第十六讲：求常数项级数和

方法一　裂项 $\frac{1}{ab}=\frac{1}{b-a}\left(\frac{1}{a}-\frac{1}{b}\right)$ 或 $\frac{b-a}{ab}=\frac{1}{a}-\frac{1}{b}$，凑差分

方法二　利用和函数

方法三　积分法

方法四　欧拉常数 $\sum_{k=1}^{n}\frac{1}{k}-\ln n\to C$

方法五　利用傅里叶级数

第十六讲：求常数项级数和 > 方法一

p是正整数，求级数$\sum_{n=1}^{\infty}\frac{1}{n(n+p)}$的和

$$\frac{1}{ab}=\frac{1}{b-a}\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$\sum_{n=1}^{N}\frac{1}{n(n+p)}=\sum_{n=1}^{N}\frac{1}{p}\left(\frac{1}{n}-\frac{1}{n+p}\right)$$

$$=\frac{1}{p}\left(\sum_{n=1}^{N}\frac{1}{n}-\sum_{n=1}^{N}\frac{1}{n+p}\right)=\frac{1}{p}\left(\frac{1}{1}+\cdots+\frac{1}{N}\right)-\frac{1}{p}\left(\frac{1}{1+p}+\cdots+\frac{1}{N+p}\right)$$

$$=\frac{1}{p}\left(\frac{1}{1}+\cdots+\frac{1}{p}\right)-\frac{1}{p}\left(\frac{1}{1+N}+\cdots+\frac{1}{N+p}\right)$$

当$N>1+p$

$$\frac{1}{1+N}+\cdots+\frac{1}{N+p}\to 0$$

$$\sum_{n=1}^{N}\frac{1}{n(n+p)}\to\frac{1}{p}\left(\frac{1}{1}+\cdots+\frac{1}{p}\right)$$

$\frac{1}{1+p}+\cdots+\frac{1}{N}$是$\frac{1}{1}+\cdots+\frac{1}{N}$与$\frac{1}{1+p}+\cdots+\frac{1}{N+p}$公共部分

第十六讲：求常数项级数和 > 方法一

求级数$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$的和

$$\frac{1}{ab}=\frac{1}{b-a}\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$\frac{1}{n(n+1)(n+2)}=\frac{1}{n}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{1}{n}\frac{1}{n+1}-\frac{1}{n}\frac{1}{n+2}=\left(\frac{1}{n}-\frac{1}{n+1}\right)-\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+2}\right)$$

$$=\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+2}-\frac{2}{n+1}\right)$$

$$\sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)}=\frac{1}{2}\sum_{n=1}^{N}\left(\frac{1}{n}+\frac{1}{n+2}-\frac{2}{n+1}\right)=\frac{1}{2}\left[\left(\frac{1}{1}+\cdots+\frac{1}{N}\right)+\left(\frac{1}{3}+\cdots+\frac{1}{N+2}\right)-2\left(\frac{1}{2}+\cdots+\frac{1}{N+1}\right)\right]$$

$$=\frac{1}{2}\left(\frac{1}{1}-\frac{1}{N+1}+\frac{1}{N+2}-\frac{1}{2}\right)\to\frac{1}{4}$$

第十六讲：求常数项级数和

凑差分$a_{n+1}-a_n$

$$\sum_{n=1}^{N}(a_{n+1}-a_n)=(a_{N+1}-a_N)+(a_N-a_{N-1})+(a_{N-1}-a_{N-2})+\cdots+(a_4-a_3)+(a_3-a_2)+(a_2-a_1)=a_{N+1}-a_1$$

极大地简化式子

利用$\frac{1}{ab}=\frac{1}{b-a}\left(\frac{1}{a}-\frac{1}{b}\right)$或$\frac{b-a}{ab}=\frac{1}{a}-\frac{1}{b}$产生差分

第十六讲：求常数项级数和 > 方法一

求级数$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$的和

$$\frac{1}{n(n+1)(n+2)}=\frac{1}{n+1}\frac{1}{n(n+2)}=\frac{1}{n+1}\cdot\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+2}\right)=\frac{1}{2}\left(\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\right)$$

$$\sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)}=\sum_{n=1}^{N}\frac{1}{2}\left(\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\right)=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{(N+1)(N+2)}\right)\to\frac{1}{4}$$

第十六讲：求常数项级数和 > 方法一

m是大于1的正整数，求级数$\sum_{n=1}^{\infty}\frac{1}{n(n+1)\cdots(n+m)}$的和

$$\frac{1}{n(n+1)\cdots(n+m)}=\frac{1}{(n+1)\cdots(n+m-1)}\cdot\frac{1}{n(n+m)}$$

$$=\frac{1}{(n+1)\cdots(n+m-1)}\left(\frac{1}{n}-\frac{1}{n+m}\right)\frac{1}{m}$$

$$=\frac{1}{m}\left(\frac{1}{n(n+1)\cdots(n+m-1)}-\frac{1}{(n+1)\cdots(n+m-1)(n+m)}\right)$$

$$\sum_{n=1}^{N}\frac{1}{n(n+1)\cdots(n+m)}=\sum_{n=1}^{N}\frac{1}{m}\left(\frac{1}{n(n+1)\cdots(n+m-1)}-\frac{1}{(n+1)\cdots(n+m-1)(n+m)}\right)$$

$$=\frac{1}{m}\left[\frac{1}{m!}-\frac{1}{(N+1)\cdots(N+m-1)(N+m)}\right]\to\frac{1}{m}\cdot\frac{1}{m!}$$

第十六讲：求常数项级数和 > 方法一

求级数 $\sum_{n=1}^{\infty}\frac{n+2}{n!+(n+1)!+(n+2)!}$ 的和

$$\sum_{n=1}^{N}\frac{n+2}{n!+(n+1)!+(n+2)!}=\sum_{n=1}^{N}\frac{n+2}{n![1+(n+1)+(n+1)(n+2)]}$$

$$=\sum_{n=1}^{N}\frac{n+2}{n!(n+2)^2}=\sum_{n=1}^{N}\frac{1}{n!(n+2)}$$

$$=\sum_{n=1}^{N}\frac{n+1}{(n+2)!}=\sum_{n=1}^{N}\frac{n+2-1}{(n+2)!}$$

$$=\sum_{n=1}^{N}\left(\frac{1}{(n+1)!}-\frac{1}{(n+2)!}\right)=\frac{1}{2!}-\frac{1}{(N+2)!}\to\frac{1}{2}$$

第十六讲：求常数项级数和 > 方法一

q是大于1的正数，求级数$\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)(1+q^{n+1})}$的和

$$\frac{1}{(1+q^n)(1+q^{n+1})}=\frac{1}{q^{n+1}-q^n}\left(\frac{1}{1+q^n}-\frac{1}{1+q^{n+1}}\right)$$

$$\frac{q^n}{(1+q^n)(1+q^{n+1})}=\frac{1}{q-1}\left(\frac{1}{1+q^n}-\frac{1}{1+q^{n+1}}\right)$$

$$\sum_{n=1}^{N}\frac{q^n}{(1+q^n)(1+q^{n+1})}=\sum_{n=1}^{N}\frac{1}{q-1}\left(\frac{1}{1+q^n}-\frac{1}{1+q^{n+1}}\right)=\frac{1}{q-1}\left(\frac{1}{1+q}-\frac{1}{1+q^{N+1}}\right)\to\frac{1}{q^2-1}$$

第十六讲：求常数项级数和 > 方法一

$b>a>0$，求级数 $\sum_{n=1}^{\infty}\frac{a^n b^n}{(a^n+b^n)(a^{n+1}+b^{n+1})}$ 的和

$$\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)(1+q^{n+1})}$$

$$\frac{1}{(a^n+b^n)(a^{n+1}+b^{n+1})}=\frac{1}{a^{n+1}+b^{n+1}-a^n-b^n}\left(\frac{1}{a^n+b^n}-\frac{1}{a^{n+1}+b^{n+1}}\right)$$

$$\frac{a^n b^n}{(a^n+b^n)(a^{n+1}+b^{n+1})}=\frac{a^n b^n}{a^{n+1}+b^{n+1}-a^n-b^n}\left(\frac{1}{a^n+b^n}-\frac{1}{a^{n+1}+b^{n+1}}\right)$$

$$\frac{a^n b^n}{(a^n+b^n)(a^{n+1}+b^{n+1})}=\frac{\left(\frac{b}{a}\right)^n\frac{1}{a}}{\left[1+\left(\frac{b}{a}\right)^n\right]\left[1+\left(\frac{b}{a}\right)^{n+1}\right]}$$

记 $q=\frac{b}{a}>1$

$$=\frac{1}{a}\frac{q^n}{(1+q^n)(1+q^{n+1})}=\frac{q^n}{a(q^{n+1}-q^n)}\left(\frac{1}{1+q^n}-\frac{1}{1+q^{n+1}}\right)=\frac{1}{a(q-1)}\left(\frac{1}{1+q^n}-\frac{1}{1+q^{n+1}}\right)$$

$$\frac{a^n b^n}{(a^n+b^n)(a^{n+1}+b^{n+1})}=\frac{1}{b-a}\left(\frac{a^n}{a^n+b^n}-\frac{a^{n+1}}{a^{n+1}+b^{n+1}}\right)$$

第十六讲：求常数项级数和 > 方法一

求级数 $\sum_{n=1}^{\infty}\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{(n+1)(n+2)}$ 的和

$$\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{(n+1)(n+2)}=\left(\frac{1}{n+1}-\frac{1}{n+2}\right)\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)=\frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-\frac{1}{n+2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

$$=\frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-\frac{1}{n+2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}\right)+\frac{1}{n+2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}\right)-\frac{1}{n+2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

$$=\frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-\frac{1}{n+2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}\right)+\frac{1}{n+2}\cdot\frac{1}{n+1}$$

$$=\frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-\frac{1}{n+2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}\right)+\frac{1}{n+1}-\frac{1}{n+2}$$

$$\sum_{n=1}^{N}\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{(n+1)(n+2)}=\frac{1}{2}-\frac{1}{N+2}\left(1+\frac{1}{2}+\cdots+\frac{1}{N}+\frac{1}{N+1}\right)+\frac{1}{2}-\frac{1}{N+2}\to 1$$

第十六讲：求常数项级数和 > 方法一

求级数$\sum_{n=1}^{\infty}\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{(n+1)(n+2)}$的和

$$\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{(n+1)(n+2)}=\left(\frac{1}{n+1}-\frac{1}{n+2}\right)\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)=\frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-\frac{1}{n+2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

$$=\frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-\frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)+\frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)-\frac{1}{n+2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

$$=\frac{1}{n+1}\cdot\frac{1}{n}+\frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)-\frac{1}{n+2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

$$=\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)-\frac{1}{n+2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

$$\sum_{n=1}^{N}\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{(n+1)(n+2)}=\frac{1}{6}+\sum_{n=2}^{N}\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{(n+1)(n+2)}$$

$$=\frac{1}{6}+\frac{1}{2}-\frac{1}{N+1}+\frac{1}{3}-\frac{1}{N+2}\left(1+\frac{1}{2}+\cdots+\frac{1}{N}\right)\to 1$$

第十六讲：求常数项级数和 > 方法一

求级数 $\sum_{n=1}^{\infty}\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{(n+1)(n+2)}$ 的和

$$\sum_{n=1}^{N}\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{(n+1)(n+2)}=\sum_{n=1}^{N}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

$$\left\{\frac{1}{n+1}\right\}\left\{1+\frac{1}{2}+\cdots+\frac{1}{n}\right\}$$

$$=\sum_{n=1}^{N}(a_n-a_{n+1})b_n=\sum_{n=2}^{N}(b_n-b_{n-1})a_n+a_1b_1-a_{N+1}b_N$$

$$=\sum_{n=2}^{N}\frac{1}{n}\cdot\frac{1}{n+1}+\frac{1}{2}-\frac{1}{N+2}\left(1+\frac{1}{2}+\cdots+\frac{1}{N}\right)$$

$$=\sum_{n=2}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right)+\frac{1}{2}-\frac{1}{N+2}\left(1+\frac{1}{2}+\cdots+\frac{1}{N}\right)$$

$$=\frac{1}{2}-\frac{1}{N+1}+\frac{1}{2}-\frac{1}{N+2}\left(1+\frac{1}{2}+\cdots+\frac{1}{N}\right)\to 1$$

第十六讲：求常数项级数和 > 方法二

$$\sum_{n=0}^{\infty}\frac{(4n+1)(4n+2)(4n+3)}{2^n}$$

求级数$\sum_{n=0}^{\infty}\frac{(4n+1)(4n+2)(4n+3)}{2^n}x^{4n}$的和函数

$$\sum_{n=0}^{\infty}\frac{(n+1)e^{n+1}}{n!}$$

求级数$\sum_{n=0}^{\infty}\frac{(n+1)x^{n+1}}{n!}$的和函数

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!(2n+3)}$$

求级数$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+3}}{(2n+1)!(2n+3)}$的和函数

$$\sum_{n=0}^{\infty}\frac{1}{(4n)!}$$

求级数$\sum_{n=0}^{\infty}\frac{x^{4n}}{(4n)!}$的和函数

第十六讲：求常数项级数和 > 方法二

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)!!}$$

求级数$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!!}$的和函数

$$\sum_{n=0}^{\infty}\frac{n^3}{n!}$$

求级数$\sum_{n=0}^{\infty}\frac{n^3x^n}{n!}$的和函数

p是正整数,$\sum_{n=1}^{\infty}\frac{1}{n(n+p)2^n}$

p是正整数，求级数$\sum_{n=1}^{\infty}\frac{x^n}{n(n+p)}$的和函数

p、q是正整数，且$q<p$,$\sum_{n=1}^{\infty}\frac{1}{n(n+q)(n+p)}$

p、q是正整数，且$q<p$，求级数$\sum_{n=1}^{\infty}\frac{x^n}{n(n+q)(n+p)}$的和函数

$$\frac{1}{1\cdot 2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{5\cdot 2^4}+\cdots+\frac{1}{(2n-1)\cdot 2^{2n-1}}-\frac{1}{(3n-1)\cdot 2^{2n}}+\cdots$$

求级数$\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{5}+\cdots+\frac{x^{2n-1}}{2n-1}-\frac{x^{2n}}{3n-1}+\cdots$的和函数

第十六讲：求常数项级数和 > 方法二

$$\sum_{n=0}^{\infty}\frac{\cos\frac{n\pi}{4}}{n!}$$

$$\sum_{n=0}^{\infty}\frac{\sin\frac{n\pi}{4}}{n!}$$

求级数 $\sum_{n=0}^{\infty}\frac{\cos\frac{n\pi}{4}x}{n!}x^n$ 的和函数

求级数 $\sum_{n=0}^{\infty}\frac{\sin\frac{n\pi}{4}x}{n!}x^n$ 的和函数

$\theta\in R,\sum_{n=0}^{\infty}\frac{\cos n\theta}{n!}$

$\theta\in R,\sum_{n=0}^{\infty}\frac{\sin n\theta}{n!}$

$\theta\in R$，求级数 $\sum_{n=0}^{\infty}\frac{\cos n\theta x}{n!}x^n$ 的和函数

$\theta\in R$，求级数 $\sum_{n=0}^{\infty}\frac{\sin n\theta x}{n!}x^n$ 的和函数

第十六讲：求常数项级数和 > 方法三

求级数 $\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}$ 的和

$$\frac{(-1)^n}{3n+1}=\frac{(-1)^n x^{3n+1}}{3n+1}\bigg|_0^1=\int_0^1\left(\frac{(-1)^n x^{3n+1}}{3n+1}\right)'dx=\int_0^1(-1)^n x^{3n}dx$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}=\sum_{n=0}^{\infty}\int_0^1(-1)^n x^{3n}dx=\int_0^1\sum_{n=0}^{\infty}(-1)^n x^{3n}dx=\int_0^1\frac{1}{1+x^3}dx$$

求级数 $\sum_{n=0}^{\infty}\frac{(-1)^n}{4n+1}$ 的和

第十六讲：求常数项级数和 > 方法三

求级数$\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\cdots+\frac{1}{4n-3}-\frac{1}{4n-1}+\cdots$的和

$$原级数=\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{9}-\frac{1}{11}\right)+\cdots+\left(\frac{1}{4n-3}-\frac{1}{4n-1}\right)+\cdots=\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}-\frac{1}{4n-1}\right)$$

$$\frac{1}{4n-3}-\frac{1}{4n-1}=\left.\left(\frac{x^{4n-3}}{4n-3}-\frac{x^{4n-1}}{4n-1}\right)\right|_0^1=\int_0^1\left(\frac{x^{4n-3}}{4n-3}-\frac{x^{4n-1}}{4n-1}\right)'dx=\int_0^1(x^{4n-4}-x^{4n-2})dx$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}-\frac{1}{4n-1}\right)=\sum_{n=1}^{\infty}\int_0^1(x^{4n-4}-x^{4n-2})dx=\int_0^1\sum_{n=1}^{\infty}(x^{4n-4}-x^{4n-2})dx=\int_0^1\left(\frac{1}{1-x^4}-\frac{x^2}{1-x^4}\right)dx=\int_0^1\frac{1}{1+x^2}dx=\frac{\pi}{4}$$

第十六讲：求常数项级数和 > 方法三

求级数$1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\cdots+\frac{1}{4n-3}+\frac{1}{4n-2}-\frac{1}{4n-1}-\frac{1}{4n}+\cdots$的和

原级数$=\left(1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}\right)+\cdots+\left(\frac{1}{4n-3}+\frac{1}{4n-2}-\frac{1}{4n-1}-\frac{1}{4n}\right)+\cdots$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}+\frac{1}{4n-2}-\frac{1}{4n-1}-\frac{1}{4n}\right)$$

$$\frac{1}{4n-3}+\frac{1}{4n-2}-\frac{1}{4n-1}-\frac{1}{4n}=\left(\frac{x^{4n-3}}{4n-3}+\frac{x^{4n-2}}{4n-2}-\frac{x^{4n-1}}{4n-1}-\frac{x^{4n}}{4n}\right)\Bigg|_0^1=\int_0^1\left(x^{4n-4}+x^{4n-3}-x^{4n-2}-x^{4n-1}\right)dx$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}+\frac{1}{4n-2}-\frac{1}{4n-1}-\frac{1}{4n}\right)=\sum_{n=1}^{\infty}\int_0^1\left(x^{4n-4}+x^{4n-3}-x^{4n-2}-x^{4n-1}\right)dx$$

$$=\int_0^1\sum_{n=1}^{\infty}\left(x^{4n-4}+x^{4n-3}-x^{4n-2}-x^{4n-1}\right)dx=\int_0^1\frac{1+x-x^2-x^3}{1-x^4}dx=\int_0^1\frac{(1+x)(1-x^2)}{(1+x^2)(1-x^2)}dx$$

$$=\int_0^1\frac{1+x}{1+x^2}dx=\frac{\pi}{4}+\frac{\ln 2}{2}$$

$r+s=1$，求级数$\frac{1}{3}-\frac{r}{5}-\frac{s}{3}+\frac{1}{5}-\frac{r}{9}-\frac{s}{7}+\frac{1}{7}-\frac{r}{13}-\frac{s}{11}+\cdots+\frac{1}{2n+1}-\frac{r}{4n+1}-\frac{s}{4n-1}+\cdots$的和

第十六讲：求常数项级数和 > 方法三

$$\lim_{\substack{m\to+\infty\\ n\to+\infty}}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{(-1)^{i+j}}{i+j}$$

$$\frac{(-1)^{i+j}}{i+j}=\left.\frac{x^{i+j}}{i+j}\right|_0^{-1}=\int_0^{-1}\left(\frac{x^{i+j}}{i+j}\right)'dx=\int_0^{-1}x^{i+j-1}dx$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{(-1)^{i+j}}{i+j}=\sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^{-1}x^{i+j-1}dx=\int_0^{-1}\sum_{i=1}^{m}\sum_{j=1}^{n}x^{i+j-1}dx$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}x^{i+j-1}=\sum_{i=1}^{m}x^{i}\left(\sum_{j=1}^{n}x^{j-1}\right)=\left(\sum_{i=1}^{m}x^{i}\right)\left(\sum_{j=1}^{n}x^{j-1}\right)=\frac{x(1-x^{m})}{1-x}\cdot\frac{1-x^{n}}{1-x}=\frac{x(1-x^{m})(1-x^{n})}{(1-x)^{2}}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{(-1)^{i+j}}{i+j}=\int_0^{-1}\frac{x(1-x^{m})(1-x^{n})}{(1-x)^{2}}dx$$

第十六讲：求常数项级数和 > 方法三

原题 $\Rightarrow \lim\limits_{\substack{m\to+\infty\\ n\to+\infty}} \int_0^{-1} \dfrac{x(1-x^m)(1-x^n)}{(1-x)^2}dx$

猜测？？？ $\lim\limits_{\substack{m\to+\infty\\ n\to+\infty}} \int_0^{-1} \dfrac{x(1-x^m)(1-x^n)}{(1-x)^2}dx = \int_0^{-1} \lim\limits_{\substack{m\to+\infty\\ n\to+\infty}} \dfrac{x(1-x^m)(1-x^n)}{(1-x)^2}dx = \int_0^{-1} \dfrac{x}{(1-x)^2}dx$

$$\int_0^{-1} \frac{x(1-x^m)(1-x^n)}{(1-x)^2}dx = \int_0^{-1} \frac{x}{(1-x)^2}dx - \int_0^{-1} \frac{x^{m+1}}{(1-x)^2}dx - \int_0^{-1} \frac{x^{n+1}}{(1-x)^2}dx + \int_0^{-1} \frac{x^{m+n+1}}{(1-x)^2}dx$$

$$\left|\int_0^{-1} \frac{x^{m+1}}{(1-x)^2}dx\right| = \left|\int_{-1}^{0} \frac{x^{m+1}}{(1-x)^2}dx\right| \le \int_{-1}^{0} \left|\frac{x^{m+1}}{(1-x)^2}\right|dx \le \int_{-1}^{0} \left|x^{m+1}\right|dx = \int_0^{1} \left|x^{m+1}\right|dx = \frac{1}{m+2} \to 0 \quad m\to+\infty，\ n\to+\infty$$

$$\lim_{\substack{m\to+\infty\\ n\to+\infty}} \int_0^{-1} \frac{x^{m+1}}{(1-x)^2}dx = 0$$

$$\lim_{\substack{m\to+\infty\\ n\to+\infty}} \int_0^{-1} \frac{x(1-x^m)(1-x^n)}{(1-x)^2}dx = \int_0^{-1} \frac{x}{(1-x)^2}dx$$

第十六讲：求常数项级数和 > 方法四

求级数$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}+\cdots$的和

求级数$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{n}}{n}$的和函数

$$S_{2N}=\sum_{n=1}^{N}\left(\frac{1}{2n-1}-\frac{1}{2n}\right)$$

$$\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{2n-1}+\frac{1}{2n}-\frac{1}{2n}-\frac{1}{2n}=\left(\frac{1}{2n-1}+\frac{1}{2n}\right)-\frac{1}{n}$$

$$\sum_{n=1}^{N}\left(\frac{1}{2n-1}-\frac{1}{2n}\right)=\sum_{n=1}^{N}\left(\frac{1}{2n-1}+\frac{1}{2n}\right)-\sum_{n=1}^{N}\frac{1}{n}$$

$$=\sum_{n=1}^{2N}\frac{1}{n}-\sum_{n=1}^{N}\frac{1}{n}=[\ln(2N)+o(1)]-[\ln N+o(1)]=\ln 2+o(1)$$

$$\lim_{N\to\infty}S_{2N}=\ln 2$$

$$S_{2N+1}=S_{2N}+a_{2N+1}\Rightarrow\lim_{N\to\infty}S_{2N+1}=\ln 2\Rightarrow\lim_{n\to\infty}S_{n}=\ln 2$$

第十六讲：求常数项级数和 > 方法四

求级数 $\frac{4}{3}-\frac{3}{2}-\frac{3}{1}+\frac{4}{5}-\frac{3}{5}-\frac{3}{4}+\frac{4}{7}-\frac{3}{8}-\frac{3}{7}+\cdots+\frac{4}{2n+1}-\frac{3}{3n-1}-\frac{3}{3n-2}+\cdots$ 的和

$$S_{3N}=\sum_{n=1}^{N}\left(\frac{4}{2n+1}-\frac{3}{3n-1}-\frac{3}{3n-2}\right)$$

$$\frac{4}{2n+1}-\frac{3}{3n-1}-\frac{3}{3n-2}=\left(\frac{4}{2n+1}+\frac{4}{2n}\right)-\left(\frac{3}{3n}+\frac{3}{3n-1}+\frac{3}{3n-2}\right)-\frac{4}{2n}+\frac{3}{3n}$$

$$=4\left(\frac{1}{2n+1}+\frac{1}{2n}\right)-3\left(\frac{1}{3n}+\frac{1}{3n-1}+\frac{1}{3n-2}\right)-\frac{1}{n}$$

$$\sum_{n=1}^{N}\left(\frac{4}{2n+1}-\frac{3}{3n-1}-\frac{3}{3n-2}\right)=4\sum_{n=1}^{N}\left(\frac{1}{2n+1}+\frac{1}{2n}\right)-3\sum_{n=1}^{N}\left(\frac{1}{3n}+\frac{1}{3n-1}+\frac{1}{3n-2}\right)-\sum_{n=1}^{N}\frac{1}{n}$$

$$=4\sum_{n=2}^{2N+1}\frac{1}{n}-3\sum_{n=1}^{3N}\frac{1}{n}-\sum_{n=1}^{N}\frac{1}{n}$$

$$=4[\ln(2N+1)-1+o(1)]-3[\ln(3N)+o(1)]-[\ln N+o(1)]$$

$$=\ln\frac{(2N+1)^4}{(3N)^3N}-4+o(1)\to\ln\frac{16}{27}-4$$

$$S_{3N+2}=S_{3N+1}+a_{3N+2}=S_{3N}+a_{3N+2}+a_{3N+2}$$

第十六讲：求常数项级数和 > 方法四

求级数$\frac{1}{3}-\frac{1}{5}-\frac{1}{3}+\frac{1}{5}-\frac{1}{9}-\frac{1}{7}+\frac{1}{7}-\frac{1}{13}-\frac{1}{11}+\cdots+\frac{1}{2n+1}-\frac{1}{4n+1}-\frac{1}{4n-1}+\cdots$的和

$$S_{3N}=\sum_{n=1}^{N}\left(\frac{1}{2n+1}-\frac{1}{4n+1}-\frac{1}{4n-1}\right)$$

$$\frac{1}{4n+2}=\frac{1}{2}\cdot\frac{1}{2n+1}=\frac{1}{2}\left(\frac{1}{2n+1}+\frac{1}{2n}-\frac{1}{2n}\right)=\frac{1}{2}\left(\frac{1}{2n+1}+\frac{1}{2n}\right)-\frac{1}{4n}$$

$$\frac{1}{2n+1}-\frac{1}{4n+1}-\frac{1}{4n-1}=\left(\frac{1}{2n+1}+\frac{1}{2n}\right)-\left(\frac{1}{4n+2}+\frac{1}{4n+1}+\frac{1}{4n}+\frac{1}{4n-1}\right)-\frac{1}{2n}+\frac{1}{4n+2}+\frac{1}{4n}$$

$$=\left(\frac{1}{2n+1}+\frac{1}{2n}\right)-\left(\frac{1}{4n+2}+\frac{1}{4n+1}+\frac{1}{4n}+\frac{1}{4n-1}\right)-\frac{1}{2n}+\frac{1}{2}\left(\frac{1}{2n+1}+\frac{1}{2n}\right)-\frac{1}{4n}+\frac{1}{4n}$$

$$=\frac{3}{2}\left(\frac{1}{2n+1}+\frac{1}{2n}\right)-\left(\frac{1}{4n+2}+\frac{1}{4n+1}+\frac{1}{4n}+\frac{1}{4n-1}\right)-\frac{1}{2n}$$

$$\sum_{n=1}^{N}\left(\frac{1}{2n+1}-\frac{1}{4n+1}-\frac{1}{4n-1}\right)=\frac{3}{2}\sum_{n=1}^{N}\left(\frac{1}{2n+1}+\frac{1}{2n}\right)-\sum_{n=1}^{N}\left(\frac{1}{4n+2}+\frac{1}{4n+1}+\frac{1}{4n}+\frac{1}{4n-1}\right)-\frac{1}{2}\sum_{n=1}^{N}\frac{1}{n}$$

$$=\frac{3}{2}\sum_{n=2}^{2N+1}\frac{1}{n}-\sum_{n=3}^{4N+2}\frac{1}{n}-\frac{1}{2}\sum_{n=1}^{N}\frac{1}{n}$$

$$=\frac{3}{2}[\ln(2N+1)-1+o(1)]-\left[\ln(4N+2)-1-\frac{1}{2}+o(1)\right]-\frac{1}{2}[\ln N+o(1)]$$

$$=\ln\frac{(2N+1)^{\frac{3}{2}}}{(4N+2)N^{\frac{1}{2}}}+o(1)\to\frac{1}{\sqrt{2}}$$

$$S_{3N+2}=S_{3N+1}+a_{3N+2}=S_{3N}+a_{3N+2}+a_{3N+2}$$

第十六讲：求常数项级数和 > 方法四

p、q是正整数，且$q<p$，求级数$\sum_{n=1}^{\infty}\frac{1}{n(n+q)(n+p)}$的和

$$\frac{1}{n(n+q)(n+p)}=\frac{1}{n(n+q)}\frac{1}{n+p}=\frac{1}{q}\left(\frac{1}{n}-\frac{1}{n+q}\right)\frac{1}{n+p}=\frac{1}{q}\left(\frac{1}{n}\frac{1}{n+p}-\frac{1}{n+q}\frac{1}{n+p}\right)$$

$$=\frac{1}{q}\left[\frac{1}{p}\left(\frac{1}{n}-\frac{1}{n+p}\right)-\frac{1}{p-q}\left(\frac{1}{n+q}-\frac{1}{n+p}\right)\right]=\frac{1}{pq}\frac{1}{n}-\frac{1}{p-q}\frac{1}{n+q}+\left(\frac{1}{p-q}-\frac{1}{pq}\right)\frac{1}{n+p}$$

$$\sum_{n=1}^{N}\frac{1}{n(n+q)(n+p)}=\frac{1}{pq}\sum_{n=1}^{N}\frac{1}{n}-\frac{1}{p-q}\sum_{n=1}^{N}\frac{1}{n+q}+\left(\frac{1}{p-q}-\frac{1}{pq}\right)\sum_{n=1}^{N}\frac{1}{n+p}$$

$$=\frac{1}{pq}[\ln N+o(1)]-\frac{1}{p-q}\left[\ln(N+q)-\sum_{n=1}^{q}\frac{1}{n}+o(1)\right]+\left(\frac{1}{p-q}-\frac{1}{pq}\right)\left[\ln(N+p)-\sum_{n=1}^{p}\frac{1}{n}+o(1)\right]$$

$$=\frac{1}{pq}[\ln N+o(1)]-\frac{1}{p-q}\left[\ln N-\sum_{n=1}^{q}\frac{1}{n}+o(1)\right]+\left(\frac{1}{p-q}-\frac{1}{pq}\right)\left[\ln N-\sum_{n=1}^{p}\frac{1}{n}+o(1)\right]$$

$$=\frac{1}{p-q}\sum_{n=1}^{q}\frac{1}{n}-\left(\frac{1}{p-q}-\frac{1}{pq}\right)\sum_{n=1}^{p}\frac{1}{n}+o(1)$$

$$\ln(N+q)-\ln N=\ln\frac{N+q}{N}=o(1)$$

$$\ln(N+p)-\ln N=\ln\frac{N+p}{N}=o(1)$$

第十六讲：求常数项级数和 > 方法五

求级数 $\sum_{n=1}^{\infty}\frac{1}{n^2}$ 的和

$f(x)$ 是周期为 2π 的周期函数，它在 $[-\pi,\pi)$ 的表达式为 $f(x)=|x|$，将 $f(x)$ 展开成傅里叶级数

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2}\qquad x\in(-\infty,+\infty)$$

令 $x=\pi\Rightarrow f(\pi)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)\pi}{(2k-1)^2}\Rightarrow\pi=\frac{\pi}{2}+\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\Rightarrow\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{\pi^2}{8}$

$$\sum_{n=1}^{2N}\frac{1}{n^2}=\sum_{n=1}^{N}\frac{1}{(2n)^2}+\sum_{n=1}^{N}\frac{1}{(2n-1)^2}=\frac{1}{4}\sum_{n=1}^{N}\frac{1}{n^2}+\sum_{n=1}^{N}\frac{1}{(2n-1)^2}$$

令 $N\to+\infty$ 记 $\sum_{n=1}^{\infty}\frac{1}{n^2}=s$

$$s=\frac{1}{4}s+\frac{\pi^2}{8}\Rightarrow s=\frac{\pi^2}{6}$$

第十六讲：求常数项级数和 > 方法五

求级数$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$的和

$$\sum_{n=1}^{2N}\frac{(-1)^{n+1}}{n^2}=\sum_{n=1}^{N}\frac{(-1)^{2n}}{(2n-1)^2}+\sum_{n=1}^{N}\frac{(-1)^{2n+1}}{(2n)^2}=\sum_{n=1}^{N}\frac{1}{(2n-1)^2}-\sum_{n=1}^{N}\frac{1}{(2n)^2}=\sum_{n=1}^{N}\frac{1}{(2n-1)^2}-\frac{1}{4}\sum_{n=1}^{N}\frac{1}{n^2}$$

令$N\to\infty$　记$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=s$

$$s=\frac{\pi^2}{8}-\frac{1}{4}\cdot\frac{\pi^2}{6}=\frac{\pi^2}{12}$$

第十六讲：求常数项级数和 > 方法五

求级数 $\sum_{n=1}^{\infty}\frac{1}{n^4}$ 的和

$f(x)$是周期为 2π的周期函数，它在 $[-\pi，\pi)$的表达式为 $f(x)=|x|$，将$f(x)$展开成傅里叶级数

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2}\qquad x\in(-\infty,+\infty)$$

$$\int_0^x f(x)dx=\int_0^x\frac{\pi}{2}dx-\frac{4}{\pi}\sum_{k=1}^{\infty}\int_0^x\frac{\cos(2k-1)x}{(2k-1)^2}dx\qquad x\in[0，\pi]$$

$$\frac{x^2}{2}=\frac{\pi}{2}x-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)x}{(2k-1)^3}\qquad x\in[0，\pi]$$

$$\int_0^x\frac{x^2}{2}dx=\int_0^x\frac{\pi}{2}xdx-\frac{4}{\pi}\sum_{k=1}^{\infty}\int_0^x\frac{\sin(2k-1)x}{(2k-1)^3}dx\qquad x\in[0，\pi]$$

$$\frac{x^3}{6}=\frac{\pi x^2}{4}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1-\cos(2k-1)x}{(2k-1)^4}\qquad x\in[0，\pi]$$

令$x=\pi\Rightarrow\frac{\pi^3}{6}=\frac{\pi^3}{4}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{2}{(2k-1)^4}\Rightarrow\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}=\frac{\pi^4}{96}$

$$\sum_{n=1}^{2N}\frac{1}{n^4}=\sum_{n=1}^{N}\frac{1}{(2n)^4}+\sum_{n=1}^{N}\frac{1}{(2n-1)^4}$$

$$=\frac{1}{16}\sum_{n=1}^{N}\frac{1}{n^4}+\sum_{n=1}^{N}\frac{1}{(2n-1)^4}$$

令$N\to+\infty$ 记$\sum_{n=1}^{\infty}\frac{1}{n^4}=s$

$$s=\frac{1}{16}s+\frac{\pi^4}{96}\Rightarrow s=\frac{\pi^4}{90}$$

第十六讲：求常数项级数和 > 方法五

求级数$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}$的和

$$\sum_{n=1}^{2N}\frac{(-1)^{n+1}}{n^4}=\sum_{n=1}^{N}\frac{(-1)^{2n}}{(2n-1)^4}+\sum_{n=1}^{N}\frac{(-1)^{2n+1}}{(2n)^4}=\sum_{n=1}^{N}\frac{1}{(2n-1)^4}-\sum_{n=1}^{N}\frac{1}{(2n)^4}=\sum_{n=1}^{N}\frac{1}{(2n-1)^4}-\frac{1}{16}\sum_{n=1}^{N}\frac{1}{n^4}$$

令$N\to\infty$ 记$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}=s$

$$s=\frac{\pi^4}{96}-\frac{1}{4}\cdot\frac{\pi^4}{90}=\frac{\pi^4}{160}$$