List-Coloring Graphs on the Torus

Félix Moreno Peñarrubia

Treball de Final de Grau

Grau en Matemàtiques, FME-UPC
Grau en Enginyeria Informàtica, FIB-UPC

May 2023

Director: Zdeněk Dvořák   *Tutor: Oriol Serra Albó*

Todo list

abstract

keywords and AMS code

Contents

1 Introduction

In this section, we lay out the basic definitions of graph theoretical and topological concepts used in this thesis, as well as the background results which contextualize our research and an outline of the results we have obtained.

1.1 Graphs and Surfaces

1.1.1 Graph Theory Terminology

A *graph* is a pair $(V(G), E(G))$ consisting of a set $V(G)$ and a set $E(G)$ of two-element subsets of $V(G)$.

(connectivity, complete graphs, etc)

1.1.2 Surfaces

A *surface* is a...

(surface classification theorem, orientability, etc)

1.1.3 Embedding Graphs in Surfaces

A graph is embedded in a surface if...

(non-contractible cicles, planarity, dual graph, etc)

references for all this

1.2 Graph Coloring

Problems related to *coloring* are a fundamental part of graph theory. Although there are many variants, the original one and the most important is *vertex coloring*.

Definition 1.1. A *vertex coloring* of a graph G is a function $f : V(G) \to \mathbb{N}$. The vertex coloring is said to be *proper* if $\forall uv \in E(G), f(u) \neq f(v)$.

We think of this as assigning one color to each vertex of the graph, so that adjacent vertices are assigned different colors. This interpretation comes from the origin of the problem in *map coloring*, in which we have to color a political map assigning colors to countries so that neighboring countries are assigned different colors in order to distinguish them. A quantity of interest is the number of colors required for a proper coloring of the graph:

Definition 1.2. A vertex coloring f is said to be a *k-coloring* if $|\Im f| = k$. A graph G is said to be *k-colorable* if it admits a proper k-coloring. The *chromatic number* $\chi(G)$ of a graph G is the minimum k such that G is k-colorable.

In the map coloring context, we study vertex coloring for *planar* graphs. The following remarkable result was the origin of this area of mathematics:

Theorem 1.3 (Four color theorem)**.** *For all planar graphs G, $\chi(G) \leq 4$.*

This theorem, originally conjectured in TODO, was proven by Appel and Haken in TODO. Their proof achieved some notoriety due to use of computers to process a lengthy case analysis.

4ct dates and references

A natural generalization of the above problem is to study the chromatic number of graphs embedded in surfaces other than the plane. The following result, due to Heawood in 1890 [1], generalizes the four color theorem to surfaces other than the plane:

Theorem 1.4 (Heawood)**.** *Let Σ be a surface with Euler genus $g(\Sigma) \geq 1$. Any graph embedded in Σ can be colored with*

$$H(\Sigma) = \left\lfloor \frac{7 + \sqrt{1 + 24g(\Sigma)}}{2} \right\rfloor$$

colors.

We call $H(\Sigma)$ the *Heawood number* of the surface.

Interestingly, this theorem has a much simpler proof than the four color theorem. For its proof, we need the following concept:

Definition 1.5. A graph is *k-degenerate* if every subgraph $H \subseteq G$ has $\delta(H) \leq k$.

Proposition 1.6. *If a graph G is k-degenerate, then it is $(k+1)$-colorable.*

Proof. For each fixed $k \geq 0$, we do induction on the size of $|V(G)|$. For $|V(G)| = 1$, it is clearly true. Assuming it is true for all k-degenerate graphs G' with $|V(G')| < n$, let G be a k-degenerate graph with $|V(G)| = n$. Let v be a vertex of G with degree at most v and consider the graph $G\ \{v\}$. It is also a k-degenerate graph since it is a subgraph of G, so it is $(k+1)$-colorable. We can then extend the coloring to G by choosing a color not in any of its at most k neighbors for v. □

Proof of Heawood's theorem

1.3 List Coloring

Ringel-Youngs

[7]

Critical graphs, Hájos construction

Theorem 1.7 (Thomassen's theorem)**.**

Theorem 1.8 (Thomassen's stronger theorem)**.**

Proof. □

Results in graphs on surfaces, Gallai, Thomassen

In coloring problems, it is useful to consider when a precoloring of a subgraph does not *extend* to the entire graph, that is, there is no coloring of the entire graph under certain constraints which agrees with the coloring of the subgraph.

Definition of List Coloring

Definition 1.9 (Extending)**.** Let G be a graph, $T \subseteq G$ a subgraph, and L a list assignment for G. For an L-coloring ϕ of T, we say that ϕ *extends* to an L-coloring of G if there exists an L-coloring ψ of G s such that $\phi(v) = \psi(v)$ for all $v \in V(T)$.

Thomassen's theorem

Criticality definition. Discuss it

It is also interesting to consider graphs which are critical in this setting. To do so, we use the following definition (from [5]):

Definition 1.10 (T-critical)**.** Let G, T, L be as above. The graph G is *T-critical with respect to L* if for every proper subgraph $G' \subset G$ such that $T \subseteq G'$, there exists an L-coloring of T that extends to an L-coloring of G', but does not extend to an L-coloring of G.

Definition 1.11 (ϕ-critical)**.** Let, G, T, L be as above. The graph G is ϕ-critical for a coloring ϕ of T if ϕ extends to every proper subgraph of G containing T but not to G.

We also have the following lemma from [5]:

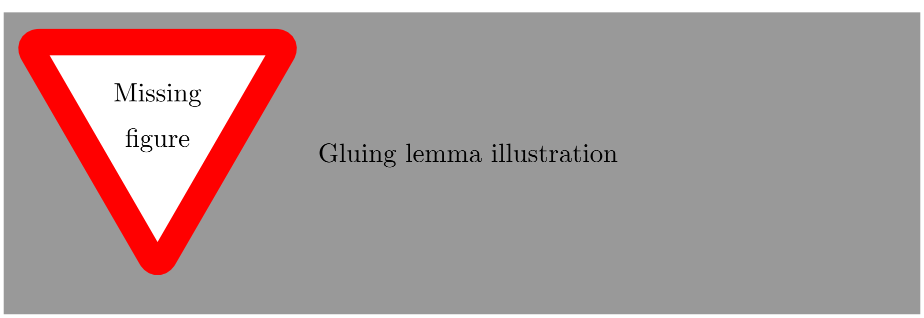
Lemma 1.12. *Let T be a subgraph of a graph G such that G is T-critical with respect to a list assignment L. Let $A, B \subseteq G$ be such that $A \cup B = G$ and $T \subseteq A$. Then $G[V(B)]$ is $A[V(A) \cap V(B)]$-critical.*

Proof. Let $G' = G[V(B)]$ and $S = A[V(A) \cap V(B)]$. If $G' = S$ there is nothing to say, suppose otherwise that $G' \neq S$ and note that therefore G' contains an edge not in S (in fact, all isolated vertices of G' must be in S). Suppose for a contradiction that G' is not S-critical. Then, by taking a maximal proper subgraph that defies the defintion, there exists an edge $e \in E(G') \setminus E(S)$ such that every L-coloring of S that extends to $G' \setminus e$ also extends to G'. Since G is T-critical and $e \notin E(T)$, there exists a coloring $\phi\restriction_T$ of T that extends to an L-coloring ϕ of $G \setminus e$, but does not extend to an L-coloring of G. However, by the choice of e, the restriction $\phi\restriction_S$ extends to an L-coloring ϕ' of G'. Let ϕ'' be such that $\phi''(v) = \phi'(v)\, \forall v \in V(G')$ and $\phi''(v) = \phi(v)\, \forall v \in V(G) \setminus V(G')$. Now, since $A \cup B = G$, ϕ'' is an L-coloring of G extending $\phi\restriction_T$, a contradiction. □

For us, it will be more useful in this form:

Lemma 1.13 (Gluing Lemma)**.** *Let T be a subgraph of a graph G such that G is T-critical with respect to a list assignment L. Let $A, B \subseteq G$ be such that $A \cup B = G$. Then $G[V(B)]$ is $(A[V(A) \cap V(B)] \cup T)$-critical.*

Proof. Apply 1.12 to $A' = A \cup T$ and $B' = B$. □



The reason we decided to name it "Gluing lemma" in this work is that it is useful to visualize the graph G as made of two separate pieces, A and B, which are glued together along $A[V(A) \cap V(B)]$. In our approach we will frequently use the fact that all T-critical graphs can be "decomposed" in this way.

Analogous results for List coloring

1.4 Goals and Results

Section on Goals and Results

2 Critical graphs on the torus

Section header

2.1 An Overview of Postle's Approach

Here we briefly explain Postle's approach in [3] to obtain the result on the finiteness of 6-list-critical graphs in general surfaces, mentioning specially those intermediate results or definitions we will also use in our approach. The results obtained by Postle are very non-explicit in the sense that the (unspecified) constant in the size bounds for the graphs he obtains is extremely large and hence useless for our purpose of finding an explicit characterization. Nevertheless, given that our approach is primarily guided by this work we consider it of interest to provide a exposition of the main argument.

The results developed in [3] in 2012 have been published successively in journal articles afterwards, often with improvements in exposition or in the strength of the result. We refer to the corresponding published article in the discussion of each particular result.

2.1.1 Notation and Terminology

Postle works mainly in a setting similar to the hypothesis of Thomassen's stronger theorem: list assignments L which have list sizes of length at least 5 for interior vertices and at least 3 for exterior vertices with some exceptions. This setting is encapsulated in the concept of *canvas*.

Definition 2.1 (Canvas)**.** We say that (G, S, L) is a *canvas* if G is a connected plane graph with outer walk C, S is a subgraph of C, and L is a list assignment such that $|L(v)| \geq 5 \, \forall v \in V(G) \setminus V(C)$ and $|L(v)| \geq 3 \, \forall v \in V(C) \setminus V(S)$. If S is a path, we say (G, S, L) is a *path-canvas* or a *wedge*. If $S = C$, then (G, C, L) is a *cycle-canvas*.

Note: in some places like [5], the term "canvas" is used for what Postle calls in [3] "cycle-canvas".

We can restate Thomassen's Stronger Theorem in these terms:

Theorem 2.2. *If (G, P, L) is a path-canvas and $|V(P)| \leq 2$, then G is L-colorable.*

Definition 2.3 (Critical canvas)**.** We say that a canvas (G, S, L) is *critical* if it is S-critical with respect to L.

2.1.2 Variations on Thomassen's Condition

Much of the technical work on Postle's thesis relies in a careful study of what happens if one varies the condition on 1.8. One of the most elegant (and also useful) results is the following alternative to Thomassen's Stronger Theorem, originally conjectured by Hutchinson in [2].

Theorem 2.4 (Two Lists of Size Two Theorem [4])**.** *If G is a plane graph with outer cycle C, $v_1, v_2 \in C$ and L is a list assignment with $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$, $|L(v)| \geq 3$ for all $v \in V(C) \setminus \{v_1, v_2\}$, and $|L(v_1)| = |L(v_2)| = 2$, then G is L-colorable.*

Or, in the language of canvases:

Theorem 2.5. *If (G, S, L) is a canvas with $|V(S)| = 2$ and $L(v) \geq 2$ for $v \in S$, then G is L-colorable.*

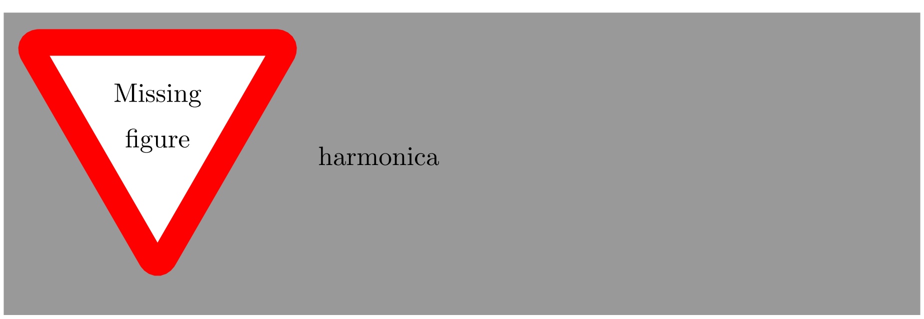
This theorem is not true when one of the two vertices has list of size 1. In fact, Postle characterizes exactly when it fails:

Definition 2.6 (Coloring Harmonica)**.** Let G be a plane graph and L a list assignment for G. Given an edge uv and a vertex w both from the outer face of G, we say that (G, L) is a *coloring harmonica from uv to w* if either:

- G is a triangle with vertex set $\{u, v, w\}$ and $L(u) = L(v) = L(w)$ with $|L(u)| = 2$, or
- There exists a vertex z incident with the outer face of G such that uvz is a triangle in G, $L(u) = L(v) \subseteq L(z)$, $|L(u)| = |L(v)| = 2$, $|L(z)| = 3$, and the pair (G', L') is a coloring harmonica from z to w, where G' is obtained by deleting **one or both** of the vertices u, v and L' is obtained from L by $L'(z) = L(z) \setminus L(u)$ and $L'(x) = L(x)$ for all other vertices $z \neq x \in V(G')$.

Given two vertices u, w in the outer face of G, we say (G, L) is a *coloring harmonica from u to w* if there exist vertices x, y incident with the outer face of G such that uxy is a triangle in G, $|L(u)| = 1$, $L(x) - L(u) = L(y) - L(u)$, $|L(x) - L(u)| = 2$, and (G', L') is a coloring harmonica from xy to w, where G' is obtained from G by removing u, and L' is obtained from L by seting $L'(x) = L'(y) = L(x) - L(u)$ and $L'(z) = L(z)$ for every $z \in V(G') \setminus \{x, y\}$.

We say that (G, L) is a coloring harmonica if it is a coloring harmonica from uv to w or a coloring harmonica from u to w for some u, v, w as specified earlier.



See the example in (reference to figure) (from [6]) for some clarity with respect to this mutually recursive definition. Note that the definition makes it clear that graphs which contain a coloring harmonica as a subgraph are not L-colorable.

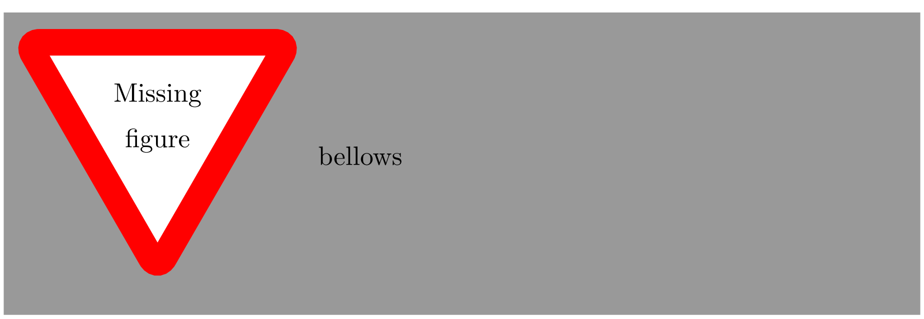
Theorem 2.7. *One List of Size One and One List of Size Two Theorem [6] Let G be a plane graph with outer cycle C, let $p_1, p_2 \in V(C)$, and let L be a list assignment with $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$, $|L(v)| \geq 3$ for all $v \in V(C) \setminus \{p_1, p_2\}$, $|L(p_1)| \geq 1$ and $|L(p_2)| \geq 2$. Then G is L-colorable if and only if the pair (G, L) does not contain a coloring harmonica from p_1 to p_2.*

Studying conditions of the sizes of the lists in the boundary in which the graph is not L-colorable like this one is also useful, because such conditions arise when dealing when reductions and therefore characterizing which are the critical graphs in such settings can give fruitful results.

Thomassen already studied when does the coloring of a path of length 2 not extend:

Definition 2.8 (Bellows)**.** We say that a path-canvas (G, P, L) with $P = p_0p_1p_2$ is a *bellows* (terminology from [3]) or a *generalized wheel* (terminology from [9]) if either:

- G has no interior vertices and its edge set consists of the edges of the outer cycle plus all edges from p_1 to vertices of the outer cycle. In this case, we say that (G, P, L) is a *fan*.
- G has one interior vertex u and its edge set consists of the edges of the outer cycle plus all edges from u to vertices of the outer cycle. In this case, we say that (G, P, L) is a *turbofan*.
- G can be formed by gluing two smaller bellows from the edges p_1p_2 and p_0p_1 respectively.



Theorem 2.9 ([9], Theorem 3)**.** *If $T = (G, P, L)$ is a path-canvas with path length 2, then G is L-colorable unless T has a bellows as a subcanvas.*

Postle studies when the coloring of two paths of length 1 does not extend. He finds the following obstruction:

Definition 2.10 (Accordion)**.** We say that a canvas $T = (G, P_1 \cup P_2, L)$ with P_1, P_2 distinct paths of length 1 is an *accordion* with *ends* P_1, P_2 if T is a bellows with $P_1 \cup P_2$ path of length 2 or T is the gluing of two smaller accordions $T_1 = (G_1, P_1 \cup U, L)$ with ends P_1, U and $T_2 = (G_2, P_2 \cup U, L)$ with ends U, P_2 along a chord $U = u_1u_2$ where $|L(u_1)|, |L(u_2)| \leq 3$.

The main result he obtains is that if the two paths are sufficiently far apart, then the graph contains a proportionally large accordion or a coloring harmonica as a subgraph.

Theorem 2.11 (Bottleneck Theorem, loosely stated)**.** *If $T = (G, P \cup P_0, L)$ is a canvas with P, P_0 distinct edges of C with $d(P, P_0) \geq 14$, then either there exists an L-coloring of G, or there exists a subcanvas $(G_0, U_1 \cup U_2, L)$ of T where $d_{G_0}(U_1, U_2) = \Omega(d_G(P, P_0))$ which is an accordion or a coloring harmonica.*

This result, along with coloring and structural properties of accordions and harmonicas, is often used as a technical lemma when proving the following results.

2.1.3 Linear Bound on Critical Cycle-Canvases

2.1.4 The Two Precolored Triangles Theorem

2.1.5 Cylinder-Canvases

2.1.6 Hyperbolicity

2.2 Critical Graphs on the Torus for (usual) Vertex Coloring

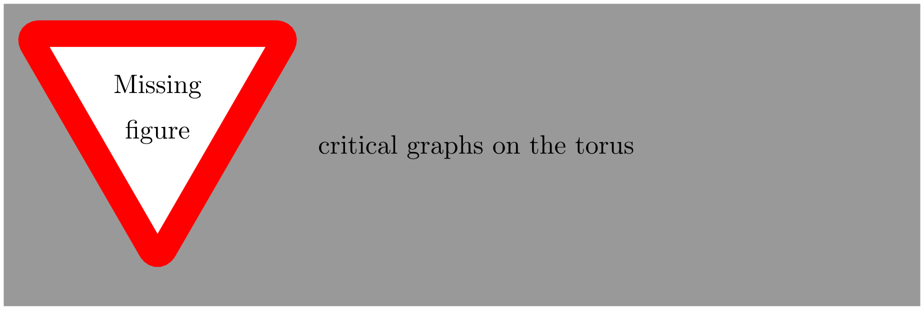
2.2.1 The Critical Graphs

Subsection introduction

Theorem 2.12 ([8])**.** *A graph G embeddable on the torus is 5-colorable if and only if it does not contain the following subgraphs:*

- K_6.
- $C_3 + C_5$.
- $K_2 + H_7$, *where H_7 is the* Moser spindle, *the graph obtained by applying the Hajós construction to a pair of K_4.*
- T_{11}, *where T_{11} is a triangulation of the torus with 11 vertices.*

Where + denotes the join of two graphs: their disjoint union with all pairs of vertices from different graphs joined by edges.



Discuss L-critical graphs

2.2.2 An Overview of Thomassen's Approach

2.3 Our Approach

3 Generation of Critical Graphs

3.1 Critical Canvases

Theorem 3.1 (Cycle chord or tripod theorem)**.** *hi*

3.2 Critical Wedges

3.3 Critical Biwedges with Bounded Distance

4 Criticality Testing

4.1 Useful Theorems

4.2 Coloring Heuristics

4.3 The Alon-Tarsi Method

4.4 The Recursive Alon-Tarsi Method

4.5 Criticality Verification

5 Approaches to the Two Precolored Triangles Theorem

5.1 Canvas Strangulation

5.2 The Forbidden 3-3 Reduction

5.3 Criticality Strength

6 Conclusions and Further Study

References

[1] P. J. Heawood. "Map colour theorem". In: *Quarterly Journal of Mathematics* 24 (1890), pp. 332–338.

[2] J. P. Hutchinson. "On list-coloring extendable outerplanar graphs". In: *ARS MATHEMATICA CONTEMPORANEA* 5.1 (2012), pp. 175–188.

[3] Luke Postle. "5-list-coloring graphs on surfaces". PhD thesis. Georgia Institute of Technology, 2012.

[4] Luke Postle and Robin Thomas. "Five-list-coloring graphs on surfaces I. Two lists of size two in planar graphs". In: (2014). DOI: `10.48550/ARXIV.1402.1813`.

[5] Luke Postle and Robin Thomas. "Five-list-coloring graphs on surfaces II. A linear bound for critical graphs in a disk". In: (2015). DOI: `10.48550/ARXIV.1505.05927`. URL: `https://arxiv.org/abs/1505.05927`.

[6] Luke Postle and Robin Thomas. "Five-list-coloring graphs on surfaces III. One list of size one and one list of size two". In: *Journal of Combinatorial Theory, Series B* 128 (2018), pp. 1–16. ISSN: 0095-8956. DOI: `https://doi.org/10.1016/j.jctb.2017.06.004`.

[7] C. Thomassen. "Every Planar Graph Is 5-Choosable". In: *Journal of Combinatorial Theory, Series B* 62.1 (1994), pp. 180–181. ISSN: 0095-8956. DOI: `https://doi.org/10.1006/jctb.1994.1062`.

[8] C. Thomassen. "Five-Coloring Graphs on the Torus". In: *Journal of Combinatorial Theory, Series B* 62.1 (1994), pp. 11–33. ISSN: 0095-8956. DOI: `https://doi.org/10.1006/jctb.1994.1052`.

[9] Carsten Thomassen. "Exponentially many 5-list-colorings of planar graphs". In: *Journal of Combinatorial Theory, Series B* 97.4 (2007), pp. 571–583. ISSN: 0095-8956. DOI: `https://doi.org/10.1016/j.jctb.2006.09.002`.