

# Problem Set 2

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## Problem 1

a)

a.1.)

We start by defining the utility functions for A and B as follows:

$$\begin{aligned}U_i(A) &:= \alpha + d_i^A a \\U_i(B) &:= \beta + d_i^B b,\end{aligned}$$

where  $\alpha, \beta \geq 0$  and  $d_i^*$  is the number of neighbours that have chosen A or B.

Using both we derive the best reply. In doing so, start with defining the following inequality (1):

$$U_i(A) > U_i(B)$$

Proceed by substituting the definition of the utility functions in the last inequality:

$$\alpha + d_i^A a > \beta + d_i^B b$$

Define  $d_i$  to be the sum of  $d_i^A$  and  $d_i^B$  and rephrase it slightly:

$$\begin{aligned}d_i &:= d_i^A + d_i^B \\d_i^B &= d_i - d_i^A\end{aligned}$$

By substituting the last identity in (1) we can reexpress it in the following way:

$$\begin{aligned}\alpha + ad_i^A &> \beta + b(d_i - d_i^A) \\d_i^A(a + b) &\geq \beta - \alpha + bd_i \\d_i^A &\geq \frac{\beta - \alpha + bd_i}{b + a},\end{aligned}$$

### Conclusion:

$\implies$  The best response is to choose A **iff** the former inequality is satisfied.

We repeat this for B. As inequality we get:

$$d_i^B \geq \frac{\alpha - \beta + ad_i}{b + a},$$

a.2.)

The Nash-Equilibrium (NE) appears when the incentive  $\beta$  is bigger than the incentive  $\alpha$  or when  $b$ , which is how much we value the friend who have chosen  $b$ .

a.3.)

We can not explain with a p-cluster, because the best response is based on  $d_i$  and therefore is not constant. If  $\alpha$ 's and  $\beta$ 's are zero then we can represent Nash-Equilibria as p-clusters.

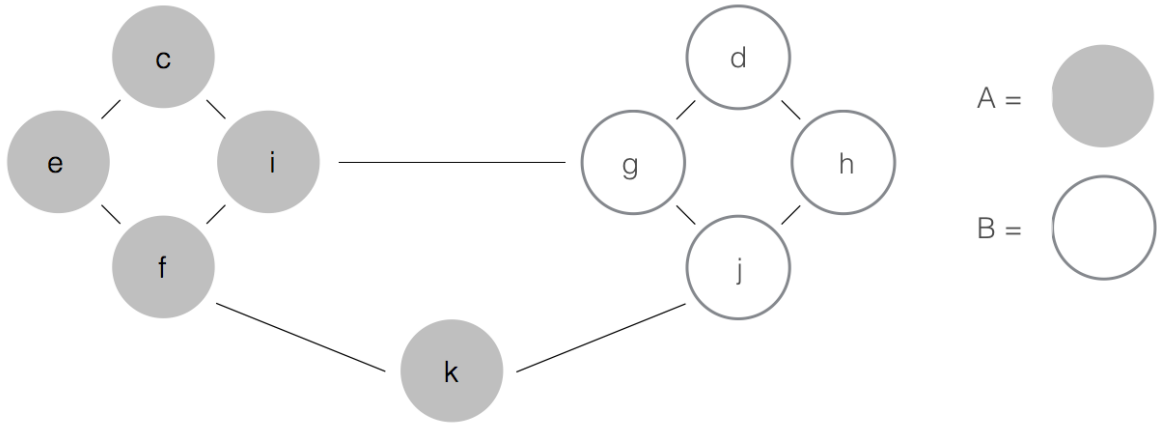
b)

Set  $\alpha = \beta = 0$ . Each nodes starts with B. Let  $t = q = \frac{1}{2}$  for switching to A.

**b.1.)**

Let  $S = \{e - f\}$ , where  $S$  is the starting set meaning that the two nodes initially adopt behaviour A.

**Figure 1**



**b.2.)**

Let  $q = \frac{2}{5}$ , belonging to  $(\frac{1}{3}, \frac{1}{2})$ , so  $p$  is going to be  $\frac{3}{5}$ . The  $p$ -cluster is  $C_p = \{g, h, d, j\}$ , with density  $D = \frac{2}{3}$ , see figure 1.

c)

Figure 2: Nash-Equilibrium 1

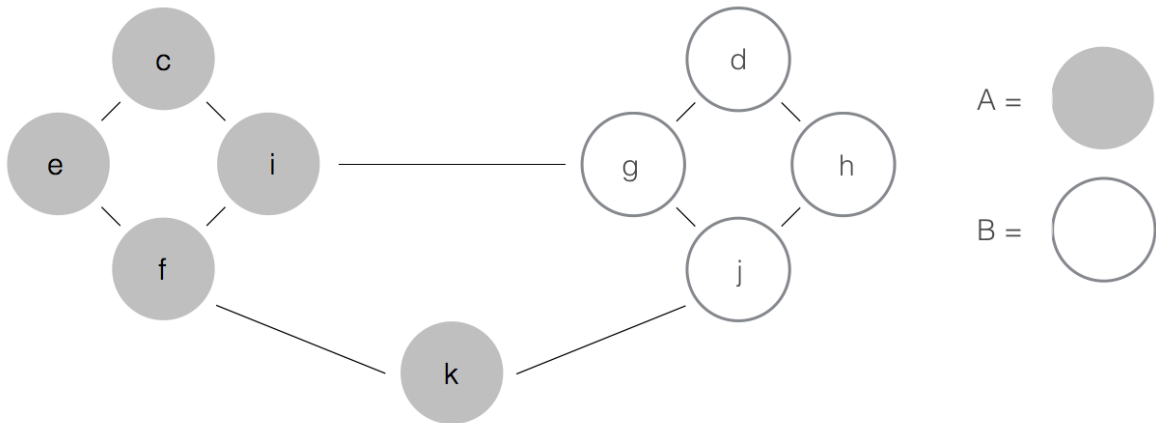
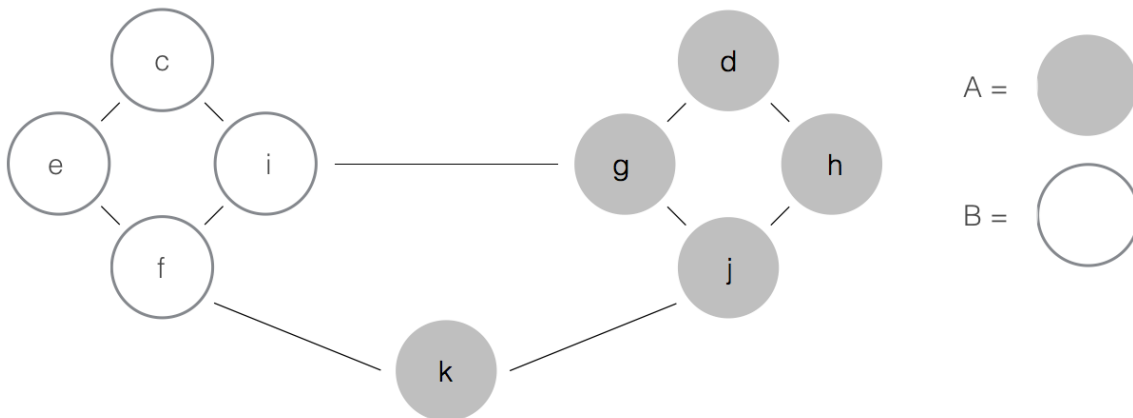
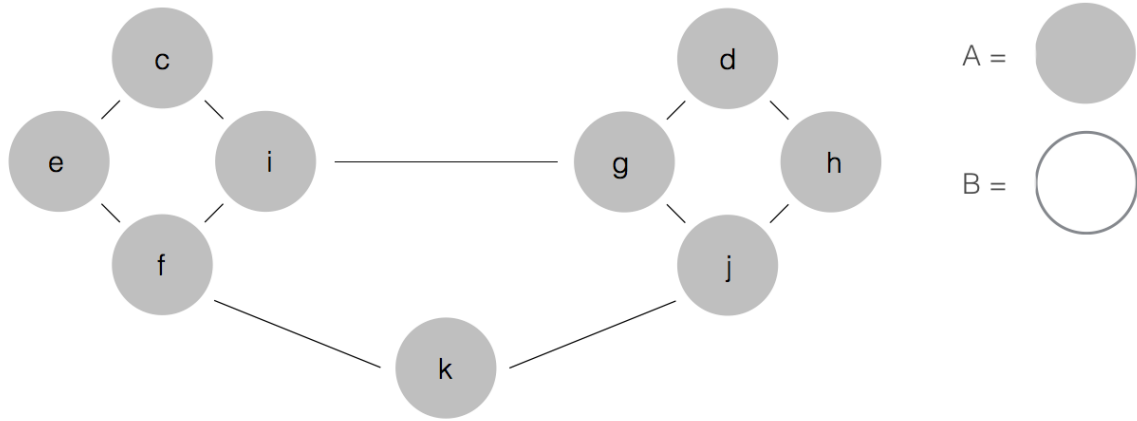


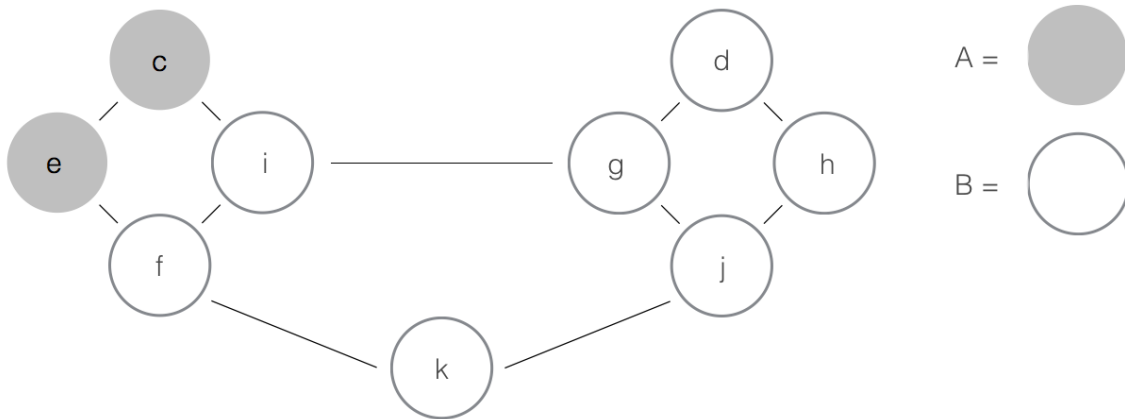
Figure 3: Nash-Equilibrium 2



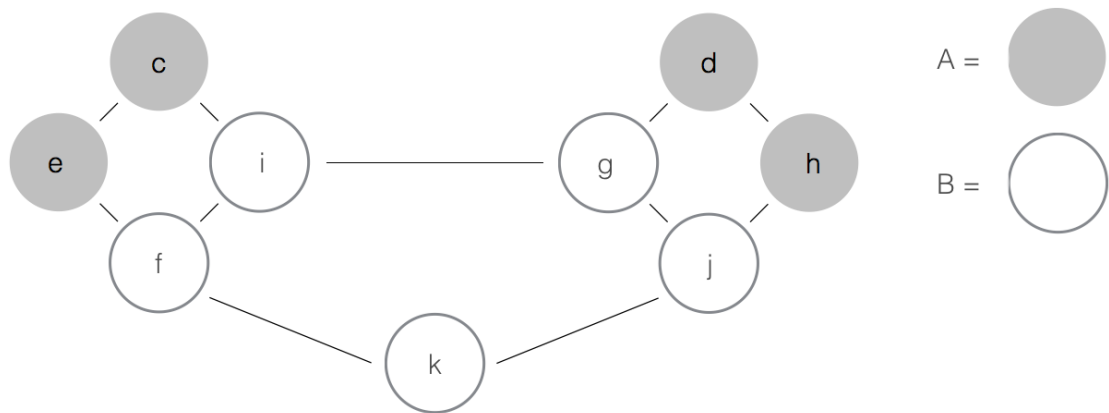
**Figure 4: Nash-Equilibrium 3**



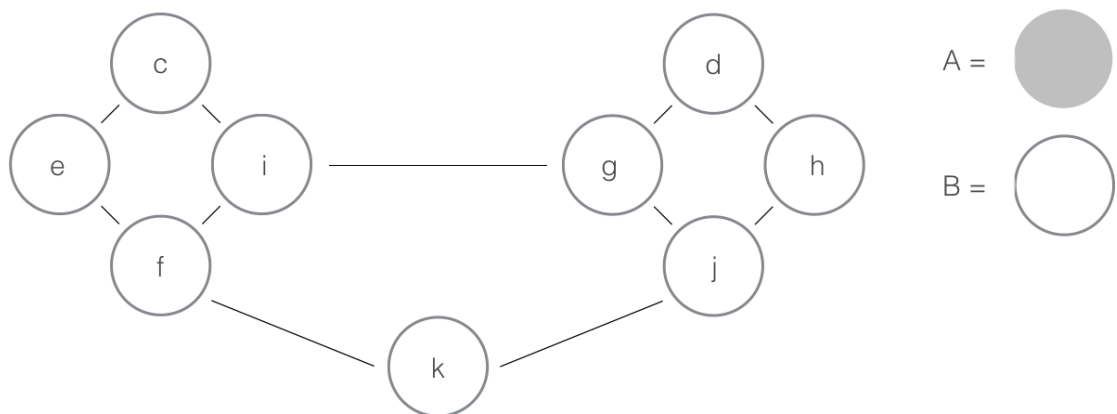
**Figure 5: Nash-Equilibrium 4**



**Figure 6: Nash-Equilibrium 5**



**Figure 7: Nash-Equilibrium 6**



**d)**

We can't have the same Nash-Equilibria. The bigger the difference between  $\alpha$  and  $\beta$  is, the more unlikely it is to subtain the initially appointed equilibria, which ever of both is bigger will point which option A or B, to choose more frequently.

## Problem 2

### 2.1)

#### 2.1.a)

We will find the answer within an iterative process:

##### Step 1:

- $h_{A_i}^{(1)} = 2h_{C_i}^{(1)} = \frac{1}{8}$
- $a_{B_i}^{(1)} = \frac{3}{8}$
- $a_D^{(1)} = \frac{5}{8}$

##### Step 2:

- $h_{A_i}^{(2)} = \frac{9}{8}$
- $a_{B_i}^{(2)} = \frac{27}{8}$
- $h_{C_i}^{(2)} = \frac{5}{8}$
- $a_D^{(2)} = \frac{25}{8}$

##### Conclusion:

From this procedure we find:

$$a_{B_i}^{(2)} > a_D^{(2)}$$

So, even from the second iteration there is a difference between the authority measures of the two different nodes.

#### 2.1.b)

We define the formulas for the two sub-graphs separatly. The **first sub-graph** consists of A's and B's. Following that, let  $b :=$  number of B's and  $a :=$  number of A's and  $c = \text{constant} = \frac{1}{8}$ . For

**For  $k = 1$ :**

- $h^{(1)} = a^0 b^0 c = c$
- $a^{(1)} = a b^0 c$

**For  $k = 2$ :**

- $h^{(2)} = a^{(1)} b = b a b^0 c = b^{(k-1)} a^{(k-1)} c$
- $a^{(2)} = a b a b^0 c = a^{(k)} b^{(k-1)} c$

**For  $k = 3$ :**

- $h^{(3)} = a^{(2)} b = a^{(k-1)} b^{(k-2)} b c = a^{(k-1)} b^{(k-1)} b c$
- $a^{(3)} = a a^{(k-1)} b^{(k-1)} c = a^{(k)} b^{(k-1)} c$

From this procedure we define for the general case:

- $h_{A_i}^{(k)} := a^{(k-1)} b^{(k-1)} b_c$
- $a_{B_i}^{(k)} := a^{(k)} b^{(k-1)} c$

We repeat this iterative procedure for the for the **second sub-graph**:

**For  $k = 1$ :**

- $h^{(1)} = c$
- $a^{(1)} = 5c$

**For  $k = 2$ :**

- $h^{(2)} = 5c = 5^{(k-1)} c$
- $a^{(2)} = 5 * 5c = 5^k c$

Also for the second graph we define for the general case:

- $h_{C_i}^{(k)} = 5^{(k-1)} c$
- $a_D^{(k)} = 5^k c$

### 2.1.c)

Firstly, let the normalizing constant be defined as follows:

$$\begin{aligned} const_h &= 3[a^{k-1} b^{k-1} c] + 5 * 5^{k-1} c \\ &= 3 * 3^{2k-2} c + 5^k c \\ &= (3^{2k-1} + 5^k) c \end{aligned}$$

Following that the normalized version of the hub measures are:

$$\begin{aligned} \widehat{h_{A_i}} &= \frac{h_{A_i}}{const_h} \propto \frac{1}{3} \frac{9^k}{(3^{2k-1} + 5^k)} \\ \widehat{h_{C_i}} &= \frac{5^k}{3^{2k-1} c + 5^k} \end{aligned}$$

The other authority normalizing constant is define as:

$$\begin{aligned} const_a &= 3a^K b^{k-1} c + 5^k c \\ &= 3^k c + 5^k c \\ \widehat{a_{B_i}} &= \frac{3^{2k-1}}{3^K b^{k-1} + 5^k} \propto \frac{1}{3} \frac{9^K}{3^K + 5^K} \\ \widehat{a_D} &= \frac{5^k}{3^K + 5^k} \end{aligned}$$

**Conclusion:** As  $k$  goes to infinity  $\widehat{a_{B_i}}$  will go slower to  $\frac{1}{3}$  and  $\widehat{a_D}$  will converge to 0, assuming that  $5^K$  is sufficiently small compared to  $3^{2K-1}$ . The same is equivalent for  $\widehat{h_{A_i}}$  than  $\widehat{h_{C_i}}$ . Therefore, the hub vector  $h$  is equal to  $h = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0, 0, 0, 0, 0, 0)$  and the authority vector is  $a = (0, 0, 0, 0, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$ , corresponding equivalently for the hub-measure of A's and the authority measure for the B's. So when pages have multiple reinforcement, meaning hub points to more than one page, then in the world web the authorities will be bigger and could more easily be found. If we increase the number of C nodes then there is the possibility that the hub-measure of the C's and the authority measure of D will converge to  $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ . This change of convergence will occur whenever the C nodes are more than nine.

## 2.2)

The result will be that the authority and the hub-measure will coincide.

$$\begin{aligned} \mathbf{h}^k &= (\mathbf{G}\mathbf{G}^T)^k \mathbf{h}^1 = \mathbf{G}^{2k} \mathbf{h}^1 \\ \mathbf{a}^k &= (\mathbf{G}^T \mathbf{G})^{k-1} \mathbf{h}^1 = \mathbf{G}^{2(k-1)} \mathbf{G} \mathbf{h}^1 = \mathbf{G}^{2k-1} \mathbf{h}^1 \approx \mathbf{G}^{2k} \mathbf{h}^1, \end{aligned}$$

when  $k$  goes to infinity.

## 2.3)

To get convergence in the HIT algorithm we should normalize at the last step.

$$\begin{aligned} \mathbf{h}^k &= (\mathbf{G}\mathbf{G}^T)^k \mathbf{h}^1 \\ \frac{\mathbf{h}^{(k)}}{\omega^{(k)}} &= (\mathbf{G}\mathbf{G}^T)^k \frac{\mathbf{h}^1}{\omega^{(k)}} \end{aligned}$$

where  $\omega^{(k)}$  is the normalizing constant at period  $k$ .

As  $k$  goes to infinity we converge to  $\mathbf{h}^*$ , but its direction shouldn't change when it is multiplied by  $\mathbf{G}\mathbf{G}^T$ .

$$\implies \mathbf{G}\mathbf{G}^T \mathbf{h}^* = k \mathbf{h}^*$$

Therefore  $\mathbf{h}^*$  is an eigenvector for  $\mathbf{G}\mathbf{G}^T$  and  $\omega$  is its corresponding eigenvalue.

By power iteration law it follows that  $\omega$  converges to the maximum eigenvalue if  $k$  goes to infinity. For any undirected bipartite graph the adjacency matrix is the following:

$$\mathbf{G} = \begin{pmatrix} 0 & \mathbf{A} \\ \mathbf{A}^T & 0 \end{pmatrix},$$

where  $\mathbf{A}$  is a symmetric matrix. Also  $\mathbf{G}$  is symmetric.

$$\begin{aligned} \mathbf{G}\mathbf{G}^T &= \begin{pmatrix} 0 & \mathbf{A} \\ \mathbf{A}^T & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{A}^T \\ \mathbf{A} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mathbf{A}^2 \\ \mathbf{A}^2 & 0 \end{pmatrix} \end{aligned}$$

This shows all paths of length two. If we assume that self loops exist then  $\mathbf{G}\mathbf{G}^T$  is actually equal to the projection matrix of the bipartite graph. Thus,  $\mathbf{G}\mathbf{G}^T \mathbf{h} = \omega \mathbf{h}^*$  is actually the eigenvector centrality measure of the projection matrix, which by definition is

$$\mathbf{c} = \frac{1}{\lambda_{max}} \mathbf{G}^2 \mathbf{c}$$

, where  $\mathbf{c}$  is the centrality vector.



### Problem 3

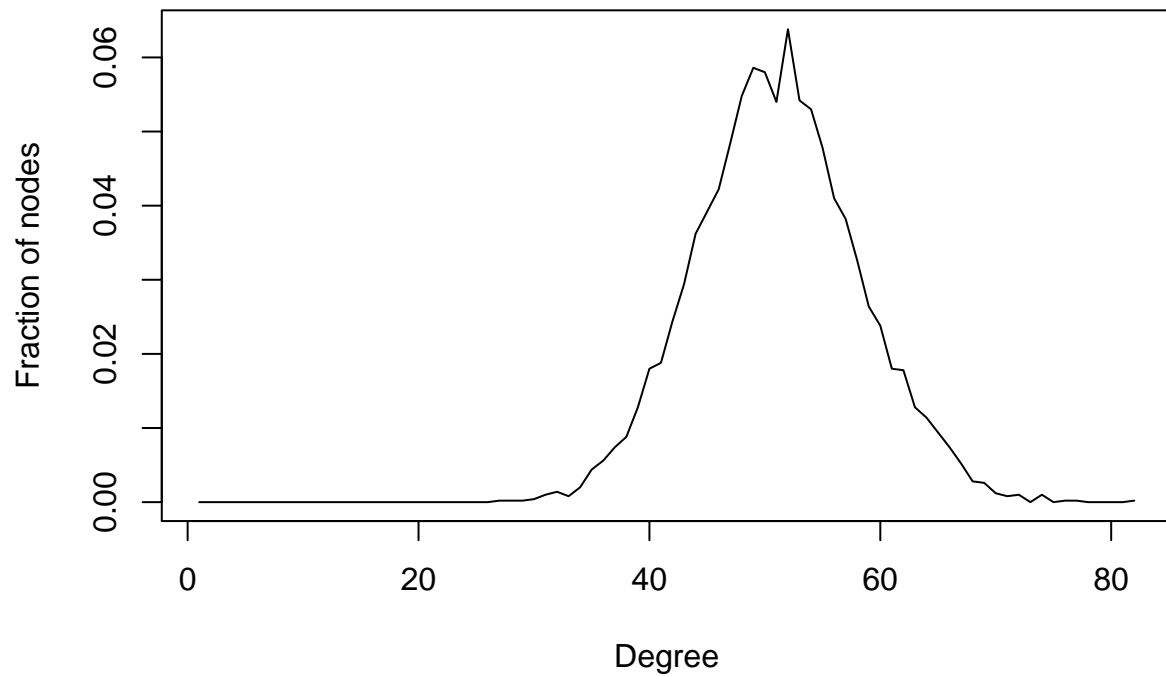
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For probability = 0.01:

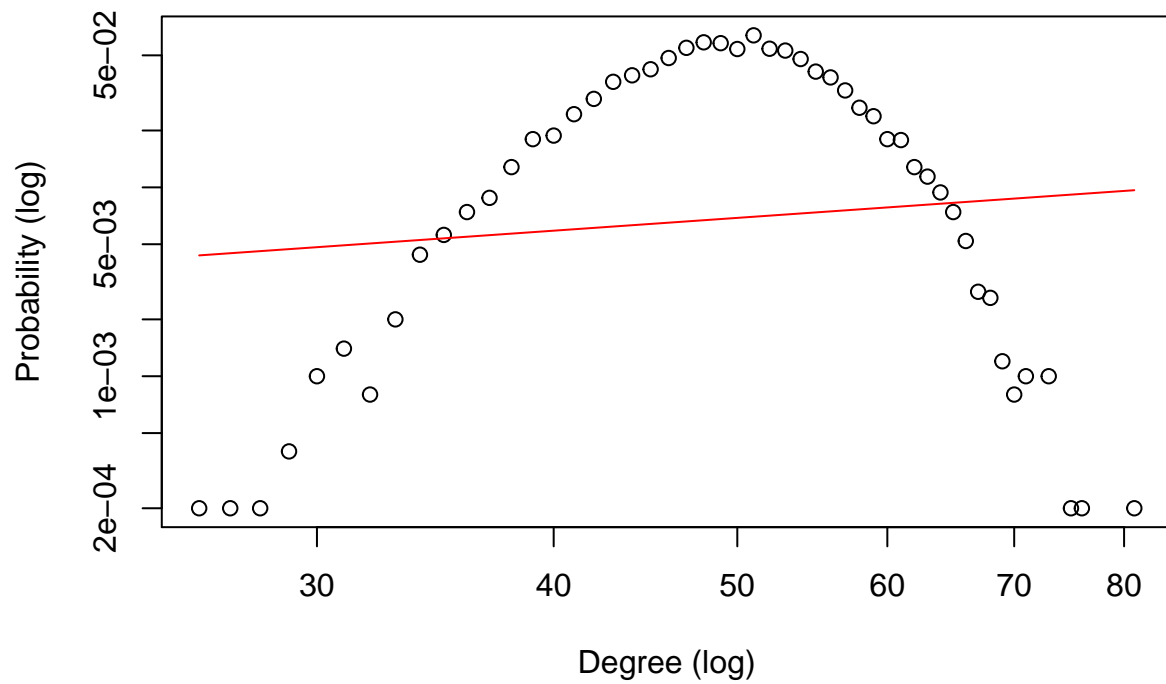
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- average distance = 2.589337
- diameter = 3
- the absolute difference between the average distance and its approximation is 0.4120418
- the absolute difference between the diameter and its approximation is 1.589337

### Degree Distribution



## Degree Distribution



- number of components = 1
- we see in the following table: 1
- ...that the largest component captures more than 80 percent of the nodes.

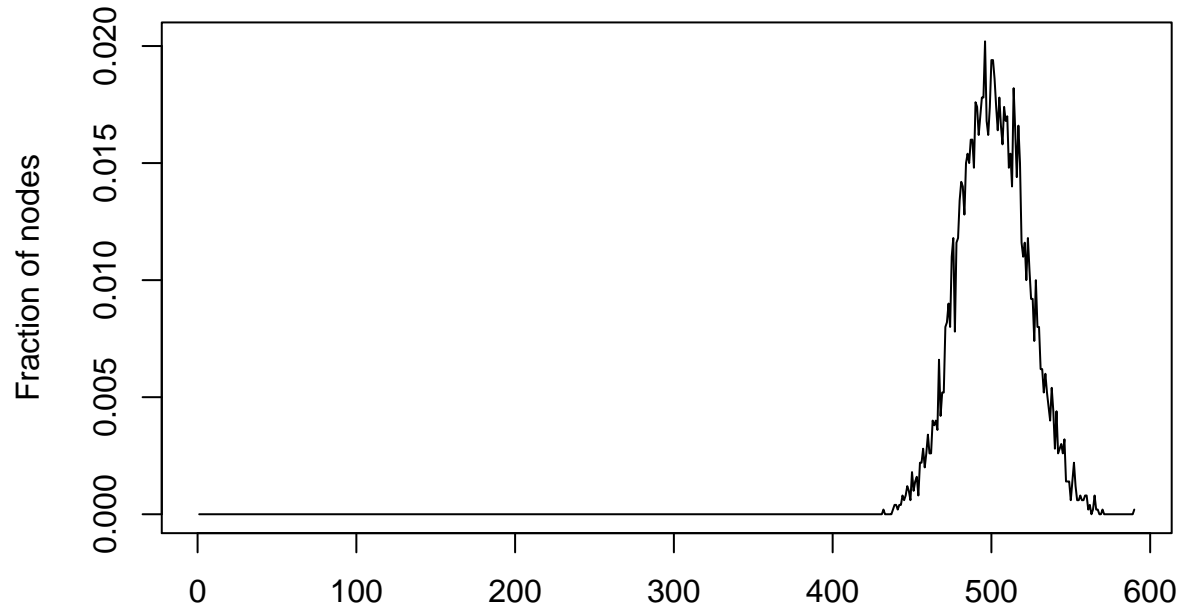
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For probability = 0.1:

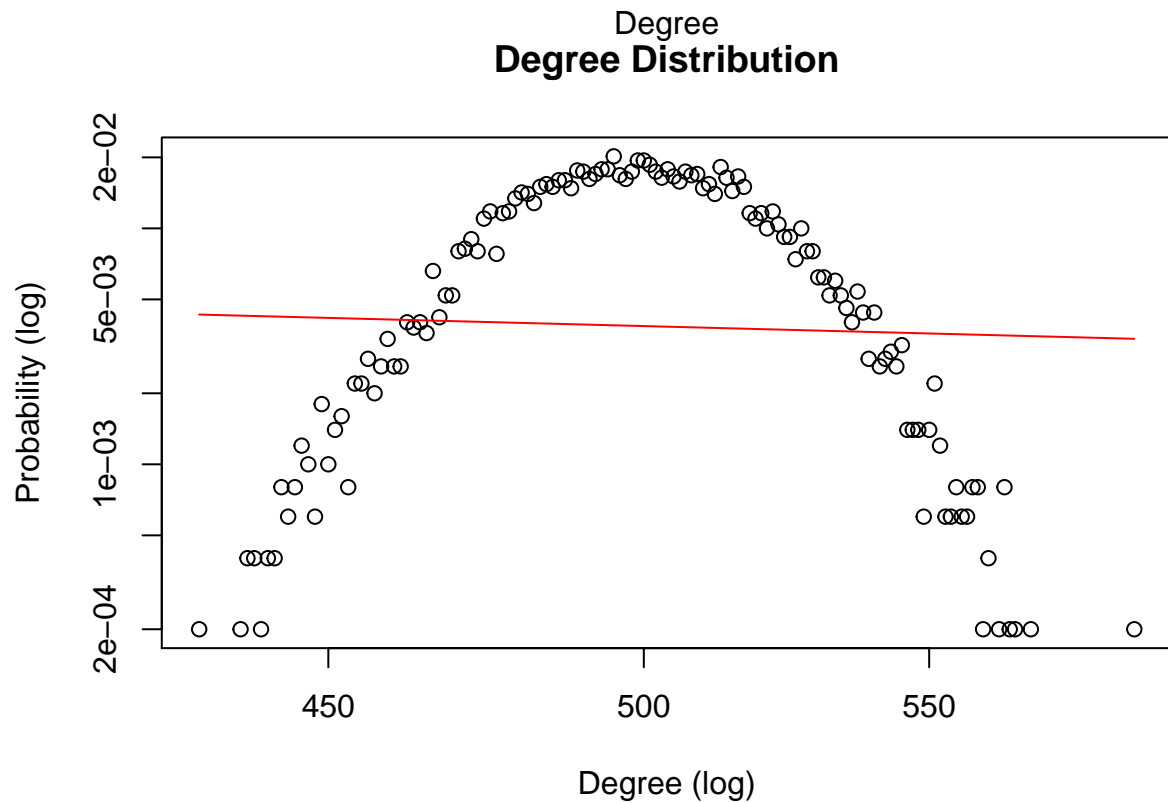
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- average distance = 1.899995
- diameter = 2
- the absolute difference between the average distance and its approximation is 0.529439
- the absolute difference between the diameter and its approximation is 0.8999948

## Degree Distribution



## Degree Distribution



- number of components = 1
- we see in the following table: 1
- ...that the largest component captures more than 80 percent of the nodes.

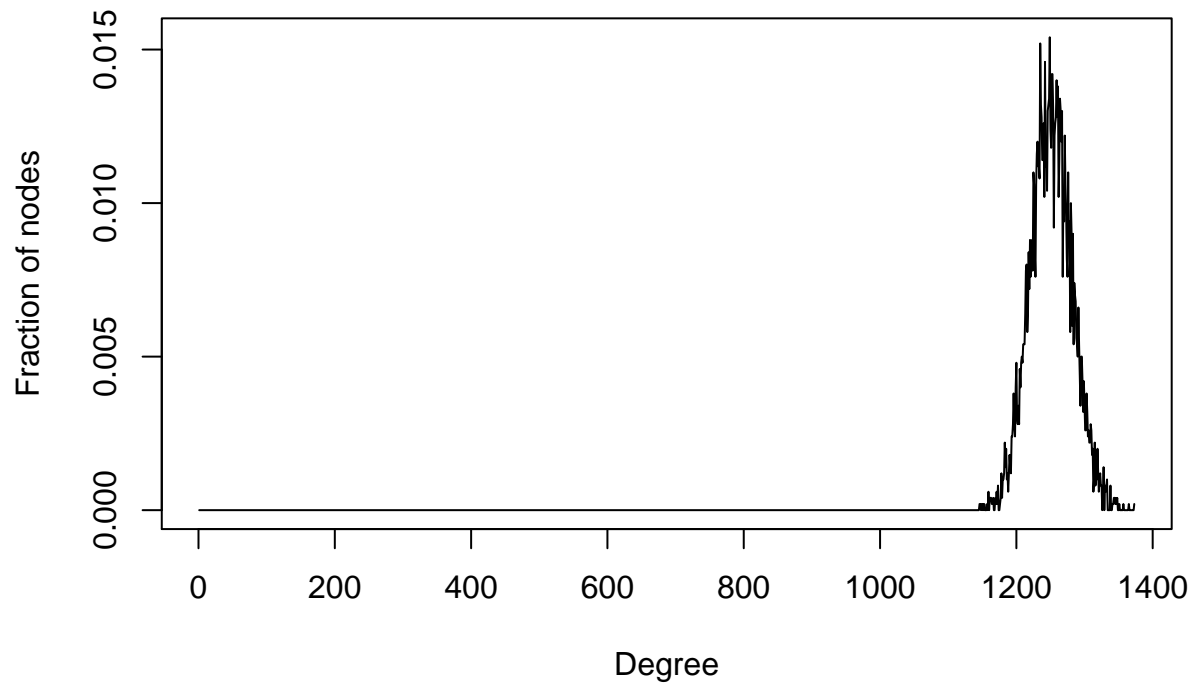
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For probability = 0.25:

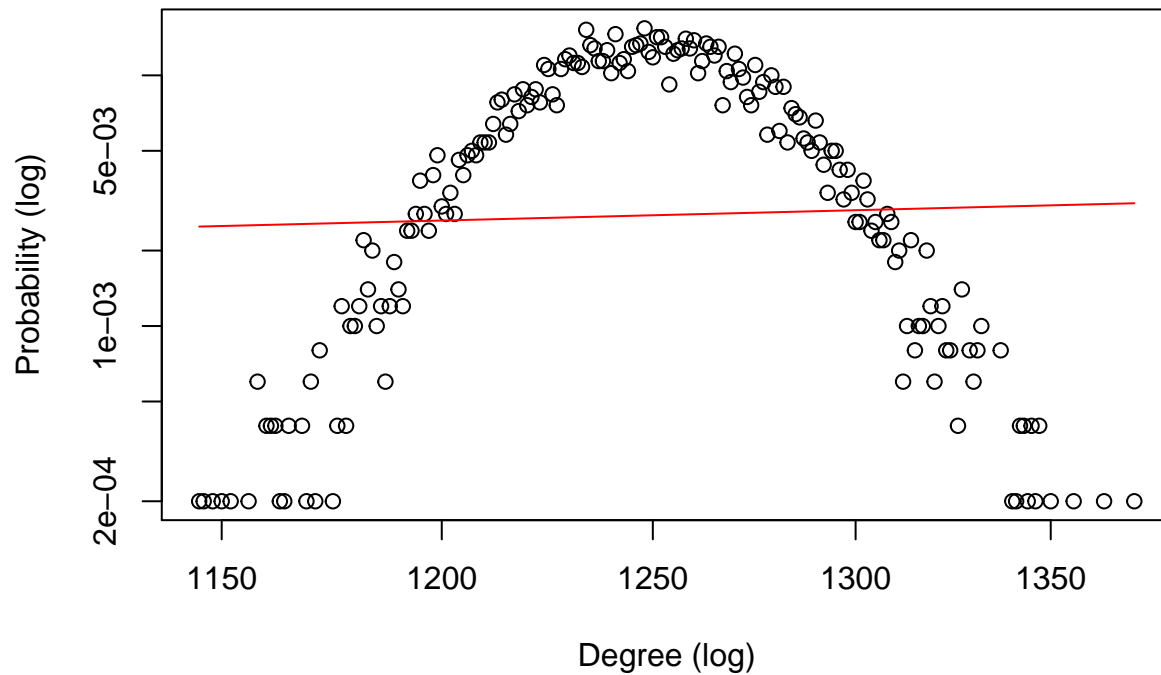
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- average distance = 1.749987
- diameter = 2
- the absolute difference between the average distance and its approximation is 0.5555473
- the absolute difference between the diameter and its approximation is 0.7499874

### Degree Distribution



## Degree Distribution



- number of components = 1
- we see in the following table: 1
- ...that the largest component captures more than 80 percent of the nodes.

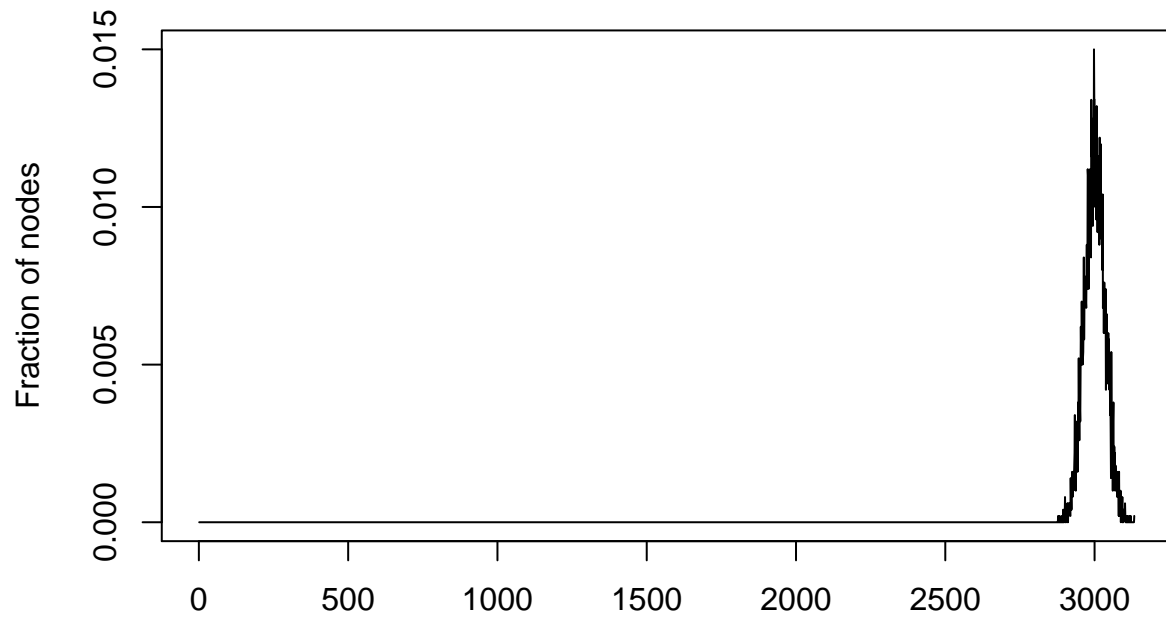
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For probability = 0.6:

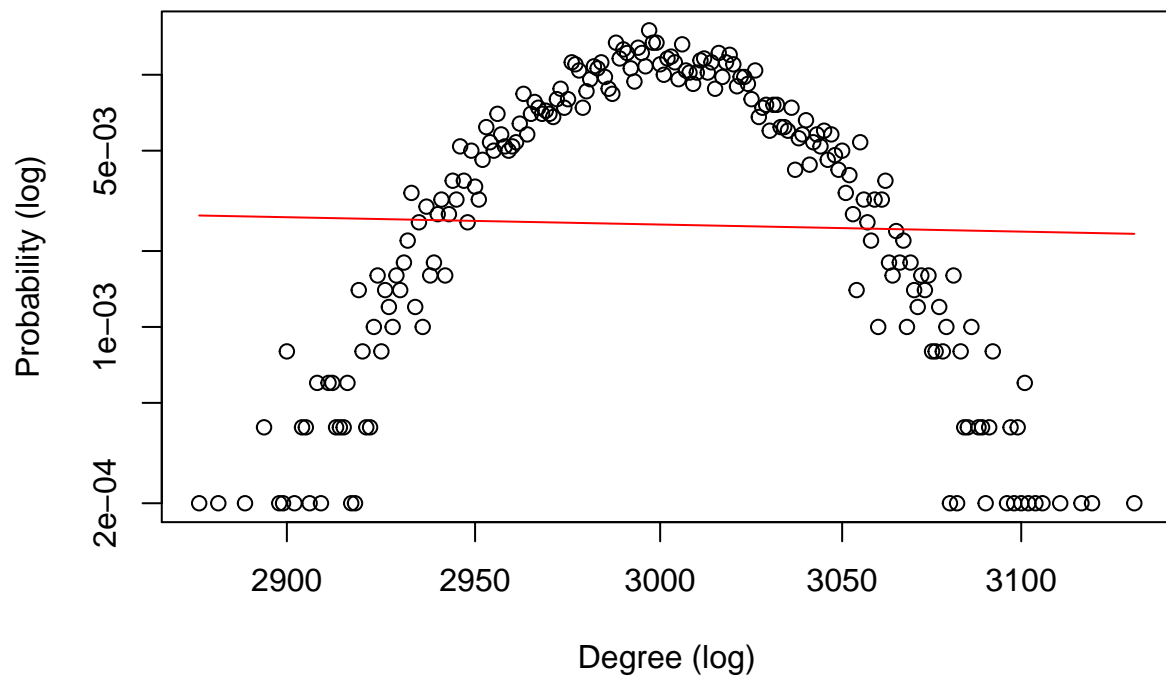
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- average distance = 1.399926
- diameter = 2
- the absolute difference between the average distance and its approximation is 0.336097
- the absolute difference between the diameter and its approximation is 0.399926

# Degree Distribution



# Degree Distribution



- number of components = 1
- we see in the following table: 1
- ...that the largest component captures more than 80 percent of the nodes.



#### Problem 4

Degree Centrality:

53213 35290 38109 62821 93504 92790 21718 1086 89732 111161

Eigen Centrality:

53213 101361 101495 101585 101446 101262 101520 101370 101558 101610

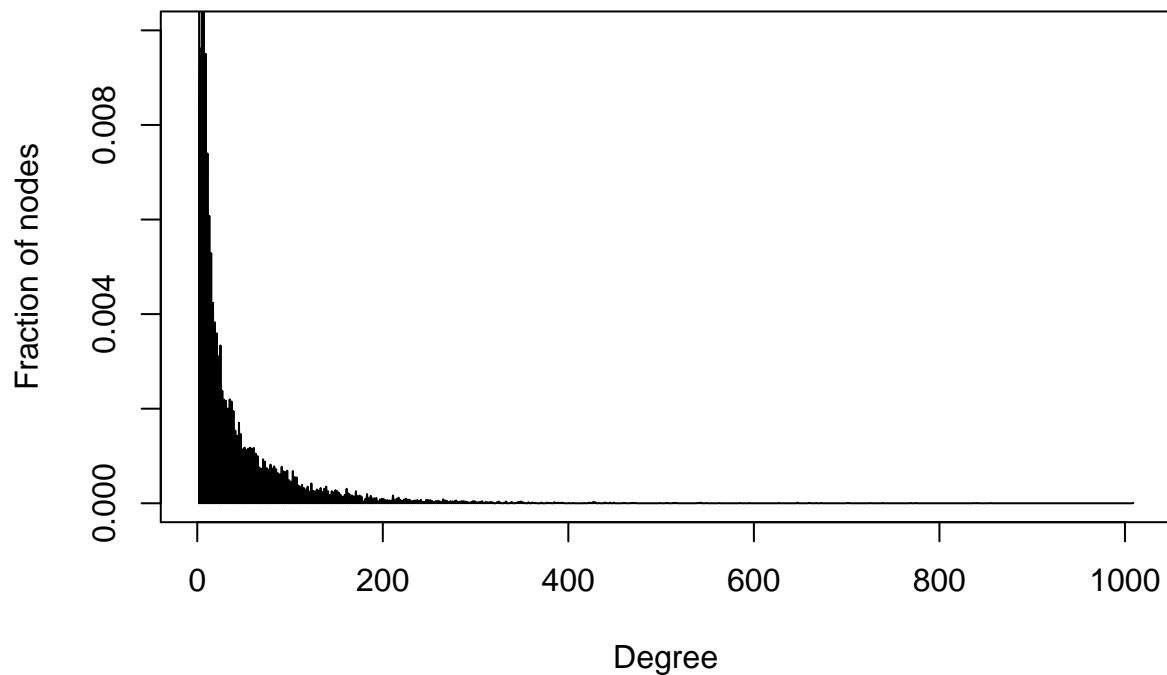
Page Rank:

105680 81761 54526 92790 50163 22653 23289 39808 92043 36907

HITS:

53213 37479 124529 84122 127886 10976 90794 91586 95036 75947

Degree Distribution



- number of components = 114797

- we see in the following table:

114508 140 84 36 14 3 3 3 1 2 1 1 1

that the largest component captures 13 percent of the nodes.

But this cluster is still much bigger than the second one.

There are 4054323 adjacent triangles.