Proof Of A Conjecture Of Matherne, Morales, And Selover On Encodings Of Unit Interval Orders

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Abstract

There are two bijections from unit interval orders on n elements to Dyck paths from $(0, 0)$ to (n, n). One is to consider the pairs of incomparable elements, which form the set of boxes between some Dyck path and the diagonal. Another is to find a particular part listing (in the sense of Guay-Paquet) which yields an isomorphic poset, and to interpret the part listing as the area sequence of a Dyck path. Matherne, Morales, and Selover conjectured that, for any unit interval order, these two Dyck paths are related by Haglund's well-known zeta bijection.

Unit interval orders

Let $\mathcal{I} = \{I_1, \ldots, I_n\}$ be a set of n intervals of unit length, numbered from left to right. $U(\mathcal{I})$ is the poset on the elements 1 to n which is defined by $i \prec j$ if and only if I_i is strictly to the left of I_j .

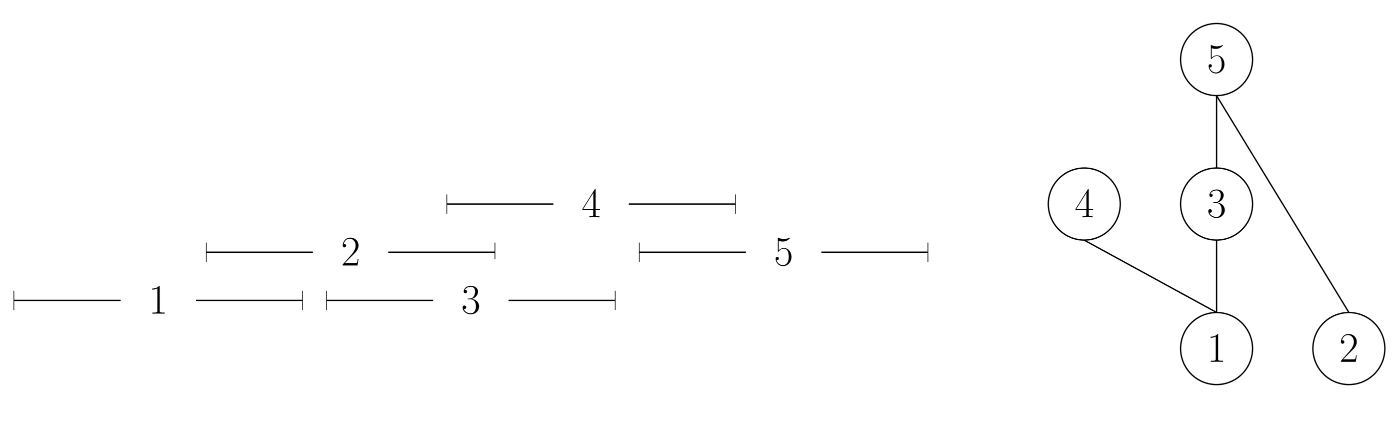
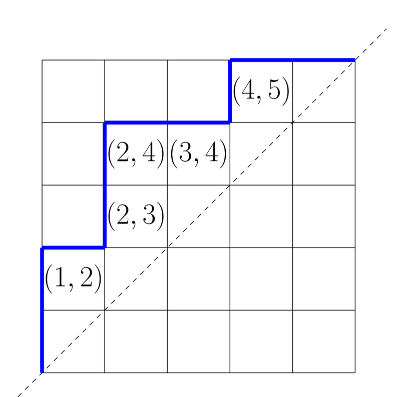


Figure 1. Unit interval order U

The map a

We define $\tilde{a}(U) = \{(x, y) | x \not\prec y, 1 \le x < y \le n\}$. The poset from the figure 1 gives us the following area set:

$$\{(1, 2), (2, 3), (2, 4), (3, 4), (4, 5)\}$$



We define $a(U)$ to be the Dyck path whose area set is given by $\tilde{a}(U)$.

The map p

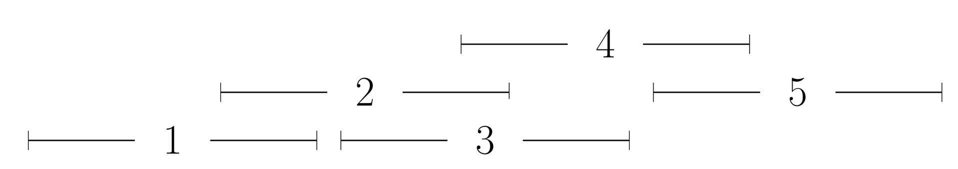
Let $w = (w_1, \ldots, w_n)$ be a sequence of non-negative integers. We define its associated poset $P(w)$ as follows [GP]. For $1 \le i, j \le n$, we set $i \prec j$ if either $w_j - w_i \ge 2$, or $w_j - w_i = 1$ and $i < j$.

For a unit interval order $U \in \mathcal{U}_n$, there is a unique part listing w such that $P(w)$ is isomorphic to U and w is the area sequence of a Dyck path. Define $\tilde{p}(U)$ to be this part listing. Define $p(U)$ to be the Dyck path whose area sequence is $\tilde{p}(U)$.

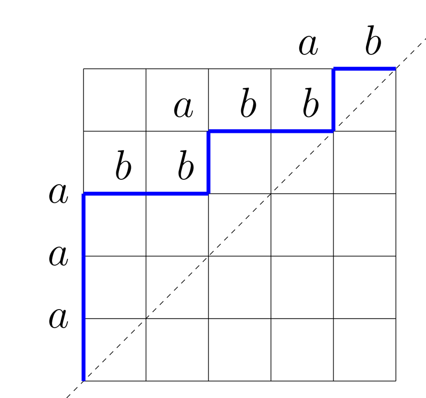
Part listings

We defined an algorithm that associate a unit interval order to an area sequence . Define $\ell(j)$ to be $\max_{i \prec j} \ell(i) + 1$. We call $\ell(i)$ the level of i (or of the interval I_i). We will successively define words $q_1, q_2, \ldots, q_n$. The word q_i is of length i, and is obtained by inserting a copy of $\ell(i)$ into q_{i-1}.

- We begin by defining $q_1 = 0$. Now suppose that q_{i-1} has already been constructed.
- Let C_i be the number of elements of level $\ell(i) - 1$ comparable to i.(Note that they are necessarily to the left of i.) The letter $\ell(i)$ is added into q_{i-1} directly after the occurrences of the letter $\ell(i)$ (if any) immediately following the C_i-th letter $\ell(i) - 1$.



$\ell(1) = 0$ $C_1 = 0$ $q_1 = (0)$
$\ell(2) = 0$ $C_2 = 0$ $q_2 = (0, 0)$
$\ell(3) = 1$ $C_3 = 1$ $q_3 = (0, 1, 0)$
$\ell(4) = 1$ $C_4 = 1$ $q_4 = (0, 1, 1, 0)$
$\ell(5) = 2$ $C_5 = 1$ $q_5 = (0, 1, 2, 1, 0)$

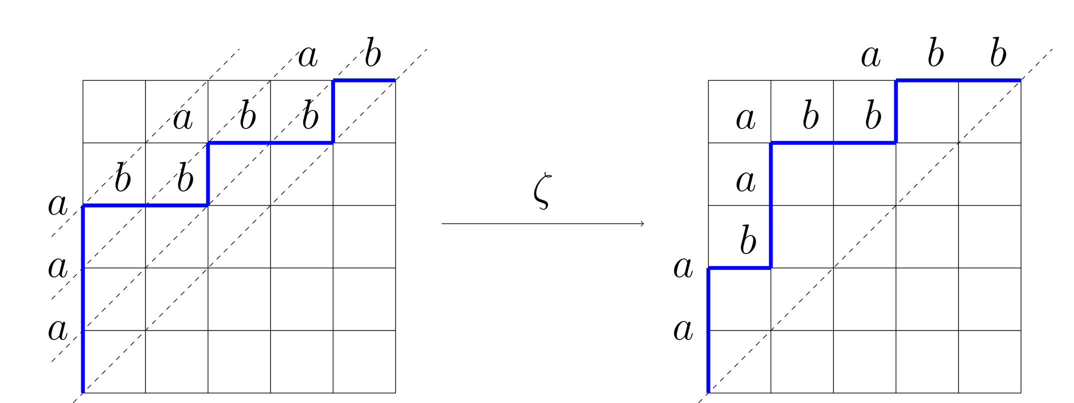


$w = a\,a\,a\,b\,b\,a\,b\,b\,a\,b$

We define $p(U)$ to be the Dyck path whose area set is given by $\tilde{p}(U) = q(U)$.

The zeta map [H]

We label the top end-point of an up step with the letter a, and we label the right endpoint of a right step with the letter b. We then read the labels: first on the line $y = x$, from bottom left to top right, then on the line $y = x + 1$, again in the same direction, then on the line $y = x + 2$, etc. Interpret b as designating an up step, and a as designating a right step. This defines a lattice path from $(0, 0)$ to (n, n). Define this to be $\zeta(D)$.



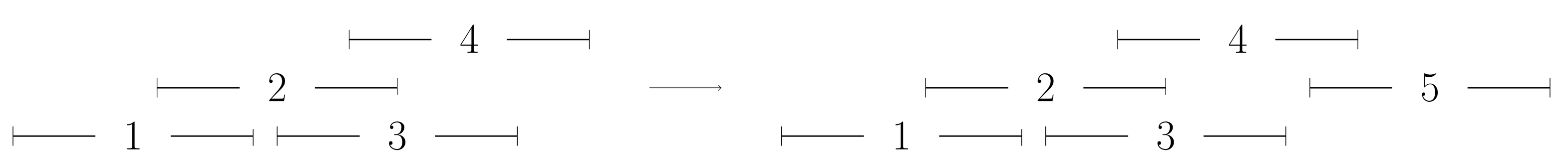
$w = a\,a\,a\,b\,b\,a\,b\,b\,a\,b$ $\qquad$ $\zeta(w) = a\,a\,b\,a\,a\,b\,b\,a\,b\,b$

Theorem (G., Segovia, Thomas)

The maps $\zeta \circ p$ and a coincide.

Proof of the theorem

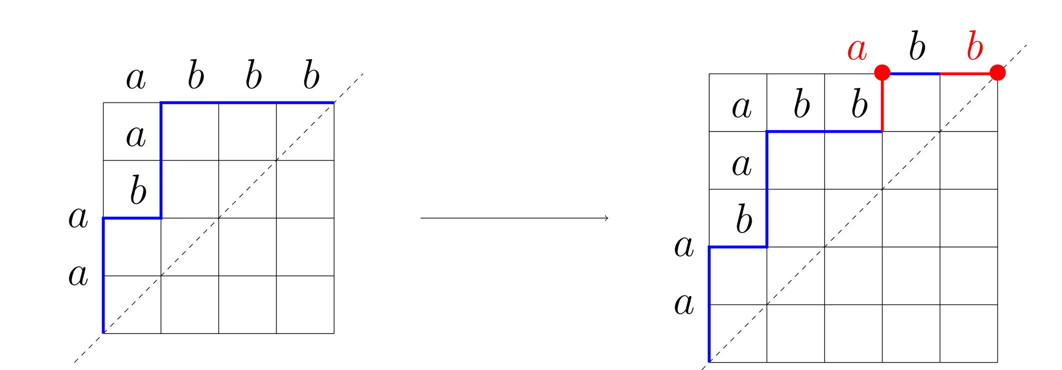
The proof of the conjecture [MMS] will proceed by induction. We suppose that for a unit interval order U, we know that $\zeta(p(U))$ and $a(U)$ coincide. We then consider what happens when we add a new rightmost interval to U.



The Dyck path $p(U')$ is obtained from $p(U)$ by adding a final maximal peak.

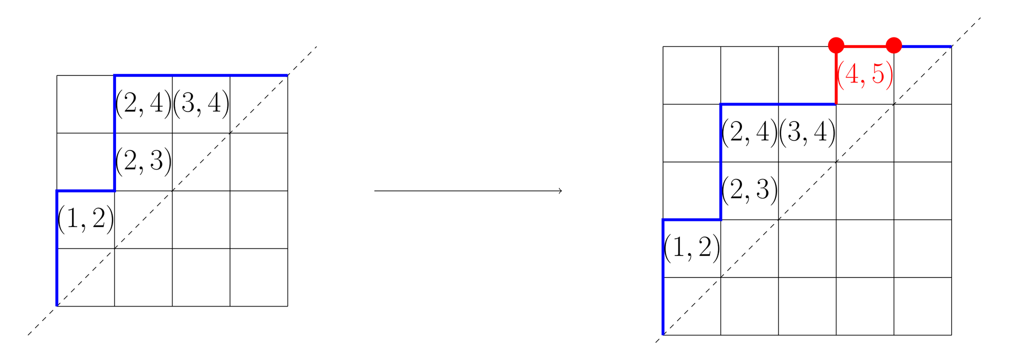
$p(U) = (0, 1, 1, 0) = a\,a\,b\,a\,b\,b\,a\,b$ $\qquad$ $p(U') = (0, 1, 2, 1, 0) = a\,a\,a\,b\,b\,a\,b\,b\,a\,b$

The Dyck path $\zeta(p(U'))$ is obtained from $\zeta(p(U))$ by adding a final peak in position $(n - r, n + 1)$, where r is the sum of the number of occurrences of the letter ℓ in $p(U)$ and of the number of occurrences of the letter $\ell - 1$ appearing after the position of the added letter ℓ in $p(U')$.



$\zeta(p(U)) = a\,a\,b\,a\,a\,b\,b\,b$ $\qquad$ $\zeta(p(U')) = a\,a\,b\,a\,a\,b\,b\,a\,b\,b$

The Dyck path $a(U')$ is obtained from $a(U)$ by adding a final peak in position $(n - s, n + 1)$, where s is the number of intervals in U' not comparable to the rightmost interval I_{n+1}.



$a(U) = a\,a\,b\,a\,a\,b\,b\,b$ $\qquad$ $a(U') = a\,a\,b\,a\,a\,b\,b\,a\,b\,b$

The theorem follows from the fact that $r = s$.

References

[GP] M. Guay-Paquet, A modular relation for the chromatic symmetric functions of (3+1)-free posets. arXiv:1306.2400.

[H] J. Haglund, The q, t-Catalan numbers and the space of diagonal harmonics, with an appendix on the combinatorics of Macdonald polynomials. University Lecture Series 41. American Mathematical Society, Providence, RI. 2008.

[MMS] J. Matherne, A. Morales, and J. Selover, The Newton polytope and Lorentzian property of chromatic symmetric functions. arXiv:2201.07333.

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