Assignment

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Execise 1
Theorem: $(\exists m \in N)(\exists n \in N)(3m + 5n = 12)$
Proof:

$$3m + 5n = 12$$
$$m = 4 - \frac{5}{3}n$$

Since m has to be a natural number, the product of $\frac{5}{3}n$ has to be a natural number, too. Therefore n has to be a multiple of 3.

$$n = 3q, q \in N$$
$$m = 4 - 5q$$

Hence, there doesn´t exist a natural number q for which m is a natural number. Therefore the theorem is false

Execise 2
Theorem: The sum of any five consecutive integer is divisible by 5 (withour remainder)

$$\forall n(\sum_{i=0}^{4} n + i = q)$$
$$q, n \in N$$
$$5/q, (q\; is\; divisible\; by\; 5)$$

Proof:

$$n + (n+1) + (n+2) + (n+3) + (n+4) = 5p, p \in N$$

$$5n + 10 = 5p$$

$$n + 2 = p$$

Hence, the theorem is true, since p is $n+2$ for every n.

Execise 3

Theorem: For any integer n, the number $n^2 + n + 1$ is odd.

Proof:

$$n^2 + n + 1 = q, q = 2p + 1$$

$$n^2 + n + 1 = 2p + 1$$

$$\frac{n^2 + n}{2} = p$$

$$\frac{n(n+1)}{2} = p$$

Since the multiplication with an even number always return an even number and the numerator must be even for every any integer n, p must be a natural number. Hence, the theorem is true.

Execise 4

Theorem: Every odd number is one of the form $4n + 1$ or $4n + 3$.

Proof: We proof this withn the division theorem The division theorem states, that every natural number can be expressed as $n = ab + r, with\, a, r \in N and\, b \in Z$ and $0 <= r < a$. Since $a = 4$ there are four possible cases, that describes any natural number. These are:

$$4b,\ 4b+1,\ 4b+2,\ 4b+3$$

Since any natural numbers of the form $4b$ or $4b + 2$ always even due to the factor 4. Hence, the only cases that a odd natural number can be expressed in is $4b + 1$ or $4b + 3$, which are the forms in the theorem.

Execise 5

Theorem: For any integer n, at least one of n, $n+2$ or $n+4$ is divisble by 3.

Proof: We proof this with the division theorem. The division theorem states, that every natural number can be expressed as $n = ab + r, with\, a, r \in N and\, b \in Z$ and $0 <= r < a$. Since $a = 3$ there are three possible cases, that describes any natural number.

$$3b,\ 3b+1,\ 3b+2$$

First case ($n = 3b$):

$$3/3b \ \vee \ 3/3b+2 \ \vee \ 3/3b+4$$

$$True \ \vee \ False \ \vee \ False = True$$

Second case ($n = 3b+1$):

$$3/3b+1 \ \vee \ 3/3b+1+2 \ \vee \ 3/3b+1+4$$

$$False \ \vee \ True \ \vee \ False = True$$

Third case ($n = 3b+2$):

$$3/3b+1 \ \vee \ 3/3b+2+2 \ \vee \ 3/3b+2+4$$

$$False \ \vee \ False \ \vee \ True = True$$

Since for every of the third cases one of n, $n+2$, $n+4$ is true. Hence the theorem is true.

Execise 6

Theorem: A classic unsolved problem in number theory asks if there are infinitely many pairs of "twin primes", pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

Proof: This theorem is partly proofed by assignment five. It states that for any integer n at least one of n, $n+2$, $n+4$ is divisble by 3. But since a prime is only divisble by himself or by 1, there can´t exist any triple prime, other than $3, 5, 7$.

Execise 7

Theorem: Prove that for any natural number $2+2^2+2^3+...2^n = 2^{n+1}-2$

Proof: We proof this theorem by induction.

$$A(n) = \sum_{i=1}^{n}(2^i), B(n) = 2^{n+1}-2$$

$$A(1) = B(1)$$
$$2^1 = 2^{1+1}-2$$
$$2 = 2$$

The theorem is true for 1. Furthermore it must be true for $n+1$ to be true.

$$A(n+1) = B(n+1)$$
$$\sum_{i=1}^{n+1}(2^i) = 2^{n+2}-2$$

Since we have a geometric series at the left hand side, we bring the sum into closed form first.

$$\frac{2(1-2^{n+2})}{1-2} = 2^{n+2}-2$$
$$2^{n+2}-2 = 2^{n+2}-2, by\ Algebra$$

Hence the theorem is true.

Execise 8

Theorem: Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, then for any fixed number $M > 0$ the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML.

Proof: We proof this theorem with the definition of a limit of a sequence. A sequence converges to a limit, if the following is true:

$$\forall n(|x_n - L| < \frac{\epsilon}{|M|}), where$$

$$\epsilon \in R, \epsilon > 0, n \in N, N \in N, n \geq N$$

Following this we get:

$$|Mx_n - ML| < M\frac{\epsilon}{|M|}$$

$$|M||x_n - L| < \epsilon$$

Hence, the sequence converges to ML.

Execise 9

Theorem: Given an infinite collection $A_n, n = 1, 2, ...$ of intervals of the real line, their intersection is defined to be $\cap_{n=1}^{\infty} A_n = \{x|(\forall n)(x \in A_n)\}$. Give an example of a family of intervals $A_n, n = 1, 2, ...$ such that $A_{n+1} \subset A_n$ for all n and $\cap_{n=1}^{\infty} A_n = \emptyset$. Prove that your example has the stated property.

Proof: A family of intervalls with the mentioned properties are the following:

$$A_n = (0, \frac{1}{n}), n = 1, 2, ...$$

We proof the the first property by taking an $n \in N$.Then every element of those two intervall $(0, \frac{1}{n})$ and $(0, \frac{1}{n+1})$ are between 0 and $\frac{1}{n}$, respectively $\frac{1}{n+1}$. Following that $\frac{1}{n+1} < \frac{1}{n}$, A_{n+1} must be a subset of A_n. Now we have to proof the second property. As we learned from the first proof that the collection A_{n+1} is always a subset of A_n, we can conclude that all A_n are an intersection of A_∞ when $n \to \infty$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, the intervall $(0, 0)$ is an empty set, which proofs the second property.

Execise 10

Theorem: Give an example of a family of intervals $A_n, n = 1, 2, ...$ such that $A_{n+1} \subset A_n$ for all n and $\cap_{n=1}^{\infty} A_n$ consists of a single real number. Prove that your example has the stated property.

Proof: This proof is similar to theorem 9 but with an closed intervall. A family of intervalls with the mentioned properties are the following:

$$A_n = [0, \frac{1}{n}], n = 1, 2, ...$$

We proof the the first property by taking an $n \in N$.Then every element of those two intervall $[0, \frac{1}{n}]$ and $[0, \frac{1}{n+1}]$ are $0 \leq x \leq \frac{1}{n}$, respectively $\frac{1}{n+1}$. Following that $\frac{1}{n+1} < \frac{1}{n}$, A_{n+1} must be a subset of A_n. Now we have to proof the second property. As we learned from the first proof that the collection A_{n+1} is always a subset of A_n, we can conclude that all A_n are an intersection of A_∞ when $n \to \infty$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, the intervall $[0, 0]$ is a set, which consists only of a singular real number 0.