Quantum Computing SoSe 2025 – Exercise Sheet 12

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1 Density operators of qubits

1. A qubit in a pure state can be represented by a state vector

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle \tag{1}$$

What is the corresponding density operator $\hat{\rho}$ for this state? Show that it can be written as

$$\hat{\rho} = \frac{1}{2}(\hat{\mathbb{1}} + \vec{r}.\hat{\vec{\sigma}}) \tag{2}$$

where $\hat{\vec{\sigma}} = (\hat{\sigma_x}, \hat{\sigma_y}, \hat{\sigma_z}) = (\hat{X}, \hat{Y}, \hat{Z})$ are the usual Pauli operators and $\vec{r} = (r_x, r_y, r_z) \in \mathbb{R}^3$. Express r_x, r_y, r_z in terms of θ and φ and show that $|\vec{r}| = 1$.

Hint: we have done some steps in one of the lectures.

2. The density matrix of a qubit in a mixed state can be written as an ensemble of density matrices of pure states, as $\hat{\rho}_{mixed} = \sum_i p_i \hat{\rho}_i$, where the $\hat{\rho}_i$ represent pure states and $\sum_i p_i = 1$. Show that $\hat{\rho}_{mixed}$ can be expressed as

$$\hat{\rho} = \frac{1}{2}(\hat{\mathbb{1}} + \vec{r}.\hat{\vec{\sigma}}) \tag{3}$$

where $|\vec{r}| < 1$.

3. Consider now a system of two qubits in the Bell state

$$|\Psi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{4}$$

(a) What is the density operator ρ_{AB} for this two-qubit state?

(b) What is the trace of ρ_{AB}^2? What does this tell us about the purity of the state?

(c) The state of qubit A alone can be characterized by the reduced density operator ρ_A, obtained by taking the partial trace of ρ_{AB} over qubit B. Calculate ρ_A and show that it represents a mixed state.

(d) Calculate the Von Neumann Entanglement entropy associated with ρ_A.

Thus, although the state of the joint system of two qubits is a pure state, that is, it is known exactly, the first qubit is in a mixed state, that is, a state about which we apparently do not have maximal knowledge. This strange property, that the joint state of a system can be completely known, yet a subsystem be in mixed states, is another hallmark of quantum entanglement.

(e) Suppose now that the two-qubit system is in a *separable* state

$$|\Phi\rangle_{AB} = |\phi\rangle_A \otimes |\chi\rangle_B \,.$$

i. Show that the reduced density operator ρ_A of system A alone is a pure state.

ii. Calculate the Von Neumann Entanglement entropy associated with ρ_A.

iii. Comment

4. The *trace distance* is a measure for quantifying how close two quantum states ρ and ρ' are. It is defined as

$$D(\rho, \rho') = \frac{1}{2}\mathrm{tr}|\rho - \rho'| \tag{5}$$

where $|A| \equiv \sqrt{A^\dagger A}$.

(a) What is the trace distance between

$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| \qquad \text{and} \qquad \rho' = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1| \qquad ?$$

(b) A good way of getting a feel for the trace distance is to understand it for the special case of a qubit, in the Bloch sphere representation. Suppose ρ and ρ' have respective Bloch vectors $\vec{r}$ and $\vec{r}'$. That is

$$\hat{\rho} = \frac{1}{2}(\hat{\mathbb{1}} + \vec{r}.\hat{\vec{\sigma}}) \qquad \text{and} \qquad \hat{\rho}' = \frac{1}{2}(\hat{\mathbb{1}} + \vec{r}'.\hat{\vec{\sigma}}) \tag{6}$$

Show that the trace distance $D(\rho, \rho')$ between the two single qubit states is equal to one half the ordinary Euclidean distance between them on the Bloch sphere.