SCHOOL OF COMPUTATION, INFORMATION AND TECHNOLOGY — INFORMATICS

TECHNISCHE UNIVERSITÄT MÜNCHEN

Master's Thesis in Informatics

Constructing Linear Types in Isabelle/HOL

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Konstruktion linearer Typen in Isabelle/HOL

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Submission Date:	13-11-2025

I confirm that this master's thesis is my own work and I have documented all sources and material used.

Munich, 13-11-2025 Felix Krayer

Acknowledgments

Abstract

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    - Datatypes in Isabelle/HOL are built on Bounded Natural Functors (BNFs) (defined in [TPB12])
    - Structure of the Thesis

2 Background

This Chapter serves to introduce BNFs and their generalization to Map-Restricted Bounded Natural Functors (MRBNFs). Note, that when we use the notion "element" of a type constructor F, we are always talking about a term of type F. In contrast to that, we call the components that make up a F element "atoms". Their type is one of the type variables of F and the structure of that F element dictates how and where they occur in it.

2.1 Bounded Natural Functors (BNFs)

As described in Chapter 1, BNFs are essential for constructing datatypes and co-datatypes in Isabelle/HOL. Especially for defining a datatype with recursion it is required that the type constructor used in that recursion is registered as a BNF, i.e., it fulfills the BNF-axioms. For example the following **datatype** command only succeeds if α list, is a BNF.

datatype α ex = A "($\alpha \times (\alpha$ ex)) list"

Since BNFs are closed under composition and fixpoints, the resulting datatype (here α ex) can be automatically registered as a BNF as well.

We write type variables as greek letters (α, β, ...) in this thesis. However, in the Isabelle proof assistant type variables are written with a "′" in front of a name, e.g., ′a list. To copy our examples to Isabelle, one has to replace these greek letters with "′" variables. Alternatively, a "′" can be prepended to the greek letters, since for example ′α is a valid type variable in Isabelle.

The type variables of a BNF are divided into two groups: *live* and *dead* variables or *lives* and *deads*. Live variables can be used for recursive datatype definitions, while dead ones do not allow for this. We take the function type $\alpha \Rightarrow \beta$ as an example. It's first type argument α is dead, while the second one β is live. Thus, of the following the first command succeeds while the second one fails

datatype α success = S1 | S2 "$\alpha \Rightarrow \alpha$ success"

datatype α fail = F1 | F2 "α fail $\Rightarrow \alpha$"

2.1.1 BNF constants

A BNF F with l live variables is characterized by one map and l set functions, a bound and a relator.

Map function and functors

The $l+1$-ary map function or *mapper* takes one function for each live of F as arguments as well as one F element. The domain types of these functions are the lives of F. These functions are recursively applied to the atoms of an element. The result is a new element of type F, where the original type variables are replaced by the range types of the mapped functions. Taking the α list type as an example, a BNF with one live α, the mapper has the type $\mathsf{map}_{\mathsf{list}} :: (\alpha \Rightarrow \alpha') \Rightarrow \alpha\ \mathsf{list} \Rightarrow \alpha'\ \mathsf{list}$.

To make F with its mapper a *functor* on the universe of all types, the mapper has to fulfill two axioms [TPB12]. First, mapping the id function on all lives over an element should leave it unchanged, which is formalized in MAP_ID. The second property MAP_COMP is concerned with mapping compositions and reads as follows: Mapping two lists of functions over an element, e.g., first $f_1 \ldots f_l$ and then $g_1 \ldots g_l$, should produce the same result as mapping the index-wise composition $(g_1 \circ f_1) \ldots (g_l \circ f_l)$ over it once. A type constructor F with a map function map_F fulfilling these two properties is considered a functor.

Set functions and naturality

A set function or *setter* is defined for each of the l live variables. Applied to an F-element, the i-th setter returns the set of all atoms that are part of the element and correspond to the i-th live. For example, the setter of the list type returns the set of elements in the list. We note here that when we write i as an index, we assume it to be in the range $1 \leq i \leq l$.

The set functions together with the mapper give rise to another property. We want the setters $\mathsf{set}_{F,i}$ to be natural transformations from F and map_F to the set and image function. Thus, they should fulfill the SET_MAP axiom. It states that taking the i-th set of an F after mapping $f_1 \ldots f_l$ to it, results in the same set as if i-th set was taken from the original F before the image of f_i was applied to it. Figure 2.1 shows a visualization of this axiom and reads as follows: Starting from an F element first applying the setter and then mapping a function (path through the top right) results in the same as first mapping the function and then applying the setter (path through the bottom left).

$$\begin{tikzcd}
(\alpha_1, \dots \alpha_l)\ F \arrow[r, "\mathsf{set}_{F,i}"] \arrow[d, "\mathsf{map}_F\ f_1\ \dots\ f_l"'] & \alpha_i\ \mathsf{set} \arrow[d, "image\ f_i"] \\
(\beta_1, \dots \beta_l)\ F \arrow[r, "\mathsf{set}_{F,i}"] & \beta_i\ \mathsf{set}
\end{tikzcd}$$

for all i where α_i is a live variable of F

Figure 2.1: $\mathsf{set}_{F,i}$ as a natural transformation

Bound and boundedness

Lastly, the BNF needs an infinite cardinal as a bound. This bound may depend on the cardinalities of the dead variables, but not on the of the live variables. In Isabelle/HOL cardinals are implemented as minimal wellorders with respect to isomorphisms [BPT14]. While details about this implementation are certainly interesting, we will not focus on these details in this thesis. For example *natLeq*, the cardinal that originates from the $\leq$ order on natural numbers, is equivalent to the smallest infinite cardinal $\aleph_0$.

Besides being a cardinal order, the bound is required to be infinite, i.e., at least $\aleph_0$ with respect to the cardinal order $\leq_o$, and regular. Regularity means that an infinite cardinal κ is stable under union, i.e., the union of any two sets of smaller cardinality than κ also has a smaller cardinality than κ.

The bound is used in the SET_BD axiom to ensure that the sets obtained by the setters are bounded. This ensures that the branching of a recursively defined datatype is also bounded and thus the resulting type F as well.

Relator and shapes

The relator is used to build a relation on F by relating the atoms of an F element. It takes one relation for each live, that relates the corresponding type variables of the two Fs that are to be related. As an example we give the type and definition of the relator for the product type as follows:

$$\mathsf{rel}_{\mathsf{prod}} ::\ (\alpha \Rightarrow \alpha' \Rightarrow \mathsf{bool}) \Rightarrow (\beta \Rightarrow \beta' \Rightarrow \mathsf{bool}) \Rightarrow (\alpha \times \beta) \Rightarrow (\alpha' \times \beta') \Rightarrow \mathsf{bool}$$
$$\mathsf{rel}_{\mathsf{prod}}\ R\ Q\ p_1\ p_2 :=\ R\ (\mathrm{fst}\ \ p_1)\ (\mathrm{fst}\ \ p_2) \land Q\ (\mathrm{snd}\ \ p_1)\ (\mathrm{snd}\ \ p_2)$$

Considering the list type again, we make an interesting observation: There are some α lists xs and ys that the relator cannot relate, no matter which α relation is chosen. The

(MAP_ID)	$\mathsf{map}_F\ \mathrm{id}^l\ x = x$
(MAP_COMP)	$\mathsf{map}_F\ g^l\ (\mathsf{map}_F\ f^l\ x) = \mathsf{map}_F\ (g \circ f)^l\ x$
(MAP_CONG)	$(\forall i.\ \forall z \in \mathsf{set}_{F,i}\ x.\ f_i\ z = g_i\ z) \Longrightarrow \mathsf{map}_F\ f^l\ x = \mathsf{map}_F\ g^l\ x$
(SET_MAP)	$\forall i.\ \mathsf{set}_{F,i}\ (\mathsf{map}_F\ f^l\ x) = f_i\ \grave{}\ \mathsf{set}_{F,i}\ x$
(BD)	$\mathsf{infinite\ bd}_F \wedge \mathsf{regular\ bd}_F \wedge \mathsf{cardinal\_order\ bd}_F$
(SET_BD)	$\forall i.\ \lvert\mathsf{set}_{F,i}\ x\rvert <_o \mathsf{bd}_F$
(REL_COMPP)	$\mathsf{rel}_F\ R^l\ \bullet\ \mathsf{rel}_F\ Q^l\ = \mathsf{rel}_F\ (R\ \bullet\ Q)^l$
(IN_REL)	$\mathsf{rel}_F\ R^l\ x\ y =$
	$\exists z.\ (\forall i.\ \mathsf{set}_{F,i}\ z \subseteq \{(a,b).\ R_i\ a\ b\}) \wedge \mathsf{map}_F\ \mathrm{fst}^l\ z = x \wedge \mathsf{map}_F\ \mathrm{snd}^l\ z = y$

where $\grave{}$ is the image function on sets, $\bullet$ is the composition of relations and $<_o$ is the less than relation on cardinals

Figure 2.2: The BNF axioms

relator on lists is index-wise defined, i.e., the α relation must relate the elements of both lists for each index. Consequently lists of different length cannot be positively related. We think of the length of a list as its *shape*. We can generalize this idea of shape to an arbitrary type constructor F. The shape of an F element is defined by the way it is constructed and the relator can only ever relate those that have the same or equivalent shape, i.e., it will always evaluate to $false$, when two elements of different shape are given, regardless of the relations given to the relator. We can think of an element of type F as a container that has a certain *shape* with slots for *atoms*. These atoms are elements of the type constructor's type arguments.

2.1.2 BNF-axioms

We formalize the BNF-axioms in Figure 2.2 where we use the notation $f^l = f_1 \ldots f_l$ for the arguments of the mapper and the relator Additionally to the axioms we already motivated in Subsection 2.1.1 (MAP_ID and MAP_COMP for the functoriality of F, SET_MAP to ensure that the setters are natural transformations and the boundedness of the setters SET_BD), we have four additional ones.

One of those is the the congruency MAP_CONG of the map function. It states that if two (lists of) functions f^l and g^l are equal when applied to the corresponding sets of all atoms of an F (obtained through the setters), then mapping these two lists of functions over the F each produces the same result. When this property holds, we can be sure, that the mapper only depends on how the functions f^l behave on the atoms of the F element.

The axiom BD just ensures that the bound bd_F is a suitable cardinal, i.e., a regular and infinite one.

Distributivity of the relator is formulated in IN_REL. We note here, that for showing that a type constructor is a BNF, it is only necessary to prove the inclusion $(\mathsf{rel}_F\ R^l \bullet \mathsf{rel}_F\ Q^l)\ x\ y \Rightarrow \mathsf{rel}_F\ (R \bullet Q)^l\ x\ y$. The other direction follows automatically from this and the next axiom, weak pullback preservation.

Lastly, weak pullback preservation IN_REL is the most abstract and complex axiom. The idea is that two elements x and y of the type $\alpha\ F$ are related through a relation R iff there exists a z that acts as a "zipped" version of x and y. The atoms of this z are R_i-related pairs of the atoms of x and y, where the first position in the pair corresponds to x and the second one to y.

2.1.3 Non-emptiness witnesses

BNF carry non-emptiness witnesses as proof that the type contains at least one element. Witnesses may depend on a subset of the BNF's live variables. For example a witness of $(\alpha_1, \ldots \alpha_l)\ F$ that depends on the first and last type variable of F, this witness has the type $\mathsf{wit}_F :: \alpha_1 \Rightarrow \alpha_l \Rightarrow (\alpha_1, \ldots \alpha_l)\ F$. It denotes that given witnesses for the types α_1 and α_l, a witness for F can be constructed.

Witnesses have to fulfill the following properties: For all type variables α_i the witness depends on, the witness may only contain the α_i elements w_i, that were given to the witness as arguments, i.e., $textscset_{F,i}$ applied to the witness evaluates to the singleton $\{w_i\}$. Furthermore, the witness must not contain any elements of the live type variables α_j, the witness does not depend on, i.e., SET$_{F,j}$ must be empty. We formalize these properties in the following where $\bar{w}$ denotes the arguments that the witness depends on.

(WITS) $\quad \forall i.\ \mathsf{set}_{F,i}\ (\mathsf{wit}_F\ \bar{w}) = (\texttt{if}\ \mathsf{wit}_F \text{ depends on } \alpha_i\ \texttt{then}\ \{w_i\}\ \texttt{else}\ \emptyset)$

If multiple types of witnesses exist for a given F, then the ones with the fewest arguments are most useful for showing non-emptiness. Concretely, we say a witness $\mathsf{wit}_{F,1}$ *subsumes* $\mathsf{wit}_{F,2}$, when $\mathsf{wit}_{F,1}$ depends on a true subset of the arguments of $\mathsf{wit}_{F,2}$. In this case we ignore the subsumed witness, as the other one is more useful. However, when two witnesses have overlapping dependencies but neither depends on a subset of the other we are interested in both, even if one has a smaller number of arguments than the other.

2.1.4 BNF examples

Further examples of BNFs are is the product type (α, β) prod, a binary type constructor with infix notation $\alpha \times \beta$, and the type of finite sets α fset. The latter is interesting for the reason that it is a subtype of the set type, which is not a BNF. By enforcing finiteness for the elements of the type it is possible to give a bound for the set function, fulfilling the SET_BD axiom, which is not possible for the unrestricted set type. Since unboundedness is the only reason that the set type is not a BNF, α fset can be shown to be a BNF.

To show, how BNFs can be combined to create new ones, we consider the type constructor (α, β) plist = $(\alpha \times \beta)$ list. We define for it a map function ($\mathsf{map}_{\mathsf{plist}}$) and two set functions ($\mathsf{set1}_{\mathsf{plist}}$ and $\mathsf{set2}_{\mathsf{plist}}$) as well as a relator $\mathsf{rel}_{\mathsf{plist}}\ R\ Q$. The exact definitions are given as such:

$$\begin{aligned} \mathsf{map}_{\mathsf{plist}}\ f\ g\ &= \mathsf{map}_{\mathsf{list}}\ (\mathsf{map}_{\mathsf{prod}}\ f\ g) \\ \mathsf{set1}_{\mathsf{plist}}\ xs &= \mathsf{set}_{\mathsf{list}}\ (\mathsf{map}_{\mathsf{list}}\ \mathrm{fst}\ \ xs) \\ \mathsf{set2}_{\mathsf{plist}}\ xs &= \mathsf{set}_{\mathsf{list}}\ (\mathsf{map}_{\mathsf{list}}\ \mathrm{snd}\ \ xs) \\ \mathsf{rel}_{\mathsf{plist}}\ R\ Q &= \mathsf{rel}_{\mathsf{list}}(\mathsf{rel}_{\mathsf{prod}}\ R\ Q) \end{aligned}$$

where we use the standard map, set and relator functions of the list and product type.

To show that (α, β) plist is a BNF, we have to prove the BNF-axioms for it. Besides the definitions above, we give $\aleph_0$ as the bound $\mathsf{bd}_{\mathsf{plist}}$.

2.2 Syntaxes with bindings

WIP: Considering a polymorphic type that is meant to represent simple λ-terms, where α is the space of variable names. If we want to substitute a free variable x in a term T by a term N, we may run into the following problem: If $T = \lambda y.T'$, we need to ensure that there are no name clashes with y in the new term N before we substitute x by N in T'. This is done by choosing a fresh y' and renaming y to y' in T'.

2.3 Map-Restricted Bounded Natural Functors (MRBNFs)

Type constructors that involve names or bindings often violate the requirements of BNFs. Considering for example the type of distinct lists α dlist, a subtype of α list that describes only lists containing pairwise distinct α atoms. The issue with this type is that the standard map function on lists cannot guarantee that the resulting list is still distinct, i.e., that it is still part of the type. Thus in BNF terms the type variable of α dlist is dead. However, by restricting the mapper to only use bijections, the distinctness of the resulting list can be ensured.

MRBNFs are a generalization of BNFs. Restricting the map function of a functor to *small-support* functions or *small-support bijections* for certain type variables allows us to reason about type constructors in terms of BNF properties, even in cases where this would not be possible otherwise. We call type variables that that are restricted to small-support functions *free* variables or *frees* and those restricted to small-support bijections *bound* variables or *bounds*. This allows us to define MRBNFs with four types of variables (lives, frees, bounds and deads) as opposed to BNFs which only distinguish between lives and deads. Our example from the beginning of this section, the distinct list α dlist is a MRBNF with α as a bound variable.

A small-support function leaves most arguments unchanged, meaning it acts like the identity function on them. Concretely defined, the cardinality of the set of arguments the function changes must be smaller than the cardinality of the argument type itself:

$$\mathsf{small\_supp}\ f = |\{x :: \alpha .\ f\ x \neq x\}| <_o |\Omega_\alpha|$$
$$\text{where } \Omega_\alpha \text{ is the universe of type } \alpha.$$

For a MRBNF F with l lives, fr frees and b bounds we define $\varpi = l + fr + b$ as the number of all non-dead type variables. With this, the mapper and setters are expanded to work for the frees and bounds just as they do for lives. Thus, F has ϖ setters and a mapper with arity $\varpi + 1$. Since the mapper takes small-support functions and bijections for the free and bound variables, which have the same type for their domain and range, this transfers to F as well. This means that the type variables for frees and bounds are the same for the F argument of the mapper and the result, while the lives can change type.

As before, we write f^l for the functions or relations of the live variables $f_1 \ldots f_l$ and analogously v^{fr} and u^b for frees and bounds. Furthermore, we write the arguments of the map function as $f^l\ v^{fr}\ u^b$. For example for the type $(\alpha,\ \beta,\ \gamma)\ F$ where α and β are free, while γ is bound, the mapper has the following type:

$$\mathsf{map}_F :: (\alpha \Rightarrow \alpha) \Rightarrow (\beta \Rightarrow \beta) \Rightarrow (\gamma \Rightarrow \gamma) \Rightarrow (\alpha,\ \beta,\ \gamma)\ F \Rightarrow (\alpha,\ \beta,\ \gamma)\ F$$

From now on we assume that the type variables of any MRBNF are ordered *lives* first, followed by *frees* and *bounds*, and *deads* at the end. This simplifies many definitions and arguments we make about MRBNFs, however these are all easily generalized to an arbitrary ordering of type variables. For example, the argument order of the mapper might be different, as the lives, bounds and frees do not have to be separated, but can be interlaced in some order. MRBNFs that are explicitly defined in terms of primitive types like list or prod are exempt from this rule.

We keep the original relator that only relates live variables with given relations and relates the free and bound variables to be with equality. Thinking in our model of F elements being shapes with atoms in slots, the regular relator rel_F requires the frees and bounds in each slot to be the same for both elements that are compared. To relate F elements that are not equal in all frees and bounds, we introduce a new map-restricted relator $\mathsf{mr\_rel}_F$. It takes a function for each free and bound - with the appropriate restrictions to small-support and bijectivity - in addition to the relations for the lives. It then uses the graphs Grp of these functions as relations for the respective free or bound variable. Transferring the ideas of bijectivity ans small-support to these graph relations, the graph of a bijective function relates each atom to exactly one other atom, while the graph of small-support function acts as equality on all the arguments that are not in its support. The new arguments of the map-restricted relator are placed in front of the relations for the live variables. It is then defined in terms of the relator as shown below. Note, that relating two elements with the graph of a function v is equivalent to mapping v over the first element and relating that to the second one by equality. Thus, we define it as follows:

$$\mathsf{mr\_rel}_F\ u^b\ v^{fr}\ R^l\ x\ y = \mathsf{rel}_F\ R^l\ (\mathsf{map}_F\ \mathrm{id}^l\ v^{fr}\ u^b\ x)\ y$$

2.3.1 MRBNF axioms

MRBNFs require the same axioms as BNFs with slight modifications. We take the formalized axioms from Figure 2.2 as a base and explain the differences.

For the MAP_COMP, MAP_CONG and SET_MAP axioms, we add the assumptions that the functions that correspond to frees and bounds are small-support functions and that the ones corresponding to bounds are additionally bijections. It means that $\mathsf{small\_supp}\ v^{fr} \land \mathsf{small\_supp}\ u^b \land \mathsf{bijective}\ u^b$ is added as assumptions to these axioms.

Furthermore, while REL_COMPP stays unchanged, using the original relator, IN_REL is changed to be defined in terms of the map-restricted relator $\mathsf{mr\_rel}_F$ as follows:

$$\begin{aligned}\mathsf{mr\_rel}_F\ u^b\ v^{fr}\ R^l\ x\ y = \exists z.\ (\forall i.\ \mathsf{set}_{F,i}\ z \subseteq \{(a,b).\ R_i\ a\ b\})\land \\ \mathsf{map}_F\ \mathrm{fst}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ z = x \land \mathsf{map}_F\ \mathrm{snd}^l\ v^{fr}\ u^b\ z = y\end{aligned}$$

2.3.2 binder datatypes

MRBNFs can be used in a **binder_datatype** command to produce a datatype with bindings.

In the resulting MRBNF the free and bound type variables are required to be *large* and *regular*. Largeness is necessary to ensure that there are always fresh elements available for renaming. It is defined as the cardinality of the type being at least $\aleph_0$ for datatypes or $\aleph_1$ for codatatypes. In Isabelle the requirements of largeness and regularity are combined in dedicated type classes, `var` and `covar` respectively.

TODO: more + cite [Bla+19]

3 Linearizing MRBNFs

3.1 Linearizing MRBNFs

In this section we define the linearization of a MRBNF F on a subset of it's *live* variables. Linearization means, that the resulting type only contains elements for which all atoms of the linearized variables are distinct. We say F is *non-repetitive* on these variables. This type is also a MRBNF with the same variable types (*live*, *free*, *bound*, *dead*), except for the linearized variables that change their type from *live* to *bound*. This means that the new map function is restricted to only allow bijective and small-support functions on these variables.

We formalize the idea of distinctness of atoms as *non-repetitiveness* on the linearized variables. In our notation we use $\mathit{lin} \leq l$ to refer to the number of linearized variables. Furthermore, we assume the variables that we linearize on to be the the *last* lin of the live variables. Consequently, the first $l' = l - \mathit{lin}$ lives of F are not linearized.

3.1.1 Non-repetitiveness

At the core of linearization lies the notion of *non-repetitiveness*. An element x of a type is considered to be non-repetitive with respect to a type variable α if it does not contain repeating α-atoms. For example, a α list is non-repetitive, if all of its α-elements it contains are pairwise distinct. To define non-repetitiveness for an arbitrary MRBNF, we have to express this property in terms of its map, set and relator functions. Considering α lists again, we can show a list xs to be distinct, iff for each other list ys of the same length, we can find a function f such that $ys = \mathsf{map}_{\mathsf{list}}\ f\ xs$. If xs were not distinct, there must exist two indices with the same α element in xs. Furthermore, there exists a ys that has different elements at these two indices and thus a function mapping xs to ys cannot exist, since it would have to map two same elements in xs to two differing ones in ys.

In Subsection 2.1.1 we proposed the idea to think about elements of a BNF (or MRBNF) F as containers with a certain shape with atoms in slots specified by the shape. Using this model, we can generalize the notion of lists having the same length to F elements having the same shape. We can express this through the relator by using the $\top$ relation that relates everything with each other as the argument. Thus, we give the definition of equivalent shape and non-repetitiveness for list:

$$\mathsf{eq\_shape}_{\mathsf{list}}\; x\; y = \mathsf{rel}_{\mathsf{list}}\; \top\; x\; y$$
$$\mathsf{nonrep}_{\mathsf{list}}\; x = \forall y.\; \mathsf{eq\_shape}_{\mathsf{list}}\; x\; y \Longrightarrow (\exists f.\; y = \mathsf{map}_{\mathsf{list}}\; f\; x)$$

Note that we use the regular relator that only relates live variables with given relations while it requires equality for all frees and bounds.

Based on this, x is a non-repetitive element, if for all other elements y with equal shape, a function exists through which x can be mapped to y. In our example of list, this holds for all lists with distinct elements (given a second list, one can easily define a function mapping the distinct elements of x to that list). It does not hold for lists with repeating elements, because no f exists that could map two equal elements at different positions in this list to distinct elements in an arbitrary second list.

More interesting is the case of $(\alpha,\ \beta)$ alist which we only want to be non-repetitive on α. For our purpose of defining non-repetitiveness on a subset of the live variables, we fix the other live variables to be equal when defining equivalent shape. For MRBNFs with more than one live variable, we can give a definitions of *non-repetitiveness* and having *equal shape* on the last *lin* live variables. In that case, we consider x and y of type F to have equal shape with respect to the variables $\alpha_{l'+1} \ldots \alpha_{lin}$, iff they are *equal* in the atoms corresponding to the non linearized lives and are related with $\top$ in on the linearized variables. Consequently for the map in the nonrep definition, the id function is applied to the non linearized lives, since they are already required to be equal.

(EQ_SHAPE) $\mathsf{eq\_shape}^{lin}_{F}\; x\; y = \mathsf{rel}_F\; \langle (=)^{l'}\; (\top)^{lin} \rangle\; x\; y$

(NONREP) $\mathsf{nonrep}^{lin}_{F}\; x = \forall y.\; \mathsf{eq\_shape}^{lin}_{F}\; x\; y \Longrightarrow (\exists f^{lin}.\; y = \mathsf{map}_F\; \langle \mathrm{id}^{l'}\; f^{lin} \rangle\; \mathrm{id}^{fr}\; \mathrm{id}^{b}\; x)$

Note, that we use $\langle \ldots \rangle$ to indicate arguments that belong together, e.g., that they are both related to lives in this case. They are just inserted to improve readability. Once again, our arguments hold for any ordering of type variables. The assumption that we linearize on the last *lin* variables only serves readability and is not a limitation in the actual implementation, where we allow an arbitrary subset of lives to be chosen for linearization.

3.1.2 Conditions for linearization

A MRBNFs has to fulfill two properties to be linearized. First, to ensure that the resulting type constructor is non-empty, it is required, that there exists a non-repetitive element (with respect to the linearized variables): $\exists x.\; \mathsf{nonrep}\; x$

Furthermore, even though MRBNFs are already required to preserve weak pullbacks defined as IN_REL in Figure 2.2, for the linearization it is required that they preserve *all* pullbacks. Formalized this means that the existence of z in the equation has to be fulfilled uniquely, i.e., for each R-related x and y there exists *exactly one* z fulfilling the property IN_REL. For example the strong pullback preservation is fulfilled by the α list and α β prod functor but not by α fset, the type constructor for finite sets of αs.

(PB_STRONG) $$\mathsf{rel}_F\ R^l\ x\ y = \exists! z.\ (\forall i.\ \mathsf{set}_{F,i}\ z \subseteq \{(a,b).\ R_i\ a\ b\}) \wedge \\ \mathsf{map}_F\ \mathrm{fst}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ z = x \wedge \mathsf{map}_F\ \mathrm{snd}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ z = y$$

We note here that the requirement of strong pullback preservation can be omitted, when the MRBNF is linearized on all its live variables, i.e., when the linearized MRBNF has no live variables. This is because in this case the REL_EXCHANGE lemma explained in Subsection 3.1.3 becomes trivial. In all other cases, that lemma is the sole reason, strong pullback preservation is required.

3.1.3 Intermediate lemmas

We want to prove the MRBNF axioms for the linearized MRBNF. For this we utilize some intermediate lemmas which we present in this section.

F is strong

From the pullback preservation with uniqueness we can prove the following lemma. In fact this notion of strength is equivalent to pullback preservation:

(F_STRONG) $$\mathsf{rel}_F\ R^l\ x\ y\ \wedge\ \mathsf{rel}_F\ Q^l\ x\ y \Longrightarrow \mathsf{rel}_F\ (\mathrm{inf}\ \ R\ Q)^l\ x\ y$$

Here the infimum inf of two relations R_i and Q_i relates exactly those elements that are related by both R_i and Q_i. To prove this lemma we first conclude that since x and y are related through some relations, they are also related with the $\top$-relation on all lives. This is the case since the relator or a MRBNF is monotonic and $\top$ relates all atoms with each other. Unfolding PB_STRONG on that, we can eliminate the subset conditions for the setters, since the right sides of the subsets are just the universe of pairs with appropriate type. The reason for this is the $\top$-relation that is fulfilled by every pair in this set. This now gives us the knowledge that there exists exactly one z that is the "zipped" version of x and y, or in other words any two z and z' fulfilling this condition must be equal.

$$\begin{aligned}
& \mathsf{rel}_F\ R^l\ x\ y \\
\Longrightarrow\ & \mathsf{rel}_F\ (\top)^l\ x\ y && \mathsf{rel}_F \text{ mono} \\
\Longrightarrow\ & \exists! z.\ (\forall i.\ \mathsf{set}_{F,i}\ z \subseteq \{(a,b).\ (\top)\ a\ b\}) \wedge && \text{PB\_STRONG} \\
& \quad \mathsf{map}_F\ \mathrm{fst}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ z = x \wedge \mathsf{map}_F\ \mathrm{snd}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ z = y \\
\Longrightarrow\ & \exists! z.\ \mathsf{map}_F\ \mathrm{fst}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ z = x \wedge \mathsf{map}_F\ \mathrm{snd}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ z = y && \top \equiv \text{True} \\
\Longrightarrow\ & \forall z\ z'.\ (\mathsf{map}_F\ \mathrm{fst}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ z = x \wedge \mathsf{map}_F\ \mathrm{snd}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ z = y\ \wedge \\
& \quad \mathsf{map}_F\ \mathrm{fst}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ z' = x \wedge \mathsf{map}_F\ \mathrm{snd}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ z' = y \Longrightarrow && \exists! \text{ alternative} \\
& \quad z = z')
\end{aligned}$$

When we unfold the MRBNF axiom IN_REL on the original formulation of the lemma, we obtain a construct with two existential quantifiers $\exists z$ in the assumptions and one in the goal. With the knowledge we gained above, we can conclude that they must all be equal

$$\begin{aligned}
& \mathsf{rel}_F\ R^l\ x\ y\ \wedge\ \mathsf{rel}_F\ Q^l\ x\ y \Longrightarrow \mathsf{rel}_F\ (\inf\ R\ Q)^l\ x\ y \\
\equiv\ & \exists z_R.\ (\forall i.\ \mathsf{set}_{F,i}\ z_R \subseteq \{(a,b).\ R_i\ a\ b\}) \wedge && \text{IN\_REL} \\
& \quad \mathsf{map}_F\ \mathrm{fst}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ z_R = x \wedge \mathsf{map}_F\ \mathrm{snd}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ z_R = y\ \wedge \\
& \exists z_Q.\ (\forall i.\ \mathsf{set}_{F,i}\ z_Q \subseteq \{(a,b).\ Q_i\ a\ b\}) \wedge \\
& \quad \mathsf{map}_F\ \mathrm{fst}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ z_Q = x \wedge \mathsf{map}_F\ \mathrm{snd}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ z_Q = y\ \Longrightarrow \\
& \exists z_{\inf}.\ (\forall i.\ \mathsf{set}_{F,i}\ z_{\inf} \subseteq \{(a,b).\ (\inf R_i\ Q_i)\ a\ b\}) \wedge \\
& \quad \mathsf{map}_F\ \mathrm{fst}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ z_{\inf} = x \wedge \mathsf{map}_F\ \mathrm{snd}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ z_{\inf} = y \\
\equiv\ & (\forall i.\ \mathsf{set}_{F,i}\ z \subseteq \{(a,b).\ R_i\ a\ b\}) \wedge && z = z' \text{ (see above)} \\
& (\forall i.\ \mathsf{set}_{F,i}\ z \subseteq \{(a,b).\ Q_i\ a\ b\}) \Longrightarrow \\
& (\forall i.\ \mathsf{set}_{F,i}\ z \subseteq \{(a,b).\ (\inf R_i\ Q_i)\ a\ b\})
\end{aligned}$$

The last step is proven by applying common rules on sets of pairs, subsets and conjunction after $(\inf R_i\ Q_i)\ a\ b$ is unfolded to $R_i\ a\ b \wedge Q_i\ a\ b$.

Relation exchange

The *exchange of relations* is a consequence of the previous property, F_STRONG: If two elements x and y are related through the relator with two different lists $R^l = R_1 \ldots R_l$ and $Q^l = Q_1 \ldots Q_l$ of atom-level relations, then x and y are also related with any index-wise combination of R^l or Q^l. For each index i either the relation R_i or Q_i is selected.

For our purpose of linearization, we are specifically interested in the case, where for all live variables that we linearize on the relation from R^l is chosen and for all others the relation from Q^l relation, i.e., $Q^{l'}$ R^{lin}. This results in the following lemma for a MRBNF F:

(REL_EXCHANGE) $$\mathsf{rel}_F\ R^l\ x\ y\ \wedge\ \mathsf{rel}_F\ Q^l\ x\ y \Longrightarrow \mathsf{rel}_F\ \langle Q^{l'}\ R^{lin}\rangle\ x\ y$$

The lemma F_STRONG states that for each type variable the atoms are related with the infimum of R_i and Q_i. This means, that the atoms of each type variable can be related with R_i and Q_i at the same time. To prove this lemma we choose the appropriate relation from each of the l infima. This informal idea is the core of this lemma's formal proof.

In the specific case, that the MRBNF is linearized on *all* of it's live variables, $l' = 0$ and $lin = l$ resulting in R^l as the combination that is chosen. Then the lemma becomes trivial, since its goal is equal to it's first assumption in this case.

As a consequence of this, the previous lemma F_STRONG is not needed to prove this lemma. Furthermore, this lemma is the sole reason why F_STRONG and strong pullback preservation are needed for the linearization. Thus the requirement of pullback preservation can be lifted, in the case that the linearization is applied to all live variables at the same time.

Mapper peresrves non-repetitiveness

An important lemma used frequently in the following proofs is that the mapper preserves non-repetitiveness. It means that given a non-repetitive element x, the result of mapping functions over it is also non-repetitive. These functions must fulfill the appropriate restrictions of bijectivity and small-support for the bounds and lives. Additionally the functions f^{lin} on the linearized lives need to be bijective.

(NONREP_MAP) $$\begin{aligned}&\mathsf{small\_supp}\ v^{fr}\ \wedge\ \mathsf{small\_supp}\ u^b\ \wedge\ \mathsf{bijective}\ u^b\ \wedge\\ &\mathsf{bijective}\ f^{lin}\ \wedge\ \mathsf{nonrep}_F^{lin}\ x\ \Longrightarrow \mathsf{nonrep}_F^{lin}\ (\mathsf{map}_F\ \langle g^{l'}\ f^{lin}\rangle\ v^{fr}\ u^b\ x)\end{aligned}$$

To give proof sketch for this lemma, we split it into two parts. First, we argue that mapping bijective f^{lin} over the linearized variables and id over all others preserves non-repetitiveness. $\mathsf{eq\_shape}_F^{lin}$ is transitive (both $=$ and $\top$ are transitive) and x and $\mathsf{map}_F\ \langle \mathrm{id}^{l'}\ f^{lin}\rangle\ \mathrm{id}^{fr}\ \mathrm{id}^b\ x$ have the same shape. Thus, we have to think about the same $\forall y$ in the NONREP definition to show non-repetitiveness for the mapped x. This means, we can fix the y and show the following, where we rename the f^{lin} in the existential quantifier to f^{lin}_{E1} and f^{lin}_{E2} to avoid naming clashes and for clarity.

$$\begin{aligned}
&\exists f^{lin}_{E1}.\ y = \mathsf{map}_F\ \langle \mathrm{id}^{l'}\ f^{lin}_{E1}\rangle\ \mathrm{id}^{fr}\ \mathrm{id}^{b}\ x \Longrightarrow \\
&\exists f^{lin}_{E2}.\ y = \mathsf{map}_F\ \langle \mathrm{id}^{l'}\ f^{lin}_{E2}\rangle\ \mathrm{id}^{fr}\ \mathrm{id}^{b}\ (\mathsf{map}_F\ \langle \mathrm{id}^{l'}\ f^{lin}\rangle\ \mathrm{id}^{fr}\ \mathrm{id}^{b}\ x)
\end{aligned}$$

From this we obtain and fix h^{lin} from the existential quantifier $\exists f^{lin}_{E1}$ in the assumption and instantiate the existential qualifier in the goal with $f^{lin}_{E2} = (h \circ (\mathrm{inv}\ f))^{lin}$. Since all f^{lin} are bijective, we know that the inverse inv for each of them exist. Using MAP_COMP, we can transform the instantiated goal to the assumption with the fixed h^{lin} as follows, which proves this part of the lemma:

$$\begin{aligned}
&y = \mathsf{map}_F\ \langle \mathrm{id}^{l'} (h \circ (\mathrm{inv}\ f))^{lin}\rangle\ \mathrm{id}^{fr}\ \mathrm{id}^{b}\ (\mathsf{map}_F\ \langle \mathrm{id}^{l'}\ f^{lin}\rangle\ \mathrm{id}^{fr}\ \mathrm{id}^{b}\ x) && \\
\equiv\ &y = \mathsf{map}_F\ \langle \mathrm{id}^{l'} ((h \circ (\mathrm{inv}\ f)) \circ f)^{lin}\rangle\ \mathrm{id}^{fr}\ \mathrm{id}^{b}\ x && \text{MAP\_COMP} \\
\equiv\ &y = \mathsf{map}_F\ \langle \mathrm{id}^{l'} (h \circ ((\mathrm{inv}\ f) \circ f))^{lin}\rangle\ \mathrm{id}^{fr}\ \mathrm{id}^{b}\ x && \circ\ \text{assoc} \\
\equiv\ &y = \mathsf{map}_F\ \langle \mathrm{id}^{l'} (h \circ \mathrm{id})^{lin}\rangle\ \mathrm{id}^{fr}\ \mathrm{id}^{b}\ x && \text{inv} \circ \\
\equiv\ &y = \mathsf{map}_F\ \langle \mathrm{id}^{l'}\ h^{lin}\rangle\ \mathrm{id}^{fr}\ \mathrm{id}^{b}\ x && \circ\ \text{id}
\end{aligned}$$

It remains to show that mapping appropriately restricted functions over the other, not linearized type variables also preserves non-repetitiveness. We informally motivate this part of the proof in terms of shapes and atoms of F elements: Consider the shape of a non-repetitive element x. From the definition NONREP we know that for any element y of equivalent shape, we can find functions f^{lin} acting only on the linearized variables to map to this other element. As we noted before in Subsection 3.1.1, we only consider elements to be of equivalent shape, if they are *equal* in all non-linearized atoms. Thus all of the other elements of equivalent shape have the same atoms of non-linearized type in the same slots of that element. Mapping function over the non-linearized atoms changes these in the exact same way for x and for any y of equivalent shape. Thus, the results x' and y' of these maps are also of equivalent shape. Furthermore, this map changes nothing about the linearized variables and thus the same f^{lin} can be used to map from the mapped x' to the mapped y'. We need to consider that there could exist a z that is equivalent to x' in terms of shape but it is not the result of a map on a y. This means however, that it is equal to x' in all non-linearized atoms and since x' is a result of mapping functions on these atoms. With this we can assume that there actually exists a y with equivalent shape of x, that would map to z through the same functions. Combining these two parts, the lemma NONREP_MAP can be shown.

Mapper reflects non-repetitiveness

Lastly we need the reflection of non-repetitiveness through map. This lemma can be seen as a partial reverse of NONREP_MAP, but only on the non-linearized lives.

(NONREP_MAP_REV) $\quad \mathsf{nonrep}_F^{lin}\ (\mathsf{map}_F\ \langle f^{l'}\ \mathrm{id}^{lin}\rangle\ \mathrm{id}^{fr}\ \mathrm{id}^{b}\ x) \Longrightarrow \mathsf{nonrep}_F^{lin}\ x$

3.1.4 Defining the subtype and its constants

Using our definition of non-repetitiveness, we carve out a subtype of F using Isabelle's **typedef** command. This subtype F' contains exactly those elements from F that are non-repetitive on the linearized variables $\alpha_{l'+1} \ldots \alpha_{lin}$. It furthermore provides us with the morphisms $\mathsf{rep}_{F'}$ to convert F' elements to the type F and $\mathsf{abs}_{F'}$ to convert F elements to F' - provided that they are non-repetitive.

In the following we specify the MRBNF constants, i.e., the mapper, setters, bound and relator for F'. We define these in terms of the base types constants and apply the morphisms to match the types: The setters stay unchanged and thus we can keep the same bound. For the relator the relations for the linearized lives are fixed to the equality relation, since in the new MRBNF these will be bounds. Lastly, for the mapper we only allow it to map bijective functions on the linearized variables in addition to the restrictions for the existing frees and bounds. This restriction is necessary to ensure that applying the map function to a F' element preserves it non-repetitiveness. If a function that violates any of the restrictions is given to the mapper, it is ignored and not applied.

As for the morphisms, concretely, we apply $\mathsf{rep}_{F'}$ to the F' arguments of the new mapper, setters and relator, and $\mathsf{abs}_{F'}$ to the result of the mapper. This leads us to the following definitions:

$$\mathsf{set}_{F',i} = \mathsf{set}_{F,i} \circ \mathsf{rep}_{F'}$$

$$\mathsf{map}_{F'}\ \langle f^{l'}\ g^{lin}\rangle\ u^{fr}\ v^{b} = \mathsf{abs}_{F'} \circ (\mathsf{map}_F\ \langle f^{l'}\ (\mathrm{asBij}\ g)^{lin}\rangle\ v^{fr}\ u^{b}) \circ \mathsf{rep}_{F'}$$

$$\mathsf{rel}_{F'}\ R^{l'}\ x\ y = \mathsf{rel}_F\ \langle R^{l'}\ (=)^{lin}\rangle\ (\mathsf{rep}_{F'}\ x)\ (\mathsf{rep}_{F'}\ y)$$

where asBij f = `if` bijective f `then` f `else` id. We note here, that in our implementation in Isabelle/HOL, we also enforce the u^b to be bijective using asBij and both v^{fr} and u^b to be small-support functions using an analogously defined asSS. We omit this here as we assume map_F to handle these cases.

3.1.5 Proving the MRBNF axioms

To show that F' is a MRBNF, we have to prove the axioms from Figure 2.2 for it. For most of the axioms this is straight forward for most of the axioms, as they only require unfolding the definitions of the new F' constants, applying the axioms of the original F and a few simple transformations. The axioms MAP_ID, MAP_CONG and SET_BD are proven this way, while MAP_COMP and SET_MAP require just a little more effort. Both contain the composition of $\mathsf{map}_{F'}$ or $\mathsf{set}_{F'}$ with $\mathsf{map}_{F'}$, respectively.

As an example we show SET_MAP for F' below. Note that we assume i to be in the range $1 \leq i \leq \varpi$ where ϖ is the number of all non-dead type variables, i.e., $\varpi = l + \mathit{fr} + b$. The proof works the same for setters of frees and bounds. Furthermore we assume all functions f^{ϖ} fulfilling their respective requirements (bijectivity and small-support) and thus all asBij and asSS evaluating to the `then` case.

$$\begin{aligned}
&\mathsf{set}_{F',i}\ (\mathsf{map}_{F'}\ f^{\varpi}\ x) && \\
&\equiv \mathsf{set}_{F,i} \circ \mathsf{rep}_{F'}\ ((\mathsf{abs}_{F'} \circ (\mathsf{map}_F\ f^{\varpi}) \circ \mathsf{rep}_{F'})\ x) && \text{unfold defs} \\
&\equiv \mathsf{set}_{F,i}\ (\mathsf{rep}_{F'}\ (\mathsf{abs}_{F'}\ (\mathsf{map}_F\ f^{\varpi}\ (\mathsf{rep}_{F'}\ x)))) && \circ \text{ application} \\
&\equiv \mathsf{nonrep}_F^{\mathit{lin}}\ (\mathsf{map}_F\ f^{\varpi}\ (\mathsf{rep}_{F'}\ x)) \Longrightarrow \mathsf{set}_{F,i}\ (\mathsf{map}_F\ f^{\varpi}\ (\mathsf{rep}_{F'}\ x)) && \textsf{abs inverse} \\
&\equiv \mathsf{nonrep}_F^{\mathit{lin}}\ (\mathsf{rep}_{F'}\ x) \Longrightarrow \mathsf{set}_{F,i}\ (\mathsf{map}_F\ f^{\varpi}\ (\mathsf{rep}_{F'}\ x)) && \text{NONREP\_MAP} \\
&\equiv \mathsf{set}_{F,i}\ (\mathsf{map}_F\ f^{\varpi}\ (\mathsf{rep}_{F'}\ x)) && \textsf{nonrep rep}_{F'} \\
&\equiv f_i\ \grave{}\ \mathsf{set}_{F,i}\ (\mathsf{rep}_{F'}\ x) && \text{SET\_MAP of } F \\
&\equiv f_i\ \grave{}\ \mathsf{set}_{F',i}\ x && \text{fold defs, } \circ
\end{aligned}$$

where "abs inverse" denotes the theorem that $\mathsf{rep}_{F'}$ is the inverse of $\mathsf{abs}_{F'}$ for arguments that are non-repetitive. Furthermore "$\mathsf{nonrep\ rep}_{F'}$" states that converting a F' element to F inherently means that the F element is non-repetitive.

The validity of the bound BD is trivially proven, since the bound is copied from F.

It remains to show REL_COMPP and IN_REL for F'. While the former is easily proven using the corresponding axiom of F and some simple properties of relational composition, the latter is certainly the most interesting axiom to show.

We don't show a full proof of this property here, but investigate an interesting step. In the proof we reach a state, where we need to show that $\mathsf{nonrep}_F^{\mathit{lin}}\ (\mathsf{map}_F\ \mathrm{fst}^{l}\ \mathrm{id}^{\mathit{fr}}\ \mathrm{id}^{b}\ z) \Longrightarrow \mathsf{nonrep}_F^{\mathit{lin}}\ (\mathsf{map}_F\ \langle \mathrm{id}^{l'}\ \mathrm{fst}^{\mathit{lin}}\rangle\ \mathrm{id}^{\mathit{fr}}\ \mathrm{id}^{b}\ z)$. To give an intuition for why this is necessary, we obtain the left side of the implication from the IN_REL axiom of F and need to show the right side to eliminate a composition $\mathsf{abs}_{F'} \circ \mathsf{rep}_{F'}$ in the goal state.

$$\begin{aligned}
&\mathsf{nonrep}_F^{\mathit{lin}}\ (\mathsf{map}_F\ \mathrm{fst}^{l}\ \mathrm{id}^{\mathit{fr}}\ \mathrm{id}^{b}\ z) \Longrightarrow \\
&\mathsf{nonrep}_F^{\mathit{lin}}\ (\mathsf{map}_F\ \langle \mathrm{fst}^{l'}\ \mathrm{id}^{\mathit{lin}}\rangle\ \mathrm{id}^{\mathit{fr}}\ \mathrm{id}^{b}\ (\mathsf{map}_F\ \langle \mathrm{id}^{l'}\ \mathrm{fst}^{\mathit{lin}}\rangle\ \mathrm{id}^{\mathit{fr}}\ \mathrm{id}^{b}\ z)) \Longrightarrow \\
&\mathsf{nonrep}_F^{\mathit{lin}}\ (\mathsf{map}_F\ \langle \mathrm{id}^{l'}\ \mathrm{fst}^{\mathit{lin}}\rangle\ \mathrm{id}^{\mathit{fr}}\ \mathrm{id}^{b}\ z)
\end{aligned}$$

The first step is reached through MAP_COMP of F, while the second one needs the NONREP_MAP_REV lemma. This is the final place, where strong pullback preservation is used and the reason why it is required.

Another interesting step in the proof of IN_REL is the conversion from $\mathsf{mr\_rel}_{F'}$ to $\mathsf{mr\_rel}_{F}$. While $\mathsf{mr\_rel}_{F'}$ takes functions for the linearized type variables - that turned to bounds, the relator of original MRBNF F takes relations for these. By explicitly specifying the following:

$$\begin{aligned}
&\mathsf{mr\_rel}_{F'}\ f^{lin}\ v^{fr}\ u^{b}\ R^{l'}\ x\ y = \\
&\mathsf{rel}_{F'}\ R^{l'}\ (\mathsf{map}_{F'}\ \langle \mathrm{id}^{l'}\ f^{lin} \rangle\ v^{fr}\ u^{b}\ x)\ y = \\
&\mathsf{rel}_{F}\ \langle R^{l'}\ (=)^{lin} \rangle\ (\mathsf{map}_{F}\ \langle \mathrm{id}^{l'}\ f^{lin} \rangle\ v^{fr}\ u^{b}\ x)\ y = \\
&\mathsf{rel}_{F}\ \langle R^{l'}\ (\mathrm{Grp}\, f)^{lin} \rangle\ (\mathsf{map}_{F}\ \mathrm{id}^{l}\ v^{fr}\ u^{b}\ x)\ y = \\
&\mathsf{mr\_rel}_{F}\ v^{fr}\ u^{b}\ \langle R^{l'}\ (\mathrm{Grp}\, f)^{lin} \rangle\ x\ y
\end{aligned}$$

3.1.6 Lifting Witnesses

Existing witnesses of the original MRBNF that do not depend on any of the linearized variables can be lifted to be witnesses of the linearized MRBNF.

For this it is necessary to show that they are non-repetitive on the linearized elements, i.e., that they are part of the new type. From WITS (Subsection 2.1.3) we know that any witness not depending on the linearized lives does not contain atoms from these lives. Thus, we can show that these witnesses are non-repetitive, since an element with no α atoms is trivially non-repetitive on α.

Other witnesses that depend on the linearized variables cannot be lifted and have to be discarded. Even if they are non-repetitive, witnesses of a MRBNF may only depend on lives and not on bounds, which the linearized lives turn into.

Additionally, new witnesses may be specified for the resulting MRBNF. For these the property WITS defined in Subsection 2.1.3 has to be proven, i.e., that they only consist of the atoms given to them as arguments.

When an liftable witness of the original MRBNF exists or a new witness fulfilling WITS is specified, the existence of a non-repetitive element we motivated in Subsection 3.1.2 is trivially proven.

3.1.7 Preservation of strength

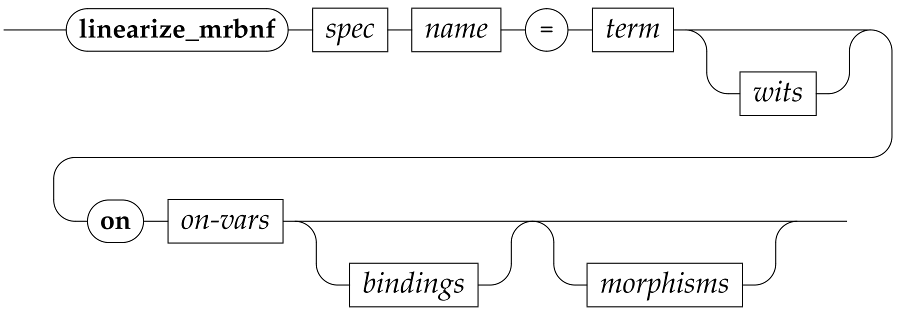
Linearizing an MRBNF preserves the strength property of it. This means, that for the new F' the following axiom holds, provided that F fulfills F_STRONG:

$$\mathsf{rel}_{F'}\ R^{l'}\ x\ y\ \wedge\ \mathsf{rel}_{F'}\ Q^{l'}\ x\ y \Longrightarrow \mathsf{rel}_{F'}\ (\inf\ R\ Q)^{l'}\ x\ ys$$

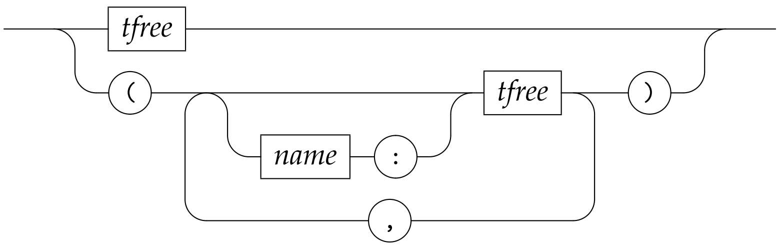
This is easily proven by unfolding the definition of $\mathsf{rel}_{F'}$, applying F_STRONG and unfolding (inf (=) (=)) $\equiv$ (=) for the linearized variables. Strength of an MRBNF is a property that often comes in useful, however in Isabelle it is not tracked in the MRBNF construct at the moment. At the very least, this proof allows us to easily linearize a linearized MRBNF again on further live variables.

3.2 Implementing the linearize_mrbnf command

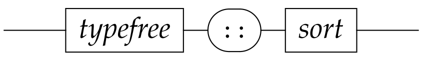
We implement a command that allows the user to linearize an existing MRBNF or BNF on one or multiple of it's live variables. The syntax of the command is given in the following:



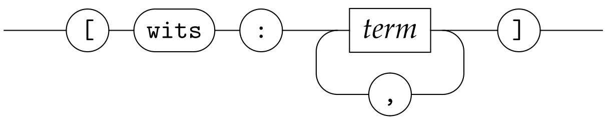
spec



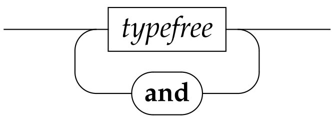
tfree



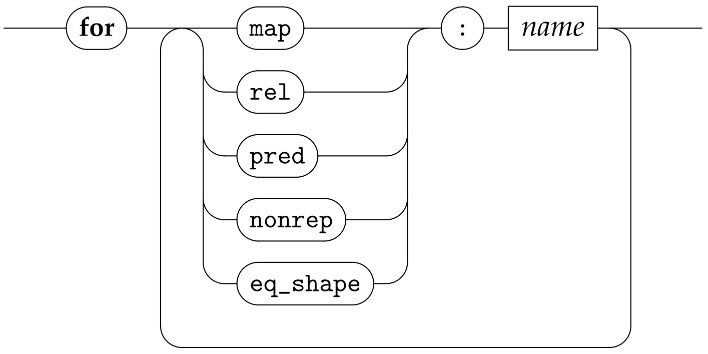
wits



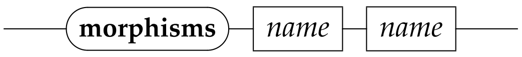
on-vars



bindings



morphisms



With this command, we can linearize our example by writing the following line in Isabelle:

$$\textbf{linearize\_mrbnf}\ (\text{keys: } \alpha :: \text{var},\ \text{vals: } \beta)\ \text{alist} = (\alpha :: \text{var} \times \beta)\ \text{list}\ \textbf{on}\ \alpha$$

Since for $(\alpha \times \beta)$ list both type variables are live and we only linearize on α, it is necessary to prove strong pullback preservation for this MRBNF.

After the user has written the command, the conditions for linearization we presented in Subsection 3.1.2 have to be proven. These are the non-emptiness of the linear type, strong pullback preservation PB_STRONG and the witness axioms WITS if applicable.

These conditions are given dynamically to the user. For example, it is only necessary to show strong pullback preservation PB_STRONG, when the resulting MRBNF has live variables remaining. Furthermore, as mentioned in Subsection 3.1.6, the non-emptiness of the non-repetitive type is easily proven when the user specified a non-emptiness witness, or a liftable witness of the original type exists. The user is not asked to show the goals that are actually necessary for a specific linearization.

Furthermore, since the original MRBNF already fulfills weak pullback preservation, we extract the uniqueness property of strong pullback preservation and require the user to prove only this. Strong pullback preservation PB_STRONG can be proven from weak pullback preservation IN_REL together with the uniqueness property we specify as follows:

$$\begin{aligned}\forall x\ y.\ (&\mathsf{map}_F\ \mathrm{fst}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ x = \mathsf{map}_F\ \mathrm{fst}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ y \wedge \\ &\mathsf{map}_F\ \mathrm{snd}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ x = \mathsf{map}_F\ \mathrm{snd}^l\ \mathrm{id}^{fr}\ \mathrm{id}^b\ y) \Longrightarrow \\ &x = y\end{aligned}$$

When the required properties are shown, the subtype and the constants for the new MRBNF are defined.

The proofs of the intermediate lemmas from Subsection 3.1.3 and MRBNF axioms are automated through `ML`-tactics. For this the existing high-level apply-style and Isar proofs had to be converted to single-step apply proofs. These proofs avoid using the automatic proof tactics of Isabelle like `metis`, `auto`, `fastforce` and even `simp`. Instead they rely on explicit rule applications, substitutions, deterministic repetitions and in certain cases instantiations of free variables in existing theorems and lemmas.

4 Examples

4.1 POPLmark challenge: Pattern

The POPLmark challenge [Ayd+05] presents a selection of problems to benchmark the progress in formalizing programming language metatheory. The challenges are built around formalizing aspects of *System* $F_{<:}$ calculus, a polymorphic typed lambda calculus with subtyping. We are interested in part 2B of this challenge, which has the goal to formalize and proof *type soundness* for terms with pattern matching over records. Type soundness is considered in terms of *preservation* (evaluating a term preserves its type) and *progress* (a term is either a value or can be evaluated).

We focus on the `record` terms `pattern-let`. A record is a term defined as a set of pairs, where the first element is a label and the second element a term: $\{(\mathtt{l}_j, \mathtt{t}_j)\}$. The labels l within a record must be pairwise distinct. A pattern is defined as either a typed variable or a set of (label, patten) pairs with pairwise distinct labels: $\mathtt{p} ::= \mathtt{x} : \mathtt{T} \mid \{(\mathtt{l}_j, \mathtt{p}_j)\}$

A formalization of part 2B of the POPLmark challenge in Isabelle/HOL is presented by Blanchette et al. [Bla+19]. They use *binder_datatypes* to abstract types, variables and terms. A central notion in this formalization is the *labeled finite set* (α, β) lfset that is used in the representation of records and patterns. This type constructor is a subtype of $(\alpha \times \beta)$ fset that only includes elements that are non-repetitive on α. This restriction is necessary, because for both records and patterns the label α must be mutually distinct, i.e., the set representing them has to be non-repetitive.

While by construction $(\alpha \times \beta)$ fset is a BNF (and an MRBNF since all BNFs are also MRBNFs) with both variables being live, (α, β) lfset is a MRBNFs with α as a bound variable, since it is non-repetitive on α. While this is a linearization, the finite set on pairs does not fulfill strong pullback preservation. Thus the approach and command we presented in Chapter 3 cannot be used here. Because of an alternate, equivalent description on non-repetitiveness specific to this type, it is still possible to manually linearize this MRBNF.

For the pattern a different type is used. It is constructed by linearizing an intermediate type prepat that is defined using the **datatype** command:

datatype $(\alpha,\ \beta)$ prepat = PPVar "α" "β typ" | PPRec "(string, $(\alpha,\ \beta)$ prepat) lfset"

Abbreviations

BNF	Bounded Natural Functor
MRBNF	Map-Restricted Bounded Natural Functor

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Bibliography

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