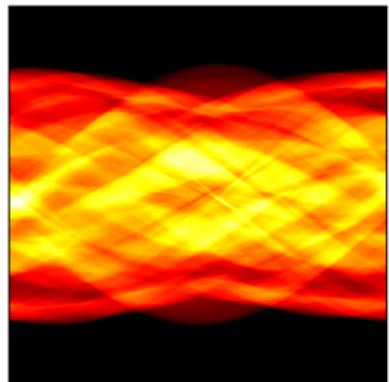
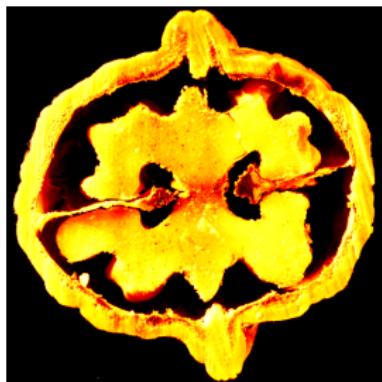


# X-Ray Computed Tomography



**Felix Lucka**

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University College London  
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**Mastermath Course**  
**"Inverse Problems and Imaging"**

March 22, 2021

# Centrum Wiskunde & Informatica (CWI)

- National research institute for mathematics and computer science, founded 1946.
- Focus: Fundamental research problems derived from societal needs.
- ~200 people working in 15 research groups on 4 research themes: Algorithms, Data & Intelligent Systems, Security & Cryptography, Quantum Computing
- National and international industry and academic collaborations.
- 27 spin-off companies
- opportunities for MSc and PhD students

## Computational Imaging @ CWI

- headed by Tristan van Leeuwen (also Utrecht Uni), 20 members
- mathematics, computer science & (medical) physics
- advanced computational techniques for 3D imaging
- (inter-)national collaborations from science, industry & medicine
- one of the two main developers of the ASTRA Toolbox
- FleX-ray Lab: custom-made, fully-automated X-ray CT scanner linked to large-scale computing hardware

# History of X-rays



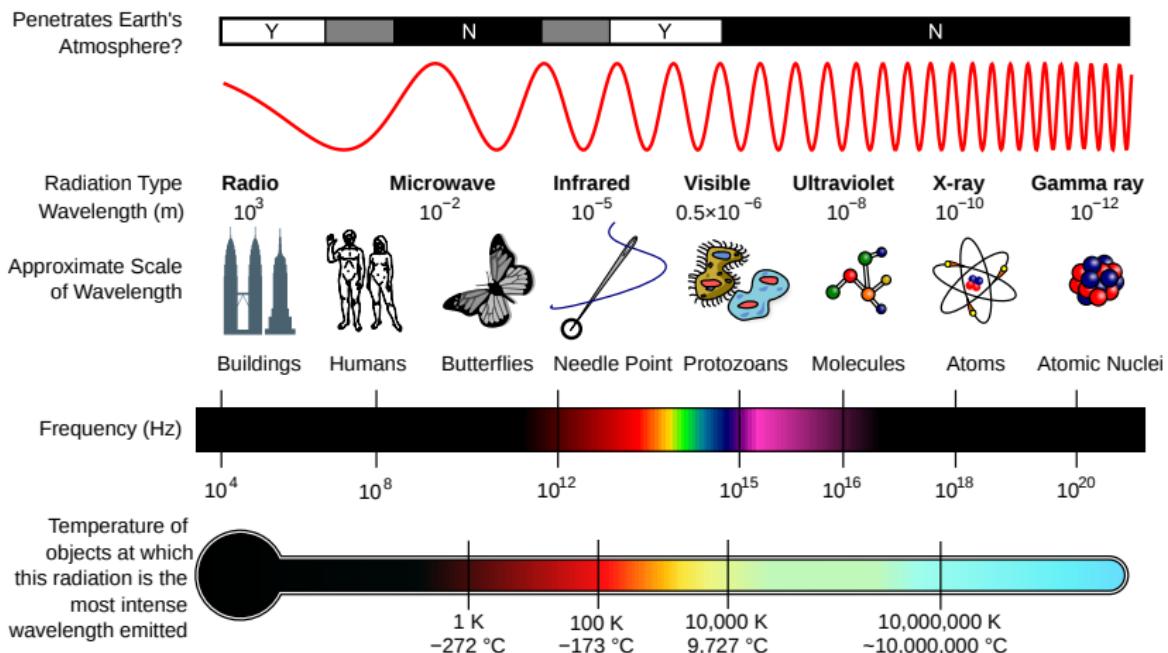
(a) Wilhelm Röntgen (1845-1923)

source: Wikimedia Commons

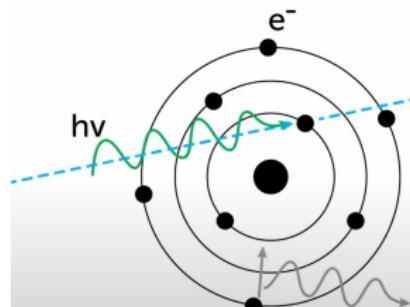


(b) First X-ray image (1895)

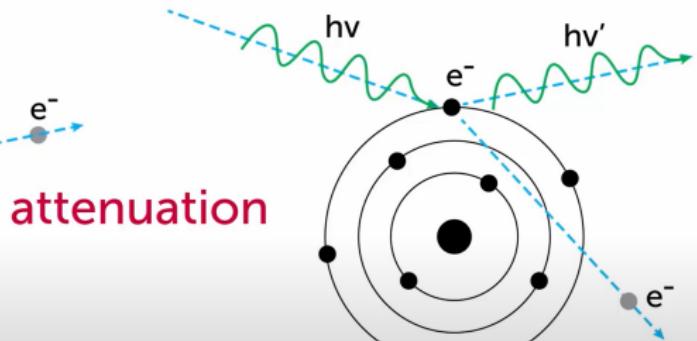
# What are X-rays?



# X-ray-matter interaction



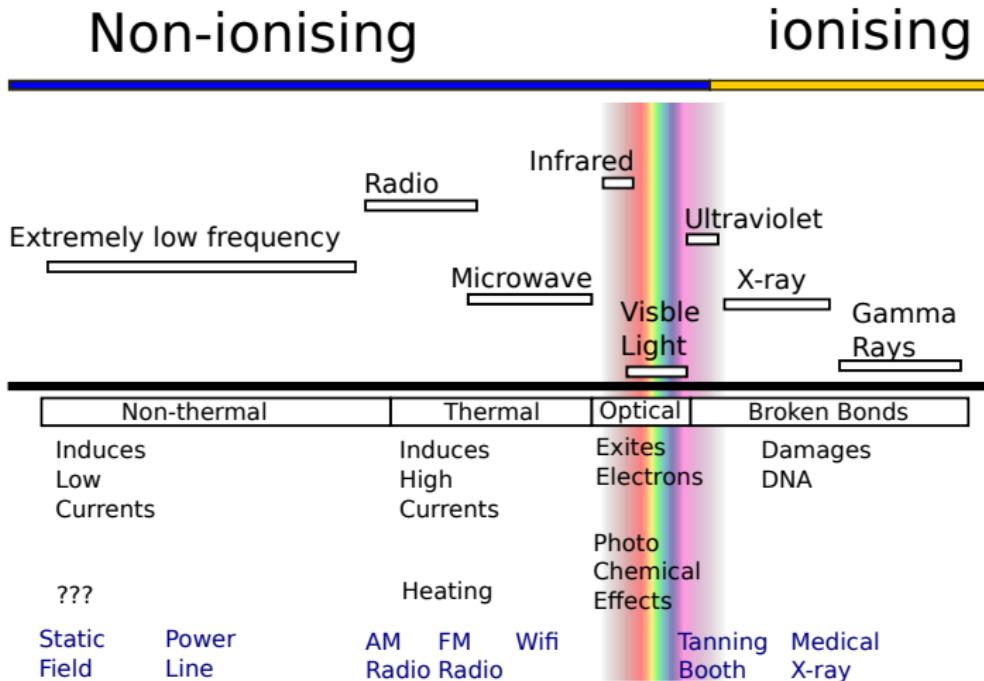
photoelectric effect



Compton scattering

Taken from corresponding video by the ASTRA toolbox team YouTube

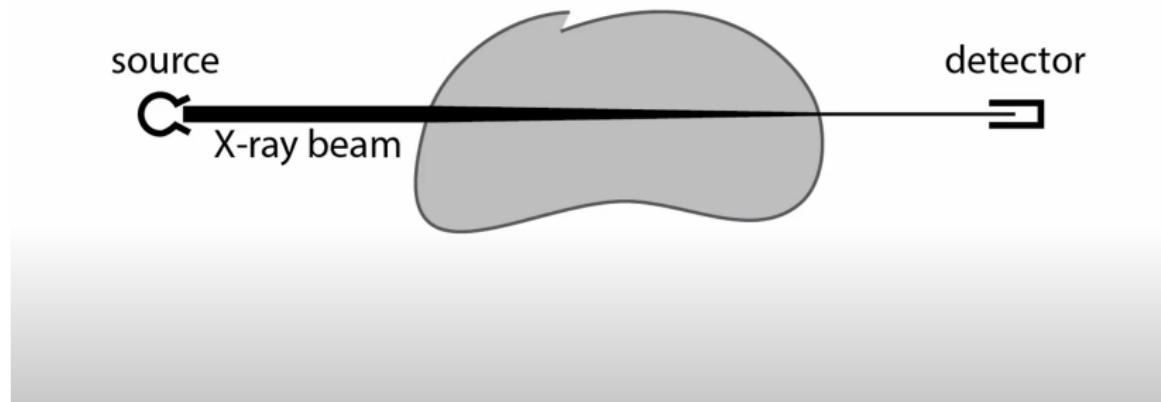
# How do X-rays interact with materials?



source: Wikimedia Commons

## Mathematics of CT 1: Beer's Law

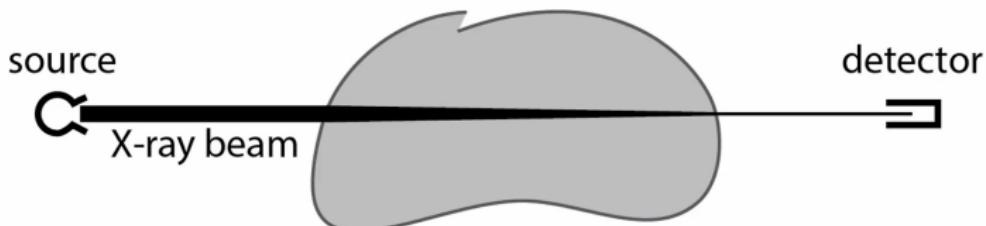
### X-ray detection



Taken from corresponding video by the ASTRA toolbox team YouTube

## Mathematics of CT 1: Beer's Law

### X-ray detection



Beam intensity at source

$$I_0(E)$$

Beam intensity at detector

$$I(E) = I_0(E) e^{-\int \mu(\xi, E) d\xi}$$

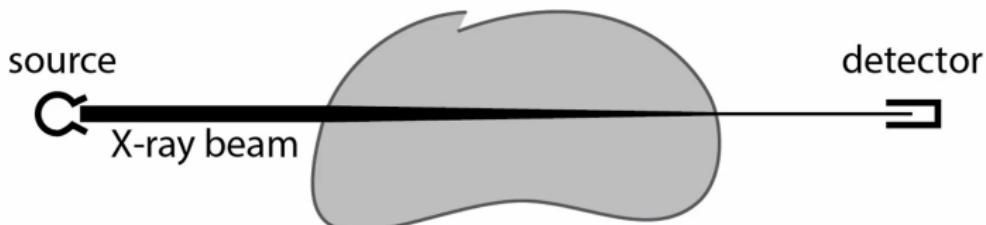
Beam intensity changes

$$I(\xi + \Delta\xi, E) = I(\xi) - \mu(\xi, E)I(\xi, E)\Delta\xi$$

Taken from corresponding video by the ASTRA toolbox team YouTube

## Mathematics of CT 1: Beer's Law

### X-ray detection



Beam intensity at source

$$I_0(E)$$

Beam intensity at detector

$$I(E) = I_0(E) e^{-\int \mu(\xi, E) d\xi}$$

Beam intensity changes

$$I(\xi + \Delta\xi, E) = I(\xi) - \mu(\xi, E)I(\xi, E)\Delta\xi$$

$$\Rightarrow P := -\log \left( \frac{I(E)}{I_0(E)} \right) = \int_{beam} \mu(\xi, E) d\xi$$

Taken from corresponding video by the ASTRA toolbox team YouTube

## Radiography

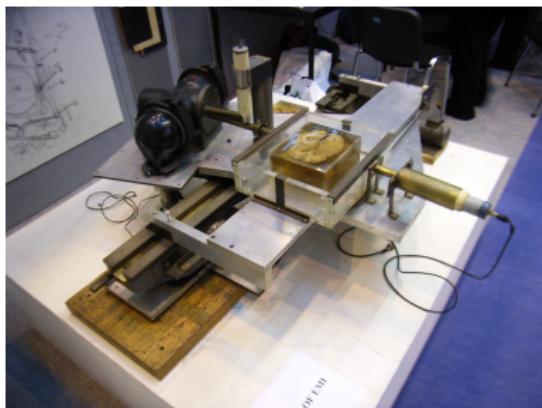
An excellent video by Samuli Siltanen:  YouTube



source: Wikimedia Commons

# History of Computed Tomography (CT)

Godfrey Hounsfield and Allan McLeod Cormack developed X-ray tomography (Nobelprize 1979)



(c) CT prototype

source: Wikimedia Commons



(d) first commercial CT head scanner

## Modern CT Scanner



a video of a scanner during rotation  YouTube

## Break & questions time



Nick Veasey, VW Camper Van , 2019

## Disclaimer

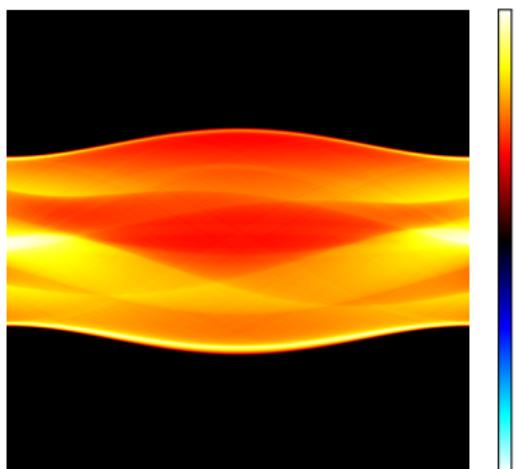
**!!! Notation on these slides varies from the script !!!**

## From projections to sinograms

Another excellent video by Samuli Siltanen: [YouTube](#)



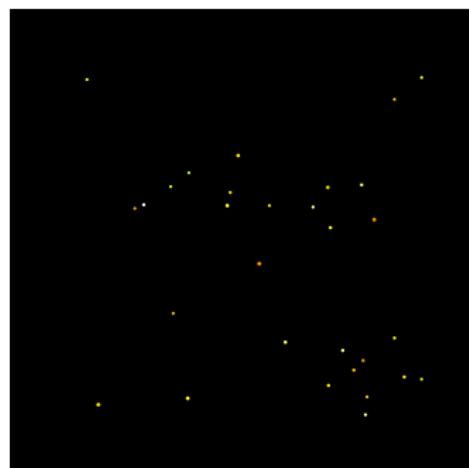
(a) image



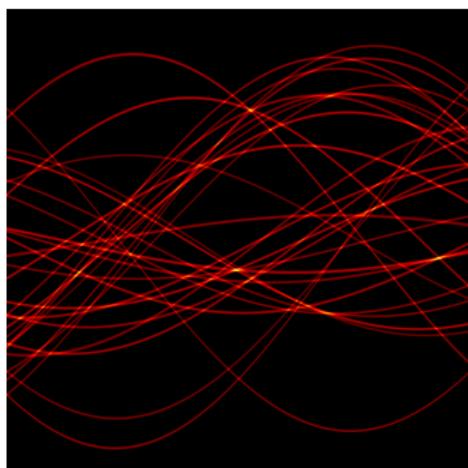
(b) sinogram

## From projections to sinograms

Another excellent video by Samuli Siltanen: [YouTube](#)



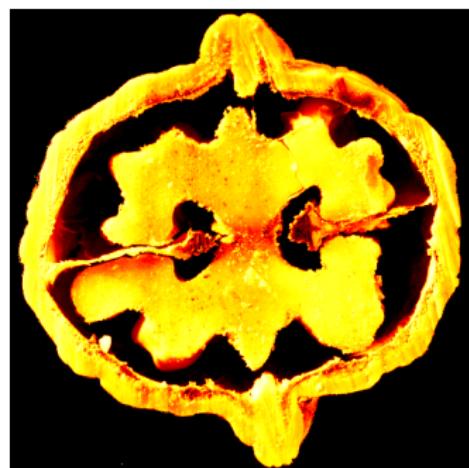
(a) image



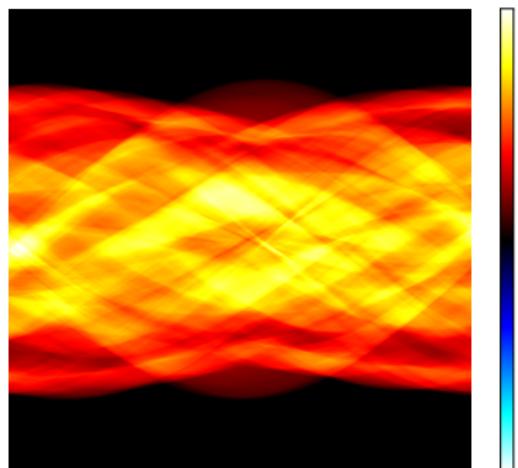
(b) sinogram

## From projections to sinograms

Another excellent video by Samuli Siltanen: [YouTube](#)

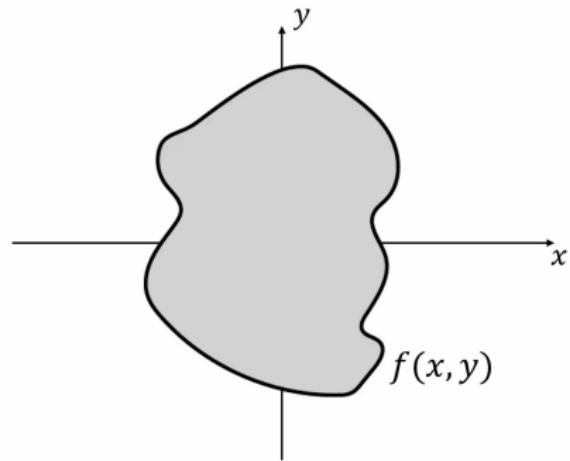


(a) image



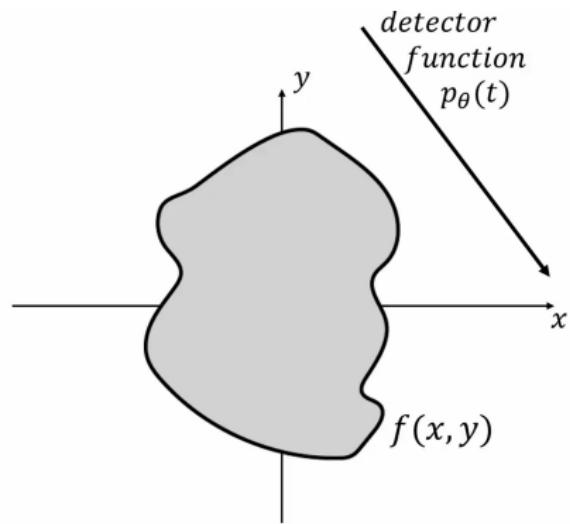
(b) sinogram

## Radon transform



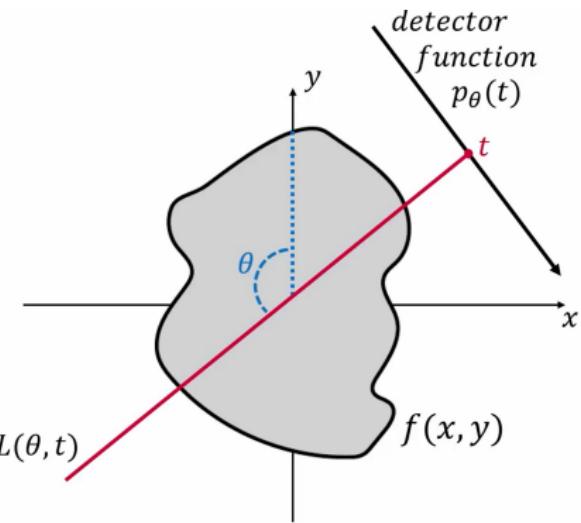
Taken from corresponding video by the ASTRA toolbox team YouTube

## Radon transform



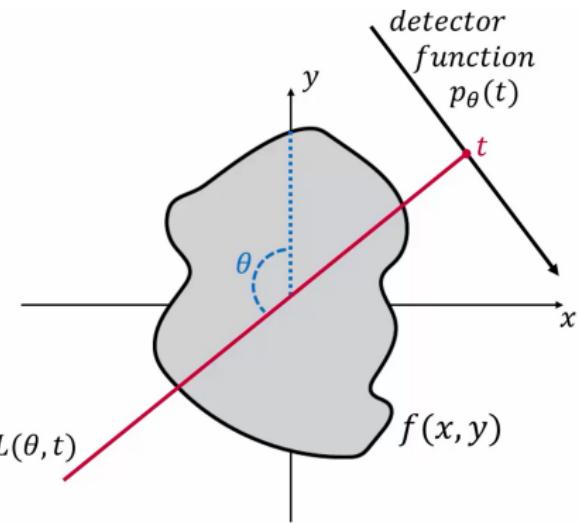
Taken from corresponding video by the ASTRA toolbox team YouTube

## Radon transform



Taken from corresponding video by the ASTRA toolbox team [YouTube](#)

## Radon transform

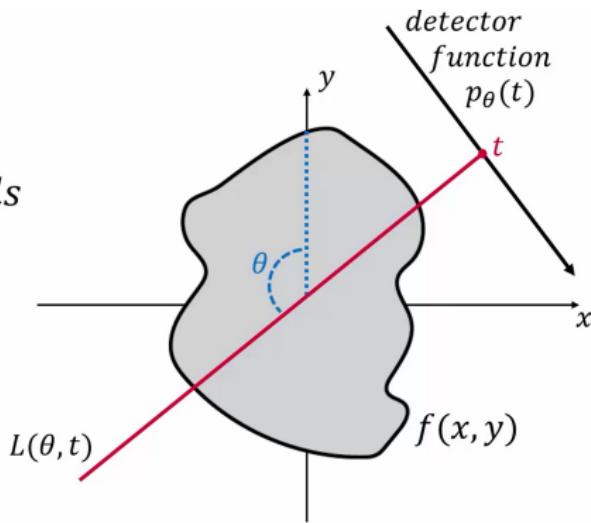


$$L(\theta, t) = \{(x, y) \in \mathbb{R} \times \mathbb{R}: x \cos \theta + y \sin \theta = t\}$$

Taken from corresponding video by the ASTRA toolbox team [YouTube](#)

## Radon transform

$$p_\theta(t) = \int_{L(\theta,t)} f(x,y) ds$$

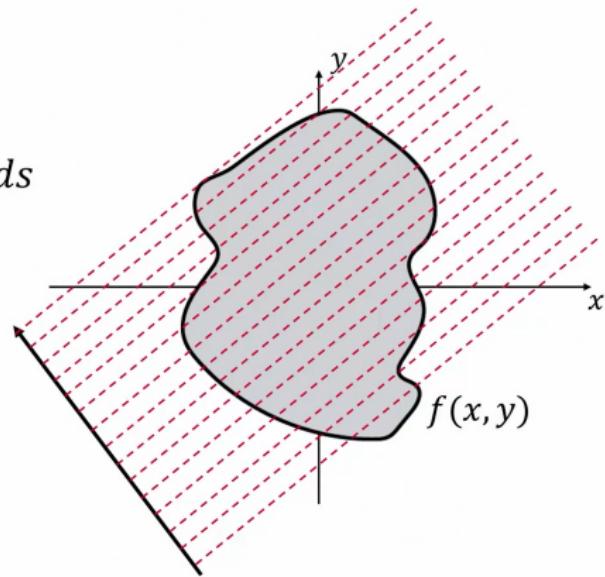
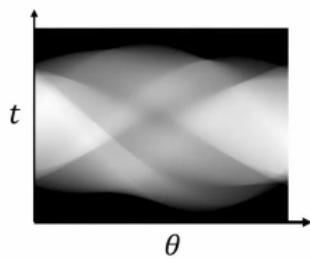


$$L(\theta, t) = \{(x, y) \in \mathbb{R} \times \mathbb{R}: x \cos \theta + y \sin \theta = t\}$$

Taken from corresponding video by the ASTRA toolbox team [YouTube](#)

## Radon transform

$$\mathcal{R}f(\theta, t) = \int_{L(\theta, t)} f(x, y) ds$$

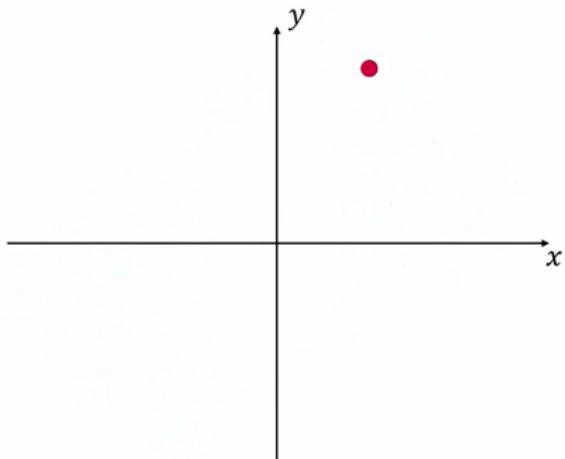
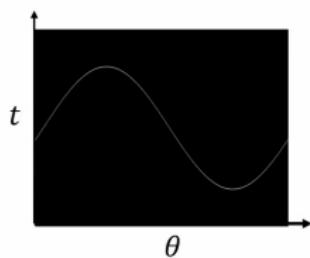


Taken from corresponding video by the ASTRA toolbox team [YouTube](#)

## Radon transform

$$\mathcal{R}f(\theta, t) = \int_{L(\theta, t)} f(x, y) ds$$

Sinogram

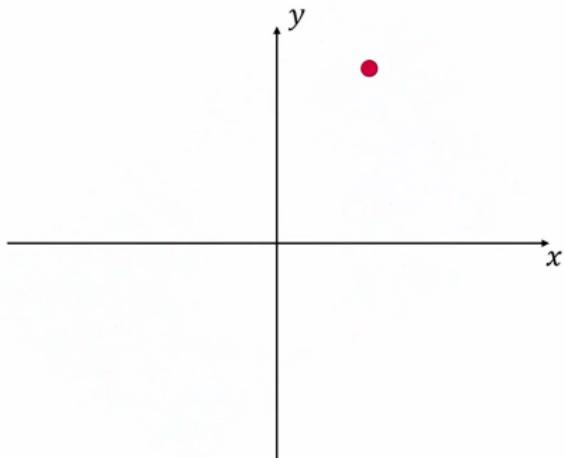
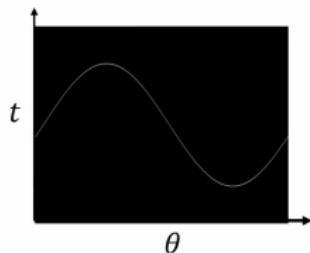


Taken from corresponding video by the ASTRA toolbox team YouTube

## Radon transform

$$\mathcal{R}f(\theta, t) = \int_{L(\theta, t)} f(x, y) ds$$

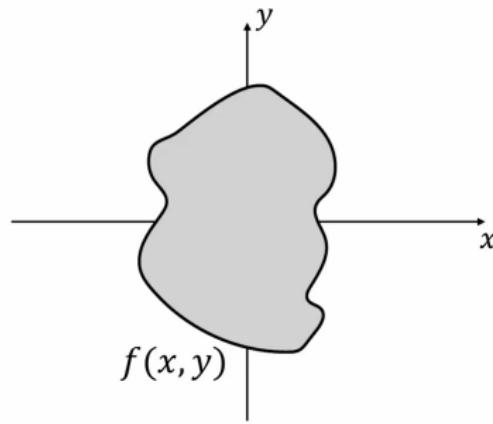
Sinogram



**$R$  is a linear operator, but is it invertible?**

Taken from corresponding video by the ASTRA toolbox team YouTube

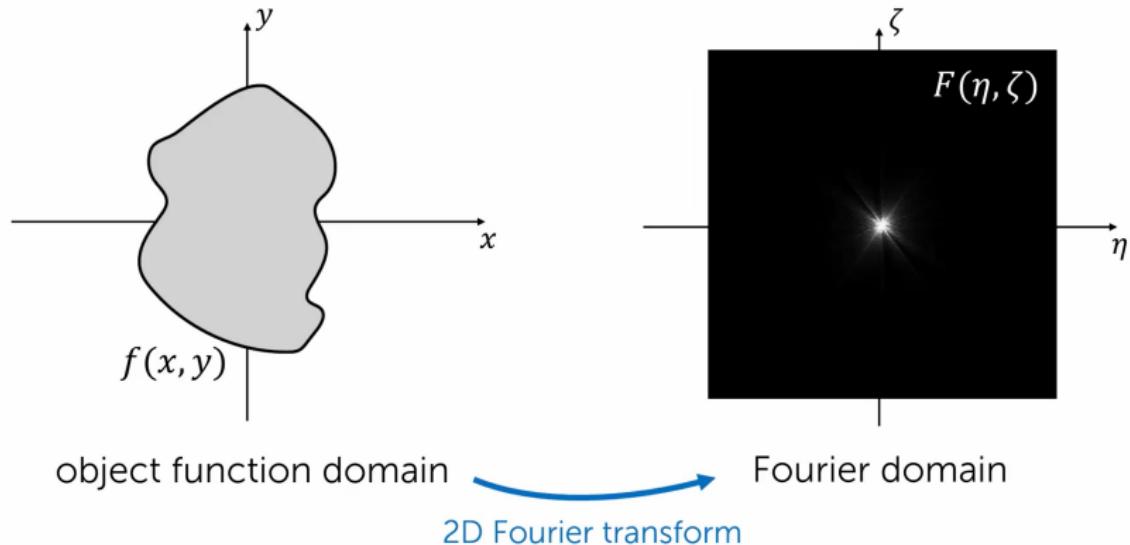
## Fourier Slice Theorem



object function domain

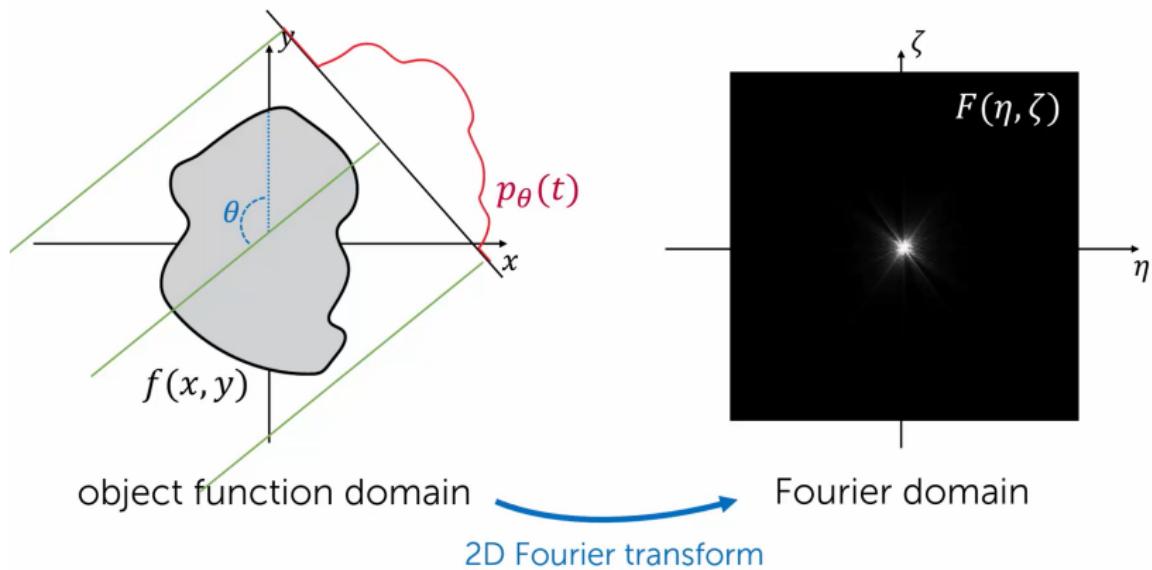
Taken from corresponding video by the ASTRA toolbox team [YouTube](#)

## Fourier Slice Theorem



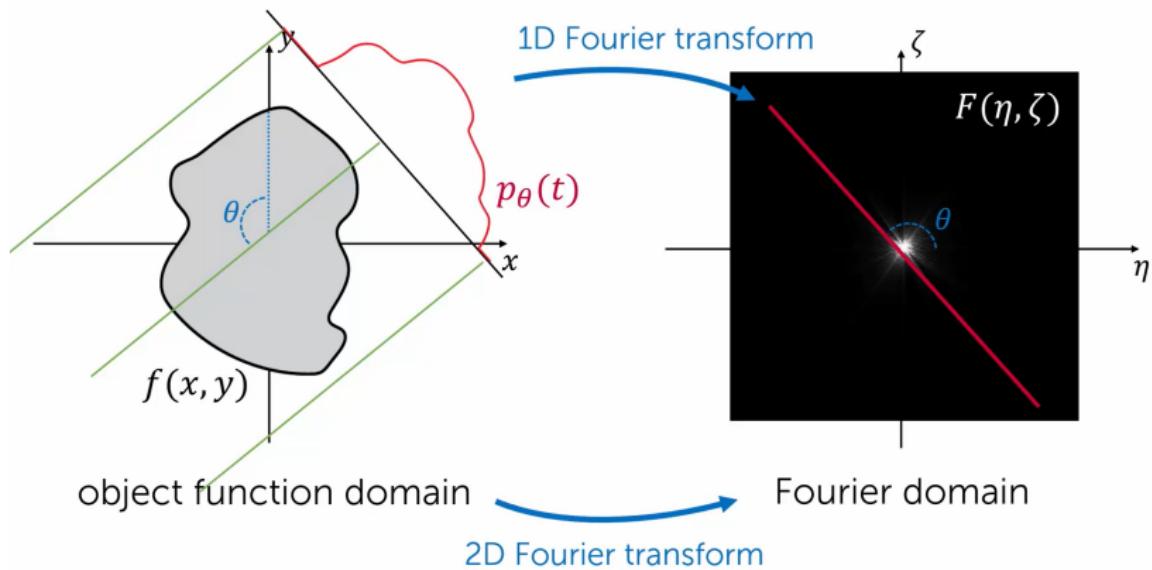
Taken from corresponding video by the ASTRA toolbox team [YouTube](#)

## Fourier Slice Theorem



Taken from corresponding video by the ASTRA toolbox team [YouTube](#)

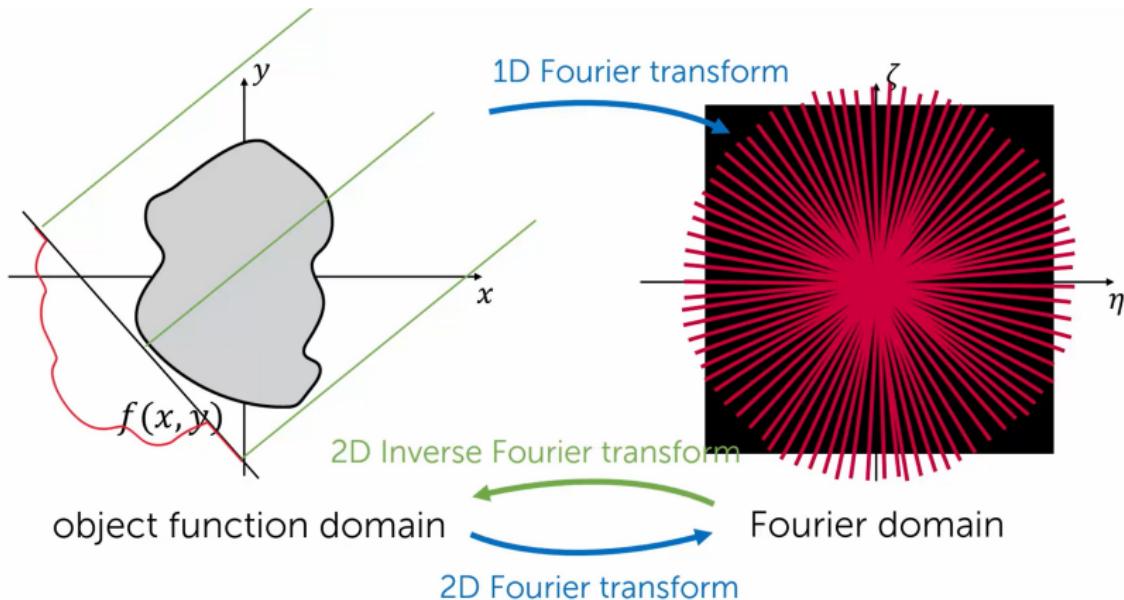
## Fourier Slice Theorem



$$\mathcal{F}_1 [\mathcal{R}_\theta f] (\omega) = \mathcal{F}_2 [f] (\omega \nu), \quad \nu = (\cos \theta, \sin \theta)$$

Taken from corresponding video by the ASTRA toolbox team YouTube

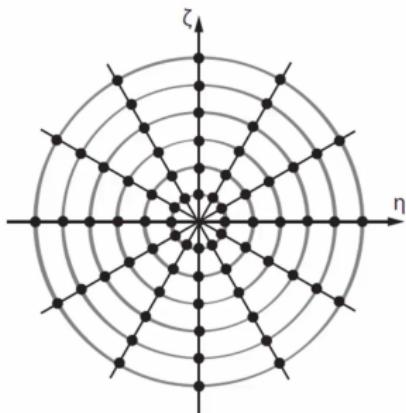
## Fourier Slice Theorem



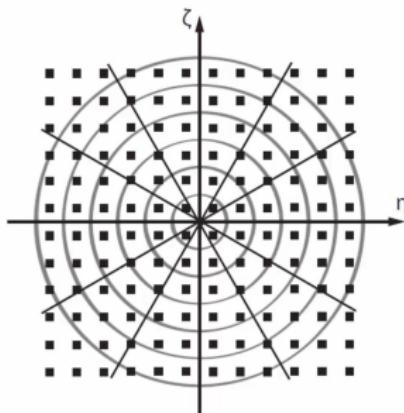
$$\mathcal{F}_1 [\mathcal{R}_\theta f] (\omega) = \mathcal{F}_2 [f] (\omega \nu), \quad \nu = (\cos \theta, \sin \theta)$$

Taken from corresponding video by the ASTRA toolbox team

## Non-uniform Fourier sampling



Fourier sampling with  
Fourier Slice Theorem



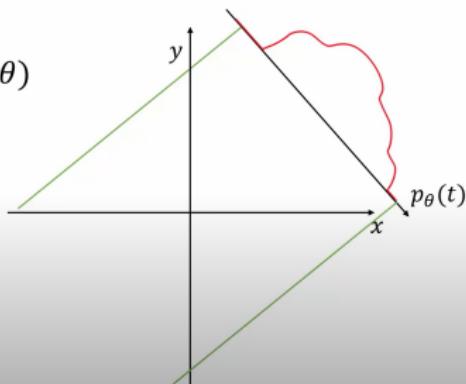
Sampling required by Fast  
Fourier Transform (FFT)

- ! high frequencies (= high resolution details) undersampled
- ! sampling non-uniform

Taken from corresponding video by the ASTRA toolbox team YouTube

## Backprojection

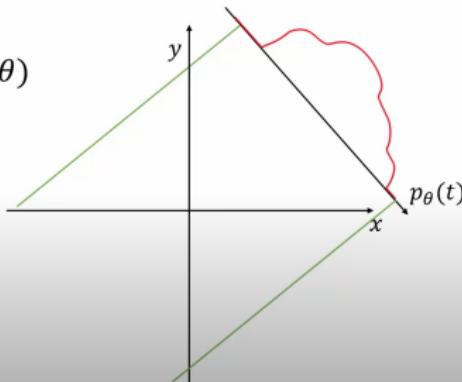
$$f_{bp}(x, y) = p_\theta(x \cos \theta + y \sin \theta)$$



$$BP [p(\theta, t)] (x, y) := \int p_\theta (x \cos \theta + y \sin \theta) d\theta$$

## Backprojection

$$f_{bp}(x, y) = p_\theta(x \cos \theta + y \sin \theta)$$



$$BP [p(\theta, t)] (x, y) := \int p_\theta (x \cos \theta + y \sin \theta) d\theta$$

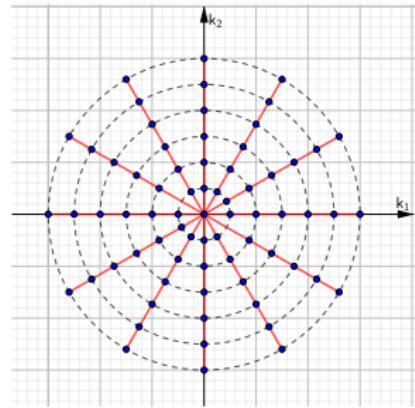
✓  $BP = \mathcal{R}^*$  and computationally efficient

!  $\mathcal{R}^* \mathcal{R} [f] \propto \frac{1}{\|x\|} * f$

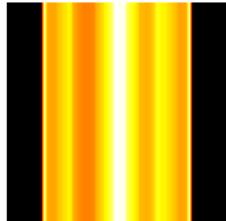
## Backprojection in action



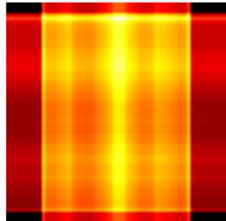
(a) true image



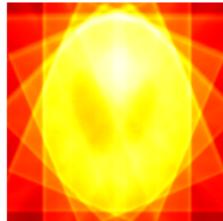
(b) Fourier sampling



(c) 1 angle



(d) 2 angles



(e) 8 angles



(f) 64 angles



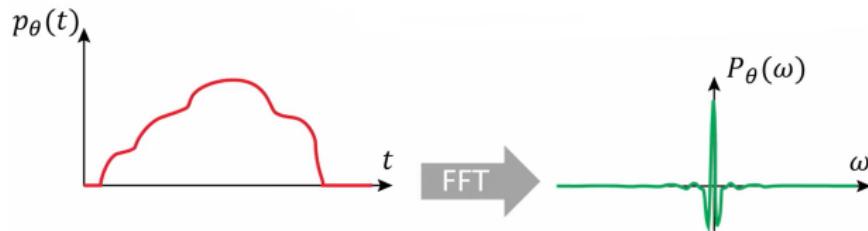
(g) 256 angles

## Filtered backprojection



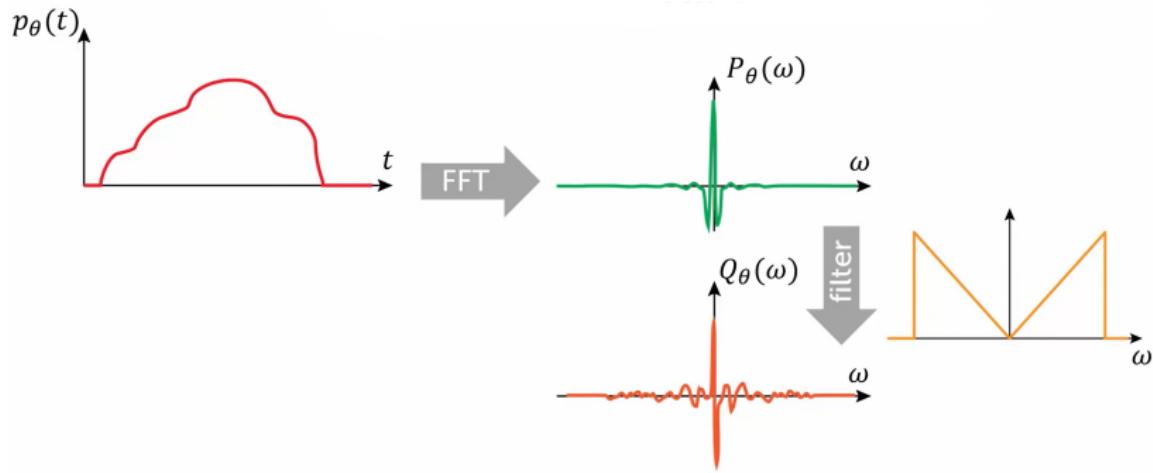
Taken from corresponding video by the ASTRA toolbox team [YouTube](#)

## Filtered backprojection



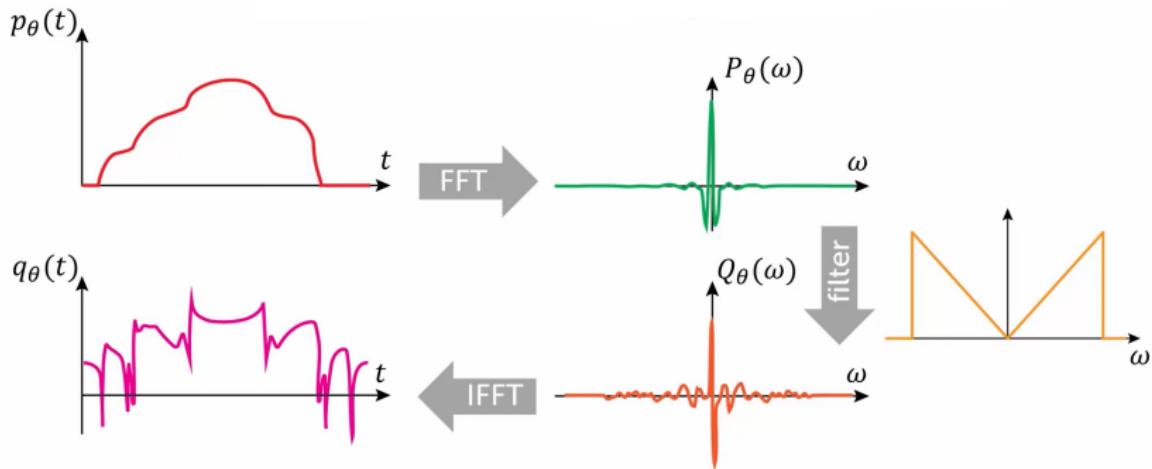
Taken from corresponding video by the ASTRA toolbox team [YouTube](#)

## Filtered backprojection



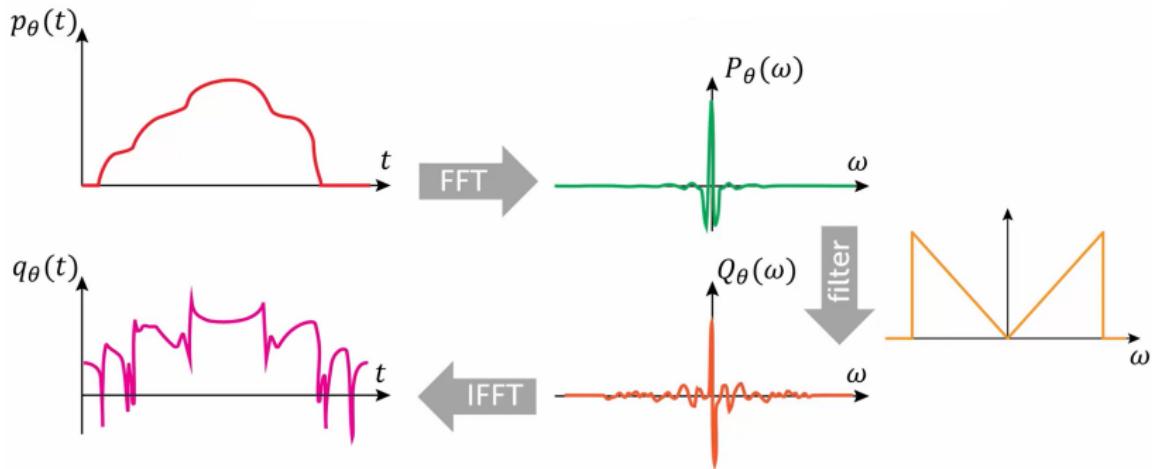
Taken from corresponding video by the ASTRA toolbox team YouTube

## Filtered backprojection



Taken from corresponding video by the ASTRA toolbox team YouTube

## Filtered backprojection

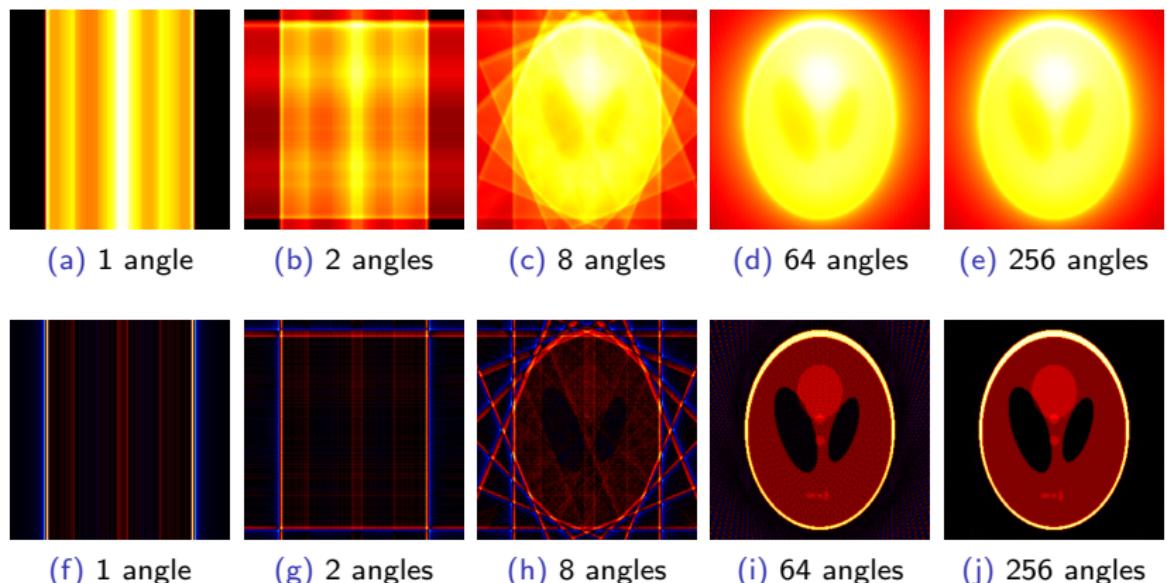


$$\begin{aligned}
 FBP[p(\theta, t)](x, y) &:= BP[q(\theta, t)](x, y) := \int q_\theta(x \cos \theta + y \sin \theta) d\theta \\
 q_\theta(t) &:= \int \mathcal{F}[p_\theta](\omega) |\omega| e^{i2\pi\omega t} d\omega
 \end{aligned}$$

Taken from corresponding video by the ASTRA toolbox team YouTube

## Filtered backprojection in action

It turns out that  $F\text{BP}(\mathcal{R}f) = \mathcal{R}^*\mathcal{H}\mathcal{R}f = f$



Just one more video by Samuli Siltanen: [YouTube](#)

## CT reconstruction methods

Analytical (or direct) methods a la filtered backprojection:

- ✓ efficient to implement and execute
- ! lack of flexibility for unconventional scanning set-ups
- ! severe artifacts for limited / sparse projection data
- ! hard to introduce a-priori knowledge

Algebraic and variational methods (iterative methods):

- ! higher computational cost
- ✓ highly flexible, arbitrary geometries
- ✓ less artifacts for limited / sparse projection data
- ✓ introduction of a-priori knowledge possible

## Algebraic Reconstructions

**Idea:** Find  $f \in \mathcal{C}$  with  $p \approx \mathcal{R}f$  as

$$f = \operatorname{argmin}_{f \in \mathcal{C}} \|\mathcal{R}f - p\|_2^2 \quad ,$$

for instance via projected gradient descent:

$$f^{k+1} = P_{\mathcal{C}}(f^k + \mathcal{R}^*(\mathcal{R}f^k - p))$$

Many variants of this exist such as ART, SART, SIRT, ...

## Variational Reconstructions

$$f = \operatorname{argmin}_{f \in \mathcal{C}} \mathcal{D}(\mathcal{R}f, p) + \lambda \mathcal{J}(f) \quad ,$$

where  $\mathcal{D}$  and  $\mathcal{J}$  are derived from **probabilistic models** for data generation (*likelihood*) and typical images (*prior*), for instance

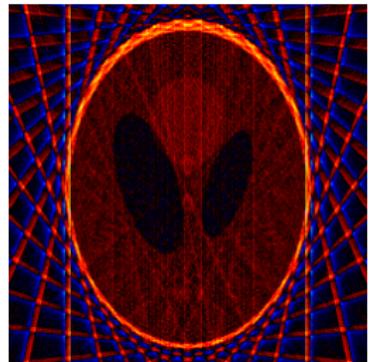
$$\mathcal{D}(\mathcal{R}f, g) := \left\| M^{-1/2} (\mathcal{R}f - p) \right\|_2^2, \quad \mathcal{J}(f) := \|\nabla f\|_1$$

Solution via **iterative optimization schemes** such as proximal gradient descent, primal-dual hybrid gradient, alternating direction method of multipliers, ...

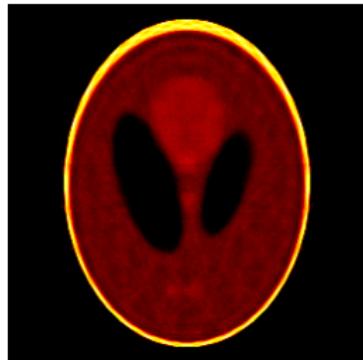
## Iterative methods in action: 15 angles



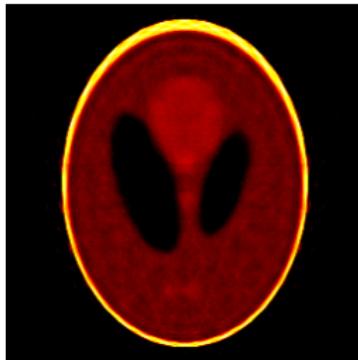
(a) true image



(b) FBP



(c) ART

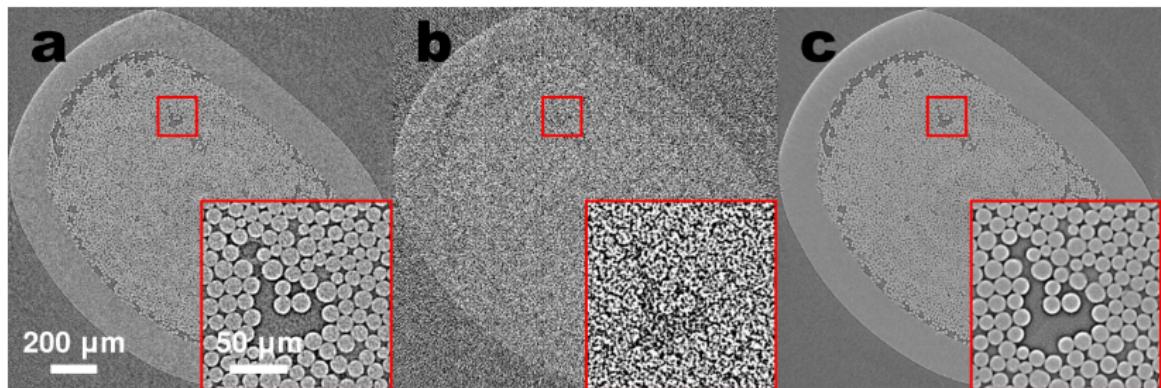


(d) SIRT



(e) TV regularization

## Deep learning for low dose CT image reconstruction



2560x2560 tomography images of fiber composite. Left: 1024 projections, middle/right: 128 projections

-  **D. Pelt, J.A. Sethian, 2018.** Mixed-scale dense network for image analysis, *PNAS* 115 (2) 254-259.

## Some current developments

- Phase contrast X-ray imaging: Exploit phase shift in X-rays caused by material interaction to gain higher soft tissue contrast.
- Dynamic X-ray: Track fast dynamic processes in 3D (4D CT).
- Spectral CT: Use energy resolved detectors to improve analysis of complex materials and tissues.
- Scan adaptation: Make best possible use of given budget of radiation.
- Machine learning: Use deep learning to improve image reconstruction and analysis.

## Break & questions time



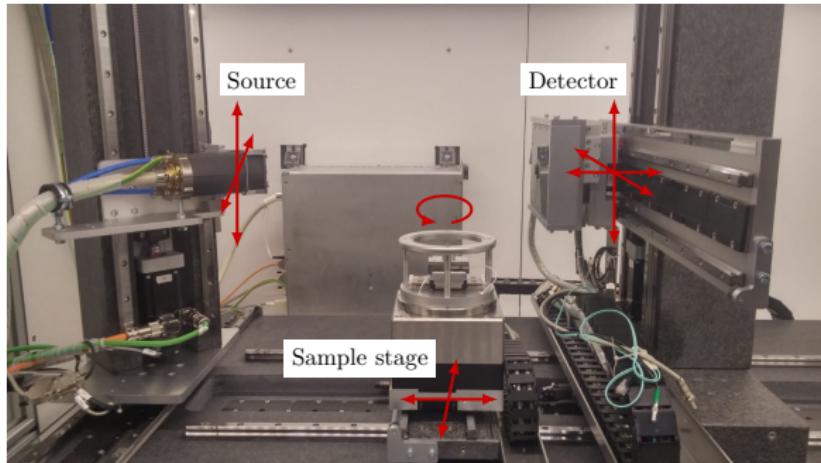
Nick Veasey, VW Camper Van , 2019

## FleX-ray Lab at CWI



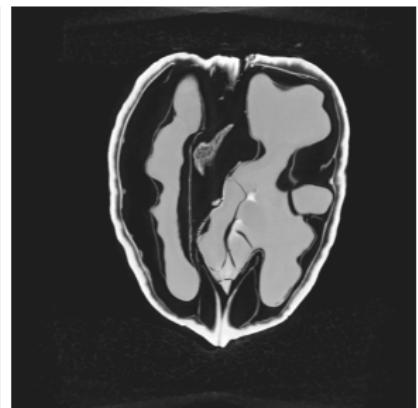
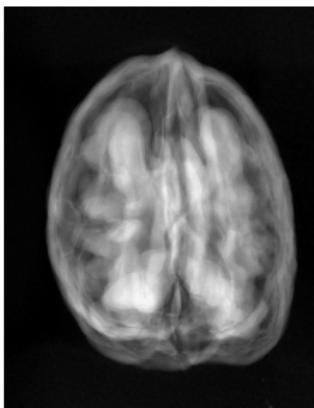
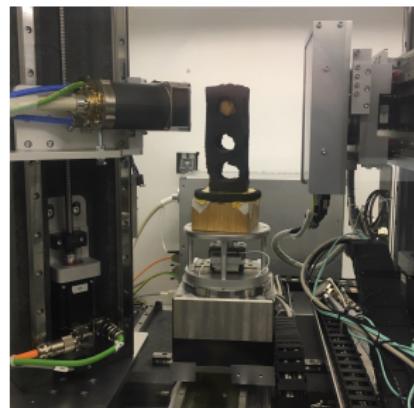
- custom-built, fully-automated, highly flexible
- linked to large-scale computing hardware
- **Aim: Proof-of-concept** experiments directly accessible to mathematicians and computer scientists.
- develop advanced computational techniques for 3D imaging

## FleX-ray Lab at CWI



- custom-built, fully-automated, highly flexible
- linked to large-scale computing hardware
- **Aim: Proof-of-concept** experiments directly accessible to mathematicians and computer scientists.
- develop advanced computational techniques for 3D imaging

## X-Ray Scan of static object

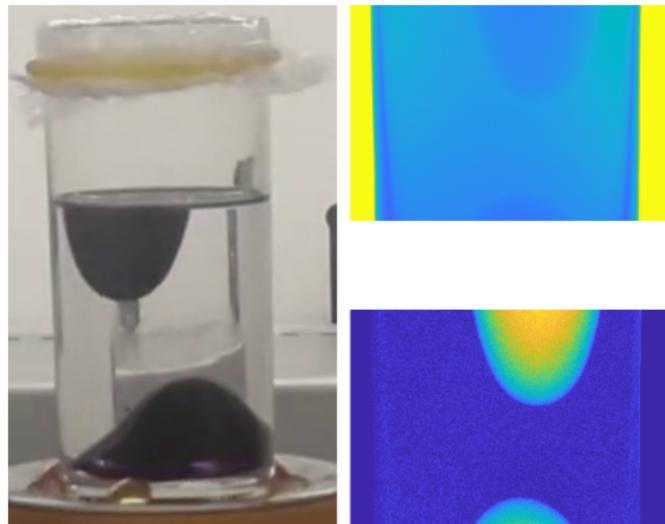


## X-Ray Scan of Dynamic Object



- canonical example of temperature-driven **two-phase flow instability**
- 120 projections per rotation → each projection averaged over  $3^\circ$
- 40ms exposure per projection → 4.8s per rotation

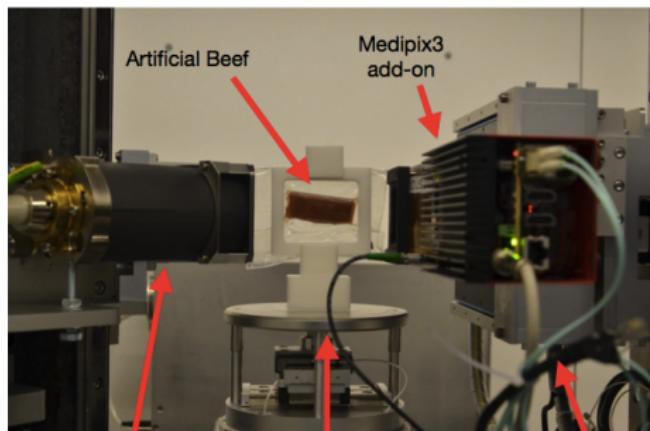
## X-Ray Scan of Dynamic Object



- canonical example of temperature-driven **two-phase flow instability**
- 120 projections per rotation → each projection averaged over 3°
- 40ms exposure per projection → 4.8s per rotation

# Foreign object detection with spectral CT

Experimental setup



Tube  
(point source)

Object mount  
platform

Detector  
(flat panel)

Meat Samples

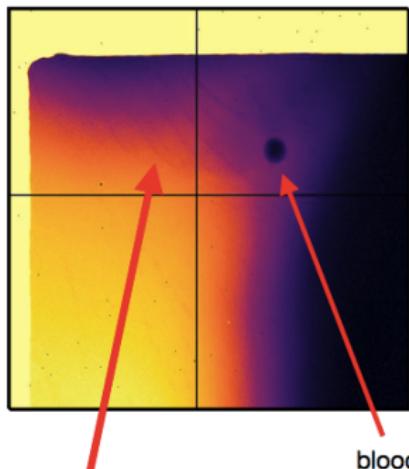


Samples included  
chicken breasts, thighs,  
skin, beef and pork meat  
and artificial meat

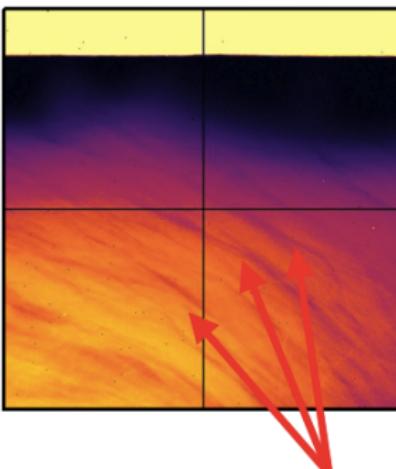
- template for many industry applications
- low quality data, high throughput

## Foreign object detection with spectral CT

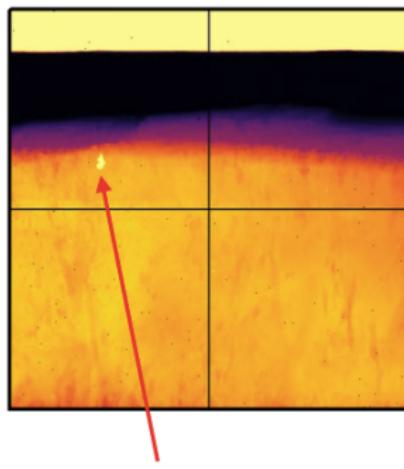
Chicken (outer breast)



Beef



Artificial beef



fat within the  
meat structure

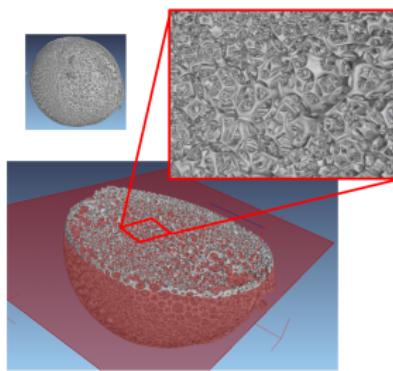
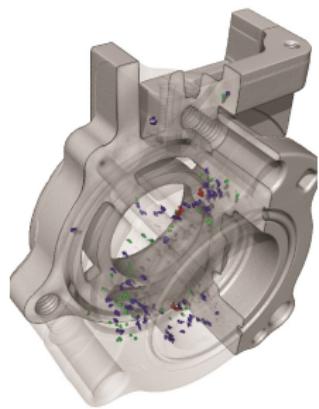
blood

fat within the  
meat structure

external object

- template for many industry applications
- low quality data, high throughput

## Example applications: Industry & security



Airport baggage screening YouTube

## Example applications: Materials science, energy & biology

- Water within porous media  [YouTube](#)
- Internal structure of Arundo donax  [YouTube](#)
- Inside live flying insects – in 3D  [YouTube](#)
- Movie of battery under load  [YouTube](#)
- Metallic foam  [YouTube](#)

## Further reading

-  **T. M. Buzug, 2008.** Computed Tomography - From Photon Statistics to Modern Cone-Beam CT, *Springer-Verlag Berlin Heidelberg*.
-  **G. T. Herman, 2009.** Fundamentals of Computerized Tomography Image Reconstruction from Projections, *Springer-Verlag London*.
-  **F. Natterer, 2001.** The Mathematics of Computerized Tomography, *Society for Industrial and Applied Mathematics*.