The Volume Polynomial of a Matroid

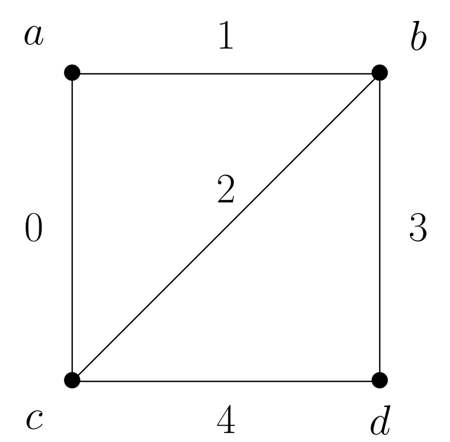
A Work of Petter Brändén and Jonathan Leake presented by Félix Gélinas

York University

Abstract

To prove the Heron-Rota-Welsh conjecture, Petter Brändén and Jonathan Leake extended the notion of hereditary Lorentzian polynomials associated with a simplicial complex to matroid theory. This poster aims to present the mathematical objects used in their proof [BL] and provide a brief overview of the concept mentioned above.

Matroid Theory



Matroid $M = (E, \mathcal{I})$

$$E = \{0,1,2,3,4\}$$

$$\mathcal{I} = \{I : I \text{ is a forest}\}$$

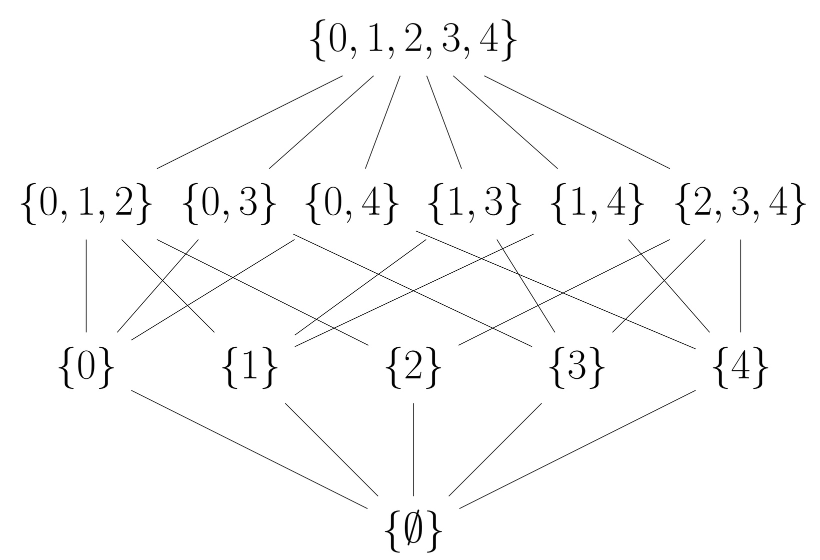
$$\mathcal{B} = \{\{0,1,3\},\{0,1,4\},\{0,2,3\},\{0,2,4\},\{0,3,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\}\}$$

{all spanning trees}

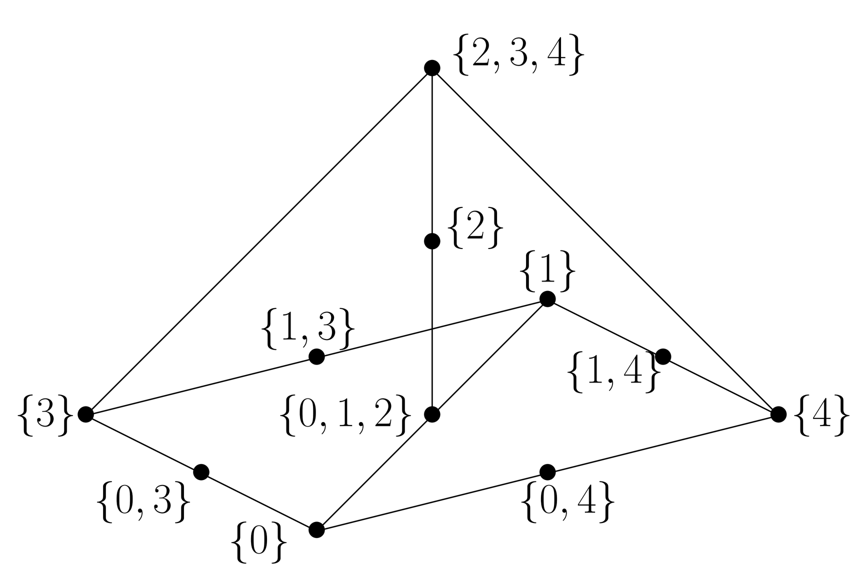
$r(M) = |I|$ for $I \in \mathcal{B}$
$r(M) = 3$

$$\mathcal{L} = \{\{0,1,2\},\{0\},\{1\},\{2\},\{3\},\{4\},\{0,3\},\{0,4\},\{1,3\},\{1,4\},\{2,3,4\}\}$$

(set of flats)



Lattice of flats of M



$\Delta(\mathcal{L})$ is the order complex of $\mathcal{L}$ i.e it is the simplicial complex of all totally ordered subsets of $\underline{\mathcal{L}} = \mathcal{L} \setminus \{K, E\}$, where K are the loops of M

Simplicial complexe $\Delta(\mathcal{L})$

Lorentzian Polynomials on Cones

Operator $D_{\mathbf{u}}$: Let us consider $D_{\mathbf{u}} = u_1\partial_1 + \dots u_n\partial_n$, where $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$ and ∂_i is the partial derivative with respect to t_i.

Lorentzian : Let $\mathcal{K}$ be an open convex cone in $\mathbb{R}^n$. A homogeneous polynomial $f \in \mathbb{R}[t_1, \dots, t_n]$ of degree $d \geq 1$ is called *$\mathcal{K}$-Lorentzian* if for all vectors $\mathbf{v}_1, \dots, \mathbf{v}_d \in \mathcal{K}$,

(P) $D_{\mathbf{v}_1} \dots D_{\mathbf{v}_d} f > 0$, and

(HR) the bilinear form on $\mathbb{R}^n$, $(\mathbf{x}, \mathbf{y}) \longmapsto D_{\mathbf{x}} D_{\mathbf{y}} D_{\mathbf{v}_3} \dots D_{\mathbf{v}_d} f$ has exactly one positive eigenvalue.

we may replace (HR) by the equivalent condition

(AF) For all vectors $\mathbf{v}_1, \dots, \mathbf{v}_d \in \mathcal{K}$,
$(D_{\mathbf{v}_1} D_{\mathbf{v}_2} D_{\mathbf{v}_3} \dots D_{\mathbf{v}_d} f)^2 \geq (D_{\mathbf{v}_1} D_{\mathbf{v}_1} D_{\mathbf{v}_3} \dots D_{\mathbf{v}_d} f) \cdot (D_{\mathbf{v}_2} D_{\mathbf{v}_2} D_{\mathbf{v}_3} \dots D_{\mathbf{v}_d} f)$.

Example of a Lorentzian polynomial

Let us respectively consider the cone $\mathcal{K} \subset \mathbb{R}^2$ and the polynomial $f \in \mathbb{R}[t_1, t_2]$

$$\mathcal{K} = \{(x_1, x_2) | x_i \geq 0\} \quad f = 4t_1t_2^2 + 4t_1^2t_2$$

It is easy to check that for any 3 vectors $\mathbf{v} = (v_1, v_2), \mathbf{u} = (u_1, u_2), \mathbf{w} = (w_1, w_2)$ in $\mathcal{K}$, we have that $D_{\mathbf{w}} D_{\mathbf{u}} D_{\mathbf{v}} f$ is strictly positive on the cone $\mathcal{K}$

$$D_{\mathbf{w}} D_{\mathbf{u}} D_{\mathbf{v}} f = w_1(8u_1v_2 + u_2(8v_1 + 8v_2)) + w_2(u_1(8v_1 + 8v_2) + 8u_2v_1)$$

and that the matrix associate to the bilinear form presented in (HR) is

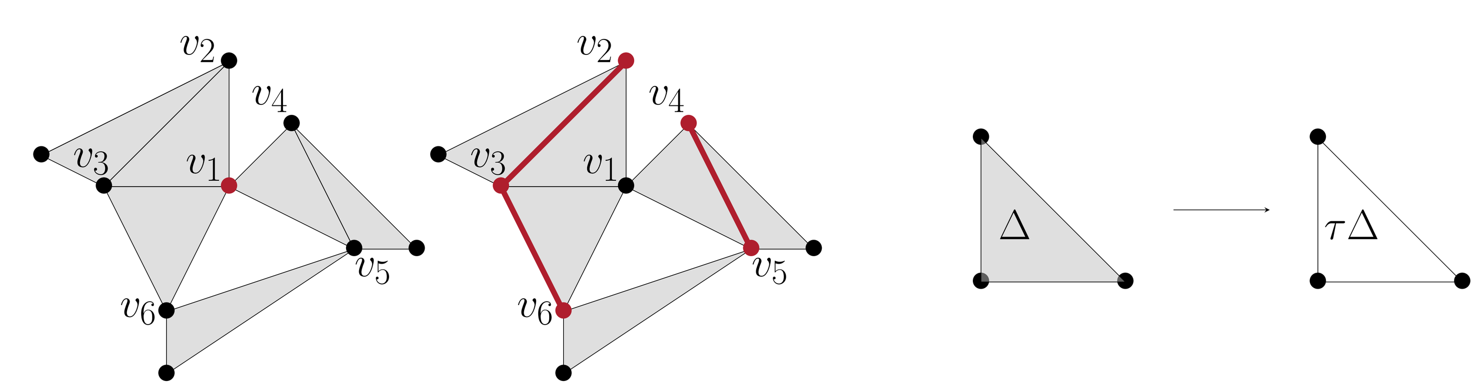
$$\begin{pmatrix} 8b & 8a+8b \\ 8a+8b & 8a \end{pmatrix} \quad \text{With exactly one positive eigenvalue } \lambda = 4\left(\sqrt{5a^2 + 6ab + 5b^2} + a + b\right)$$

where in this example $\mathbf{v}_3 = (a, b) \in \mathcal{K}$.

Polytopes

Let Δ be an abstract simplicial complex on a finite set V.

Link : $lk_\Delta(S) = \{T | T \cap S = \emptyset \text{ and } S \cup T \in \Delta\}$ where $S \in \Delta$ **Skeleton** : $\{S \in \Delta | \dim(S) < \dim(\Delta)\}$



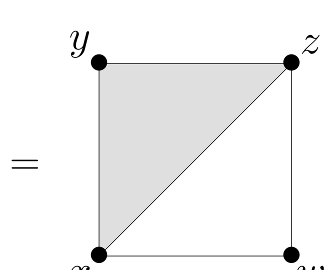
Link of the vertex v_1 Skeleton $\tau\Delta$ of the simplicial complex Δ

Hereditary Simplicial Complex

Let L be a linear subspace $\mathbb{R}^V$. We say that the pair (Δ, L) is hereditary if $\{(\ell_i)_{i \in T} | \ell \in L\} = \mathbb{R}^T$, for all $T \in \Delta$.

Let us consider the following simplicial complex Δ:

$\Delta =$



Non-Example: If we have $l_x = (0,2,2,0), l_y = (0,1,1,0), l_w = (1,0,0,0), l_z = (0,0,1,0)$ and we consider the linear subspace $L = \langle l_x, l_y, l_w, l_z \rangle$ we have that (Δ, L) is not hereditary.

Example: If $L = \langle l_x = (1,0,0,0), l_y = (0,1,0,0), l_w = (0,0,1,0), l_z = (0,0,0,1) \rangle$ then the pair (Δ, L) is hereditary.

Hereditary Polynomials

Lineality space of a polynomial : We define the lineality L_f space of f to be the set of all $\mathbf{v} \in \mathbb{R}^n$ for which $D_{\mathbf{v}} f \equiv 0$. That is the set of all $\mathbf{v}$ for which $f(\mathbf{t} + \mathbf{v}) = f(\mathbf{t})$, for all $\mathbf{t} \in \mathbb{R}^n$.

Let $f \in \mathbb{R}[t_i | i \in V]$ be a homogeneous polynomial of degree $d \geq 1$, and let L_f be its lineality space. Define a simplicial complex

$$\Delta_f = \{S \subseteq V | \partial^S f \not\equiv 0\}$$

We say that f is hereditary if $(\tau\Delta_f, L_f)$ is hereditary, and we say that f is strongly hereditary if (Δ_f, L_f) is hereditary.

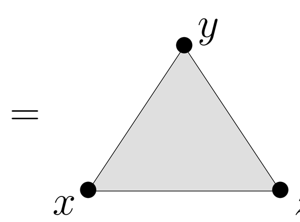
Example of a Hereditary Polynomial

Let us consider the following polynomial $f \in \mathbb{R}[x, y, z]$

$$f = 2x^2z + 2xz^2 + 2x^2y + 2xy^2 + 2y^2z + 2yz^2 + \frac{2x^3}{3} + \frac{2z^3}{3} + \frac{2y^3}{3} + 4xyz$$

We have the following simplicial complex Δ_f associated to it:

$\Delta_f =$



Since $D_{\mathbf{v}=(v_1,v_2,v_3)} f = v_1(4xz + 4yz + 4xy + 2x^2 + 2y^2 + 2z^2) + v_2(4xz + 4yz + 4xy + 2x^2 + 2y^2 + 2z^2) + v_3(4xz + 4yz + 4xy + 2x^2 + 2y^2 + 2z^2)$ we have that the lineality space $L_f = \langle (1,-1,0), (1,0,-1), (0,1,-1) \rangle$. Hence if $l_x = (1,-1,0), l_y = (1,0,-1), l_z = (0,1,-1)$ we have that the pair (Δ_f, L_f) is hereditary.

Volume Polynomials of Matroids

For a matroid M with lattice of flats $\mathcal{L} = [K, E]$, there is a unique strongly hereditary polynomial $\text{pol}_{\mathcal{L}} \in \mathbb{R}[t_G | G \in \underline{\mathcal{L}}]$ of degree $d = r - 1$ (where rk is the rank of $\mathcal{L}$) satisfying

$$\Delta_{\text{pol}_{\mathcal{L}}} = \Delta(\mathcal{L}), \; L(\mathcal{L}) \subseteq L_{\text{pol}_{\mathcal{L}}}, \; \partial^S \text{pol}_{\mathcal{L}} = 1$$

for all facets $S \in \Delta(\mathcal{L})$, where $L(\mathcal{L})$ be the linear subspace of $\mathbb{R}^{\underline{\mathcal{L}}}$ consisting of all modular $(y_F)_{F \in \underline{\mathcal{L}}}$.

The polynomial $\text{pol}_{\mathcal{L}}$ happens to correspond to the volume polynomial of Chow ring of a matroid. More specifically, This polynomial is the cogenerator of the Chow ring of a matroid. For a rank 3 matroid we have the formula

$$2 \cdot \text{pol}_{\mathcal{L}}(t) = \left(\sum_F t_F\right)^2 - \sum_G \left(t_G - \sum_{F<G} t_F\right)^2$$

where F denotes the flats of rank one, and G denotes the flats of rank two. Considering the matroid M described previously, this gives us the following polynomial

$$\begin{aligned} 2 \cdot \text{pol}_{\mathcal{L}}(t) = & -2t_0^2 + 2t_{0,1,2}t_0 + 2t_{0,3}t_0 + 2t_{0,4}t_0 - 2t_1^2 - t_2^2 - 2t_3^2 - 2t_4^2 - t_{0,1,2}^2 - t_{0,3}^2 - t_{0,4}^2 \\ & - t_{1,3}^2 - t_{1,4}^2 - t_{2,3,4}^2 + 2t_1t_{0,1,2} + 2t_2t_{0,1,2} + 2t_3t_{0,3} + 2t_4t_{0,4} + 2t_1t_{1,3} + 2t_3t_{1,3} \\ & + 2t_1t_{1,4} + 2t_4t_{1,4} + 2t_2t_{2,3,4} + 2t_3t_{2,3,4} + 2t_4t_{2,3,4} \end{aligned}$$

It is easy to check that $\Delta_{\text{pol}_{\mathcal{L}}} = \Delta(\mathcal{L})$ and that $\partial^S \text{pol}_{\mathcal{L}} = 1$.

Heron-Rota-Welsh Conjecture

Theorem [BL] : The absolute values of the coefficients of the reduced characteristic polynomial of a matroid $\overline{\chi}_M(t) = \sum_{F \in \mathcal{L}} \mu(K, F) t^{r([F,E])}/(t-1)$ form a log-concave sequence.

Idea of the proof: The crux of the proof is to see that the coefficients of the reduced characteristic polynomial of a matroid $\mu(K, F)$ is equal to $D_\alpha^k D_\beta^{d-k} \text{pol}_{\mathcal{L}}$, where $d = d(\mathcal{L}) = r(\mathcal{L}) - 1$ and $k = d([F, E])$. Since $\text{pol}_{\mathcal{L}}$ is $\mathcal{K}_{\text{pol}_{\mathcal{L}}}$-Lorentzian, where $\mathcal{K}_{\text{pol}_{\mathcal{L}}}$ is a well-defined non empty cone, the condition (AF) gives us that the sequence $a_k = D_\alpha^k D_\beta^{d-k} \text{pol}_{\mathcal{L}}$ is log-concave.

References

[A] A. Ashraf, Of matroid polytopes, chow rings and character polynomials . Electronic Thesis and Dissertation Repository. 6492. https://ir.lib.uwo.ca/etd/6492, 2019.

[AHK] K. Adiprasito, J. Huh and E. Katz, Hodge Theory for Combinatorial Geometries, arXiv : 1511.02888, 2018.

[BL] P. Brändén and J. Leake, Lorentzian polynomials on cones. arXiv:2304.13203, 2023.

[O] J.G. Oxley, Matroid Theory, Oxford graduate texts in mathematics, Oxford University Press, 2006.

felixgel@yorku.ca