SCHOOL OF COMPUTATION,
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TECHNISCHE UNIVERSITÄT MÜNCHEN

Master's Thesis in Informatics

Constructing Linear Types in Isabelle/HOL

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Konstruktion linearer Typen in Isabelle/HOL

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I confirm that this master's thesis is my own work and I have documented all sources and material used.

Munich, 13-11-2025 Felix Krayer

Acknowledgments

Abstract

Contents

1 Introduction

- Datatypes in general
- Datatypes in Isabelle/HOL are built on Bounded Natural Functors (BNFs) (defined in [TPB12])
- Structure of the Thesis

2 Background

This Chapter serves to introduce BNFs and their generalization to Map-Restricted Bounded Natural Functors (MRBNFs).

2.1 Bounded Natural Functors (BNFs)

As described in Chapter 1, BNFs are essential for constructing datatypes and co-datatypes in Isabelle/HOL. Especially for defining a recursive datatype like

$$\textbf{datatype}\ 'a\ \mathsf{ex} = \mathrm{A}\ ''('a\ \times\ 'a\ \mathsf{ex})\ \mathsf{list}''$$

it is required that the type constructor $'a$ list is registered as a BNF, i.e., the BNF-axioms have been shown for it. Non-BNF types like $'a$ set may be used in a **datatype** command, but they cannot be used to recurse. Since BNFs are closed under composition and fixpoints, the resulting datatype (in the example $'a$ ex) can be automatically registered as a BNF as well.

2.1.1 BNF-axioms

A BNF is characterized by map and set functions, a relator and a bound. We consider a n-ary BNF F and use the notation $\overline{f} = f_1 \dots f_n$. $\dot{G}$. Furthermore, when we write i as an index, we assume it to be in the range $1 \leq i \leq n$. We give the BNF-axioms as follows:

$$\text{MAP\_ID: } \mathsf{map}_F\ \overline{id}\ x = x \tag{2.1}$$

$$\text{MAP\_COMP: } \mathsf{map}_F\ \overline{g}\ (\mathsf{map}_F\ \overline{f}\ x) = \mathsf{map}_F\ \overline{(g \circ f)}\ x \tag{2.2}$$

$$\text{MAP\_CONG: } (\forall i.\ \forall z \in \mathsf{set}_{F,i}\ x.\ f_i\ z = g_i\ z) \Longrightarrow \mathsf{map}_F\ \overline{f}\ x = \mathsf{map}_F\ \overline{g}\ x \tag{2.3}$$

$$\text{SET\_MAP: } \forall i.\ \mathsf{set}_{F,i}(\mathsf{map}_F\ \overline{f}\ x) = f_i\ \grave{}\ \mathsf{set}_{F,i}\ x \tag{2.4}$$

$$\text{BD: } \mathsf{infinite}\ \mathsf{bd}_F \wedge \mathsf{regular}\ \mathsf{bd}_F \wedge \mathsf{cardinal\_order}\ \mathsf{bd}_F \tag{2.5}$$

$$\text{SET\_BD: } \forall i.\ |\mathsf{set}_{F,i}\ x| <_o \mathsf{bd}_F \tag{2.6}$$

$$\text{REL\_COMPP\_LEQ: } \mathsf{rel}_F\ \overline{R}\ \bullet\ \mathsf{rel}_F\ \overline{Q}\ = \mathsf{rel}_F\ \overline{(R\ \bullet\ Q)} \tag{2.7}$$

$$\text{IN\_REL: } \text{Weak Pullback Preservation WP} \tag{2.8}$$

where ` is the image function on sets and $\bullet$ is the composition of relations. Furthermore $<_o$ is the less than relation on the level of cardinals

While most of these properties are straightforward, we want to explain the preservation of weak pullbacks in more detail.

$$\mathsf{rel}_F\ \overline{R}\ x\ y = \exists z.\ (\forall i.\ \mathsf{set}_{F,i}\ z \subseteq \{(a,b).\ R_i\ a\ b\}) \wedge \mathsf{map}_F\ \overline{fst}\ z = x \wedge \mathsf{map}_F\ \overline{snd}\ z = y \quad (2.9)$$

The idea is that two elements x and y of the type $\alpha\ F$ are related through a relation R iff there exists a z that acts as a "zipped" version of x and y. The atoms of this z are the atoms of x and y, that are organized in pairs of R-related with the x as the first and y as the second position in the pair.

2.1.2 BNF examples

Further examples of BNFs are is the product type $('a,\ 'b)$ prod, a binary type constructor with infix notation $'a\ \times\ 'b$, and the type of finite sets $'a$ fset. The latter is interesting for the reason that it is a subtype of the set type, which is not a BNF. By enforcing finiteness for the elements of the type it is possible to give a bound for the set function, fulfilling the SET_BD axiom, which is not possible for the unrestricted set type. Since unboundedness is the only reason that the set type is not a BNF, $'a$ fset can be shown to be a BNF.

To show, how BNFs can be combined to create new ones, we consider the type constructor $('a,\ 'b)$ plist = $('a\ \times\ 'b)$ list. We define for it a map function ($\mathsf{map}_{\mathsf{plist}}$) and two set functions ($\mathsf{set1}_{\mathsf{plist}}$ and $\mathsf{set2}_{\mathsf{plist}}$) as well as a relator $\mathsf{rel}_{\mathsf{plist}}\ R\ S$. The exact definitions are given as such:

$$\begin{aligned}\mathsf{map}_{\mathsf{plist}}\ f\ g\ &= \mathsf{map}_{\mathsf{list}}\ (\mathsf{map}_{\mathsf{prod}}\ f\ g)\\ \mathsf{set1}_{\mathsf{plist}}\ xs &= \mathsf{set}_{\mathsf{list}}\ (\mathsf{map}_{\mathsf{list}}\ fst\ xs)\\ \mathsf{set2}_{\mathsf{plist}}\ xs &= \mathsf{set}_{\mathsf{list}}\ (\mathsf{map}_{\mathsf{list}}\ snd\ xs)\\ \mathsf{rel}_{\mathsf{plist}}\ R\ S &= \mathsf{rel}_{\mathsf{list}}(\mathsf{rel}_{\mathsf{prod}}\ R\ S)\end{aligned}$$

where we use the standard map, set and relator functions of the list and product type.

To show that $('a,\ 'b)$ plist is a BNF, we have to prove the BNF-axioms for it. Besides the definitions above, a bound $\mathsf{bd}_{\mathsf{plist}}$ is needed. We chose *natLeq*.

2.2 Subtype

We can carve out a subtype from a type constructor using the **typedef** command.

2.3 Map-Restricted Bounded Natural Functors (MRBNFs)

MRBNFs are a generalization of BNFs. Restricting the map function of a functor to *small-support* functions or *small-support bijections* for certain type variables allows us to reason about type constructors in terms of BNF properties, even in cases where this would not be possible otherwise. We call type variables that that are restricted to small-support functions *free* variables and those restricted to small-support bijections *bound* variables. This allows us to define MRBNFs with four types of variables (live, free, bound and dead) as opposed to BNFs which only distinguish between lives and deads.

MRBNFs can be used in a **binder_datatype** command to produce a datatype with bindings.

Consequently, for type constructors with variables that are considered dead in BNF terms, we can declare some of them as free or bound variables, depending on the type.

- cite [Bla+19]

3 Linearizing MRBNFs

3.1 Linearization of MRBNFs (In theory)

In this section we define the linearization of an MRBNF on a subset of it's *live* variables. The result of the linearization is a new MRBNF with the same variable types (*live*, *dead*, *bound*, *free*), except for the linearized variables that change their type from *live* to *bound*. This means that the map function is now restricted to only allow bijective and small-support functions on these variables. Apart from this change, it is ensured that the MRBNF is *non-repetitive* with respect to the linearized variables. We give a definition non-repetitiveness in the following Subsection 3.1.1. Intuitively it means that the atoms of that type cannot occur multiple times in an element of the type.

3.1.1 Non-repetitiveness

At the core of linearization lies the notion of *non-repetitiveness*. We think of an element x of a type $\alpha\ F$ as being non-repetitive if all its α atoms are distinct from another. We give an exact definition nonrep of non-repetitiveness in relation to all other elements of $\alpha\ F$ that are of *equal shape*:

$$\mathsf{eq\_shape}_F\ x\ y = \mathsf{rel}_F\ top\ x\ y \tag{3.1}$$

$$\mathsf{nonrep}_F\ x = \forall y.\ \mathsf{eq\_shape}\ x\ y \longrightarrow (\exists f.\ y = \mathsf{map}_F\ f\ x) \tag{3.2}$$

We consider two $\alpha\ F$ elements x and y to have *equal shape*, if they are related with the *top* relation, i.e., the relation that relates all α atoms to all others. This is true, if the relator rel_F can relate the elements. In the case of list, this is the case when two lists are equal in length but have possibly different content.

Based on this, x is a non-repetitive element, if for all other elements y with equal shape, a function exists through which x can be mapped to y. In our example of list, this holds for all lists with distinct elements (given a second list, one can easily define a function mapping the distinct elements of x to that list). It does not hold for lists with repeating elements, because no f exists that could map two equal elements at different positions in this list to distinct elements in an arbitrary second list.

For MRBNFs with more than one live variable, we can give a definitions of *non-repetitiveness* and having *equal shape* on a subset of the live variables. In that case, we consider x and y of type $(\alpha, \beta)\ G$ to have equal shape with respect to α, iff they are *equal* in their β atoms and are related with *top* in α as before. Consequently for the map in the nonrep definition, the *id* function is applied to the β atoms, since they are already required to be equal.

$$\mathsf{eq\_shape}^1_G\ x\ y = \mathsf{rel}_G\ top\ (=)\ x\ y \tag{3.3}$$

$$\mathsf{nonrep}^1_G\ x = \forall y.\ \mathsf{eq\_shape}\ x\ y \Longrightarrow (\exists f.\ y = \mathsf{map}_G\ f\ id\ x) \tag{3.4}$$

3.1.2 Conditions for linearization

A MRBNFs has to fulfill two properties to be linearized. First, to ensure that the resulting type constructor is non-empty, it is required, that there exists a non-repetitive element (with respect to the linearized variables): $\exists x.\ \mathsf{nonrep}\ x$

Furthermore, even though MRBNFs are already required to preserve weak pullbacks as defined in Equation 2.9, for the linearization it is required that they preserve *all* pullbacks. Formalized this means that the existance of z in the equation has to be fulfilled uniquely, i.e., for each R-related x and y there existes *exactly one* z fulfilling the property in Equation 2.9. For example the strong pullback preservation is fulfilled by the α list and $\alpha\ \beta$ prod functors but not by α fset, the type constructor for finite sets of αs.

We note here that the requirement of strong pullback preservation can be omitted, when the MRBNF is linearized on all its live variables, i.e., when the linearized MRBNF has no live variables. This is because in this case the *relation exchange* lemma explained in Subsection 3.1.3 becomes trivial. In all other cases, that lemma is the sole reason, strong pullback preservation is required.

3.1.3 Intermediate lemmas

We want to prove the MRBNF axioms for the linearized MRBNF. For this we utilize some intermediate lemmas which we present in this section.

F strong From the pullback preservation with uniqueness we can prove the following lemma. In fact this notion of strongness is equivalent to pullback preservation:

$$[\![\mathsf{rel}_F\ R\ x\ y;\ \mathsf{rel}_F\ Q\ x\ y]\!] \Longrightarrow \mathsf{rel}_F\ (inf\ R\ Q)\ x\ y$$

where the infimum inf of two relations R and Q relates exactly those elements that are related by both R and Q.

Relation exchange The *exchange of relations* is a consequence of the previous property, *F strong*: If two elements x and y are related through the relator with two different lists $\overline{R} = R_1 \dots R_n$ and $\overline{Q} = Q_1 \dots Q_n$ of atom-level relations, then x and y are also related with any index-wise combination of $\overline{R}$ or $\overline{Q}$. For each index $1 \leq i \leq n$ either the relation R_i or Q_i is selected.

For our purpose of linearization, we are specifically interested in the case, where for all live variables that we linearize on the relation from $\overline{R}$ is chosen and for all others the Q relation. As an example, this results in the following lemma for $(\alpha, \beta)\ G$ from Subsection 3.1.1:

$$[\![\mathsf{rel}_G\ R_1\ R_2\ x\ y;\ \mathsf{rel}_G\ Q_1\ Q_2\ x\ y]\!] \Longrightarrow \mathsf{rel}_G\ R_1\ Q_2\ x\ y$$

In the specific case, that the MRBNF is linearized on *all* of it's live variables, the goal of the lemma is equal to it's first assumption. Thus, the lemma becomes trivial since exactly the list of relations $\overline{R}$ is chosen.

As a consequence of this, the previous lemma *F strong* is not needed to prove this lemma. Furthermore, this lemma is the sole reason why *F strong* and strong pullback preservation are needed for the linearization. Thus the requirement of pullback preservation can be lifted, in the case that the linearization is applied to all live variables at the same time.

map peresrving non-repetitiveness

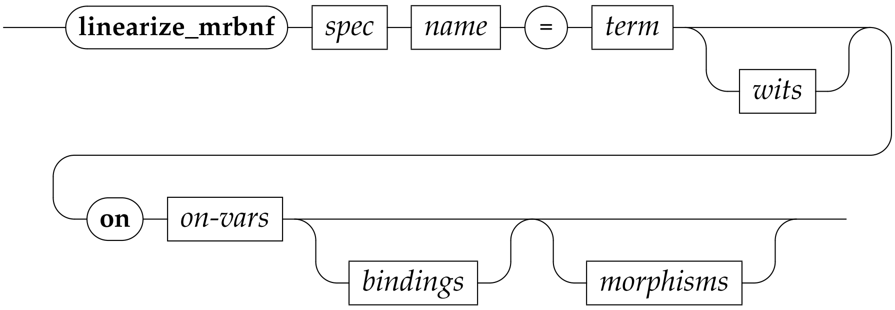
$$[\![\mathsf{nonrep}^1_G\ x;\ \mathsf{bijective} f]\!] \Longrightarrow \mathsf{nonrep}^1_G\ (\mathsf{map}_G\ f\ g\ x)$$

3.1.4 Proving the MRBNF axioms

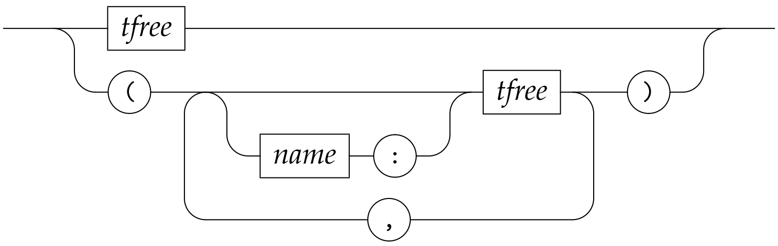
3.1.5 Non-emptines Witnesses

3.2 Linearization of MRBNFs (In Isabelle)

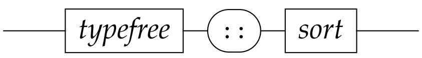
We implement a command that allows the user to linearize an existing MRBNF or BNF on one or multiple of it's live variables. The syntax of the command is given in the following:



spec



tfree



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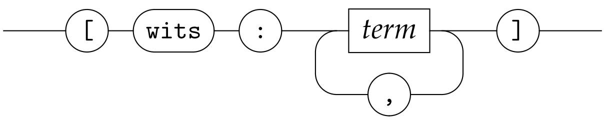
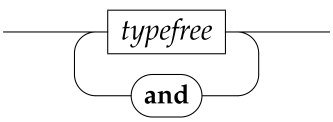
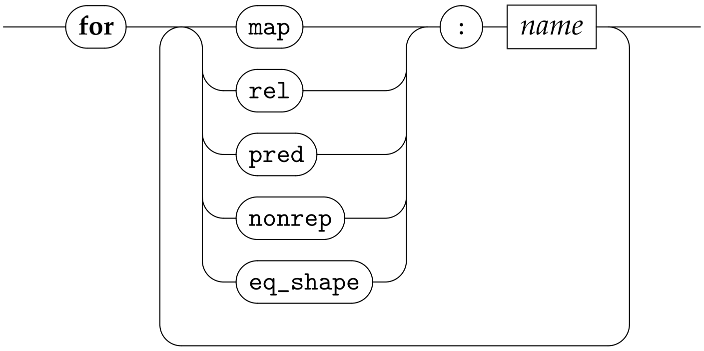


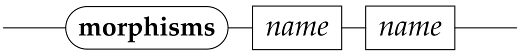
on-vars



bindings



morphisms



With this command, we can linearize our example by writing the following line in Isabelle:

$$\textbf{linearize\_mrbnf}\ (\text{keys}: {'k} :: \text{var},\ \text{vals}: {'v})\ \text{alist} = ({'k} :: \text{var}\ \times\ {'v})\ \text{list}\ \textbf{on}\ {'a}$$

Since for $('k \ \times \ 'v)$ list both type variables are live and we only linearize on $'k$, it is necessary to prove strong pullback preservation for this MRBNF.

After the user has written the command, the conditions for linearization we presented in Subsection 3.1.2 have to be shown, i.e., non-emptines and strong pullback preservation.

These conditions are given dynamically to the user. For example, strong pullback preservation only has to be shown, when the resulting MRBNF has live variables remaining. Furthermore, the non-emptines of the non-repetitive type is easily proven when the user specified a non-emptines witness, or a preservable witness of the original type exists. Thus, the user is not asked to show the existance of a non-repetitive element in these cases.

4 Examples

4.1 POPLmark challenge: Pattern

The POPLmark challenge [Ayd+05] presents a selection of problems to benchmark the progress in formalizing programming language metatheory. The challenges are built around formalizing aspects of *System* $F_{<:}$ calculus, a polymorphic typed lambda calculus with subtyping. We are interested in part 2B of this challenge, which has the goal to formalize and proof *type soundness* for terms with pattern matching over records. Type soundness is considered in terms of *preservation* (evaluating a term preserves its type) and *progress* (a term is either a value or can be evaluated).

We focus on the `record` terms `pattern-let`. A record is a term defined as a set of pairs, where the first element is a label and the second element a term: $\{(\mathtt{l}_j, \mathtt{t}_j)\}$. The labels l within a record must be pairwise distinct. A pattern is defined as either a typed variable or a set of (label, patten) pairs with pairwise distinct labels: $\mathtt{p} ::= \mathtt{x} : \mathtt{T} \mid \{(\mathtt{l}_j, \mathtt{p}_j)\}$

A formalization of part 2B of the POPLmark challenge in Isabelle/HOL is presented by Blanchette et al. [Bla+19]. They use *binder_datatypes* to abstract types, variables and terms. A central notion in this formalization is the *labeled finite set* $('a, 'b)$ lfset that is used in the representation of records and patterns. This type constructor is a subtype of $('a \times 'b)$ fset that only includes elements that are non-repetitive on $'a$. This restriction is necessary, because for both records and patterns the label $'a$ must be mutually distinct, i.e., the set representing them has to be non-repetitive.

While by construction $('a \times 'b)$ fset is a BNF (and an MRBNF since all BNFs are also MRBNFs) with both variables being live, $('a, 'b)$ lfset is a MRBNFs with $'a$ as a bound variable, since it is non-repetitive on $'a$. While this is a linearization, the finite set on pairs does not fulfill strong pullback preservation. Thus the approach and command we presented in Chapter 3 cannot be used here. Because of an alternate, equivalent description on non-repetitiveness specific to this type, it is still possible to manually linearize this MRBNF.

For the pattern a different type is used. It is constructed by linearizing an intermediate type prepat that is defined using the **datatype** command:

datatype $('v,\ 'tv)$ prepat $=$ PPVar "$'v$" "$'tv$ typ" | PPRec "(string, $('v,\ 'tv)$ prepat) lfset"

Abbreviations

BNF Bounded Natural Functor

MRBNF Map-Restricted Bounded Natural Functor

List of Figures

List of Tables

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