'''(\xi_1)]=\frac{f'''(\xi_1)+f'''(\xi_2)}{48}$

$\Rightarrow 12=\frac{f'''(\xi_1)+f'''(\xi_2)}{2}$

1° $f'''(\xi_1)=f'''(\xi_2)$，$\xi=\xi_1$ 或 $\xi=\xi_2$ $f'''(\xi)=12$

2° $f'''(\xi_1)<f'''(\xi_2)\Rightarrow f'''(\xi_1)<\frac{f'''(\xi_1)+f'''(\xi_2)}{2}<f'''(\xi_2)$

$\exists\xi\in(\xi_1,\xi_2)\Rightarrow f'''(\xi)=\frac{f'''(\xi_1)+f'''(\xi_2)}{2}$

例3（2013）：设 $f(x)$ 在 $[1,+\infty)$ 连续可导，且有

$$f'(x)=\frac{1}{1+f^2(x)}\left[\sqrt{\frac{1}{x}}-\sqrt{\ln\left(1+\frac{1}{x}\right)}\right]$$

证明：$\lim\limits_{x\to+\infty}f(x)$ 存在。

$|f'(x)|>1 \quad x\in[1,+\infty)$

$f(x)-f(1)=f'(\xi)(x-1)$

$|f(x)|=|f(1)+f'(\xi)\cdot(x-1)|$

$>|f'(\xi)|\cdot|x-1|-|f(1)|$

$\ln(1+\frac{1}{x})\Rightarrow$ 考虑函数 $f(t)=\ln(1+t)$

$\Rightarrow f'(t)=\frac{1}{1+t}\Rightarrow \ln(1+t)-\ln 1=f'(\xi)\cdot t \quad \xi\in(0,t)$

$\Rightarrow \ln(1+t)=\frac{t}{1+\xi} \quad \xi\in(0,t)$

$$\Rightarrow \frac{t}{1+t} < \frac{t}{1+\xi} < t$$

$$\Rightarrow \frac{t}{1+t} < \ln(1+t) < t$$

$$\Rightarrow \frac{\frac{1}{x}}{1+\frac{1}{x}} < \ln\left(1+\frac{1}{x}\right) < \frac{1}{x}$$

$$\Rightarrow \frac{1}{x+1} < \ln\left(1+\frac{1}{x}\right) < \frac{1}{x}$$ $f'(x)>0$

$$f'(x) = \frac{1}{1+f^2(x)}\left[\sqrt{\frac{1}{x}} - \sqrt{\ln\left(1+\frac{1}{x}\right)}\right] \le \sqrt{\frac{1}{x}} - \sqrt{\ln\left(1+\frac{1}{x}\right)}$$

$$\le \sqrt{\frac{1}{x}} - \sqrt{\frac{1}{x+1}} = \frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x}\cdot\sqrt{x+1}} = \frac{1}{\sqrt{x}\cdot\sqrt{x+1}\left(\sqrt{x+1}+\sqrt{x}\right)}$$

$$= \frac{1}{\sqrt{x}\cdot\sqrt{x+1}\cdot\sqrt{x+1}+\sqrt{x}\cdot\sqrt{x+1}\cdot\sqrt{x}} \le \frac{1}{2\sqrt{x^3}}$$

$f(x)$ · $f'(x)$ $\Rightarrow f(x)-f(1) = \int_1^x f'(t)\,dt$

$$f(x) = \int_1^x f'(t)\,dt + f(1)$$

$$\le \int_1^x \frac{1}{2\sqrt{t^3}}\,dt + f(1) = \int_1^x \frac{1}{2}\cdot t^{-\frac{3}{2}}\,dt + f(1)$$

$$= -t^{-\frac{1}{2}}\Big|_1^x + f(1) = 1 - \frac{1}{\sqrt{x}} + f(1)$$

$\Rightarrow \lim\limits_{x\to\infty} f(x) \le 1+f(1)$ 有界 单增 $\Rightarrow$ 极限存在

例4（2013）：设函数$f(x)$在$[-2,2]$上二阶可导，且$|f(x)|<1$，又 $f^2(0)+[f'(0)]^2=4$，试证明在$(-2,2)$内至少存在一点 ξ，使得$f(\xi)+f''(\xi)=0$。

$$\left[f^2(x) + (f'(x))^2\right]'$$

$$= 2f(x)\cdot f'(x) + 2f'(x)\cdot f''(x)$$

至少存在一点 ξ ，使得 $f(\xi)+f''(\xi)=0$ 。

$= 2f(x)\cdot f'(x) + 2f'(x)\cdot f''(x)$

$= 2f'(x)[f(x)+f''(x)]$

$$\frac{f(0)-f(-2)}{0+2} = f'(\xi_1) \quad \xi_1 \in (-2,0)$$

$$\frac{f(2)-f(0)}{2} = f'(\xi_2) \quad \xi_2 \in (0,2)$$

由 $|f|\le 1$ $\quad |f'(\xi_1)| = \left|\frac{f(0)-f(-2)}{2}\right| \le \frac{|f(0)|+|f(-2)|}{2} \le 1$

$|f'(\xi_2)| \le 1$

令 $F(x) = f^2(x) + [f'(x)]^2$ $\quad f\in[-2,2]$ 二阶可导

$|F(\xi_1)| = |f^2(\xi_1)| + |f'(\xi_1)|^2 \le 2 \quad |F(\xi_2)| \le 2$

$F(0) = f^2(0) + [f'(0)]^2 = 4$

F 在 $[\xi_1,\xi_2]$ 连续，取最大值 $\max\limits_{x\in[\xi_1,\xi_2]} F(x) = M$

$F(x)$ 最大值一定不会在 ξ_1、ξ_2 处取，在 (ξ_1,ξ_2) 内取 F 可导

$4 = F(0) \le M$

$\Rightarrow$ 极大值点一定在内部取且 $F'(\xi)=0$

$F'(\xi) = 2f'(\xi)[f(\xi)+f''(\xi)] = 0$

$M = F(\xi) = f^2(\xi) + [f'(\xi)]^2 \ge 4 \quad |f|\le 1$

$[f'(\xi)]^2 \ge 3 \Rightarrow f'(\xi) \ne 0$

$\Rightarrow f(\xi)+f''(\xi)=0$

例5（2014）： 设 $f(x)\in C^4[-\infty,+\infty]$, $f(x)$ 满足

$$f(x+h)=f(x)+f'(x)h+\frac{1}{2}f''(x+\theta h)h^2$$ 2阶

其中 θ 是与 x,h 无关的常数，证明 f 是不超过三次

$\boxed{\begin{array}{l} f(x)\ge 0 \quad [a,b] \\ f(c)>0 \\ \int_a^b f(x)dx>0 \end{array}}$

$\int_a^b f(x)dx > 0$

其中 θ 是与 x, h 无关的常数，证明 f 是不超过三次的多项式。

Δ $f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}\cdot h^2 + \frac{f'''(x)}{6}\cdot h^3 + \frac{f^{(4)}(\xi)}{24}\cdot h^4$

$\boxed{f''(x+\theta h) = f''(x) + f'''(x)\theta h + \frac{f^{(4)}(\eta)}{2}\cdot \theta^2 h^2}$

$h \to 0$ $\theta h \to 0$ $\xi \to x$ $\eta \to x$

$f(x+h) = f(x) + f'(x)h + \frac{1}{2}h^2\left[f''(x) + f'''(x)\theta h + \frac{f^{(4)}(\eta)}{2}\cdot\theta^2 h^2\right]$

$= f(x) + f'(x)\cdot h + \frac{1}{2}h^2 f''(x) + \frac{1}{2}h^3\cdot\theta f'''(x) + \frac{1}{4}f^{(4)}(\eta)\cdot\theta^2 h^4$

$\frac{f'''(x)}{6}h^3 + \frac{f^{(4)}(\xi)}{24}h^4 = \frac{1}{2}h^3\theta f'''(x) + \frac{1}{4}f^{(4)}(\eta)\cdot\theta^2 h^4$

$f'''(x)\left[\frac{1}{6} - \frac{1}{2}\theta\right] = \frac{1}{24}h\left[6f^{(4)}(\eta)\cdot\theta^2 - f^{(4)}(\xi)\right]$

$4(1-3\theta)f'''(x) = [6f^{(4)}(\eta)\theta^2 - f^{(4)}(\xi)]h$ 右 $h \to 0$

① $\theta = \frac{1}{3}$ 左边 $=0$. 右边 $= [6f^{(4)}(\eta)\theta^2 - f^{(4)}(\xi)]h = 0$

$\frac{2}{3}f^{(4)}(\eta) = f^{(4)}(\xi) \xrightarrow{h\to 0}$ $f \in C^4[a,b]$

$\frac{2}{3}f^{(4)}(x) = f^{(4)}(x) \Rightarrow f^{(4)}(x) = 0$

$\Rightarrow$ f 最高只有三次.

② $\theta \neq \frac{1}{3}$ 右边 $\to 0$ $\Rightarrow$ $\boxed{4(1-3\theta)}f'''(x) \to 0 \Rightarrow f'''(x) \equiv 0$

$\Rightarrow$ f 最高只有两次

∴ 综上：f 最高不超过三次的多项式

例6（2017）：设 $0<x<\frac{\pi}{2}$，证明：

$$\frac{4}{\pi^2}<\frac{1}{x^2}-\frac{1}{\tan^2 x}<\frac{2}{3}$$

设 $F(x)=\frac{1}{x^2}-\frac{1}{\tan^2 x}$ $x\in(0,\frac{\pi}{2})$

$$F'(x)=-\frac{2}{x^3}+\frac{-\cos x \cdot \sin x \cdot \sin^2 x-\cos^2 x\cdot \sin x\cdot\cos x}{(\sin x)^4}=-\frac{2}{x^3}+\frac{2\cos x}{\sin^3 x}$$

$$=\frac{2(x^3\cos x-\sin^3 x)}{x^3\cdot\sin^3 x}$$

$x^3\cos x-\sin^3 x$ 符号决定

$x^3=\frac{\sin^3 x}{\cos x}$

$x=\sqrt[3]{\frac{\sin^3 x}{\cos x}}$

$$\varphi(x)=\sqrt[3]{\frac{\sin^3 x}{\cos x}}-x=\frac{\sin x}{\sqrt[3]{\cos x}}-x$$

$\varphi(0)=0$

$$\varphi'(x)=\frac{\cos x\cdot\sqrt[3]{\cos x}+\sin x\cdot\frac{1}{3}\cos^{-\frac{2}{3}}x\cdot\sin x}{(\cos x)^{\frac{2}{3}}}-1$$

$$=(\cos x)^{\frac{2}{3}}+\frac{1}{3}(\sin^2 x\cdot\cos^{-\frac{4}{3}}x)-1$$

$$=(\cos x)^{\frac{2}{3}}+\frac{1}{3}(1-\cos^2 x)\cos^{-\frac{4}{3}}x-1$$

$$=\underline{(\cos x)^{\frac{2}{3}}}+\frac{1}{3}\cos^{-\frac{4}{3}}x-\underline{\frac{1}{3}\cos^{\frac{2}{3}}x}-1$$

$$=\underline{\frac{2}{3}(\cos x)^{\frac{2}{3}}+\frac{1}{3}\cos^{-\frac{4}{3}}x}-1$$

$$=\frac{1}{3}(\underline{\cos^{\frac{2}{3}}x+\cos^{\frac{2}{3}}x+\cos^{-\frac{4}{3}}x})-1$$

$$>\sqrt[3]{\cos^{\frac{2}{3}}x\cdot\cos^{\frac{2}{3}}x\cdot\cos^{-\frac{4}{3}}x}-1=1-1=0$$

$\Rightarrow \varphi'(x)>0$ 单增 $x\in(0,\frac{\pi}{2})$ $\varphi(0)=0$

$\Rightarrow F'(x) < 0$ 下降减

$$\lim_{x\to\frac{\pi}{2}^-} F(x) = \lim_{x\to\frac{\pi}{2}^-}\left(\frac{1}{x^2}-\frac{1}{\tan^2 x}\right)=\frac{4}{\pi^2}$$

$$\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+}\left(\frac{1}{x^2}-\frac{1}{\tan^2 x}\right)$$

$\tan x \sim x$

$$=\lim_{x\to 0^+}\frac{\tan^2 x - x^2}{x^2\cdot\tan^2 x} = \lim_{x\to 0^+}\underbrace{\frac{\tan x + x}{x}}_{2}\cdot\frac{\tan x - x}{x\cdot\tan^2 x}$$

$$=\lim_{x\to 0^+} 2\cdot\underbrace{\frac{\tan x - x}{x^3}}_{\frac{1}{3}}\cdot\frac{x^2}{\tan^2 x}\to 1$$

$$=\frac{2}{3}$$

$1-\cos x$

$$\frac{\frac{1}{\cos^2 x}-1}{3x^2} = \frac{1}{3\cos^2 x}\cdot\frac{\sin^2 x}{x^2}\to 1$$

$\to x^2$

$\to\frac{1}{3}$

$$\lim_{x\to 0^+} f(x)=\frac{2}{3}$$

$f'(x)<0$

$$\lim_{x\to\frac{\pi}{2}^-} f(x)=\frac{4}{\pi^2}$$

$$\frac{4}{\pi^2}=\lim_{x\to\frac{\pi}{2}^-} f(x) < f(x) < \lim_{x\to 0^+} f(x) < \frac{2}{3}$$

例7（2018）：设函数$f(x)$在区间$(0,1)$内连续，且存在两两互异的点$x_1,x_2,x_3,x_4\in(0,1)$，使得

$$\alpha=\frac{f(x_1)-f(x_2)}{x_1-x_2}<\frac{f(x_3)-f(x_4)}{x_3-x_4}=\beta$$

证明：对任意$\lambda\in(\alpha,\beta)$，存在互异的两个点$x_5,x_6\in(0,1)$

使得：$\lambda=\frac{f(x_5)-f(x_6)}{x_5-x_6}$

$$F(t)=\frac{f[(1-t)x_2+tx_4]-f[(1-t)x_1+tx_3]}{(1-t)(x_2-x_1)+t(x_4-x_3)}$$ 关于t的连续函数

$$F(0)=\frac{f(x_2)-f(x_1)}{x_2-x_1}=\alpha \qquad \alpha=F(0)<\lambda<F(1)=\beta$$

$$F(1)=\frac{f(x_4)-f(x_3)}{x_4-x_3}=\beta$$

$$F\in C^1(t) \qquad F(0)<\lambda<F(1)$$

$$\Rightarrow \exists\ t_0 \qquad F(t_0)=\lambda$$

$$F(t_0)=\frac{f[(1-t_0)x_2+t_0x_4]-f[(1-t_0)x_1+t_0x_3]}{(1-t_0)(x_2-x_1)+t_0(x_4-x_3)}$$

$$\lambda=\frac{f(x_5)-f(x_6)}{x_5-x_6} \qquad 令\ x_5=(1-t_0)x_1+t_0x_3$$

$$x_6=(1-t_0)x_2+t_0x_4$$

$x_1,x_2\ x_3\ x_4\in(0,1)$ (~~互~~两两不同)

$x_5\ x_6\in(0,1) \qquad x_1<x_2<x_3<x_4$

$$|x_5-x_6|=|(1-t_0)x_1-(1-t_0)x_2+t_0x_3-t_0x_4|$$

$$=|(1-t_0)(x_1-x_2)+t_0(x_3-x_4)|>0$$

例8（2018）： 设函数$f(x)$在区间$[0,1]$内连续且$\int_0^1 f(x)dx\neq 0$

证明：在区间$[0,1]$上存在三个不同的点x_1,x_2,x_3，使得

$$\frac{\pi}{8}\int_0^1 f(x)dx=\left[\frac{1}{1+x_1^2}\int_0^{x_1}f(t)dt+f(x_1)\arctan x_1\right]x_3$$
$$=\left[\frac{1}{1+x_2^2}\int_0^{x_2}f(t)dt+f(x_2)\arctan x_2\right](1-x_3)$$

$$F(x)=\arctan x\cdot\int_0^x f(t)dt$$

$$F(0)=0 \qquad F(1)=\frac{\pi}{4}\cdot\int_0^1 f(t)dt$$

$F(0)=0 \qquad \underline{F(1)=\frac{\pi}{4}\cdot\int_0^1 f(t)dt}$

$$F(x)=\frac{4\arctan x\int_0^x f(t)dt}{\pi\cdot\int_0^1 f(t)dt} \Rightarrow \begin{array}{l} F(0)=0 \\ F(1)=1 \end{array}$$

$\exists\, x_3 \quad \underline{F(x_3)}=\frac{1}{2} \qquad [0,x_3) \quad (x_3,1)$考虑

$(0,x_3) \qquad \underline{F(x_3)-F(0)}=F'(\xi_1)(x_3-0)$

$(x_3,1) \qquad F(x_3)-F(1)=F'(\xi_2)(x_3-1)$

$$F'(x)=\left(\frac{1}{1+x^2}\int_0^x f(t)dt+f(x)\cdot\arctan x\right)\frac{4}{\pi}\cdot\frac{1}{\int_0^1 f(t)dt}$$

$$\frac{\pi}{8}\int_0^1 f(t)dt=\underline{\left[\frac{1}{1+\xi_1^2}\int_0^{\xi_1} f(t)dt+f(\xi_1)\arctan\xi_1\right]x_3}$$

$$\frac{\pi}{8}\int_0^1 f(t)dt=\underline{\left[\frac{1}{1+\xi_2^2}\int_0^{\xi_2} f(t)dt+f(\xi_2)\arctan\xi_2\right]\cdot(1-x_3)}$$

令 $x_1=\xi_1 \quad x_2=\xi_2$

$2\cdot\frac{\pi}{\cancel{8}}\int_0^1 f(t)dt=[$