



Centrum Wiskunde & Informatica



Challenges of Mathematical Image Reconstruction

Felix Lucka

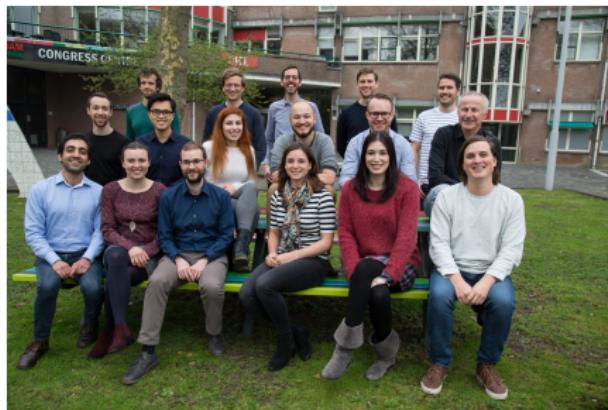
Mathematics Colloquium

Groningen

27 November 2018

Introduction and Overview

Computational Imaging @ CWI



- headed by **Joost Batenburg**, 18 members
- mathematics, computer science & (medical) physics
- advanced computational techniques for 3D imaging
- (inter-)national collaborations from science, industry & medicine
- one of the two main developers of the **ASTRA Toolbox**
- **FleX-ray lab:** custom-made, fully-automated **X-ray CT** scanner linked to large-scale computing hardware

X-ray Computed Tomography (CT)



- X-rays (high-energy photons) get **attenuated** by matter
- 3D attenuation image **computed** from different 2D projections

X-ray Computed Tomography (CT)



(a) Modern CT scanner



(b) CT scan of a patient's lung

Source: Wikimedia Commons

Imaging Across Disciplines

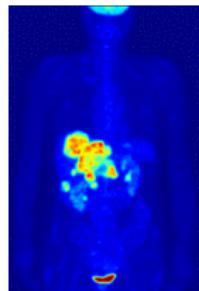
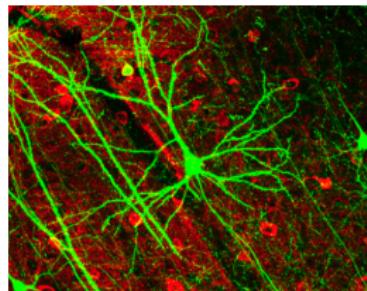
Observational astronomy

Life and material science

microscopy

Medical imaging

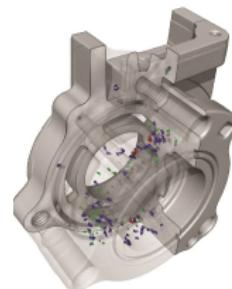
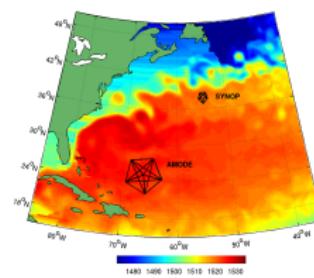
CT, MRI, PET, SPECT, US...



Geophysical imaging

(electrical) resistivity, seismic

(ground-penetrating) radar, ...



Remote sensing

earth science, military & intelligence

Industrial process imaging

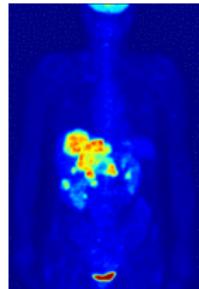
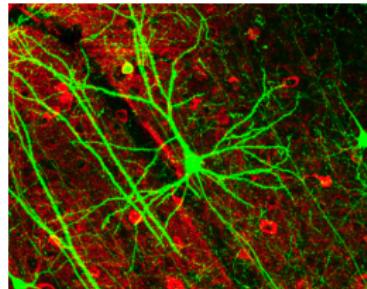
Source: Wikimedia Commons

Imaging Across Disciplines

Observational astronomy

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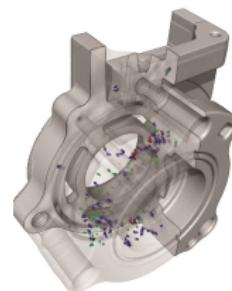
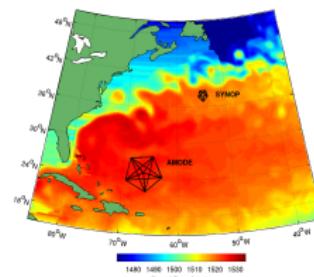
Medical imaging

CT, MRI, PET, SPECT, US...

Geophysical imaging

(electrical) resistivity, seismic

(ground-penetrating) radar, ...



Remote sensing

earth science, military & intelligence

Industrial process imaging

Source: Wikimedia Commons

Mathematical Imaging: *Reconstruct spatially distributed quantities of interest from indirect observations through algorithms derived from rigorous mathematics.*

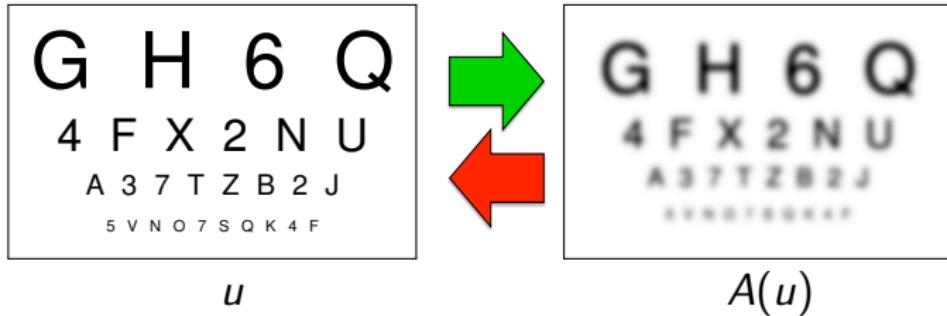
Imaging: An Inverse Problem

Inverse problem: Given **data** f recover **unknowns** u (image) from

$$f = A(u) + \varepsilon$$

- **Forward operator** A solution of **PDE** modelling underlying physics.

Imaging: An Inverse Problem

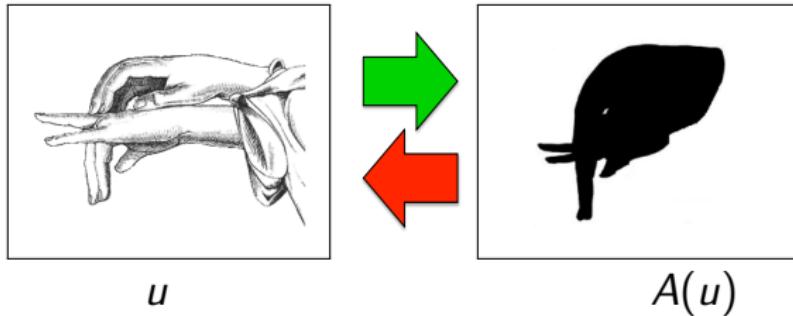


Inverse problem: Given **data** f recover **unknowns** u (image) from

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- **Forward operator** A a solution of **PDE** modelling underlying physics.
- Typical inverse problems are **ill-posed**.

Imaging: An Inverse Problem



Inverse problem: Given **data** f recover **unknowns** u (image) from

$$f = A(u) + \varepsilon$$

- **Forward operator** A solution of **PDE** modelling underlying physics.
- Typical inverse problems are **ill-posed**.
- Stable solution requires **a-priori information** on u .

Inverse Problems / Imaging Workflow

mathematical modeling

physics, PDEs, approximations

theoretical analysis

uniqueness, recovery conditions,
stability

reconstruction/inference approach

regularization, statistical inference,
machine learning

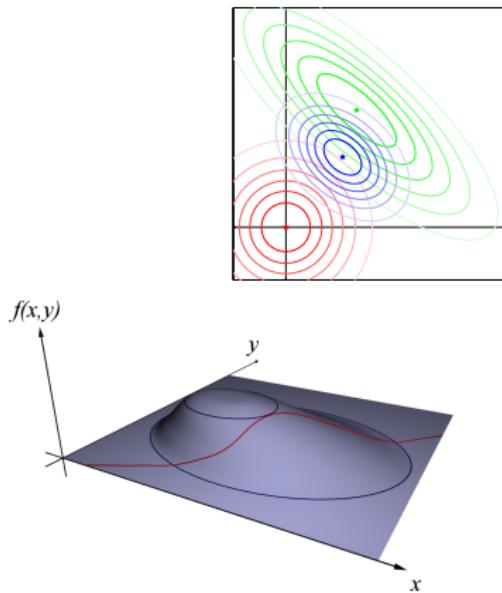
reconstruction algorithm

numerical linear algebra, PDEs,
optimization, MCMC

large-scale computing

parallel computing, GPU computing

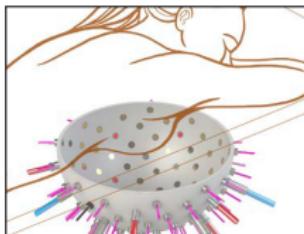
$$(s \cdot \nabla + \mu_a(x) + \mu_s(x)) \phi(x, s) \\ = q(x, s) + \mu_s(x) \int \Theta(s, s') \phi(x, s') ds'$$



Current Challenges in Computational Imaging

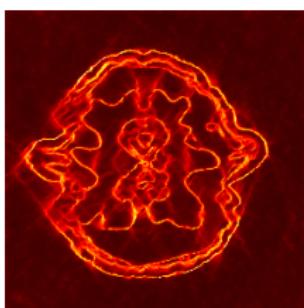
core development for new modalities:

hybrid imaging



more from more:

multi-spectral, multi-modal, high resolution



same from less:

low-dose, limited-view, compressed, dynamic

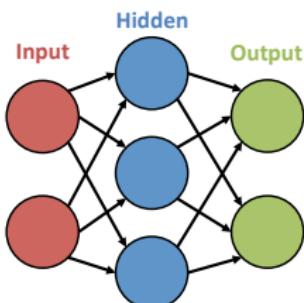
break the routine:

real-time, dose adaptation, zooming

uncertainty quantification & quantitative imaging

machine learning:

embedding, networks for 3D/4D, clinical training data



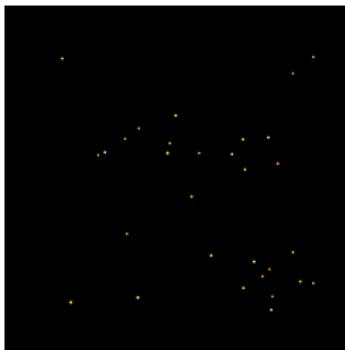
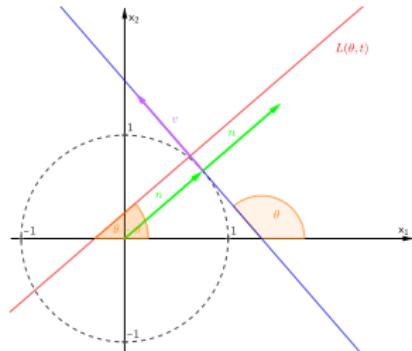
X-ray Computed Tomography

Mathematics of X-ray Computed Tomography (CT)

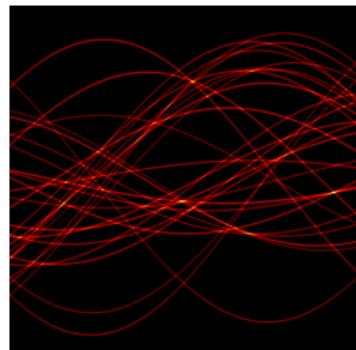
Beer-Lambert's law: Intensity of monochromatic ray passing through heterogeneous medium described by $\log(I_1/I_0) = - \int_I u(x)dx$.

→ **integral geometry problem**, A reduces to **Radon transform**:

$$f(\theta, t) = \int_{L(\theta, t)} u(x)dx, \quad L(\theta, t) = \{x \in \mathbb{R} \mid x_1 \cos(\theta) + x_2 \sin(\theta) = t\}$$



u



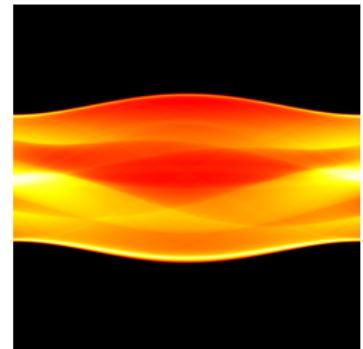
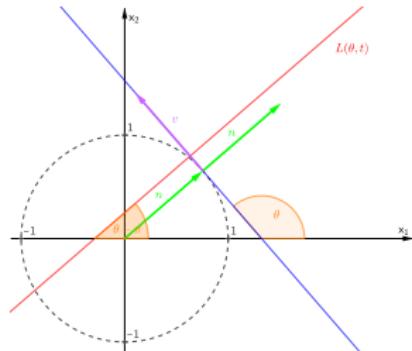
Au

Mathematics of X-ray Computed Tomography (CT)

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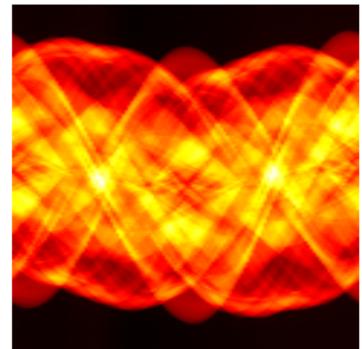
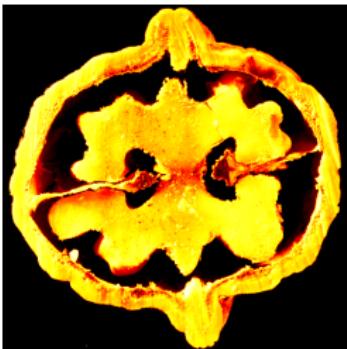
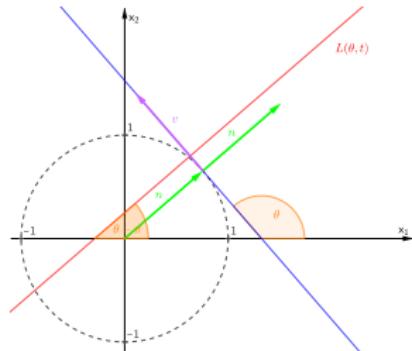


Image Reconstruction Approaches

$$f = Au + \varepsilon$$

Analytical - determine A^{-1} , regularize it, discretize it

$$\hat{u} = A^* \mathcal{H} f \quad (\textit{filtered backprojection} - FBP)$$

- ✓ efficient to implement and execute
- ! lack of flexibility for unconventional scanning set-ups
- ! severe artifacts for limited / sparse projection data
- ! hard to introduce a-priori knowledge

Image Reconstruction Approaches

$$f = Au + \varepsilon$$

Analytical - determine A^{-1} , regularize it, discretize it

Algebraic / variational - discretize and optimize via iterative scheme

$$\hat{u}_\lambda = \operatorname{argmin}_{u \in \mathcal{U}} \left\{ \frac{1}{2} \|Au - f\|_2^2 + \lambda \mathcal{J}(u) \right\}$$

- ! higher computational cost
- ✓ highly flexible, arbitrary geometries
- ✓ less artifacts for limited / sparse projection data
- ✓ introduction of a-priori knowledge possible

Image Reconstruction Approaches

$$f = Au + \varepsilon$$

Analytical - determine A^{-1} , regularize it, discretize it

Algebraic / variational - discretize and optimize via iterative scheme

Bayesian / statistical - explicit uncertainty modeling

$$p_{post}(u|f) = \frac{p_{like}(f|u)p_{prior}(u)}{p(f)}$$

- !! even higher computational cost
- ✓ rigorous assessment of solution's uncertainties

Image Reconstruction Approaches

$$f = Au + \varepsilon$$

Analytical - determine A^{-1} , regularize it, discretize it

Algebraic / variational - discretize and optimize via iterative scheme

Bayesian / statistical - explicit uncertainty modeling

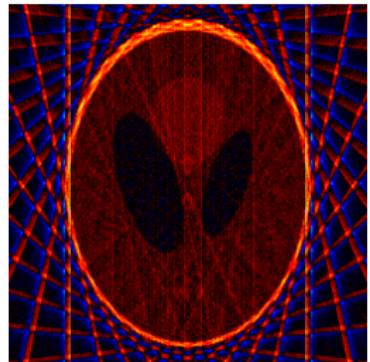
Deep learning - improve everything by trained DNNs

- ✓ extremely promising
- ✓ can be fast
- ! not well understood (yet)
- ! training data

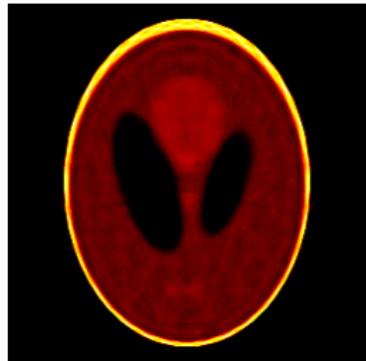
Illustration of Different Reconstruction Methods



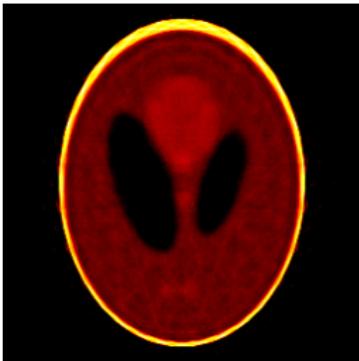
(a) true image



(b) FBP



(c) ART



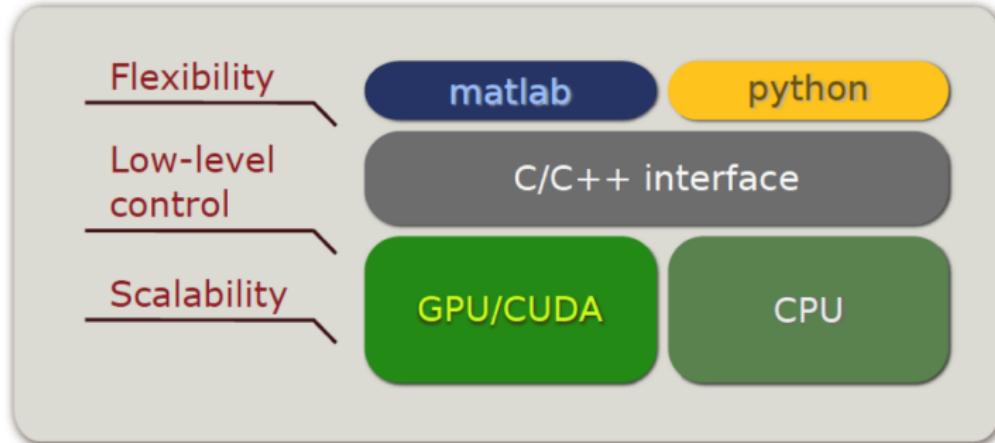
(d) SIRT



(e) TV regularization

ASTRA Toolbox

- open source software, developed by CWI and Univ. Antwerp
- provides scalable, high-performance GPU primitives for tomography
- flexible with respect to projection geometry
- featured in the NVIDIA CLARA Platform



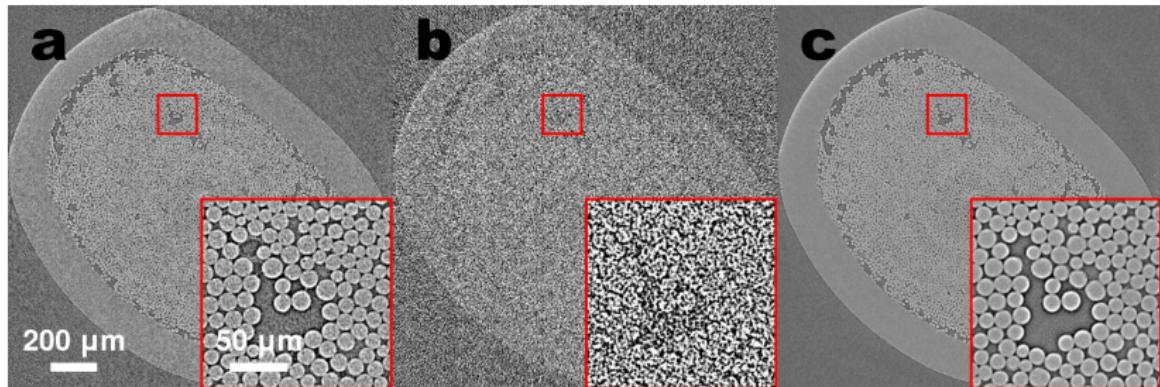
Approximate function $v = G(u)$ by **neuronal network** G_θ :

- G_θ : composition of **many computational units (layers)**
- layers: $y = \sigma(Wx + b)$
- W is convolution: **convolutional neuronal network (CNN)**
- θ : all free parameters
- **learning**: from **training set** $\{(u_i, v_i)\}_{i=1}^m$

$$\hat{\theta} = \operatorname{argmin}_{\theta \in \Theta} \left\{ \sum_i^m \text{Loss}(G_\theta(u_i), v_i) + \lambda \mathcal{J}(\theta) \right\}$$

- (stochastic) gradients via **backpropagation**

DNN for Removal of FBP Artefacts



2560x2560 tomography images of fiber composite.

Left: 1024 projections, middle/right: 128 projections

 **Pelt, Batenburg, Sethian, 2018.** Improving Tomographic Reconstruction from Limited Data Using Mixed-Scale Dense Convolutional Neural Networks, *Journal of Imaging* 4 (11), 128.

 **Pelt, Sethian, 2018.** Mixed-scale dense network for image analysis, *PNAS* 115 (2) 254-259.

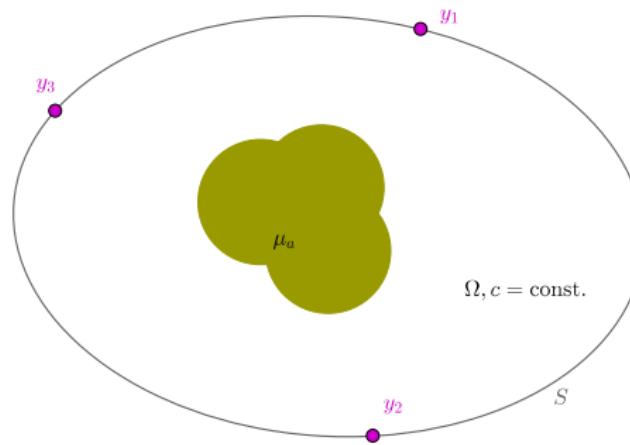
Photoacoustic and Ultrasound Tomography

Photoacoustic Imaging: Physical Principles

Optical Part

optical absorption coefficient: μ_a

Acoustic Part



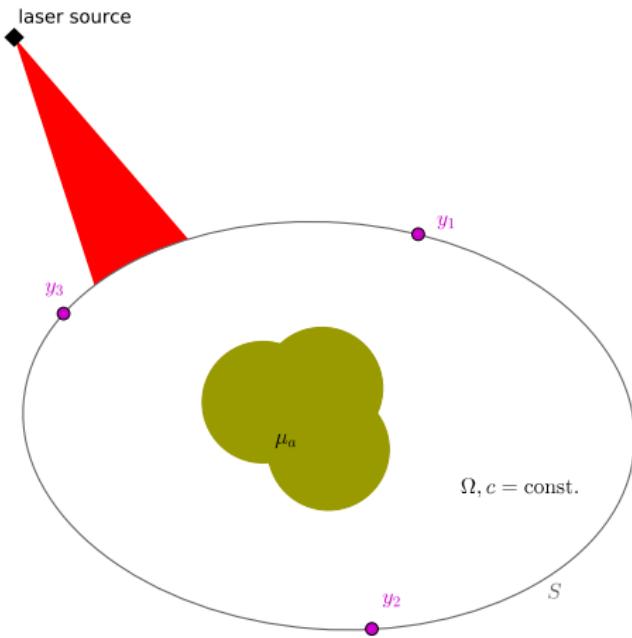
Photoacoustic Tomography: Physical Principles

Optical Part

optical absorption coefficient: μ_a

pulsed laser excitation: Φ

Acoustic Part



Photoacoustic Tomography: Physical Principles

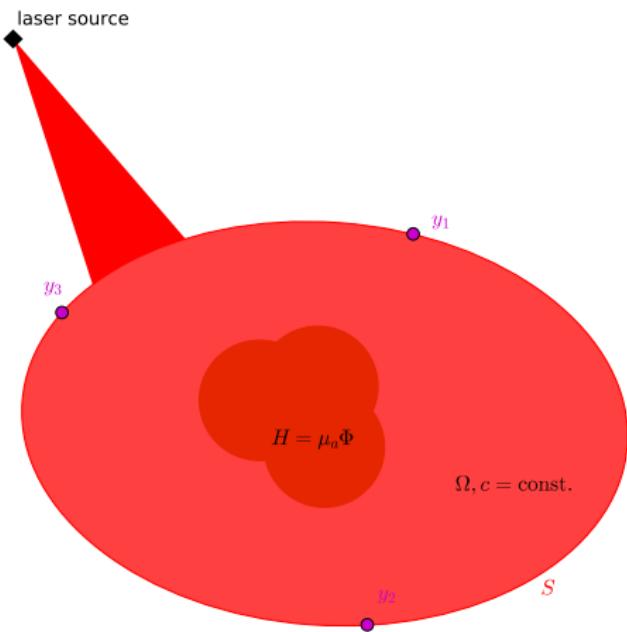
Optical Part

optical absorption coefficient: μ_a

pulsed laser excitation: Φ

thermalization by chromophores: $H = \mu_a \Phi$

Acoustic Part



Photoacoustic Tomography: Physical Principles

Optical Part

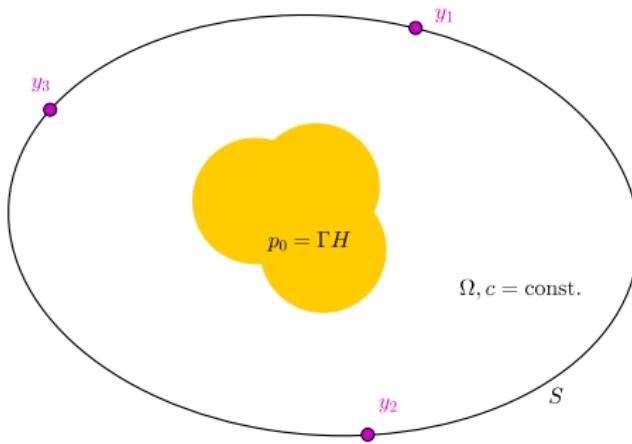
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Acoustic Part

local pressure increase: $p_0 = \Gamma H$



Photoacoustic Tomography: Physical Principles

Optical Part

optical absorption coefficient: μ_a

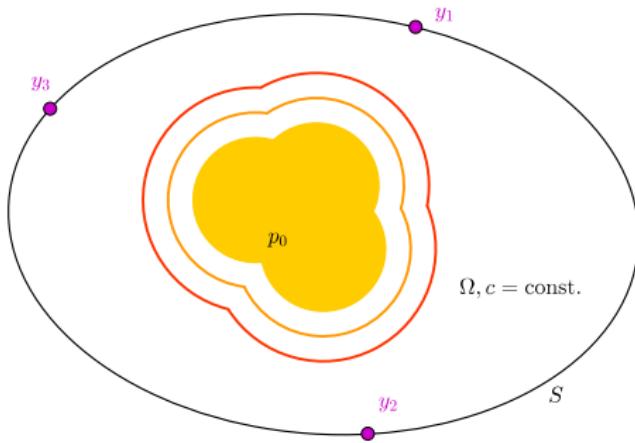
pulsed laser excitation: Φ

thermalization by chromophores: $H = \mu_a \Phi$

Acoustic Part

local pressure increase: $p_0 = \Gamma H$

elastic wave propagation: $p(x, t)$



Photoacoustic Tomography: Physical Principles

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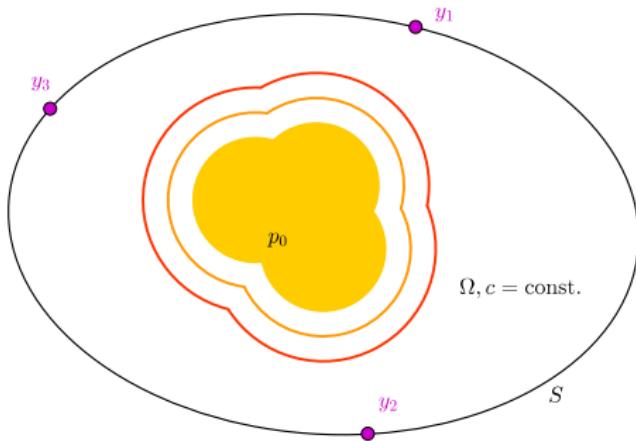
Acoustic Part

local pressure increase: $p_0 = \Gamma H$

elastic wave propagation: $p(x, t)$

measurement of pressure time courses:

$$f_i(t) = p(y_i, t)$$



Photoacoustic Tomography: Physical Principles

Optical Part

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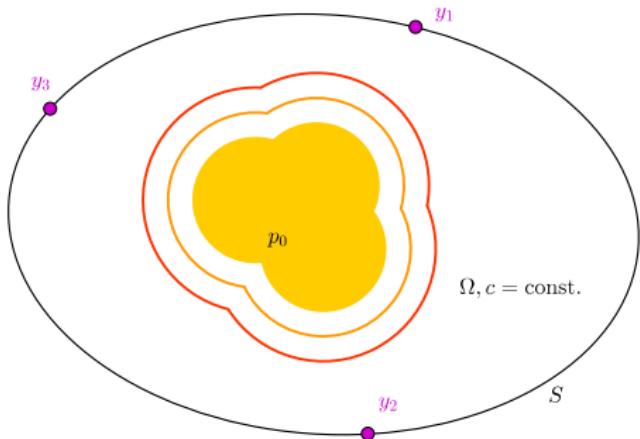
Acoustic Part

local pressure increase: $p_0 = \Gamma H$

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Photoacoustic effect

- **coupling** of optical and acoustic modalities.
- "hybrid imaging"
- **high optical contrast** sensed by **high-resolution** ultrasound.

Photoacoustic Tomography: Mathematical Formulation

(stationary) radiative transfer equation (RTE)

$$(s \cdot \nabla + \mu_a(x) + \mu_s(x)) \phi(x, s) = q(x, s) + \mu_s(x) \int \Theta(s, s') \phi(x, s') ds',$$

coupled with **acoustic wave equation**

$$\begin{aligned} p(x, t=0) &= p_0 := \Gamma(x) \mu_a(x) \int \phi(x, s) ds, & \partial_t p(x, t=0) &= 0 \\ (c(x)^{-2} \partial_t^2 - \Delta) p(x, t) &= 0, & f &= Sp|_{M \times [0, T]} \end{aligned}$$

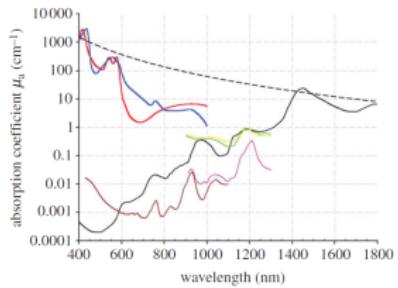
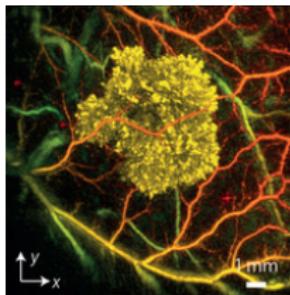
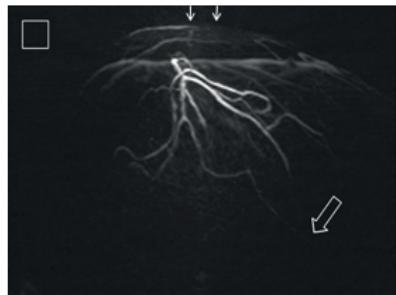
Hybrid inverse problem:

- ✓ acoustic initial value problem with boundary data
- ✓ optical parameter identification problem with internal data

vs. *diffuse optical tomography (DOT)*:

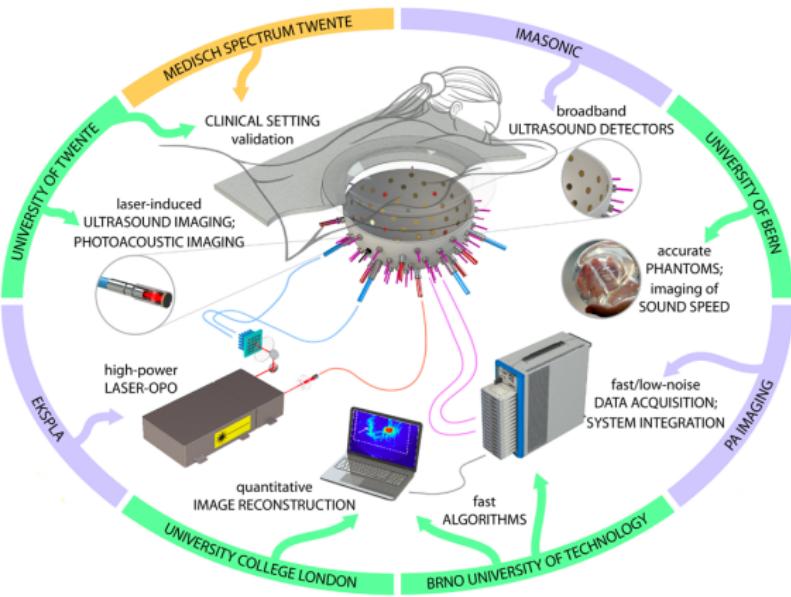
! optical parameter identification problem with boundary data

Photoacoustic Tomography: Applications



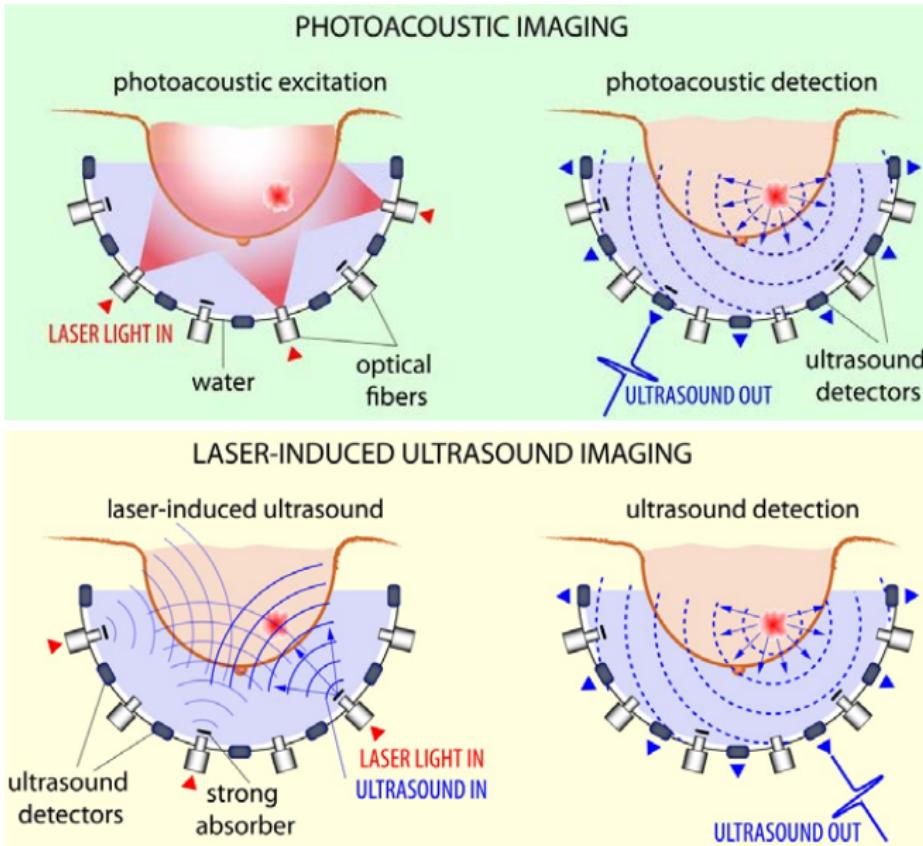
- High contrast for light-absorbing structures in soft tissue.
- Gap between oxygenated and deoxygenated blood.
- Different wavelengths allow **quantitative spectroscopic examinations**.
- Use of contrast agents for **molecular imaging**.
- **Extremely promising future imaging technique!**

H2020 Project: Photoacoustic Mammography Scanner



- Real-time photoacoustic imaging
- Multi-modal: joint **ultrasound CT (USCT)** and PA imaging.
- Multi-spectral: quantitative **sO₂** imaging.

H2020 Project: Photoacoustic Mammography Scanner



3D Wave Propagation Methods for PAT and USCT

k-space pseudospectral time domain method:

B. Treeby and B. Cox, 2010. k-Wave:

MATLAB toolbox for the simulation and reconstruction of photoacoustic wave fields,

Journal of Biomedical Optics.



derivation and discretization of **adjoint PAT operator A^* :**

- Arridge, Betcke, Cox, L, Treeby, 2016. On the Adjoint Operator in Photoacoustic Tomography, *Inverse Problems* 32(11).

approximation via deep learning:

- Hauptmann, Cox, L, Huynh, Betcke, Beard, Arridge, 2018. Approximate k-space models and Deep Learning for fast photoacoustic reconstruction, *MLMIR 2018*.

Radiative Transfer Equation in 3D

$$(s \cdot \nabla + \mu_a + \mu_s) \phi(x, s) = q + \mu_s \int \Theta(s, s') \phi(x, s') ds', \quad \Phi(x) = \int \phi(x, s) ds$$

! $(x, s) \in \mathbb{R}^5 \rightsquigarrow$ direct FEM infeasible.

Diffusion approximation:

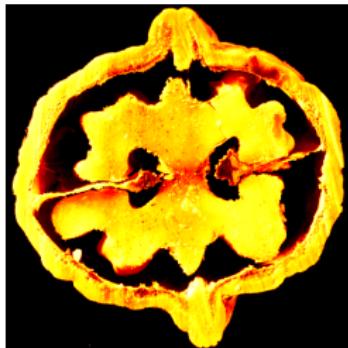
$$(\mu_a - \nabla \cdot \kappa(x) \nabla) \Phi(x) = \int q(x, s) ds, \quad \kappa = \frac{1}{3(\mu_a + \mu_s(1 - g))}$$

Schweiger, Arridge, 2014. The Toast++ software suite for forward and inverse modeling in optical tomography, *Journal of Biomedical Optics*.

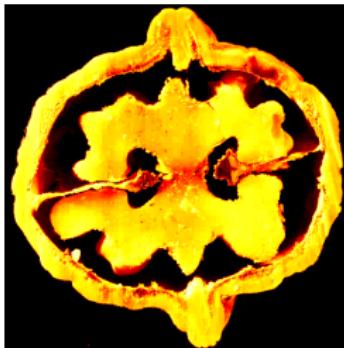
Alternative: GPU-based **Monte Carlo** estimate of transport density

Compressed Sensing and Dynamic Imaging

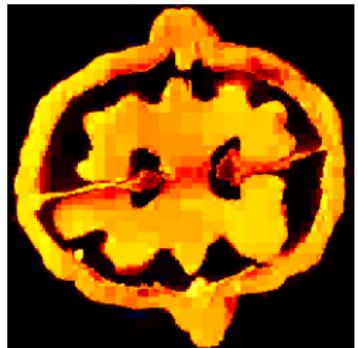
Sparsity & Compressed Sensing



(a) 100%



(b) 10%



(c) 1%

- **sparsity** traditionally used for compression of **Nyquist data**.
- Nyquist sampling: too much time/radiation!
- directly sense non-redundant information? → **compressed sensing**

Accelerated Imaging via Compressed Sensing

Beat Nyquist for objects with **low spatio-temporal complexity** by **incoherent sub-sampling**,

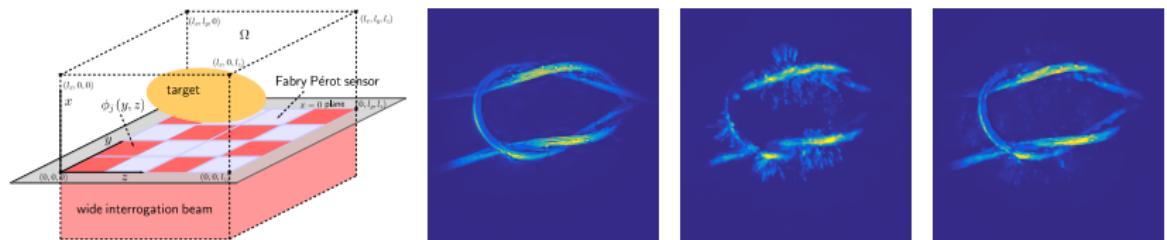
$$f^c = Cf = C(Au + \varepsilon)$$

combined with **sparsity-constrained variational image reconstruction**:

$$\hat{u}_\lambda = \operatorname{argmin}_{u \in \mathcal{U}} \left\{ \frac{1}{2} \|CAu - f\|_2^2 + \lambda \mathcal{J}(u) \right\}$$

- ! Development of novel acquisition systems.
- ! Iterative, first-order methods for non-smooth optimization.
- ! Matrix-free implementation of A , A^* .

Accelerated 3D PAT via Compressed Sensing



- ✓ development of compressed sensing PAT scanners
- ✓ implementation of **sparse regularization** schemes
- ✓ realistic simulated, experimental and *in-vivo* data
- ✓ **significant acceleration** with minor loss of quality
- ✓ further improvement through **deep learning**

Arridge, Beard, Betcke, Cox, Huynh, L, Ogunlade, Zhang, 2016. Accelerated High-Resolution Photoacoustic Tomography via Compressed Sensing, *PMB*.

Hauptmann, L, Betcke, Huynh, Adler, Cox, Beard, Ourselin, Arridge, 2018. Model-Based Learning for Accelerated, Limited-View 3-D Photoacoustic Tomography, *IEEE-TMI*.

Spatio-Temporal Reconstruction: 4D PAT

Dynamic compressed sensing:

$$f_t^c = C_t f_t = C_t(Au_t + \varepsilon_t)$$

Limitations of frame-by-frame →

full data

16x acc. (6.25%)

Spatio-Temporal Reconstruction: 4D PAT

Dynamic compressed sensing:

$$f_t^c = C_t f_t = C_t(A u_t + \varepsilon_t)$$

Limitations of frame-by-frame →

Spatio-temporal image reconstruction: full data 16x acc. (6.25%)

Parametric models (shift, stretch, etc.): simple and nice if applicable.

Non-parametric models, e.g., spatio-temporal variational schemes:

$$\hat{u} = \operatorname{argmin}_{u \in \mathcal{U}} \left\{ \sum_t^T \frac{1}{2} \|C_t A u_t - f_t^c\|_2^2 + \lambda \mathcal{R}(u) \right\}$$

- space-time decomposition (structured low-rank)
- more sophisticated: joint reconstruction of image and dynamics.

Spatio-Temporal Reconstruction: 4D PAT

Dynamic compressed sensing:

$$f_t^c = C_t f_t = C_t(A u_t + \varepsilon_t)$$

Limitations of frame-by-frame →

full data 16x acc. (6.25%)

$$(\hat{u}, \hat{v}) = \underset{u \in \mathcal{U}, v \in \mathcal{V}}{\operatorname{argmin}} \left\{ \sum_t^T \frac{1}{2} \|C_t A u_t - f_t^c\|_2^2 + \alpha \mathcal{J}(u_t) + \beta \mathcal{H}(v_t) + \gamma \mathcal{S}(u, v) \right\}$$

$\mathcal{S}(u, v)$ enforces PDE model of dynamics, e.g., optical flow equation:

$$\partial_t u(x, t) + (\nabla_x u(x, t)) v(x, t) = 0$$



Burger, Dirks, Schönlieb, 2018. A Variational Model for Joint Motion Estimation and Image Reconstruction, .

Dynamic Compressed Sensing with Optical Flow Constraints

X maxIP

Y maxIP

Z maxIP

full data, TV-fbf

16x, TV-fbf

16x, TTVL2

- ✓ Proof-of-concept for 4D CS PAT data.
- ! High dimensional, **non-smooth, bi-convex** optimization problem.



L, Huynh, Betcke, Zhang, Beard, Cox, Arridge, 2018. Enhancing Compressed Sensing 4D Photoacoustic Tomography by Simultaneous Motion Estimation, *SIAM Journal on Imaging Sciences* 11:4, 2224-2253.

Dynamic Compressed Sensing with Deep Learning



Hauptmann, Arridge, L, Muthurangu, Steeden, 2018. Realtime cardiovascular MR with spatiotemporal artifact suppression using deep learning - proof of concept in congenital heart disease, *Magnetic Resonance in Medicine*.

Summary

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- imaging has broad range of applications
- mathematically: **inverse problem** of reconstructing distributed quantities from indirect observations
- stable solution requires **a-priori information**
- **mathematical modeling**, (solving) **PDEs, numerical optimization**
- 3D: high performance computing
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Thank you for your attention!

-  **Arridge, Betcke, Cox, L, Treeby, 2016.** On the Adjoint Operator in Photoacoustic Tomography, *Inverse Problems* 32(11).
-  **Arridge, Beard, Betcke, Cox, Huynh, L, Ogunlade, Zhang, 2016.** Accelerated High-Resolution Photoacoustic Tomography via Compressed Sensing, *PMB* 61(24).
-  **Hauptmann, L, Betcke, Huynh, Adler, Cox, Beard, Ourselin, Arridge, 2018.** Model-Based Learning for Accelerated, Limited-View 3-D Photoacoustic Tomography, *IEEE-TMI* 37(6).
-  **L, Huynh, Betcke, Zhang, Beard, Cox, Arridge, 2018.** Enhancing Compressed Sensing 4D Photoacoustic Tomography by Simultaneous Motion Estimation, *SIAM-IS* 11(4).
-  **Hauptmann, Arridge, L, Muthurangu, Steeden, 2018.** Realtime cardiovascular MR with spatiotemporal artifact suppression using deep learning - proof of concept in congenital heart disease, *Magnetic Resonance in Medicine*.