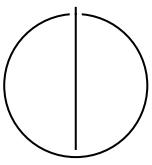
SCHOOL OF COMPUTATION,
INFORMATION AND TECHNOLOGY —
INFORMATICS

TECHNISCHE UNIVERSITÄT MÜNCHEN

Master's Thesis in Informatics

Constructing Linear Types in Isabelle/HOL

Felix Krayer



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Constructing Linear Types in Isabelle/HOL

Konstruktion linearer Typen in Isabelle/HOL

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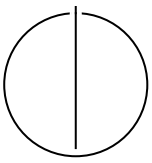


I confirm that this master's thesis is my own work and I have documented all sources and material used.

Munich, 13-11-2025 Felix Krayer

Acknowledgments

Abstract

Contents

1 Introduction

- Datatypes in general
  - Datatypes in Isabelle/HOL are built on Bounded Natural Functors (BNFs) (defined in [TPB12])
  - Structure of the Thesis

2 Background

This Chapter serves to introduce BNFs and their generalization to Map-Restricted Bounded Natural Functors (MRBNFs).

2.1 Bounded Natural Functors (BNFs)

As described in Chapter 1, datatypes in Isabelle are implemented using BNFs, meaning that the type constructors of polymorphic types (types with type variables) can be described through the properties of a BNF.

Examples of one such types are α list and α β prod (infix notation $\alpha \times \beta$). These are unary and binary type constructors, meaning that they can be applied to other types to form a new type. Thus, type constructors are *functors*.

Furthermore, they have a well-defined set and map function. The exact properties are listed below. Thus they behave *natural*. Lastly, they are *bounded*, since the set of elements that the type can describe is bounded by a (possibly transfinite) cardinal.

$$\textit{map\_id}: \mathsf{map}_F\ id\ x = x \tag{2.1}$$

$$\textit{map\_comp}: \mathsf{map}_F\ g\ (\mathsf{map}_F\ f\ x) = \mathsf{map}_F\ (g \circ f)\ x \tag{2.2}$$

$$\textit{map\_cong}: (\forall z \in \mathsf{set}_F\ z.\ f\ z = g\ z) \Longrightarrow \mathsf{map}_F\ f\ x = \mathsf{map}_F\ g\ x \tag{2.3}$$

$$\textit{set\_map}: \mathsf{set}_F(\mathsf{map}_F\ f\ x) = f\text{`}\mathsf{set}_F\ x \tag{2.4}$$

$$\textit{infinite\_regular\_card\_order}: \mathsf{infinite}\ \mathsf{bound}_F \wedge \mathsf{regular}\ \mathsf{bound}_F \wedge \mathsf{cardinal\_order}\ \mathsf{bound}_F \tag{2.5}$$

$$\textit{set\_bd}: |\mathsf{set}_F x| <_o \mathsf{bound}_F \tag{2.6}$$

$$\textit{rel\_compp\_leq}: \mathsf{rel}_F\ R\ \mathsf{OO}\ \mathsf{rel}_F\ Q\ = \mathsf{rel}_F\ (R\ \mathsf{OO}\ Q) \tag{2.7}$$

$$\textit{in\_rel}: \text{Weak Pullback Preservation WP} \tag{2.8}$$

Where` is the image function on sets and OO is the composition of relations.

While most of these properties are straightforward, we want to explain the preservation of weak pullbacks in more detail.

$$\mathsf{rel}_F\ R\ x\ y = \exists z.\ \mathsf{set}_F\ z \subseteq \{(a,b).\ R\ a\ b\} \wedge \mathsf{map}_F\ fst\ z = x \wedge \mathsf{map}_F\ snd\ z = y \tag{2.9}$$

The idea is that two elements x and y of the type $\alpha\ F$ are related through a relation R iff there exists a z that acts as a "zipped" version of x and y. The atoms of this z are the atoms of x and y, that are organized in pairs of R-related with the x as the first and y as the second position in the pair.

2.2 Map-Restricted Bounded Natural Functors (MRBNFs)

MRBNFs are a generalization of BNFs. By restricting the map function of a functor to *small support* functions or *small support bijections* for certain type variables, it is possible to reason about certain polymorphic types in ways that are not possible with just BNFs.

- Explain all the BNF theorems
- cite [Bla+19]

3 Linearizing MRBNFs

3.1 Linearization of MRBNFs (In theory)

In this section we define the linearization of an MRBNF on a subset of it's *live* variables. The result of the linearization is a new MRBNF with the same variable types (*live*, *dead*, *bound*, *free*), except for the linearized variables that change their type from *live* to *bound*. This means that the map function is now restricted to only allow bijective and small-support functions on these variables. Apart from this change, it is ensured that the MRBNF is *nonrepetitive* with respect to the linearized variables. We give a definition nonrepetitiveness in the following Subsection 3.1.1. Intuitively it means that the atoms of that type cannot occur multiple times in an element of the type.

3.1.1 Nonrepetitiveness

At the core of linearization lies the notion of *nonrepetitiveness*. We think of an element x of a type $\alpha\ F$ as being nonrepetitive if all its α atoms are distinct from another. We give an exact definition nonrep of nonrepetitiveness in relation to all other elements of $\alpha\ F$ that are of *equal shape*:

$$\mathsf{eq\_shape}_F\ x\ y = \mathsf{rel}_F\ top\ x\ y \tag{3.1}$$

$$\mathsf{nonrep}_F\ x = \forall y.\ \mathsf{eq\_shape}\ x\ y \longrightarrow (\exists f.\ y = \mathsf{map}_F\ f\ x) \tag{3.2}$$

We consider two $\alpha\ F$ elements x and y to have *equal shape*, if they are related with the *top* relation, i.e., the relation that relates all α atoms to all others. This is true, if the relator rel_F can relate the elements. In the case of list, this is the case when two lists are equal in length but have possibly different content.

Based on this, x is a nonrepetitive element, if for all other elements y with equal shape, a function exists through which x can be mapped to y. In our example of list, this holds for all lists with distinct elements (given a second list, one can easily define a function mapping the distinct elements of x to that list). It does not hold for lists with repeating elements, because no f exists that could map two equal elements at different positions in this list to distinct elements in an arbitrary second list.

For MRBNFs with more than one live variable, we can give a definitions of *nonrepetitiveness* and having *equal shape* on a subset of the live variables. In that case, we consider x and y of type $(\alpha, \beta)\ G$ to have equal shape with respect to α, iff they are *equal* in their β atoms and are related with *top* in α as before. Consequently for the map in the nonrep definition, the *id* function is applied to the β atoms, since they are already required to be equal.

$$\mathsf{eq\_shape}^1_G\ x\ y = \mathsf{rel}_G\ top\ (=)\ x\ y \tag{3.3}$$

$$\mathsf{nonrep}^1_G\ x = \forall y.\ \mathsf{eq\_shape}\ x\ y \Longrightarrow (\exists f.\ y = \mathsf{map}_G\ f\ id\ x) \tag{3.4}$$

3.1.2 Required properties

A MRBNFs has to fulfill two properties to be linearized. First, to ensure that the resulting type constructor is non-empty, it is required, that there exists a nonrepetitive element (with respect to the linearized variables): $\exists x.\ \mathsf{nonrep}\ x$

Furthermore, even though MRBNFs are already required to perserve weak pullbacks as defined in Equation 2.9, for the linearization it is required that they perserve *all* pullbacks. Formalized this means that the existance of z in the equation has to be fulfilled uniquely, i.e., for each R-related x and y there existes *exactly one* z fulfilling the property in Equation 2.9. For example the strong pullback preservation is fulfilled by the α list and $\alpha\ \beta$ prod functors but not by α set.

We note here that the requirement of strong pullback preservation can be omitted, when the MRBNF is linearized on all its live variables, i.e., when the linearized MRBNF has no live variables. This is because in this case the *relation exchange* lemma explained in Subsection 3.1.3 becomes trivial. In all other cases, that lemma is the sole reason, strong pullback preservation is required.

3.1.3 Intermediate lemmas

We want to prove the MRBNF axioms for the linearized MRBNF. For this we utilize some intermediate lemmas which we present in this section.

F strong From the pullback preservation with uniqueness we can prove the following lemma. In fact this notion of strongness is equivalent to pullback preservation:

$$[\![\mathsf{rel}_F\ R\ x\ y;\ \mathsf{rel}_F\ Q\ x\ y]\!] \Longrightarrow \mathsf{rel}_F\ (inf\ R\ Q)\ x\ y$$

where the infimum inf of two relations R and Q relates all elements that are related by both R and Q.

Relation exchange The *exchange of relations* is a consequence of the previous property, *F strong*: If two elements x and y are related through the relator with two different lists $\overline{R} = R_1 \dots R_n$ and $\overline{Q} = Q_1 \dots Q_n$ of atom-level relations, then x and y are also related with any index-wise combination of $\overline{R}$ or $\overline{Q}$. For each index $1 \leq i \leq n$ either the relation R_i or Q_i is selected.

For our purpose of linearization, we are specifically interested in the case, where for all live variables that we linearize on the relation from $\overline{R}$ is chosen and for all others the Q relation. As an example, this results in the following lemma for (α, β) G from Subsection 3.1.1:

$$[\![\mathsf{rel}_G\ R_1\ R_2\ x\ y;\ \mathsf{rel}_G\ Q_1\ Q_2\ x\ y]\!] \Longrightarrow \mathsf{rel}_G\ R_1\ Q_2\ x\ y$$

In the specific case, that the MRBNF is linearized on *all* of it's live variables, the goal of the lemma is equal to it's first assumption. Thus, the lemma becomes trivial since exactly the list of relations $\overline{R}$ is chosen.

As a consequence of this, the previous lemma *F strong* is not needed to prove this lemma. Furthermore, this lemma is the sole reason why *F strong* and strong pullback preservation are needed for the linearization. Thus the requirement of pullback preservation can be lifted, in the case that the linearization is applied to all live variables at the same time.

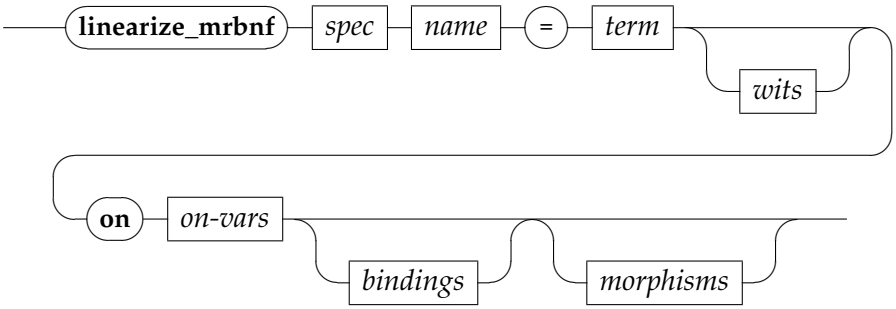
map peresrving nonrepetitiveness

$$[\![\mathsf{nonrep}^1_G\ x;\ \mathsf{bijective} f]\!] \Longrightarrow \mathsf{nonrep}^1_G\ (\mathsf{map}_G\ f\ g\ x)$$

3.1.4 Proving the MRBNF axioms

3.2 Linearization of MRBNFs (In Isabelle)

We implement a command that allows the user to linearize an existing MRBNF or BNF on one or multiple of it's live variables. The syntax of the command is given in the following:



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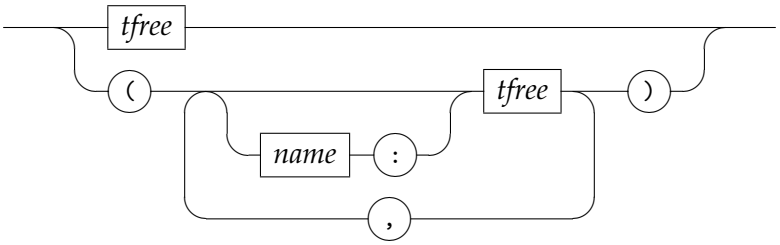

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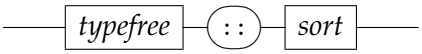

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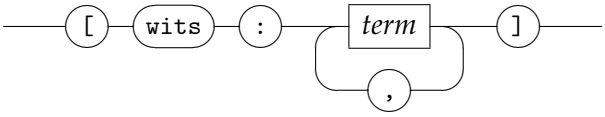

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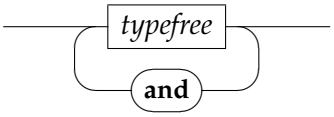
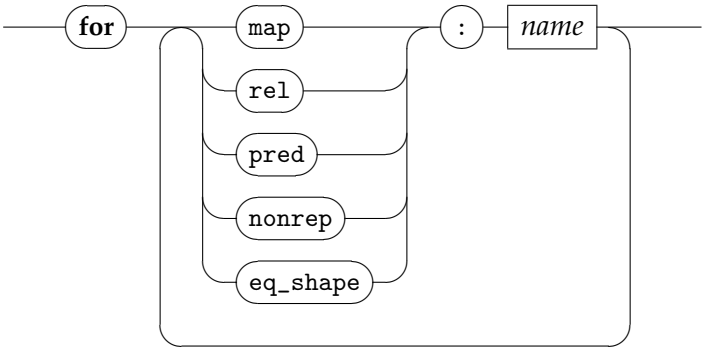
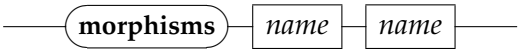


bindings



morphisms



With this command, we can linearize our example by writing the following line in Isabelle:

linearize_mrbnf (keys: $'k$ :: var, vals: $'v$) alist = ($'k$ :: var $\times$ $'v$) list **on** $'a$

Since for ($'k$ $\times$ $'v$) list both type variables are live and we only linearize on $'k$, it is necessary to prove strong pullback preservation for this MRBNF.

Abbreviations

BNF Bounded Natural Functor

MRBNF Map-Restricted Bounded Natural Functor

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