List-Coloring Graphs on the Torus

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Todo list

abstract

keywords and AMS code

Contents

fix function restrictions

1 Introduction

In this section, we lay out the basic definitions of graph theoretical and topological concepts used in this thesis, as well as the background results which contextualize our research and an outline of the results we have obtained.

1.1 Graphs and Surfaces

1.1.1 Graph Theory Terminology

A *graph* is a pair $(V(G), E(G))$ consisting of a set $V(G)$ and a set $E(G)$ of two-element subsets of $V(G)$.

(connectivity, complete graphs, etc)

1.1.2 Surfaces

A *surface* is a...

(surface classification theorem, orientability, etc)

1.1.3 Embedding Graphs in Surfaces

A graph is embedded in a surface if...

(non-contractible cicles, planarity, dual graph, etc)

references for all this

1.2 Graph Coloring

Problems related to *coloring* are a fundamental part of graph theory. Although there are many variants, the original one and the most important is *vertex coloring*.

Definition 1.1. A *vertex coloring* of a graph G is a function $f : V(G) \to \mathbb{N}$. The vertex coloring is said to be *proper* if $\forall uv \in E(G), f(u) \neq f(v)$.

We think of this as assigning one color to each vertex of the graph, so that adjacent vertices are assigned different colors. This interpretation comes from the origin of the problem in *map coloring*, in which we have to color a political map assigning colors to countries so that neighboring countries are assigned different colors in order to distinguish them. A quantity of interest is the number of colors required for a proper coloring of the graph:

Definition 1.2. A vertex coloring f is said to be a *k-coloring* if $|\Im f| = k$. A graph G is said to be *k-colorable* if it admits a proper k-coloring. The *chromatic number* $\chi(G)$ of a graph G is the minimum k such that G is k-colorable.

In the map coloring context, we study vertex coloring for *planar* graphs. The following remarkable result was the origin of this area of mathematics:

Theorem 1.3 (Four color theorem)**.** *For all planar graphs G, $\chi(G) \leq 4$.*

This theorem, originally conjectured in TODO, was proven by Appel and Haken in TODO. Their proof achieved some notoriety due to use of computers to process a lengthy case analysis.

4ct dates and references

A natural generalization of the above problem is to study the chromatic number of graphs embedded in surfaces other than the plane. The following result, due to Heawood in 1890 [1], generalizes the four color theorem to surfaces other than the plane:

Theorem 1.4 (Heawood)**.** *Let Σ be a surface with Euler genus $g(\Sigma) \geq 1$. Any graph embedded in Σ can be colored with*

$$H(\Sigma) = \left\lfloor \frac{7 + \sqrt{1 + 24g(\Sigma)}}{2} \right\rfloor$$

colors.

We call $H(\Sigma)$ the *Heawood number* of the surface.

Interestingly, this theorem has a much simpler proof than the four color theorem. For its proof, we need the following concept:

Definition 1.5. A graph is *k-degenerate* if every subgraph $H \subseteq G$ has $\delta(H) \leq k$.

Proposition 1.6. *If a graph G is k-degenerate, then it is $(k+1)$-colorable.*

Proof. For each fixed $k \geq 0$, we do induction on the size of $|V(G)|$. For $|V(G)| = 1$, it is clearly true. Assuming it is true for all k-degenerate graphs G' with $|V(G')| < n$, let G be a k-degenerate graph with $|V(G)| = n$. Let v be a vertex of G with degree at most v and consider the graph $G \ \{v\}$. It is also a k-degenerate graph since it is a subgraph of G, so it is $(k+1)$-colorable. We can then extend the coloring to G by choosing a color not in any of its at most k neighbors for v. □

Proof of Heawood's theorem

1.3 List Coloring

Ringel-Youngs

Critical graphs, Hájos construction

figure explaining pictorial notation for list sizes

Results in graphs on surfaces, Gallai, Thomassen

[7]

Definition of List Coloring

Theorem 1.7 (Thomassen's theorem)**.** *For all planar graphs G, $\chi_\ell(G) \leq 5$.*

Thomassen's theorem introduction

Thomassen actually proved a stronger theorem:

Theorem 1.8 (Thomassen's stronger theorem)**.** *Let G be a plane (embedded) graph with outer walk C, and let L be a list assignment satisfying:*

- $|L(v)| \geq 5$ *for all internal vertices.*
- $|L(v)| \geq 3$ *for all* $v \in V(C) \setminus \{x, y\}$ *where* x, y *are a pair of adjacent vertices.*
- $|L(x)| = |L(y)| = 1$, $L(x) \neq L(y)$.

Then G is L-colorable.

Proof. Suppose we have a counterexample with minimal $|V(G)|$. It is clear that for $|V(G)| \leq 3$ the theorem is true, so we assume $|V(G)| \geq 4$.

First we prove that G is 2-connected. Assume it is not. Then, we have two subgraphs $G_1, G_2 \subset G$ with $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \{v\}$, with v a cutvertex. Assume, without loss of generality, that $x, y \in G_1$. By minimality of G, G_1 is $L\restriction_{G_1}$ colorable. Let ϕ_1 be a coloring of G_1. Now, let w be a neighbor of v in the outer face of G_2 and consider the list assignment L' for G_2 for which $L'(v) = \{\phi_1(v)\}$, $L'(w) = c$ for some arbitrary $c \in L(w)$, and $L'(u) = L(u)\ \forall u \in V(G_2) \setminus \{v, w\}$. Note that G_2 and L' satisfy the hypothesis of the theorem. Therefore, by minimality of our counterexample, G_2 has a L'-coloring ϕ_2. But now note that, since $\phi_1(v) = \phi_2(v)$, the coloring $\phi(u) = \phi_i(u)$ if $u \in V(G_i)$ is well-defined and is an L-coloring of G, contradiction.

Hence, G is 2-connected and the outer walk C is a cycle. Now we prove that there is no chord in C. The proof is similar to the above argument. Assume there is a chord vw. Then, we have two subgraphs $G_1, G_2 \subset G$ with $G_1 \cup G_2 = G$, $G_1 \cap G_2 = \{v, w\}$ and $x, y \subset G_1$. By minimality of G, G_1 has an L-coloring ϕ_1. If we set $L'(v) = \{\phi_1(v)\}$, $L'(w) = \{\phi_1(w)\}$, and $L'(u) = L(u)$ for all other vertices $u \in V(G_2) \setminus \{v, w\}$, then G_2 is L'-colorable and a coloring of G can be constructed.

Now we have that G has no chords or cutvertices. Let u be the neighbor of y in the outer face other than x and let v be the neighbor of u in the outer face other than y (possibly $v = x$). Let $\{c_1, c_2\} \subseteq L(u)\ L(y)$. Now, let G' be the graph obtained by removing u from G and let L' be the list assignment for G' in which $\{c_1, c_2\}$ are removed from the lists of the neighbors of u other than v. G' satisfies the hypothesis of the theorem: every vertex in the outer face has list size at least 3, since each of those vertices is either a vertex previously in the outer face of G all of which have their previous lists (the only neighbors of u in the outer face are u and y, since G has no chords), or a previously interior vertex, which has had at most 2 of its ≥ 5 colors removed. So G' has an L'-coloring, which can be extended to an L-coloring of G by coloring u with one of c_1 or c_2 (whichever is not in use by v), contradiction.

$\square$

Criticality definition. Discuss it

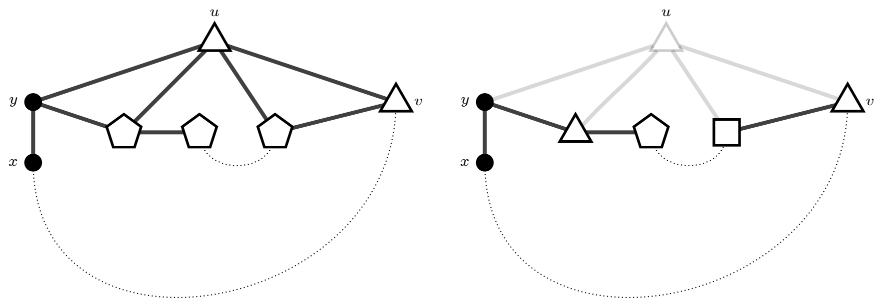


Figure 1: Illustration of Thomassen's reduction. At most 2 colors are erased from the lists of u's neighbors.

In coloring problems, it is useful to consider when a precoloring of a subgraph does not *extend* to the entire graph, that is, there is no coloring of the entire graph under certain constraints which agrees with the coloring of the subgraph.

Definition 1.9 (Extending)**.** Let G be a graph, $T \subseteq G$ a subgraph, and L a list assignment for G. For an L-coloring ϕ of T, we say that ϕ *extends* to an L-coloring of G if there exists an L-coloring ψ of G s such that $\phi(v) = \psi(v)$ for all $v \in V(T)$.

It is also interesting to consider graphs which are critical in this setting. To do so, we use the following definition (from [5]):

Definition 1.10 (T-critical)**.** Let G, T, L be as above. The graph G is *T-critical with respect to L* if for every proper subgraph $G' \subset G$ such that $T \subseteq G'$, there exists an L-coloring of T that extends to an L-coloring of G', but does not extend to an L-coloring of G. If the list assignment L is clear from context, we just say *T-critical.*

Definition 1.11 (ϕ-critical)**.** Let, G, T, L be as above. The graph G is ϕ-critical for a coloring ϕ of T if ϕ extends to every proper subgraph of G containing T but not to G.

In a way similar to the general notion of criticality, we have that graphs for which colorings of T do not extend contain a non-trivial T-critical subgraph:

Lemma 1.12. *Let G be a graph, T a subgraph, and L a list assignment for G. If there is an L-coloring ϕ of T that does not extend to G, then G contains a subgraph H with $T \subsetneq H$ which is ϕ-critical, and hence also T-critical with respect to $L\restriction_H$.*

Proof. Let ϕ be the coloring of T that does not extend and let H be a minimal subgraph of G for which ϕ does not extend. Now note that H is ϕ-critical by construction. $\square$

Lemma 1.13. *Let G be a graph, T a subgraph, L a list assignment for G, and $H \supseteq T$ a subgraph of G which is minimal with respect to the following property: for every L-coloring ϕ of T that extends to H, ϕ also extends to G. Then H is T-critical.*

Proof. Suppose not. Then, H contains a proper subgraph H' so that every L-coloring ϕ that extends to H' also extends to H and hence to G. But then H' is a smaller subgraph with that property, contradiction. $\square$

We also have the following lemma from [5]:

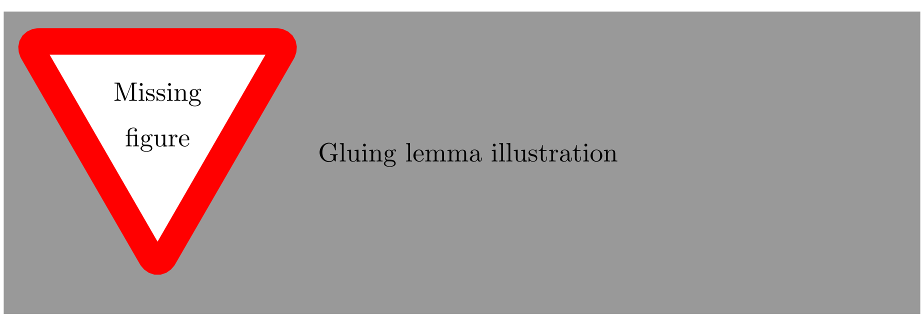
Lemma 1.14. *Let T be a subgraph of a graph G such that G is T-critical with respect to a list assignment L. Let $A, B \subseteq G$ be such that $A \cup B = G$ and $T \subseteq A$. Then $G[V(B)]$ is $A[V(A) \cap V(B)]$-critical.*

Proof. Let $G' = G[V(B)]$ and $S = A[V(A) \cap V(B)]$. If $G' = S$ there is nothing to say, suppose otherwise that $G' \neq S$ and note that therefore G' contains an edge not in S (in fact, all isolated vertices of G' must be in S). Suppose for a contradiction that G' is not S-critical. Then, by taking a maximal proper subgraph that defies the defintion, there exists an edge $e \in E(G') \setminus E(S)$ such that every L-coloring of S that extends to $G' \setminus e$ also extends to G'. Since G is T-critical and $e \notin E(T)$, there exists a coloring $\phi \restriction_T$ of T that extends to an L-coloring ϕ of $G \setminus e$, but does not extend to an L-coloring of G. However, by the choice of e, the restriction $\phi \restriction_S$ extends to an L-coloring ϕ' of G'. Let ϕ'' be such that $\phi''(v) = \phi'(v)\, \forall v \in V(G')$ and $\phi''(v) = \phi(v)\, \forall v \in V(G) \setminus V(G')$. Now, since $A \cup B = G$, ϕ'' is an L-coloring of G extending $\phi \restriction_T$, a contradiction. $\square$

For us, it will be more useful in this form: check if this is the best phrasing

Lemma 1.15 (Gluing Lemma)**.** *Let T be a subgraph of a graph G such that G is T-critical with respect to a list assignment L. Let $A, B \subseteq G$ be such that $A \cup B = G$. Then $G[V(B)]$ is $(A[V(A) \cap V(B)] \cup T)$-critical.*

Proof. Apply 1.14 to $A' = A \cup T$ and $B' = B$. $\square$



The reason we decided to name it "Gluing lemma" in this work is that it is useful to visualize the graph G as made of two separate pieces, A and B, which are glued together along $A[V(A) \cap V(B)]$. In our approach we will frequently use the fact that all T-critical graphs can be "decomposed" in this way. Analogous results for List coloring

1.4 Goals and Results

Section on Goals and Results

2 Critical graphs on the torus

Section header

2.1 An Overview of Postle's Approach

Here we briefly explain Postle's approach in [3] to obtain the result on the finiteness of 6-list-critical graphs in general surfaces, mentioning specially those intermediate results or definitions we will also use in our approach. The results obtained by Postle are very non-explicit in the sense that the (unspecified) constant in the size bounds for the graphs he obtains is extremely large and hence useless for our purpose of finding an explicit characterization. Nevertheless, given that our approach is primarily guided by this work we consider it of interest to provide a exposition of the main argument.

The results developed in [3] in 2012 have been published successively in journal articles afterwards, often with improvements in exposition or in the strength of the result. We refer to the corresponding published article in the discussion of each particular result.

2.1.1 Notation and Terminology

Postle works mainly in a setting similar to the hypothesis of Thomassen's stronger theorem: list assignments L which have list sizes of length at least 5 for interior vertices and at least 3 for exterior vertices with some exceptions. This setting is encapsulated in the concept of *canvas*.

Definition 2.1 (Canvas)**.** We say that (G, S, L) is a *canvas* if G is a connected plane graph with outer walk C, S is a subgraph of C, and L is a list assignment such that $|L(v)| \geq 5\, \forall v \in V(G) \setminus V(C)$ and $|L(v)| \geq 3\, \forall v \in V(C) \setminus V(S)$. If S is a path, we say (G, S, L) is a *path-canvas* or a *wedge*. If $S = C$ and C is a cycle, then (G, C, L) is a *cycle-canvas*.

Note: in some places like [5], the term "canvas" is used for what Postle calls in [3] "cycle-canvas".

We can restate Thomassen's Stronger Theorem in these terms:

Theorem 2.2. *If (G, P, L) is a path-canvas and $|V(P)| \leq 2$, then G is L-colorable.*

Definition 2.3 (Critical canvas)**.** We say that a canvas (G, S, L) is *critical* if it is S-critical with respect to L.

It is interesting to study in which circumstances can a critical canvas contain chords or cutvertices.

determine if I am going to use essential definition

Definition 2.4.

Lemma 2.5. *If $T = (G, S, L)$ is a critical canvas, then:*

1. *Every cutvertex of G is essential.*
2. *Every chord of the outer walk of G is essential.*

Proof. □

2.1.2 Variations on Thomassen's Condition

Much of the technical work on Postle's thesis relies in a careful study of what happens if one varies the condition on 1.8. One of the most elegant (and also useful) results is the following strengthening to Thomassen's Stronger Theorem, originally conjectured by Hutchinson in [2].

Theorem 2.6 (Two Lists of Size Two Theorem [4])**.** *If G is a plane graph with outer cycle C, $v_1, v_2 \in C$ and L is a list assignment with $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$, $|L(v)| \geq 3$ for all $v \in V(C) \setminus \{v_1, v_2\}$, and $|L(v_1)| = |L(v_2)| = 2$, then G is L-colorable.*

Or, in the language of canvases:

Theorem 2.7. *If (G, S, L) is a canvas with $|V(S)| = 2$ and $L(v) \geq 2$ for $v \in S$, then G is L-colorable.*

This theorem is not true when one of the two vertices has list of size 1. In fact, Postle characterizes exactly when it fails:

Definition 2.8 (Coloring Harmonica)**.** Let G be a plane graph and L a list assignment for G. Given an edge uv and a vertex w both from the outer face of G, we say that (G, L) is a *coloring harmonica from uv to w* if either:

- G is a triangle with vertex set $\{u, v, w\}$ and $L(u) = L(v) = L(w)$ with $|L(u)| = 2$, or
- There exists a vertex z incident with the outer face of G such that uvz is a triangle in G, $L(u) = L(v) \subseteq L(z)$, $|L(u)| = |L(v)| = 2$, $|L(z)| = 3$, and the pair (G', L') is a coloring harmonica from z to w, where G' is obtained by deleting **one or both** of the vertices u, v and L' is obtained from L by $L'(z) = L(z) \setminus L(u)$ and $L'(x) = L(x)$ for all other vertices $z \neq x \in V(G')$.

Given two vertices u, w in the outer face of G, we say (G, L) is a *coloring harmonica from u to w* if there exist vertices x, y incident with the outer face of G such that uxy is a triangle in G, $|L(u)| = 1$, $L(x) - L(u) = L(y) - L(u)$, $|L(x) - L(u)| = 2$, and (G', L') is a coloring harmonica from xy to w, where G' is obtained from G by removing u, and L' is obtained from L by seting $L'(x) = L'(y) = L(x) - L(u)$ and $L'(z) = L(z)$ for every $z \in V(G') \setminus \{x, y\}$.

We say that (G, L) is a coloring harmonica if it is a coloring harmonica from uv to w or a coloring harmonica from u to w for some u, v, w as specified earlier.

See the example in (reference to figure) (from [6]) for some clarity with respect to this mutually recursive definition. Note that the definition makes it clear that graphs which contain a coloring harmonica as a subgraph are not L-colorable.

Theorem 2.9. *One List of Size One and One List of Size Two Theorem [6] Let G be a plane graph with outer cycle C, let $p_1, p_2 \in V(C)$, and let L be a list assignment with $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$, $|L(v)| \geq 3$ for all $v \in V(C) \setminus \{p_1, p_2\}$, $|L(p_1)| \geq 1$ and $|L(p_2)| \geq 2$. Then G is L-colorable if and only if the pair (G, L) does not contain a coloring harmonica from p_1 to p_2.*

Studying conditions of the sizes of the lists in the boundary in which the graph is not L-colorable like this one is also useful, because such conditions arise when dealing when reductions and therefore characterizing which are the critical graphs in such settings can give fruitful results.

Thomassen already studied when does the coloring of a path of length 2 not extend:

Definition 2.10 (Bellows)**.** We say that a path-canvas (G, P, L) with $P = p_0p_1p_2$ is a *bellows* (terminology from [3]) or a *generalized wheel* (terminology from [10]) if either:

- G has no interior vertices and its edge set consists of the edges of the outer cycle plus all edges from p_1 to vertices of the outer cycle. In this case, we say that (G, P, L) is a *fan*.
- G has one interior vertex u and its edge set consists of the edges of the outer cycle plus all edges from u to vertices of the outer cycle. In this case, we say that (G, P, L) is a *turbofan*.
- G can be formed by gluing two smaller bellows from the edges p_1p_2 and p_0p_1 respectively.

Theorem 2.11 ([10], Theorem 3)**.** *If $T = (G, P, L)$ is a path-canvas with path length 2, then G is L-colorable unless T has a bellows as a subcanvas.*

Postle studies when the coloring of two paths of length 1 does not extend. He finds the following obstruction:

Definition 2.12 (Accordion)**.** We say that a canvas $T = (G, P_1 \cup P_2, L)$ with P_1, P_2 distinct paths of length 1 is an *accordion* with *ends* P_1, P_2 if T is a bellows with $P_1 \cup P_2$ path of length 2 or T is the gluing of two smaller accordions $T_1 = (G_1, P_1 \cup U, L)$ with ends P_1, U and $T_2 = (G_2, P_2 \cup U, L)$ with ends U, P_2 along a chord $U = u_1u_2$ where $|L(u_1)|, |L(u_2)| \leq 3$.

The main result he obtains is that if the two paths are sufficiently far apart, then the graph contains a proportionally large accordion or a coloring harmonica as a subgraph.

Theorem 2.13 (Bottleneck Theorem, loosely stated)**.** *If $T = (G, P \cup P_0, L)$ is a canvas with P, P_0 distinct edges of C with $d(P, P_0) \geq 14$, then either there exists an L-coloring of G, or there exists a subcanvas $(G_0, U_1 \cup U_2, L)$ of T where $d_{G_0}(U_1, U_2) = \Omega(d_G(P, P_0))$ which is an accordion or a coloring harmonica.*

This result, along with coloring and structural properties of accordions and harmonicas, is often used as a technical lemma when proving the following results.

2.1.3 Linear Bound on Critical Cycle-Canvases

Postle proves the following result:

Theorem 2.14 ([5])**.** *Let G be a plane graph with outer cycle C and L a 5-lis-assignment for G. Let H be a minimal subgraph of G such that every L-coloring of C that extends to an L-coloring of H also extends to an L-coloring of G. Then H has at most $19|V(C)|$ vertices.*

Or, equivalently stated in the language of critical canvases:

Theorem 2.15. *If (G, C, L) is a critical cycle-canvas, then $|V(G)| \leq 19|V(C)|$.*

The equivalence of the two statements is given by 1.13. This result is interesting in its own right because by 1.15, all faces of a T-critical graph which do not separate vertices from T are in fact critical cycle-canvases, and therefore what the result tells us is that for such graphs there is only finitely many kinds of faces that can appear for each given cycle length. This gives us a lot of information of how critical graphs look like.

A first observation that can be made is that critical cycle-canvases (in which C is indeed a simple cycle) are 2-connected, so each face is bounded by a cycle:

Lemma 2.16. *If (G, C, L) is a critical cycle-canvas, then it is 2-connected.*

Proof. If G is not 2-connected, then there exist subgraphs A, B such that $A \cup B = G$ with $|V(A) \cap V(B)| \leq 1$ and $|V(B) \setminus V(A)| \geq 1$. Assume $C \subseteq A$ and apply 1.15 to get that B is $A[V(A) \cap V(B)]$-critical, contradicting 1.8. □

The key result in Postle's proof of the linear bound for cycles is the following theorem about the structure of critical cycle-canvases:

Theorem 2.17 (Cycle Chord or Tripod Theorem)**.** *If (G, C, L) is a nontrivial critical cycle-canvas, then either*

1. *C has a chord in G, or*
2. *there exists a vertex $v \in V(G)\ V(C)$ with at least three neighbors on C such that at most one of the faces of $G[\{v\} \cup V(C)]$ includes a vertex or edge of G.*

think if I should include proof of CCTT

Using this result, Postle carefully examines what happens near the boundary cycle in order to define some quantities related to sums of lengths of faces and proves that certain inequalities with those quantites are mantained when adding tripods in critical canvases.

2.1.4 The Two Precolored Triangles Theorem

Next, Postle proves the following theorem:

Theorem 2.18. *There exists d such that the following holds. Let G be a planar graph and T_1, T_2 triangles in G at distance at least d. Let L be a 5-list-assignment of G. Then, every L-coloring of $T_1 \cup T_2$ extends to an L-coloring of G.*

The value of d that Postle obtains is not explicitly stated, but it is on the order of 100. However, we conjecture that 4 or 5 suffices.

Conjecture 2.19. *Let G be a planar graph and T_1, T_2 triangles in G at distance at least 5. Let L be a 5-list-assignment of G. Then, every L-coloring of $T_1 \cup T_2$ extends to an L-coloring of G.*

The argument that Postle uses to prove his result is as follows. First, he proves that one can precolor a path between the two triangles in such a way that, when deleting the path and deleting the corresponding colors from the lists of neighboring vertices, all remaining non-precolored vertices have lists of size at least 3. The proof of this begins with the simple observation that each vertex outside a shortest path has at most 3 neighbors inside the path. Using planarity properties, a shortest path can be found so that it can be colored in such a way that the vertices with 3 neighbors inside the path only see two different colors from their lists.

After precoloring and deleting the path between the two triangles, a canvas $(G, P_1 \cup P_2, L)$ is obtained. If there was a precoloring of the triangles that did not extend, then the canvas contains a critical canvas, and by 2.13 it contains a proportionally long accordion or harmonica. Postle proves that this (together with some technical details related to how the path between the triangles was chosen) implies that in the original graph there must be a long chain of separating triangles so that the graph between each separating triangle pertains to one of three very specific types, which he calls tetrahedral, octahedral or hexadecahedral bands.

check if I defined separating triangle in introduction

Finally, he proves that for a sufficiently long chain of this type, any precoloring of the innermost and outermost triangles extends to the whole chain. This proves the theorem, because of the following observation:

Proposition 2.20. *Let G be a plane graph with L a list assignment, T_1, T_2 two facial triangles with T_1 bounding the infinite face of G, and T_1', T_2' two triangles such that T_1' is a separating triangle between T_1 and T_2' and T_2' is a separating triangle between T_1' and T_2. Denote by $G[T_1', T_2']$ the subgraph comprised between the two triangles T_1', T_2'. If there exists some L-coloring of $T_1 \cup T_2$ that does not extend to G, then there exists some L-coloring of $T_1' \cup T_2'$ that does not extend to $G[T_1', T_2']$.*

Proof. By 1.8, the coloring on T_1 extends to $G[T_1, T_1']$ and the coloring on T_2 extends to $G[T_2', T_2]$. The coloring of $T_1' \cup T_2'$ given by this extensions can not extend to $G[T_1', T_2']$ by the assumption that the original coloring of $T_1 \cup T_2$ did not extend to G. □

2.1.5 Hyperbolicity

hyperbolicity and cylinder canvases section

2.2 Critical Graphs on the Torus for (usual) Vertex Coloring

In this section we discuss the result from Thomassen in [8] that characterizes the critical graphs for 5-coloring (not 5-list-coloring) on the torus.

2.2.1 The Critical Graphs

Theorem 2.21 ([8])**.** *A graph G embeddable on the torus is 5-colorable if and only if it does not contain the following subgraphs:*

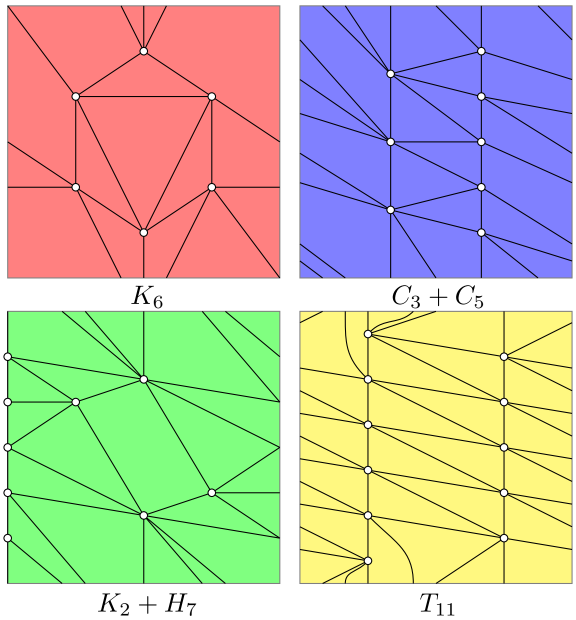


Figure 2: 6-critical graphs embedded on the torus.

- K_6.
- $C_3 + C_5$.
- $K_2 + H_7$, *where* H_7 *is the* Moser spindle, *the graph obtained by applying the Hajós construction to a pair of* K_4.

check if I explained Hajós construction in original

- T_{11}, *where* T_{11} *is a triangulation of the torus with* 11 *vertices.*

Where + *denotes the join of two graphs: their disjoint union with all pairs of vertices from different graphs joined by edges.*

If a graph is not 5-colorable, it is not 5-list-colorable, so all graphs that contain any of the above subgraphs are not 5-list-colorable. We conjecture that this characterizes the 5-list-colorable graphs on the torus too:

Conjecture 2.22. *A graph* G *embeddable on the torus is* 5*-list-colorable if and only if it does not contain the following subgraphs:* K_6, $C_3 + C_5$, $K_2 + H_7$, T_{11}.

This means that those are the minimal 6-list-critical graphs on the torus. Note that there may be additional 6-list-critical graphs embeddable on the torus, but what we are conjecturing is that they all contain those subgraphs. For example:

Observation 2.23. K_7 *is* 6*-list-critical.*

Proof. Consider the following 5-list-assignment for K_7: $L(v_1) = L(v_2) = L(v_3) = L(v_4) = L(v_5) = \{1,2,3,4,5\}$, $L(v_6) = L(v_7) = \{1,2,3,4,6\}$. K_7 is not L-colorable, since there are only 6 available colors. But any subgraph is L-colorable. Let's give a coloring ϕ for $K_7 \setminus v_iv_j$. If $i, j \leq 5$, then setting $\phi(v_i) = \phi(v_j) = 5$ and $\phi(v_7) = 6$ leaves 4 vertices to be colored with 4 colors. If $i \leq 5$ and $j \geq 6$, then setting $\phi(v_i) = \phi(v_j) = 1$, $\phi(v_{13-j}) = 6$ leaves 4 vertices to be colored with 4 colors. If $\{i, j\} = \{6, 7\}$, then $\phi(v_i) = \phi(v_j) = 6$ leaves 5 vertices to be colored with 5 colors.

Hence, K_7 is L-critical for a 5-list-coloring L, and is therefore 6-list-critical.

□

2.2.2 An Overview of Thomassen's Approach

Thomassen's article where he characterizes the graphs on the torus ([8]) predates his result on finitely many 6-critical graphs for all surfaces ([9]). For the characterization of 6-critical graphs on the torus, he only uses elementary, relatively straighforward arguments that work on specifically in the torus. We briefly summarize his approach here in order to discuss which arguments can be reused for the list-coloring case.

First, Thomassen considers the case when the minimum degree is at least 6. He obtains this result:

Theorem 2.24. *Let G be a graph embedded on the torus with $\delta(G) \geq 6$. Then G is 5-colorable unless $G = K_7$ or $G = T_{11}$.*

We will discuss later how to arrive at this result, because this is the part of the proof that can be adapted for list-coloring. But let us first describe Thomassen's argument for general graphs.

He assumes a minimum counterexample G_0 to 2.21 (the counterexample has minimum number of vertices, maximum number of edges restricted to that, and some other assumptions about details we will not discuss here). By the previous result, there must be a vertex $v_0 \in V(G_0)$ with degree ≤ 5, and the degree of v_0 is in fact equal to 5 by minimality of the counterexample.

Consider two vertices $x, y \in N(v_0)$ which are not adjacent (if all the vertices of $N(v_0)$ were adjacent, then G_0 would contain K_6, a contradiction). Let G_{xy} be the graph obtained from $G_0 \setminus v_0$ by identifying the vertices x and y. G_{xy} can be embedded in the torus by modifying the embedding of G_0. If G_{xy} were 5-colorable, then we would have a 5-coloring of G_0 by assigning the same color to x and y and coloring v_0 with a color not appearing in its 5 neighbors. Hence, G_{xy} is not 5-colorable and by minimality of our counterexample it contains K_6, $C_3 + C_5$, $K_2 + H_7$ or T_{11}.

The above argument works for all pairs x, y of non-adjacent vertices in $N(v_0)$, so potentially we can have many different obstructions for each of the corresponding G_{xy} subgraphs. But we can prove that, by minimality, there can not be much else in G_0 apart from these obstructions arising from all the G_{xy} subgraphs. More precisely:

Proposition 2.25. *For any non-adjacent $x, y \in N(v_0)$, let G'_{xy} a copy of K_6, $C_3 + C_5$, $K_2 + H_7$ or T_{11} in G_{xy}, and let G''_{xy} be the induced subgraph of G_{xy} by the vertex set of G'_{xy}. Then G_0 consists of v_0, $N(v_0)$, the edges between vertices of $\{v_0\} \cup N(v_0)$, and the union over all non-adjacent $x, y \in N(v_0)$ of the graph obtained from G''_{xy} by splitting the contracted vertex into x and y.*

Proof. We will prove that the graph described above, which is a subgraph of G_0, is not 5-colorable. This means, by the assumptions of minimality of vertices and maximality of edges, that G_0 is in fact equal to that subgraph.

If the subgraph had a 5-coloring, then two non-adjacent vertices x, y of $N(v_0)$ would have the same color. But by then identifying the two vertices we can get a 5-coloring of G''_{xy}, which contains the non-5-colorable subgraph G'_{xy}, contradiction.

□

Note that, since the maximum number of vertices in a critical graph is 11, this means that G_0 has at most $(11-1) \cdot \binom{5}{2} + 6 = 106$ vertices, and hence what remains is a finite problem.

Thomassen uses some more arguments to narrow down the remaining possibilities for G_0, but we can already see an important point of failure of this argument for list-coloring: in the proof of 2.24, it is used that a necessary and sufficient condition for a coloring of $G_0 \setminus v_0$ to extend to v_0 is that two neighbors of v_0 have the same color. In list coloring, this condition is not necessary. So we cannot conclude that the minimum counterexample is the union of the graphs induced by the obstructions in G_{xy} and the argument breaks down here.

2.2.3 6-Regular 6-Critical Graphs

As we said before, Thomassen's argument for graphs with minimum degree 6 can be reused for list-coloring. This is because these graphs are very restricted, and therefore their structure can be completely characterized and a 5-(list)-coloring can be explicitly exhibited for the ones that are colorable. The basic result is this.

Proposition 2.26. *If a graph G embedded on the torus has $\delta(G) \geq 6$, then:*

1. *G is 6-regular.*

2. *G is a triangulation of the torus.*

Proof. We apply Euler's formula: let V, E, F be the number of vertices, edges and faces in the embedding, respectively. We have that $\delta(G) \geq 6 \implies V \leq \frac{1}{3}E$ with equality iff G is 6-regular, and $F \leq \frac{2}{3}E$ with equality iff G is a triangulation. Then $0 = V - E + F \leq \frac{1}{3}E - E + \frac{2}{3}E = 0$, so we have equality on both inequalities. □

Using the proposition above, Thomassen then proves the following:

Proposition 2.27 (3.2 in [8]). *Let G be a 6-regular graph on the torus. If G contains a vertex v, such that $\{v\} \cup N(v)$ induces a nonplanar graph, then $G = K_7$ or G is obtained from K_8 or K_9 by deleting the edges of a 1-regular or 2-regular subgraph.*

The study of 6-regular graphs on the torus without vertices whose neighborhood induces a nonplanar graph was already done by Thomassen in his previous paper [**thomassentilings**], in the context of finding all tilings of the torus in order to prove a conjecture by Babai about vertex-transitive graphs.

finish 6-regular graphs section

see if it is worth it to include, maybe it is nontrivial to actually prove 5-list-colorability

2.3 Our Approach

3 Generation of Critical Graphs

In this section we describe algorithms for processing and generating critical canvases via computer search.

3.1 Representation of Canvases

The first step is deciding how to represent canvases in our algorithms. Recall that a canvas T is a tuple (G, S, L) where G is a plane graph, S is a subgraph of the outer face and L is a list assignment for G satisfying some conditions. Here, in each algorithm we will usually be working with one particular family of canvases at a time, for example cycle-canvases or path-canvases with a fixed size of cycle or path, so the information about the subgraph S can be "implicit" in each different representation for each different algorithm instead of working with a general representation that allows all canvases. Also, in some scenarios we will be working with conditions on the list assignment L which are different from the ones in the definition of canvas. In this section we intend to just expose some general ideas about how the representation of graphs in this context can be done, which will be afterwards applied in different scenarios.

The most important thing to state is that we will not be interested in storing the list assignment L at all. This is because there is a significant combinatorial explosion in the number of list assignments to be considered and we are interested in the graphs themselves, not the list assignments. Also, most of the results we will be using such as 1.15 or 2.17 are directly related to subgraphs and not list assignments, and while they are in theory stated with respect to a fixed list assignment it is more useful to not consider the list assignment at all.

Thus, when we generate all critical canvases (G, S, L), what we will actually be doing is generate all pairs (G, S) such that there exists some list assignment L so that (G, S, L) is a critical canvas. In some scenarios, we will also be interested in storing the prescribed size of the list assignment for each vertex: that is, we will be storing a tuple (G, S, f) with $f : V(G) \to \mathbb{N}$ so that we will only be considering list assignments L with $|L(v)| = f(v)$, but other than that we will not store information about the actual list assignment.

Plane graph canonization

3.2 Generation of Critical Cycle-Canvases

Our algorithm for the generation of critical cycle-canvases is based on 2.17. This theorem says that every critical cycle-canvas can either be decomposed into two smaller critical cycle-canvases through a chord in the outer face, it can be decomposed into a "tripod", a vertex v with at least 3 neighbors in C, and a smaller critical cycle-canvas contained in the only nonempty face incident with v. In these decompositions, it is possible that instead of a smaller critical canvas we get an empty canvas, which is technically not critical.

check that all definitions of critical canvases are consistent with this

This implies that we can generate all critical cycle-canvases from smaller cycle-canvases by gluing cycle-canvases through outer face edges to get a canvas with a chord, or by adding a tripod to the outside of a cycle-canvas. We then have to check whether the resulting canvas is indeed critical, since the decomposition into two smaller critical cycle-canvases is a necessary but not sufficient condition for criticality. We will see how to do this in Section 4. .

references for sections

The three cases for generation of critical cycle-canvases

figure for generation of critical cycle-canvases and references to figure

If we are generating cycle-canvases with cycle length ℓ, then a chord partitions the cycle-canvas into two cycle-canvases of length a, b with $a, b \geq 3$ and $a + b = \ell + 2$ (see figure (a)), so $a, b \leq \ell - 1$ and therefore if we have generated all cycle-canvases with cycle length $< \ell$ we can generate all cycle-canvases with cycle length ℓ with a chord. In the case of adding a tripod, though, if the vertex v of the tripod is adjacent to only three adjacent vertices in the outer face, then the smaller cycle-canvas has the same cycle length as the larger cycle-canvas (see figure (c)).

In order to resolve this, what we do is first generate all the cycle-canvases obtained from cycle-canvases with smaller cycle size (as in figure (a) and (b)), enqueue the resulting critical canvases, and then process the canvases from the queue and add tripods to three consecutive vertices in all possible ways, enqueueing the new critical cycle-canvases that are found. Here is the description of the algorithm:

Algorithm 1: Generation of Critical Cycle-Canvases.

```
/* Generate critical canvases of cycle size ℓ, including
empty one                                                  */
function generateCriticalCycleCanvases(ℓ)
    for i = 3, ..., ℓ − 1 do
        S_i ← generateCriticalCycleCanvases(i);
    end
    S ← {emptyCycle(ℓ)};
    for a = 3, ..., ℓ − 1 do
        b ← ℓ − a + 2;
        for G_1 ∈ S_a do
            for G_2 ∈ S_b do
                T ← fuseChordSet(G_1, G_2);
                ; /* Set of cycle-canvases obtained by fusing G_1
                 and G_2 along outer cycle edges in all possible
                 ways */
                for G ∈ T do
                    if G ∉ S AND isCritical(G) then
                        S ← S ∪ {G};
                    end
                end
            end
        end
    end
    for k = 3, ..., ℓ − 1 do
        for G_1 ∈ S_k do
            T ← addTripodSet(G_1, ℓ − k + 3, 3);
            ;     /* Set of cycle-canvases obtained by adding a
             tripod with 3 neighbors in the outer face to get a
             cycle-canvas of length ℓ in all possible ways */
            for G ∈ T do
                if G ∉ S AND isCritical(G) then
                    S ← S ∪ {G};
                end
            end
        end
    end
    Q ← Queue(S);
    while Q is not empty do
        G_1 ← first(Q);
        dequeue(Q);
        T ← addTripodSet(G_1, 3, 3);
        for G ∈ T do
            if G ∉ S AND isCritical(G) then
                S ← S ∪ {G};
                enqueue(Q, G);
            end
        end
    end
    return S;
end
```


fix algorithm size

Note that we only need to add tripods with 3 adjacent neighbors since vertices with a larger number of neighbors in the outer face can be obtained by first adding chords and then adding finally adding a tripod with 3 neighbors. However, often we are interested in just generating chordless critical canvases. In that case, we do need to add tripods of all sizes. The modified algorithm for chordless critical cycle-canvases is the following:

Algorithm 2: Generation of Chordless Critical Cycle-Canvases.

```
/* Generate critical canvases of cycle size ℓ, including
empty one                                                  */
function generateCriticalCycleCanvases(ℓ)
    for i = 3,...,ℓ−1 do
        S_i ← generateCriticalCycleCanvases(i);
    end
    S ← {emptyCycle(ℓ)};
    for k = 3,...,ℓ−1 do
        for j = 3,...,ℓ−k+3 do
            for G_1 ∈ S_k do
                T ← addTripodSet(G_1, ℓ−k+3, j);
                for G ∈ T do
                    if G ∉ S AND isCritical(G) then
                        S ← S ∪ {G};
                    end
                end
            end
        end
    end
    Q ← Queue(S);
    while Q is not empty do
        G_1 ← first(Q);
        dequeue(Q);
        T ← addTripodSet(G_1, 3, 3);
        for G ∈ T do
            if G ∉ S AND isCritical(G) then
                S ← S ∪ {G};
                enqueue(Q, G);
            end
        end
    end
    return S;
end
```

3.3 Generation of Critical Wedges

Generation of critical wedges: chord or tripod theorem, etc

4 Criticality Testing

4.1 Useful Theorems

4.2 Coloring Heuristics

4.3 The Alon-Tarsi Method

4.4 Recursive Colorability Testing

Algorithm 3: Recursive Colorability Testing.

```
function containsColorableSubgraph(G)
    if G is empty then
        return false;
    end
    if alonTarsi(G) then
        return true;
    end
    H ← minimalNonColorable(G);
    return containsColorableSubgraph(G \ H);
end
function minimalNonColorable(G)
    for v ∈ V(G) do
        if not alonTarsi(removeVertex(G,v)) then
            return minimalNonColorable(removeVertex(G,v));
        end
    end
    return G;
end
```

4.5 Criticality Verification

5 Approaches to the Two Precolored Triangles Theorem

5.1 Canvas Strangulation

5.2 The Forbidden 3-3 Reduction

5.3 Criticality Strength

6 Conclusions and Further Study

fix bibliography (not arxiv version, extraneous urls, etc

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