Sample-based Bayesian Inference in Inverse Problems

Felix Lucka
University College London
f.lucka@ucl.ac.uk

Centre for Medical Image Computing

Warwick, February 13, 2015

Ill-posed inverse problems with additive Gaussian noise:

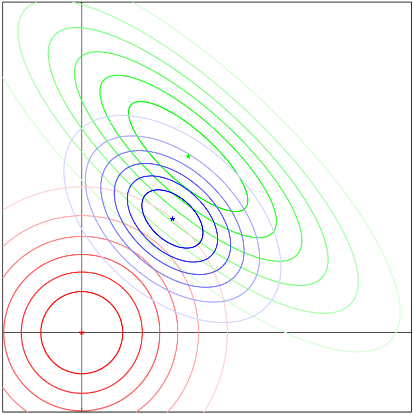
$$\boxed{f = \mathcal{A}(u) + \varepsilon}$$

My focus: Biomedical imaging.

$p_{like}(f|u) \propto$
$$\exp\left(-\frac{1}{2}\|f - A\,u\|^2_{\Sigma_\varepsilon^{-1}}\right)$$

$p_{prior}(u) \propto$
$$\exp\left(-\lambda\,\|D^T u\|_2^2\right)$$

$p_{post}(u|f) \propto$
$$\exp\left(-\frac{1}{2}\|f - A\,u\|^2_{\Sigma_\varepsilon^{-1}} - \lambda\,\|D^T u\|_2^2\right)$$



Probabilistic representation of solution allows for a rigorous quantification of its uncertainties.

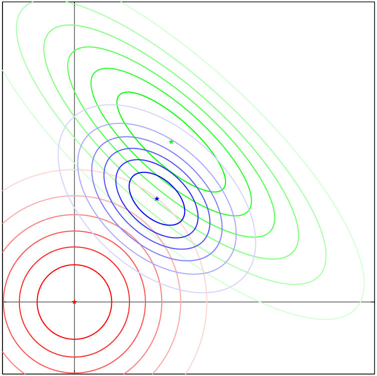
Inverse problems in the Bayesian framework
edited by Daniela Calvetti, Jari P Kaipio and Erkki Somersalo.
Special issue of *Inverse Problems*, November 2014.

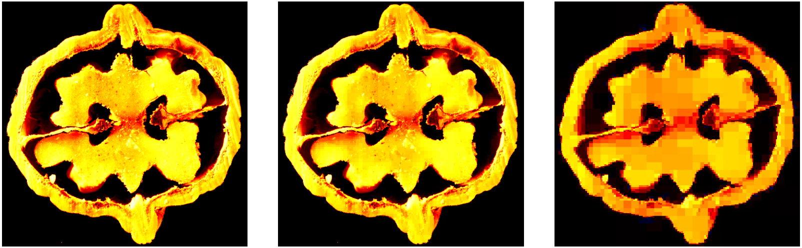
UQ and a Model Inverse Problem
Marco Iglesias and Andrew M. Stuart
SIAM News, July/August 2014.

Advantageous for high uncertainties:

- Strongly non-linear problems.
- Severely ill-posed problems.
- Little analytical structure
- Additional model uncertainties.

- Uncertainty quantification of inverse solutions.
- Dynamic Bayesian inversion for prediction or control $\rightsquigarrow$ dynamical systems
- Infinite dimensional Bayesian inversion.
- Incorporating model uncertainties.
- Large-scale posterior sampling techniques.
- Using sparsity as a-priori information.



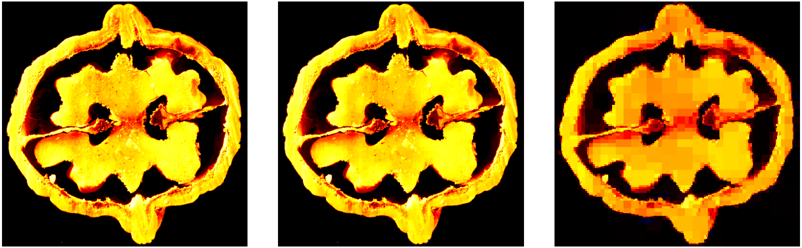


(a) 100% (b) 10% (c) 1%

Sparsity a-priori constraints are used in variational regularization, compressed sensing and ridge regression:

$$\hat{u}_\lambda = \operatorname*{argmin}_u \left\{ \tfrac{1}{2} \| f - A\, u \|_2^2 + \lambda \| D^T u \|_1 \right\}$$

(e.g. *total variation*, *wavelet shrinkage*, *LASSO*,...)



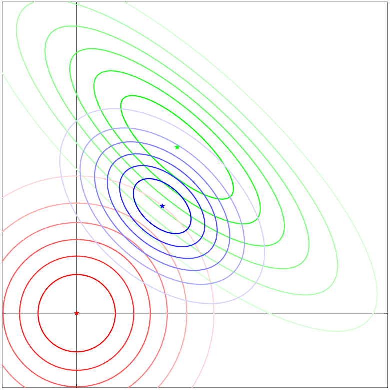
(a) 100% (b) 10% (c) 1%

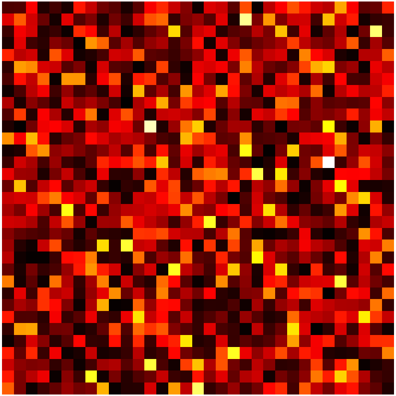
Sparsity a-priori constraints are used in variational regularization, compressed sensing and ridge regression:

$$\hat{u}_\lambda \quad = \underset{u}{\operatorname{argmin}} \left\{ \tfrac{1}{2} \| f - A\,u \|_2^2 + \lambda \| D^T u \|_1 \right\}$$

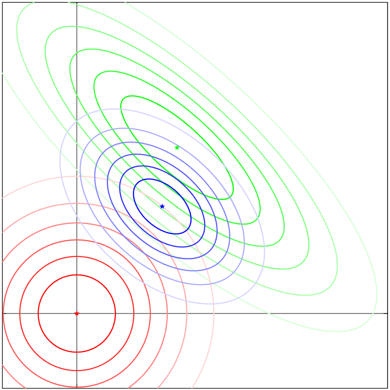
(e.g. *total variation*, *wavelet shrinkage*, *LASSO*,...)

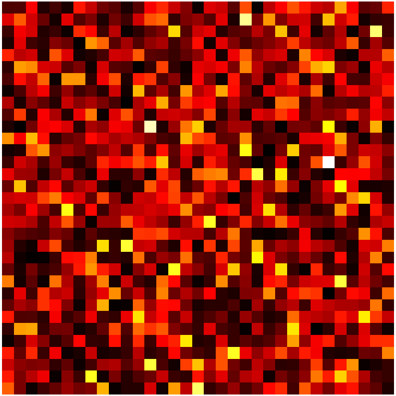
How about sparsity as a-priori information in the Bayesian approach?





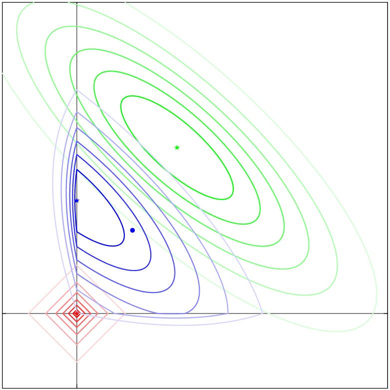
$$p_{prior}(u) \propto \exp\left(-\lambda \, \|D^T u\|_2^2\right)$$

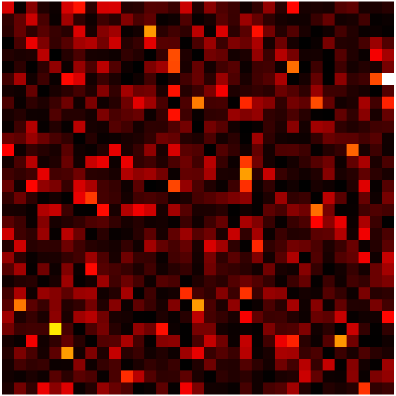




Decrease p from 2 to 0 and stop at $p = 1$ for convenience:

$$p_{prior}(u) \propto \exp\left(-\lambda \, \|D^T u\|_p^p\right)$$



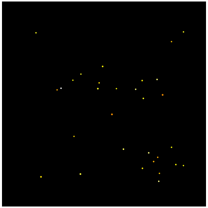


Decrease p from 2 to 0 and stop at $p = 1$ for convenience:

$$p_{prior}(u) \propto \exp\left(-\lambda \, \|D^T u\|_1\right)$$

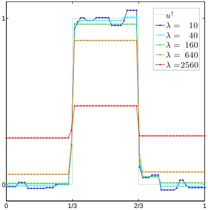
Bayesian Inference with ℓ_1 Priors

$$p_{post}(u|f) \propto \exp\left(-\frac{1}{2}\|f - A\,u\|^2_{\Sigma_\varepsilon^{-1}} - \lambda\,\|D^T u\|_1\right)$$



Aims: Bayesian inversion in high dimensions ($n \to \infty$).

Priors: Simple ℓ_1, total variation (TV), Besov space priors.



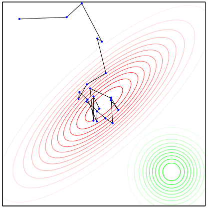
Starting points:

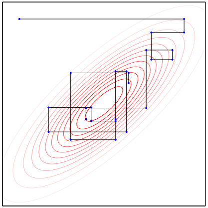
- M. Lassas, S. Siltanen, 2004. *Can one use total variation prior for edge-preserving Bayesian inversion?* *Inverse Problems*, 20.

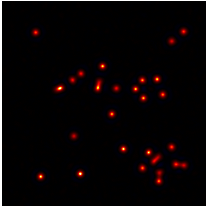
- M. Lassas, E. Saksman, S. Siltanen, 2009. *Discretization invariant Bayesian inversion and Besov space priors.* *Inverse Problems and Imaging*, 3(1).

- V. Kolehmainen, M. Lassas, K. Niinimäki, S. Siltanen, 2012. *Sparsity-promoting Bayesian inversion* *Inverse Problems*, 28(2).

Efficient MCMC Techniques for ℓ_1 Priors







Task: Monte Carlo integration by samples from

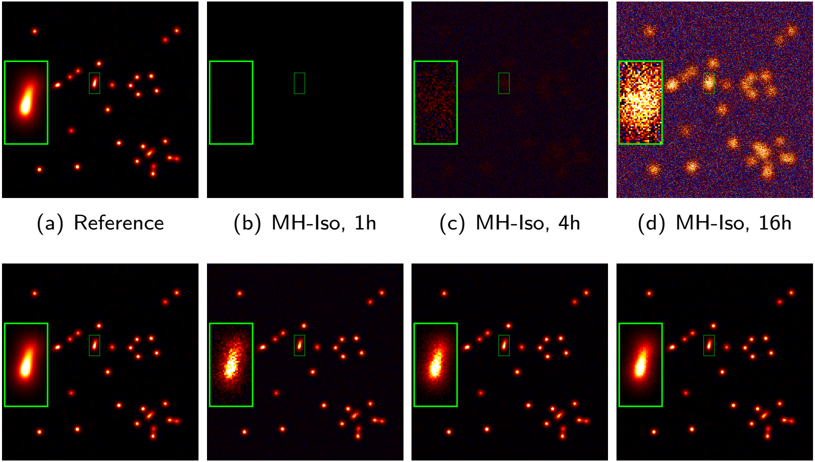
$$p_{post}(u|f) \propto \exp\left(-\frac{1}{2}\|f - A\,u\|^2_{\Sigma_\varepsilon^{-1}} - \lambda \, \|D^T u\|_1\right)$$

Problem: Standard Markov chain Monte Carlo (MCMC) sampler (Metropolis-Hastings) inefficient for large n or λ.

Contributions:

- Development of explicit single component Gibbs sampler.
- Tedious implementation for different scenarios ($n > 10^6$).
- Still efficient in high dimensions.
- Speed-up by ordered overrelaxation (Neal, 1995).
- Detailed evaluation and comparison to MH.

F.L., 2012. *Fast Markov chain Monte Carlo sampling for sparse Bayesian inference in high-dimensional inverse problems using L1-type priors.* *Inverse Problems*, 28(12):125012.



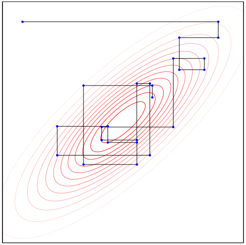
(a) Reference (b) MH-Iso, 1h (c) MH-Iso, 4h (d) MH-Iso, 16h

(e) Reference (f) SC Gibbs, 1h (g) SC Gibbs, 4h (h) SC Gibbs, 16h

Deconvolution, simple ℓ_1 prior, $n = 513 \times 513 = 263\,169$.

Recent Generalization: Slice-Within-Gibbs Sampling

$$p_{prior}(u) \propto \exp\left(-\lambda \|D^T u\|_1\right)$$



Limitations:

- D must be diagonalizable (synthesis priors):

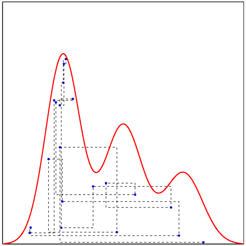
$$D^T V = W = \mathrm{diag}(w), \quad u = V\xi$$

$$p_{post}(\xi|f) \propto \exp\left(-\tfrac{1}{2}\|f - AV\xi\|^2_{\Sigma_\varepsilon^{-1}} - \lambda \|W\xi\|_1\right)$$

- ℓ_p^q-prior: $\exp\left(-\lambda \|D^T u\|_p^q\right)$? TV in 2D/3D?

Contributions:

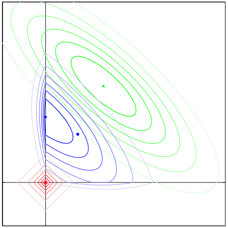
- Replace explicit by generalized slice sampling.
- Implementation & evaluation for most common priors.



R.M. Neal, 2003. *Slice Sampling.* Annals of Statistics 31(3)

F.L., 2015. *Fast Gibbs sampling for high-dimensional Bayesian inversion.* (in preparation)

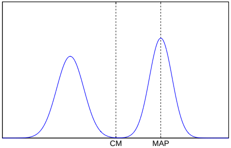
$$\hat{u}_{\mathrm{MAP}} := \underset{u \in \mathbb{R}^n}{\operatorname{argmax}} \{ \, p_{post}(u|f) \} \quad \text{OR} \quad \hat{u}_{\mathrm{CM}} := \int u \; p_{post}(u|f) \, \mathrm{d}u$$

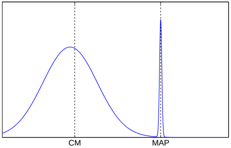


Classical Bayes cost formalism discriminates MAP (= variational regularization) and advocates CM.

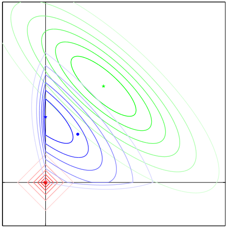
However...

- Theoretical argument has a logical flaw.
- Discrimination of MAP estimate is not intuitive.
- Gaussian priors: MAP = CM. Funny coincidence?
- Non-Gaussian priors: Poor computational validation!





$$\hat{u}_{\text{MAP}} := \underset{u \in \mathbb{R}^n}{\text{argmax}} \{ p_{post}(u|f) \} \quad \text{OR} \quad \hat{u}_{\text{CM}} := \int u \; p_{post}(u|f) \, \mathrm{d}u$$

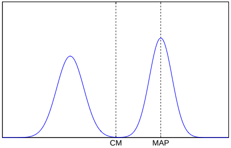


Classical Bayes cost formalism discriminates MAP (= variational regularization) and advocates CM.

However...

- Theoretical argument has a logical flaw.
- Discrimination of MAP estimate is not intuitive.
- Gaussian priors: MAP = CM. Funny coincidence?
- Non-Gaussian priors: Poor computational validation!

$\Longrightarrow$ Let's compute some examples!



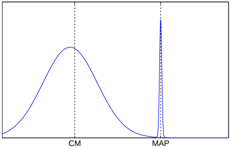
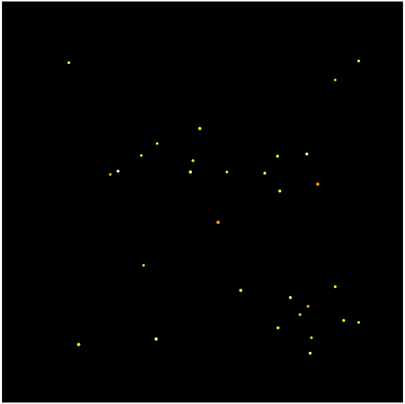
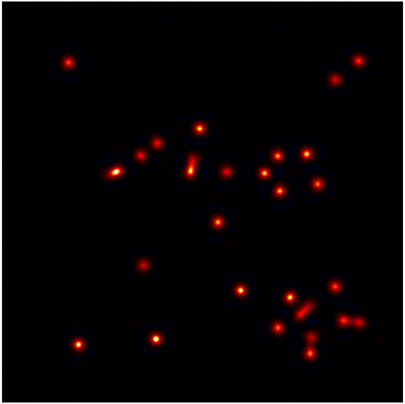


Image Deblurring Example in 2D

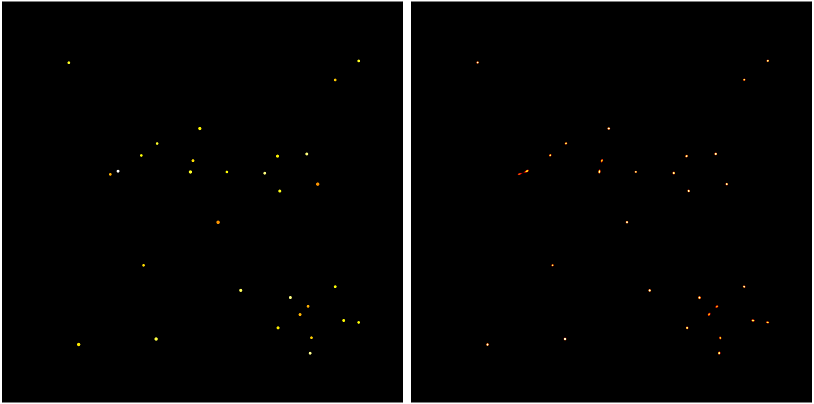


(a) Unknown function $\tilde{u}$



(b) CM estimate by our Gibbs sampler

Deconvolution, simple ℓ_1 prior, $n = 1023 \times 1023 = 1\,046\,529$.

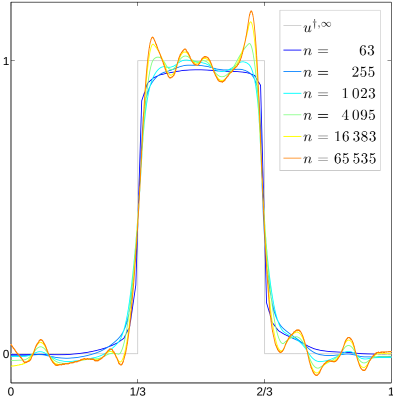


(a) Unknown function $\tilde{u}$ (b) MAP estimate by ADMM

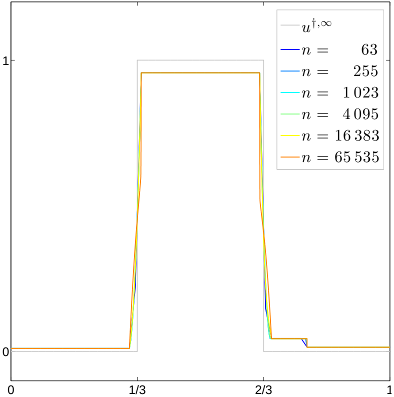
Deconvolution, simple ℓ_1 prior, $n = 1023 \times 1023 = 1\,046\,529$.

"Can one use total variation prior for edge-preserving Bayesian inversion?"

- For $\lambda_n = const.$ and $n \longrightarrow \infty$ the TV prior diverges.
- CM diverges.
- MAP converges to edge-preserving limit.



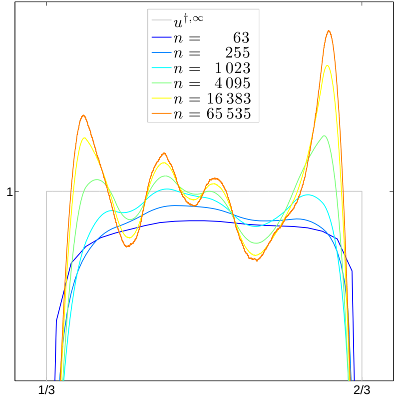
(a) CM by our Gibbs Sampler



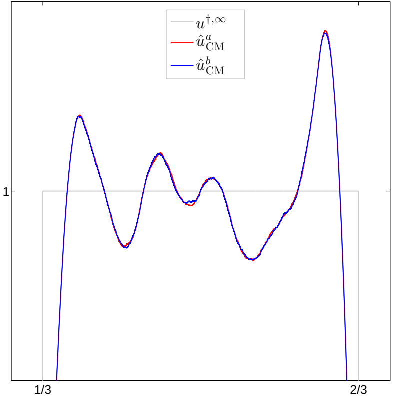
(b) MAP by ADMM

"Can one use total variation prior for edge-preserving Bayesian inversion?"

- For $\lambda_n = const.$ and $n \longrightarrow \infty$ the TV prior diverges.
- CM diverges.
- MAP converges to edge-preserving limit.



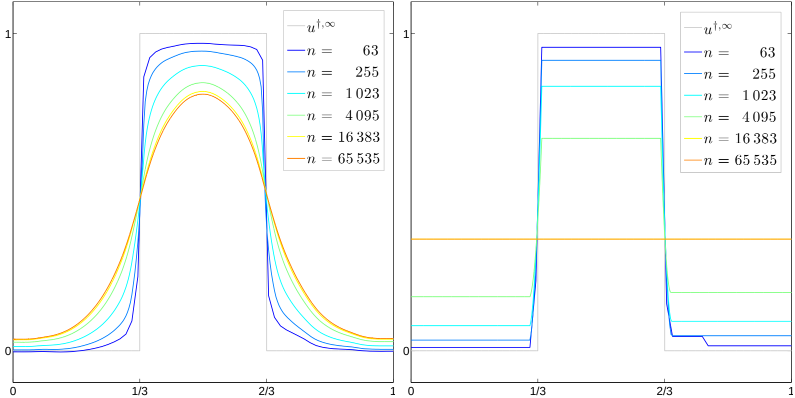
(a) Zoom into CM estimates



(b) MCMC convergence check

"Can one use total variation prior for edge-preserving Bayesian inversion?"

- For $\lambda_n \propto \sqrt{n+1}$ and $n \longrightarrow \infty$ the TV prior converges to a smoothness prior.
- CM converges to smooth limit.
- MAP converges to constant.

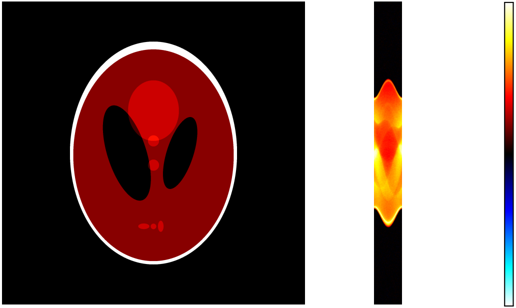


(a) CM by our Gibbs Sampler　　(b) MAP by ADMM

Sparse-Angle-CT with Discretization Invariant Besov Priors

M. Lassas, E. Saksman, S. Siltanen, 2009. *Discretization invariant Bayesian inversion and Besov space priors.* *Inverse Probl Imaging*, 3(1).

V. Kolehmainen, M. Lassas, K. Niinimäki, S. Siltanen, 2012. *Sparsity-promoting Bayesian inversion* *Inverse Probl*, 28(2).



real solution $\tilde{u}$ data f colormap

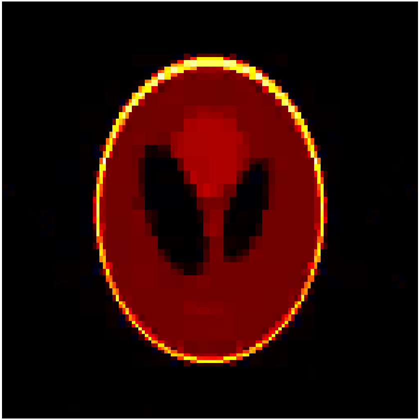
- CT using only 45 projection angles and 500 measurement pixel.
- Besov space priors using Haar wavelets.

Reconstructions for $\lambda = 2\text{e}4$, $n = 64 \times 64 = 4.096$

MAP estimate (by ADMM)

CM estimate (by our Gibbs sampler)

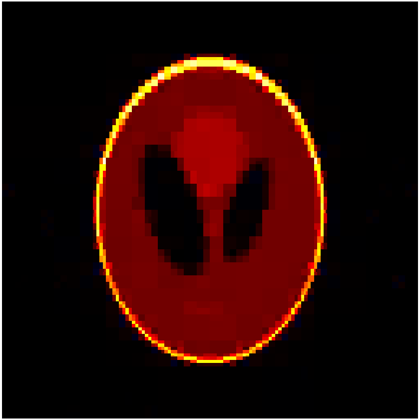
Reconstructions for $\lambda = 2e4$, $n = 128{\times}128 = 16.384$



MAP estimate (by ADMM)

CM estimate (by our Gibbs sampler)

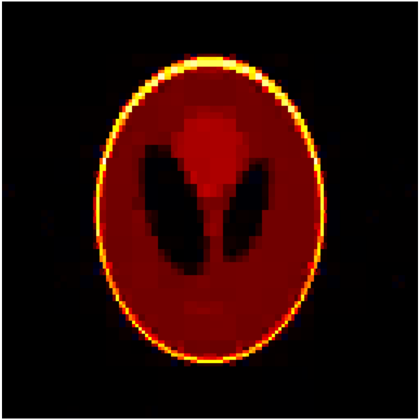
Reconstructions for $\lambda = 2e4$, $n = 256 \times 256 = 65.536$



MAP estimate (by ADMM)

CM estimate (by our Gibbs sampler)

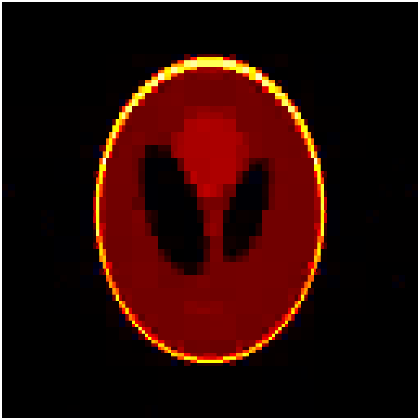
Reconstructions for $\lambda = 2e4$, $n = 512 \times 512 = 262.144$



MAP estimate (by ADMM)

CM estimate (by our Gibbs sampler)

Reconstructions for $\lambda = 2e4$, $n = 1024 \times 1024 = 1.048.576$

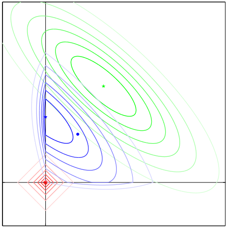


MAP estimate (by ADMM)

CM estimate (by our Gibbs sampler)

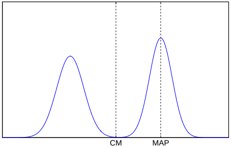
New Light on an Old Debate: MAP vs CM Estimates

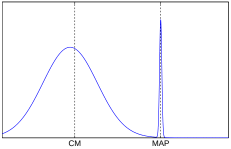
$$\hat{u}_{\text{MAP}} := \underset{u \in \mathbb{R}^n}{\operatorname{argmax}} \{ \, p_{post}(u|f) \} \quad \text{OR} \quad \hat{u}_{\text{CM}} := \int u \; p_{post}(u|f) \, \mathrm{d}u$$



Summary:

- Gaussian priors: MAP $=$ CM. Funny coincidence?
- For reasonable priors, CM and MAP look quite similar. Fundamentally different?
- If the CM estimate looks good, it looks like the MAP.
- MAP estimates are sparser, sharper, look and perform better...
- Gribonval, Marchart, Louchet and Moisan, 2011-2013: CM are MAP estimates for different priors.

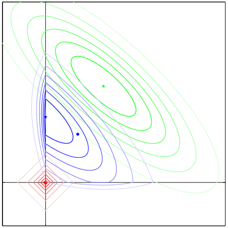




$\Longrightarrow$ Classical theory cannot be complete!

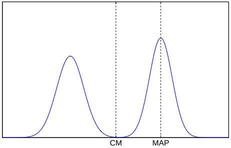
New Light on an Old Debate: MAP vs CM Estimates

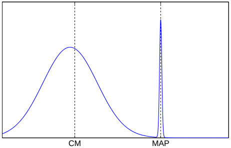
$$\hat{u}_{\text{MAP}} := \underset{u \in \mathbb{R}^n}{\operatorname{argmax}} \left\{ \, p_{post}(u|f) \right\} \quad \text{OR} \quad \hat{u}_{\text{CM}} := \int u \; p_{post}(u|f) \, \mathrm{d}u$$



We developed new Bayes cost functions such that

- Both MAP and CM are proper Bayes estimators for proper, convex cost functions.
- Key ingredient: Bregman distances.
- Gaussian case is no strange exception but consistent in this framework.

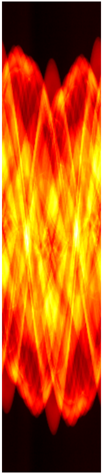


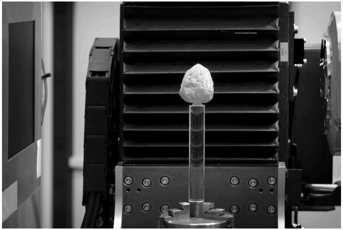


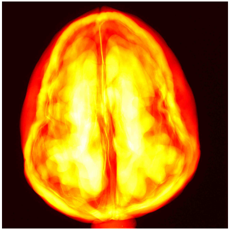
M. Burger, F.L., 2014. *Maximum a posteriori estimates in linear inverse problems with log-concave priors are proper Bayes estimators.* *Inverse Problems*, 30(11):114004.

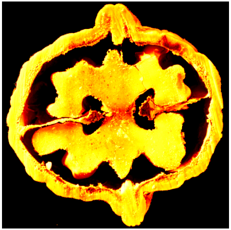
T. Helin, M. Burger, 2015. *Maximum a posteriori probability estimates in infinite-dimensional Bayesian inverse problems.* arXiv:1412.5816v2

- Cooperation with Samuli Siltanen et al. (University Helsinki).
- Implementation of MCMC methods for Fanbeam-CT.
- Besov and TV prior; non-negativity constraints.
- Stochastic noise modeling.
- Bayesian perspective on limited angle CT.



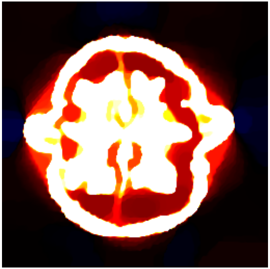






Walnut-CT with TV Prior: Full Angle

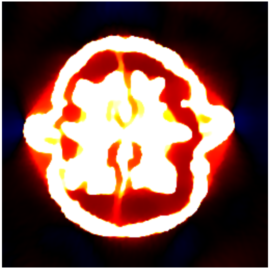
(a) MAP



(b) MAP, special color scale

(c) CStd

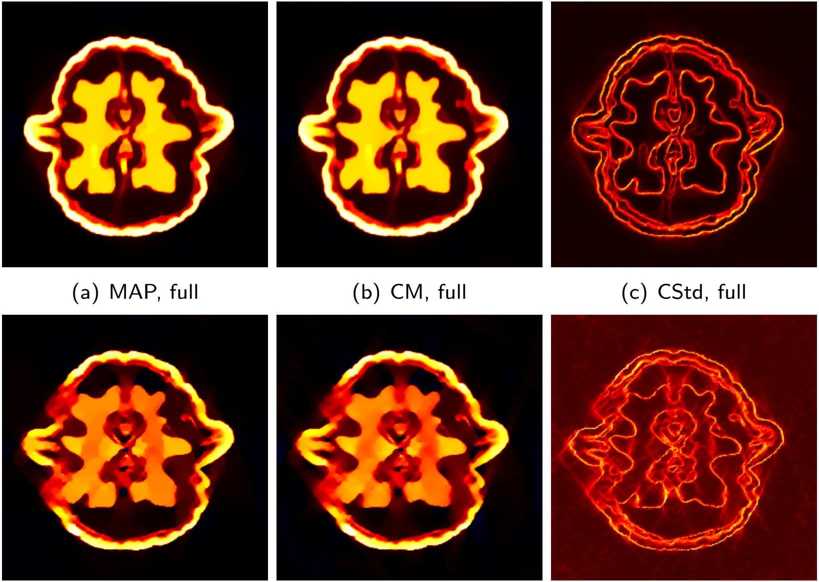
(d) CM



(e) CM, special color scale

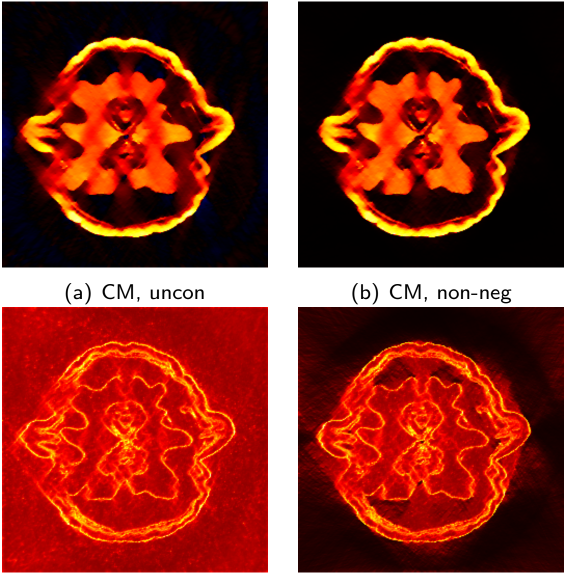
(f) CM of $\|\nabla u\|_2$

Walnut-CT with TV Prior: Full vs. Limited Angle



(a) MAP, full (b) CM, full (c) CStd, full

(d) MAP, limited (e) CM, limited (f) CStd, limited



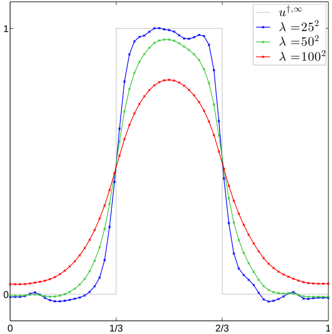
(a) CM, uncon

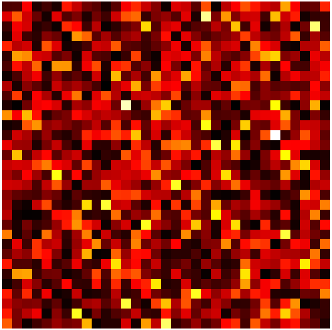
(b) CM, non-neg

(c) CStd, uncon

(d) CStd, non-neg

Outline

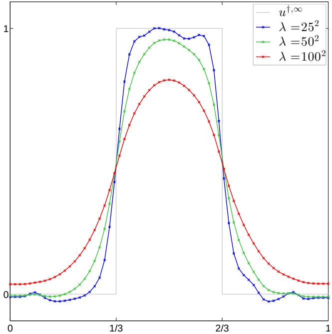


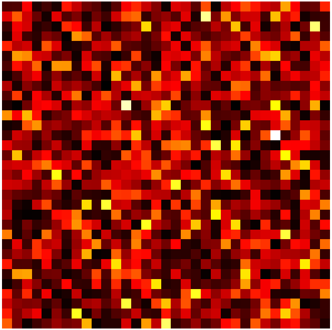


$$p_{prior}(u) \propto \prod \exp\left(-\lambda(u_{i+1} - u_i)^2\right)$$

- Gaussian variables take values on a characteristic scale, determined by $\gamma = 1/(2\lambda)$.
- Similar amplitudes are likely, sparsity (= outliers) is unlikely.

Hierarchical Bayesian Modeling (HBM) of Sparsity





$$p_{prior}(u) \propto \prod \exp\left(-\lambda (u_{i+1} - u_i)^2\right)$$

- Gaussian variables take values on a characteristic scale, determined by $\gamma = 1/(2\lambda)$.
- Similar amplitudes are likely, sparsity (= outliers) is unlikely.

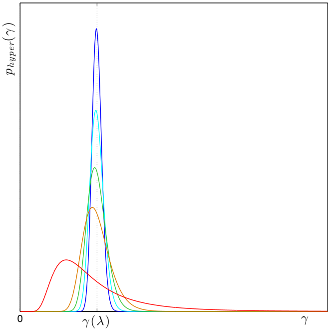
$\Longrightarrow$ Let's introduce individual γ_i's! (*hyperparameter/latent variables*)

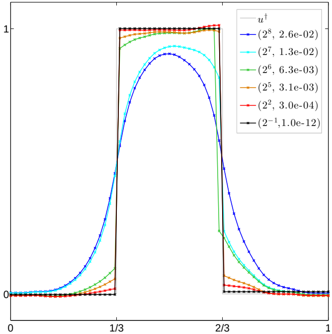
Conditional prior:

$$p_{prior}(u|\gamma) \propto \prod_i \exp\left(-\frac{(u_{i+1} - u_i)^2}{\gamma_i}\right)$$

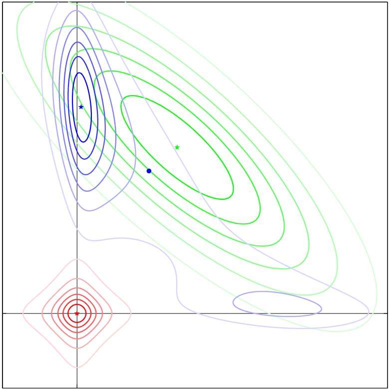
Scale-invariant hyperprior to approximate un-informative γ_i^{-1} prior:

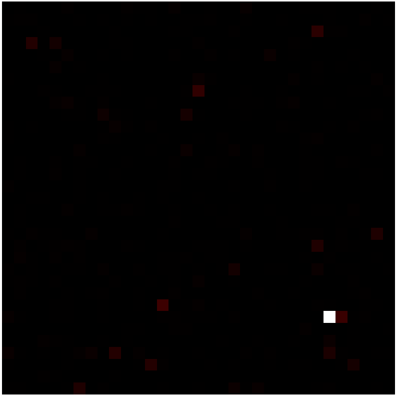
$$p_{hyper}(\gamma_i) \propto \gamma_i^{-(\alpha+1)} \exp\left(-\frac{\beta}{\gamma_i}\right), \quad \textit{inverse gamma distribution}$$





Hierarchical Bayesian Modeling (HBM) of Sparsity





Implicit prior is a Student's t-prior with $\nu = 2\alpha, \theta = \beta/(2\alpha)$:

$$p_{prior}(u) \propto \prod_i \left(1 + \frac{u_i^2}{\nu\theta}\right)^{-\frac{\nu-1}{2}}$$

$$p_{post}(u|f) \propto \exp\left(-\frac{1}{2}\|f - A\,u\|^2_{\Sigma_\varepsilon^{-1}} - \frac{\nu-1}{2}\sum_i \log\left(1 + \frac{u_i^2}{\nu\theta}\right)\right)$$

$$p_{post}(u, \gamma | f) \propto \exp\left(-\frac{1}{2} \| f - A\, u \|^2_{\Sigma_\varepsilon^{-1}} - \sum_i^n \left(\frac{u_i^2 + 2\beta}{2\gamma_i} + (\alpha + 1/2) \log(\gamma_i) \right) \right)$$

All computational approaches (optimization or sampling) exploit the conditional structure:

- Fix γ and update u by solving n-dim linear problem.
- Fix u and update γ by solving n 1-dim non-linear problems.

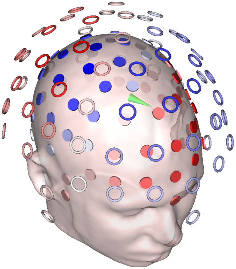
Major difficulty: Multimodality of posterior.

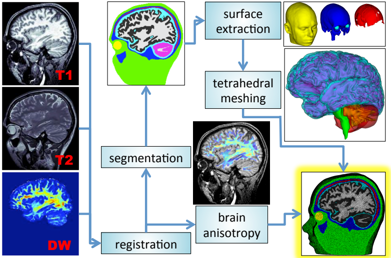
Heuristic Full-MAP computation:

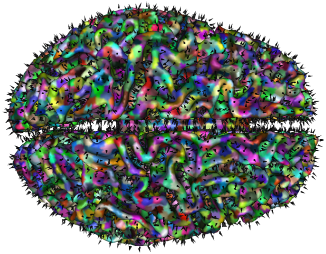
- Use MCMC to explore posterior (avoids very sub-optimal local modes).
- Initialize alternating optimization by local MCMC averages to compute local modes.

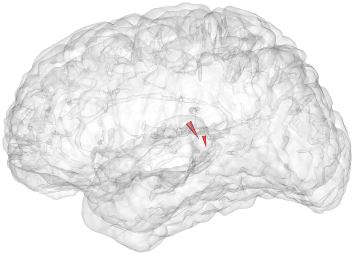
No guarantee for finding highest mode but usually an acceptable result.

Why HBM? EEG/MEG Source Reconstruction









Notoriously ill-posed problem

Why HBM? EEG/MEG Source Reconstruction

- Inversion with log-concave priors suffers from systematic depth miss-localization, HBM does not.
- HBM shows promising results for focal brain networks with simulated and real data.

F.L., Ü. Aydin, J. Vorwerk, M. Burger, C.H. Wolters, 2013. *Hierarchical Fully-Bayesian Inference for Combined EEG/MEG Source Analysis of Evoked Responses: From Simulations to Real Data*.
BaCI 2013, Geneva.

F.L., S. Pursiainen, M. Burger, C.H. Wolters, 2012. *Hierarchical Fully-Bayesian Inference for EEG/MEG combination: Examination of Depth Localization and Source Separation using Realistic FE Head Models*.
Biomag 2012, Paris

F.L., S. Pursiainen, M. Burger, C.H. Wolters, 2012. *Hierarchical Bayesian inference for the EEG inverse problem using realistic FE head models: Depth localization and source separation for focal primary currents*.
NeuroImage, 61(4):1364–1382.

Take Home Messages

- Sparsity can be modeled in different ways.
- The elementary MCMC schemes may show very different performance.
- Contrary to common beliefs they are not in general slow and scale bad with increasing dimension.
- Sample-based Bayesian inversion in high dimensions is feasible if tailored samplers are developed.
- MAP estimates are proper Bayes estimators.
- But "MAP or CM?" is NOT the key question in Bayesian inversion.
- Everything beyond point-estimates is far more interesting and can really complement variational approaches.
- Hierarchical Bayesian modeling leads to multimodal posteriors but MCMC sampling can be utilized to cope with it.
- HBM gives promising results for a notoriously ill-posed problem.

- Fast samplers can be used for simulated annealing.
- Reason for the efficiency of the Gibbs samplers is unclear.
- Adaptation, parallelization, multimodality, multi-grid.
- HBM and ℓ_p approaches can be combined: ℓ_p-hypermodels.
- Application studies had proof-of-concept character up to now.
- Specific UQ task to explore full potential of the Bayesian approach.

F.L., 2014. *Bayesian Inversion in Biomedical Imaging*
PhD Thesis, University of Münster.

M. Burger, F.L., 2014. *Maximum a posteriori estimates in linear inverse problems with log-concave priors are proper Bayes estimators*
Inverse Problems, 30(11):114004.

F.L., 2012. *Fast Markov chain Monte Carlo sampling for sparse Bayesian inference in high-dimensional inverse problems using L1-type priors.*
Inverse Problems, 28(12):125012.

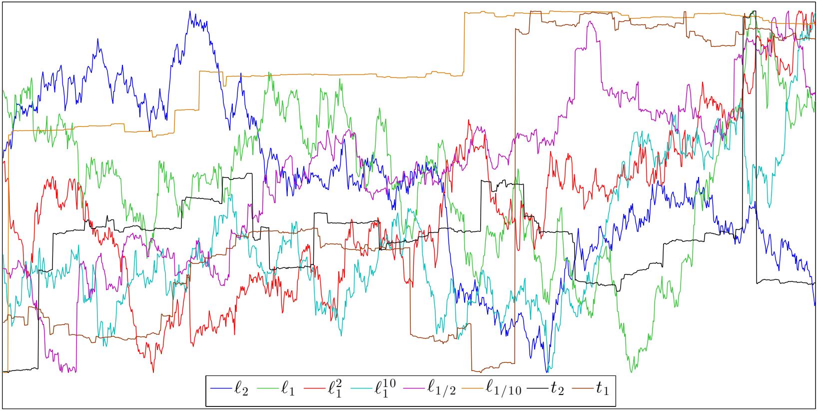
F.L., S. Pursiainen, M. Burger, C.H. Wolters, 2012. *Hierarchical Bayesian inference for the EEG inverse problem using realistic FE head models: Depth localization and source separation for focal primary currents.*
NeuroImage, 61(4):1364–1382.

Thank you for the invitation
and for your attention!

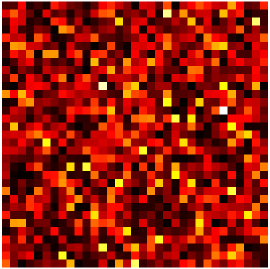
- F.L., 2014. *Bayesian Inversion in Biomedical Imaging*
  PhD Thesis, University of Münster.
- M. Burger, F.L., 2014. *Maximum a posteriori estimates in linear inverse problems with log-concave priors are proper Bayes estimators*
  Inverse Problems, 30(11):114004.
- F.L., 2012. *Fast Markov chain Monte Carlo sampling for sparse Bayesian inference in high-dimensional inverse problems using L1-type priors.*
  Inverse Problems, 28(12):125012.
- F.L., S. Pursiainen, M. Burger, C.H. Wolters, 2012. *Hierarchical Bayesian inference for the EEG inverse problem using realistic FE head models: Depth localization and source separation for focal primary currents.*
  NeuroImage, 61(4):1364–1382.

Appendix

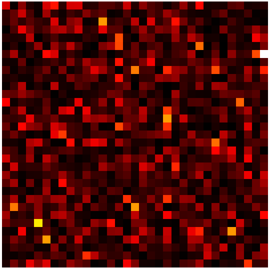
Prior Samples in 1D



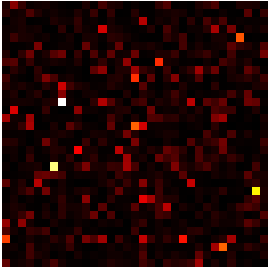
Prior Samples in 2D



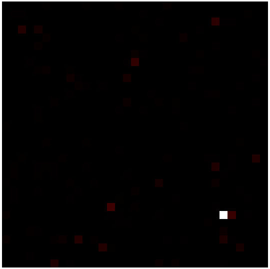
(a) ℓ_2-prior



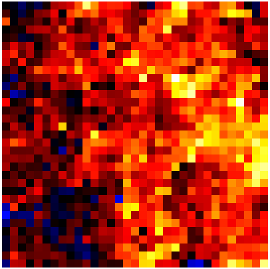
(b) ℓ_1-prior



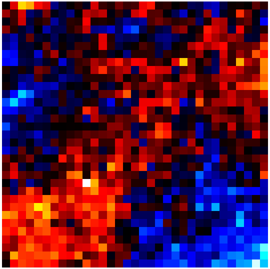
(c) $\ell_{1/2}$-prior



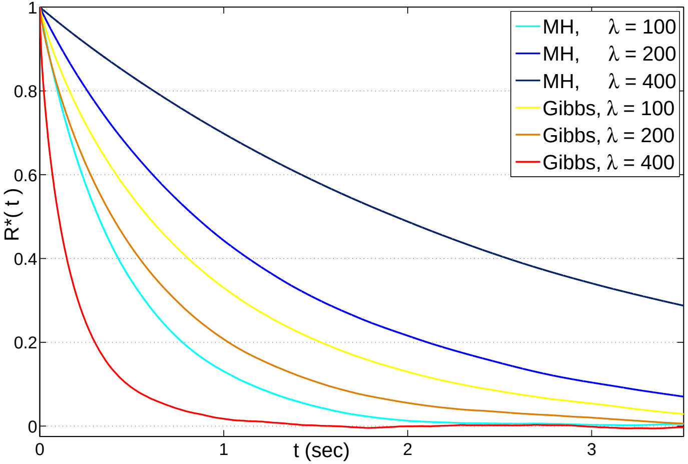
(d) Cauchy prior



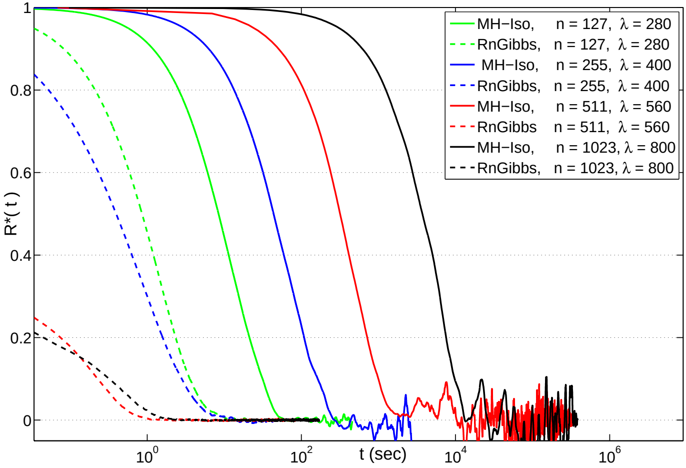
(e) TV prior, sample 1



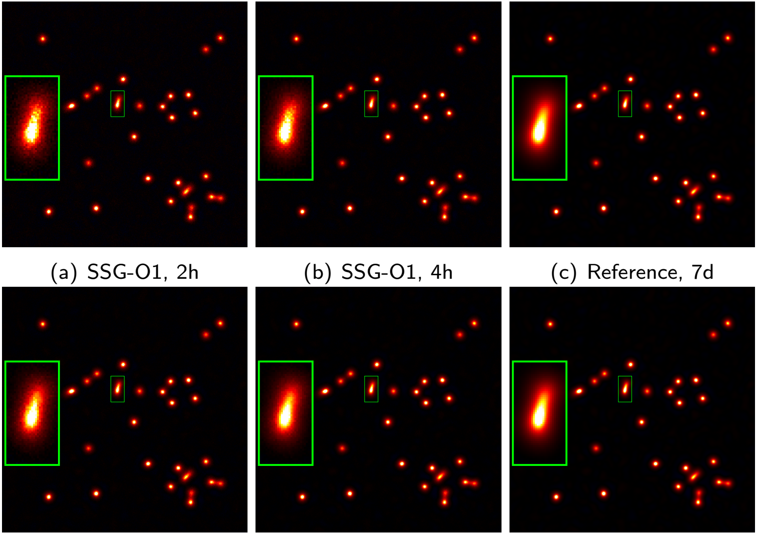
(f) TV prior, sample 2



Temporal autocorrelation $R^*(t)$ for 1D TV-deblurring, $n = 63$.



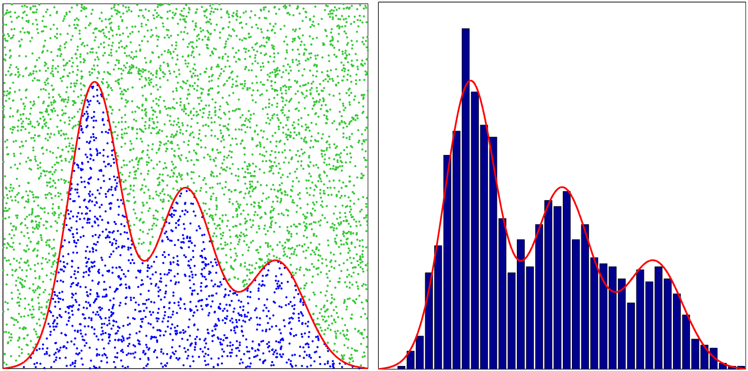
Temporal autocorrelation $R^*(t)$ for 1D TV-deblurring.



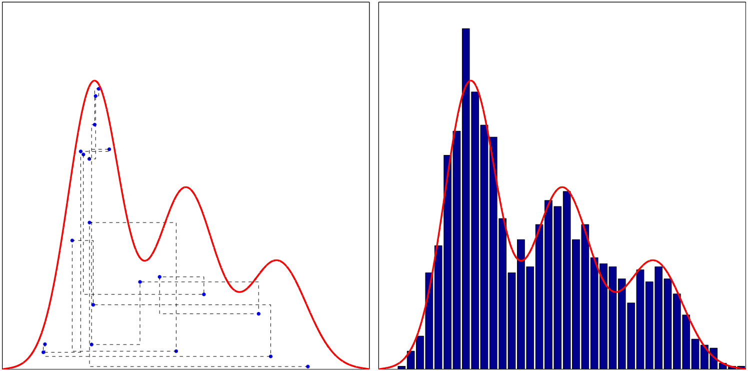
(a) SSG-O1, 2h (b) SSG-O1, 4h (c) Reference, 7d

(d) SSG-O7, 2h (e) SSG-O7, 4h (f) Reference, 7d

"Fundamental Theorem of Simulation"



Basic Slice Sampling



Gibbs sampler for $\tilde{p}(x, y) \propto \mathbb{1}_{\mathcal{G}_p}(x, y)$, keep only x values:

Draw y uniformly from $[0, p(x^i)]$ (vertical move)

Draw x^{i+1} uniformly from $\mathcal{S}^y := \{z | p(z) \geqslant y\}$ (horizontal move)

Horizontal move not easy to implement for our problems, but...

...slice sampling is a variant of auxiliary variables algorithms:

- Introduce additional variable y with suitable $p(y|x)$.
- Sample $p(x,y) = p(y|x)p(x)$ by a Gibbs sampler (needs $p(y|x), p(x|y)$)
- Keep only $\{x^i\}$

Basic slice sampler:

$$p(x)$$

$$p(y|x) = \frac{\mathbb{1}_{[0,p(x)]}(y)}{p(x)}$$

$$\Longrightarrow p(x,y) = p(x)\frac{\mathbb{1}_{[0,p(x)]}(y)}{p(x)}$$

$$\Longrightarrow p(x|y) \propto \mathbb{1}_{[0,p(x)]}(y) = \mathbb{1}_{\{x|p(x) \geqslant y\}}(x)$$

Factorizing slice sampler:

$$p(x) = p_1(x)p_2(x)$$

$$p(y|x) = \frac{\mathbb{1}_{[0,p_2(x)]}(y)}{p_2(x)}$$

$$\Longrightarrow p(x,y) = p_1(x)\mathbb{1}_{[0,p_2(x)]}(y)$$

$$\Longrightarrow p(x|y) \propto p_1(x)\mathbb{1}_{\{x|p_2(x) \geqslant y\}}(x)$$

For linear inverse problems, most SC densities have the form

$$p(x) \propto \exp\left(-ax^2 + bx - \phi(x)\right) = \underbrace{\exp\left(-ax^2 + bx\right)}_{p_1(x)} \underbrace{\exp\left(-\phi(x)\right)}_{p_2(x)}$$

For $i = 1, \ldots, K$ do:

draw y uniformly from $[0, p_2(x^i)]$ (vertical move)

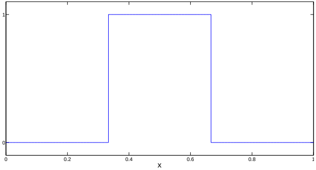
draw x uniformly from $\mathcal{S}_{\in}{}^{y} := \{z | p_2(z) \geqslant y\}$ (weighted horizontal move)

Requires

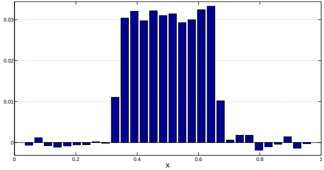
- Explicit or numerical computation of $\mathcal{S}_{\in}{}^{y} := \{z | p_2(z) \geqslant y\}$.
- Fast and robust sampling from truncated Gaussians (Chopin, 2010)
- Hard-constraints can be easily incorporated.

- 1D toy model of a *charge coupled device* (*CCD*).
- Unknown light intensity $\tilde{u}$ is indicator on $[\frac{1}{3}, \frac{2}{3}]$.
- Integrated into $m = 30$ CCD pixels + noise.
- Reconstruction on n-dim. grid.
- D^T is the forward finite difference operator with NB cond.

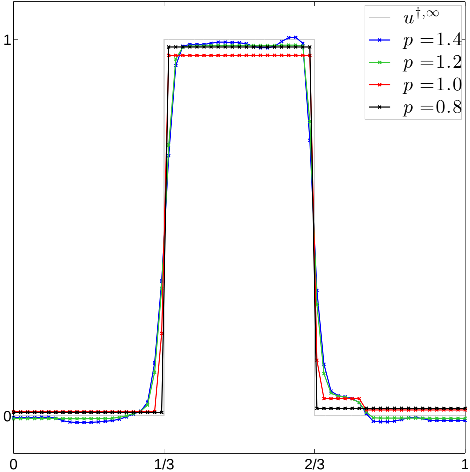
$$p_{post}(u|f) \propto \exp\left(-\frac{1}{2}\|f - A\,u\|^2_{\Sigma_\varepsilon^{-1}} - \lambda\,\|D^T u\|_1\right)$$



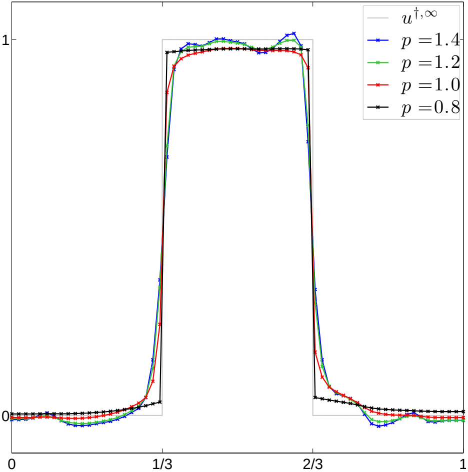
(a) The unknown function $\tilde{u}(t)$



(b) The measurement data f

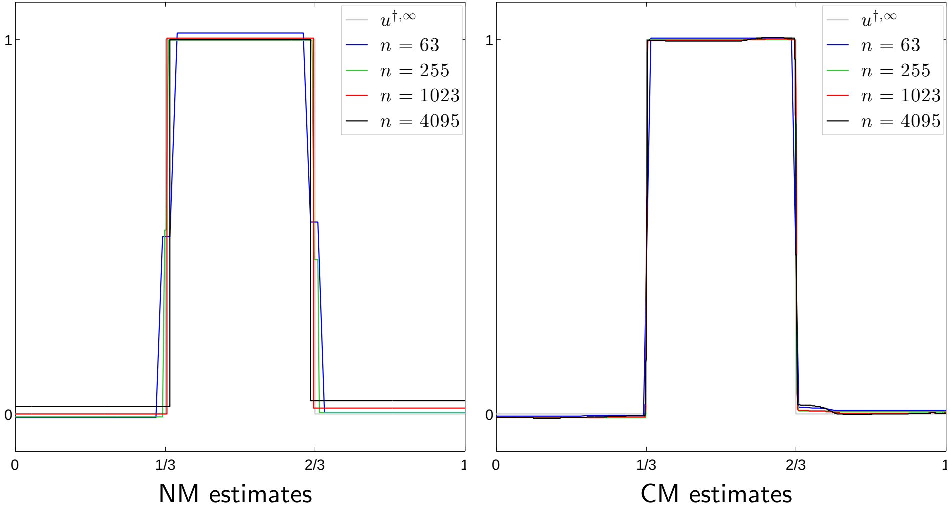


MAP estimates (SA)



CM estimates (Gibbs-MCMC)

HBM Prior in 1D: MAP vs. CM



A theoretical argument "decides" the conflict: The Bayes cost formalism.

- An estimator is a random variable, as it relies on f and u.
- How does it perform on average? Which estimator is "best"?
- $\rightsquigarrow$ Define a cost function $\Psi(u, v)$.
- Bayes cost is the expected cost:

$$BC(\hat{u}) = \iint \Psi(u, \hat{u}(f))\, p_{like}(f|u)\, \mathrm{d}f\; p_{prior}(u)\, \mathrm{d}u$$

- Bayes estimator $\hat{u}_{BC}$ for given Ψ minimizes Bayes cost. Turns out:

$$\hat{u}_{BC}(f) = \underset{\hat{u}}{\operatorname{argmin}} \left\{ \int \Psi(u, \hat{u}(f))\, p_{post}(u|f)\, \mathrm{d}u \right\}$$

Main classical arguments pro CM and contra MAP estimates:

- CM is Bayes estimator for $\Psi(u, \hat{u}) = \|u - \hat{u}\|_2^2$ (MSE).
- Also the minimum variance estimator.
- The mean value is intuitive, it is the "center of mass", the known "average".
- MAP estimate can be seen as an asymptotic Bayes estimator of

$$\Psi_\epsilon(u, \hat{u}) = \begin{cases} 0, & \text{if} \quad \|u - \hat{u}\|_\infty \leqslant \epsilon \\ 1 & \text{otherwise,} \end{cases}$$

  for $\epsilon \to 0$ (uniform cost). $\Longrightarrow$ It is not a proper Bayes estimator.

- MAP and CM seem theoretically and computationally fundamentally different $\Longrightarrow$ one should decide.
- *"A real Bayesian would not use the MAP estimate"*
- People feel "ashamed" when they have to compute MAP estimates (even when their results are good).

"A real Bayesian would not use the MAP estimate as it is not a proper Bayes estimator".

"MAP estimate can be seen as an asymptotic Bayes estimator of

$$\Psi_\epsilon(u, \hat{u}) = \begin{cases} 0, & \text{if} \quad \|u - \hat{u}\|_\infty < \epsilon \\ 1 & \text{otherwise,} \end{cases}$$

for $\epsilon \to 0$.
???$\Longrightarrow$??? It is not a proper Bayes estimator."

"MAP estimator is asymptotic Bayes estimator for some degenerate Ψ"
$\not\Rightarrow$ "MAP can't be Bayes estimator for some proper Ψ" !!!!

Define

(a) $\Psi_{\text{LS}}(u, \hat{u}) := \|A(\hat{u} - u)\|^2_{\Sigma_\varepsilon^{-1}} + \beta \|L(\hat{u} - u)\|_2^2$

(b) $\Psi_{\text{Brg}}(u, \hat{u}) := \|A(\hat{u} - u)\|^2_{\Sigma_\varepsilon^{-1}} + \lambda D_{\mathcal{J}}(\hat{u}, u)$

for a regular L and $\beta > 0$.

Properties:

- Proper, convex cost functions
- For $\mathcal{J}(u) = \beta / \lambda \|Lu\|_2^2$ (Gaussian case!) we have $\lambda D_{\mathcal{J}}(\hat{u}, u) = \beta \|L(\hat{u} - u)\|_2^2$, and $\Psi_{\text{LS}}(u, \hat{u}) = \Psi_{\text{Brg}}(u, \hat{u})$!

Theorems:

(I) The CM estimate is the Bayes estimator for $\Psi_{\text{LS}}(u, \hat{u})$

(II) The MAP estimate is the Bayes estimator for $\Psi_{\text{Brg}}(u, \hat{u})$

feature	ℓ_p prior	HBM
$\mathcal{J}(u)$	$\|u\|_p^p$	$\frac{\nu+1}{2}\sum \log\left(1+\frac{u^2}{\nu\theta}\right)$
sparsifying parameter	$p>0$	$\nu>0$
quadratic limit	$p=2$	$\nu\rightarrow\infty$
sparse limit	$p\rightarrow 0$	$\nu\rightarrow 0$
limit functional	$\|u\|_0$	$\sum_i^n \log(\|u_i\|)$ if all $u_i \neq 0$, $-\infty$ else
solutions	sparse	compressible
differentiable	$p>1$	always
convex	everywhere for $p\geqslant 1$	$\|u\|_\infty < \sqrt{\nu\theta}$
homogeneous	yes	no