Quantum Computing SoSe 2025 – Exam

1 Faster-than-light communication?

A useful application of the reduced density operator is the analysis of quantum teleportation that we studied during the course. We remind that quantum teleportation is a procedure for sending quantum information from Alice to Bob (or vice versa), given that Alice and Bob share an EPR pair (Bell state)

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \tag{1}$$

and have a classical communication channel.
More precisely, Alice was able to send a qubit in a superposition state $|\psi\rangle = a|0\rangle + b|1\rangle$ to Bob, using their shared entanglement. At first sight it appears as though teleportation can be used to do faster-than-light communication, which is impossible according to the theory of relativity. What prevents it is the need for Alice to communicate her measurement result to Bob. The formalism of the reduced density operator allows us to make this rigorous, as will be shown below.

After she entangled her qubit $|\psi\rangle$ with the Bell state $|\Phi^+\rangle_{AB}$, the quantum state of the three qubits was:

$$\begin{aligned} |\Psi\rangle \quad = \quad & \frac{1}{2}|00\rangle \otimes (a|0\rangle + b|1\rangle) + \frac{1}{2}|01\rangle \otimes (a|1\rangle + b|0\rangle) \\ & + \frac{1}{2}|10\rangle \otimes (a|0\rangle - b|1\rangle) + \frac{1}{2}|11\rangle \otimes (a|1\rangle - b|0\rangle) \end{aligned} \tag{2}$$

where the two left qubits are in Alice's possession, while the right qubit in a superposition state is with Bob.

1. How can we characterize the state of Bob's qubit alone?
2. What entanglement measure can quantify how entangled is Bob's qubit with Alice's qubits? Calculate this measure.
3. Now Alice measures her two qubits in the computational basis. What happens to Bob's qubit? Is it possible for Bob to infer the state $|\psi\rangle$, without having Alice communicate the result of her measurement to him?

2 Quantum Simulation of the Fermi-Hubbard model

The Fermi-Hubbard model is one of the simplest models of interacting fermions (electrons) that exhibit non-trivial correlations, and is used in condensed matter physics to study collective physical phenomena, such as Mott insulators [1].

The model consists of a system of interacting electrons distributed on a lattice with L sites, where each lattice site can be occupied by zero, one or two electrons (one with spin up and one with spin down). In the present study we will consider a one-dimensional lattice, as shown in the top panel of Fig. 1. The Hamiltonian of the system is given by

$$\hat{H} = -J \sum_{l=0}^{L-2} \sum_{s=\uparrow,\downarrow} \left(a^{\dagger}_{\ell,s} a_{\ell+1,\, s} + a^{\dagger}_{\ell+1,\, s} a_{\ell,\, s} \right) + U \sum_{\ell=0}^{L-1} \hat{n}_{\ell,\uparrow} \hat{n}_{\ell,\downarrow} \tag{3}$$

where ℓ denotes the electron sites, and $s = \uparrow, \downarrow$ denotes the spin projection of the electron. The number operator $\hat{n}_{\ell,s} \equiv a^{\dagger}_{\ell,s} a_{\ell,\, s}$ counts the number of electrons at site ℓ with spin projection s.

This Hamiltonian shows that each electron is subject to a kinetic energy term (or "hopping term") J which allows them to hop to neighbouring sites, and an on-site potential $U > 0$, mimicking a Coulomb repulsion, which discourages electrons to occupy the same sites.

In the present problem we will restrict ourselves to the "half-filling" approximation, i.e. to the case where the number of electrons N_e is equal to the number of sites: $\boldsymbol{N_e = L}$.

2.1 Mapping of the system to qubits

The states i of single electrons are labeled by both the site index and the spin projection: $i \equiv (\ell, s)$, and can be occupied by zero or one electron.

We will consider a simple Jordan-Wigner (JW) mapping where each electron state i is mapped onto a qubit. We will use the convention where the state $|0\rangle_i$ of the qubit corresponds to the electron state i being empty, while $|1\rangle_i$ corresponds to i being occupied. This mapping requires $N_q = 2L$ qubits, as shown in the bottom panel of Fig. 1.

1. Using the JW mapping of the creation and annihilation operators $a^{\dagger}_i$ and a_j to qubit operators, write the Hamiltonian in Eq. (3) in terms of tensor products of Pauli operators $\hat{X}$, $\hat{Y}$, $\hat{Z}$ and identity $\hat{\mathbb{1}}$.

[1] A Mott insulator is a material in which strong electron-electron correlations prevent it from conducting electricity

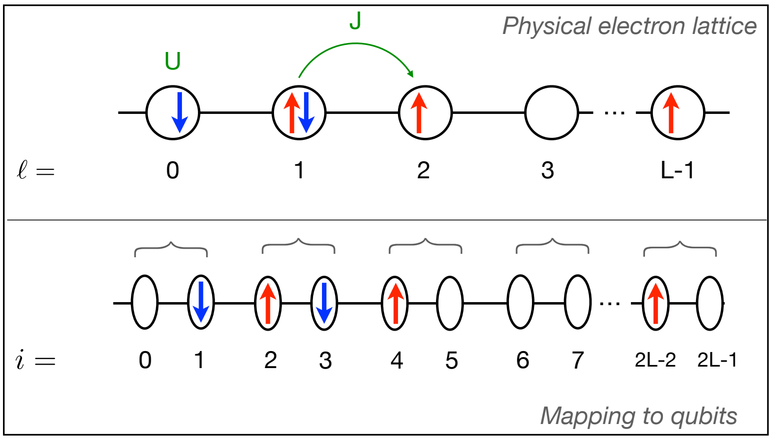

Fig. 1: Top panel: Physical lattice of interacting electrons with L sites. Bottom panel: Mapping of the lattice to a system of $2L$ qubits.

2.2 Time-Evolution

Preliminaries: In the following we will fix the number of sites to $L = 4$, which means we have $N_e = 4$ electrons and $N_q = 8$ qubits.
For the time evolution we will consider a simple initial state. For example:

$$|\Psi_0\rangle = |00100111\rangle \tag{4}$$

which contains one fully occupied site (site 0) (one pair of electron $\uparrow, \downarrow$) and two unpaired electrons occupying different sites (site 1 and 2) [2].
Finally, we will consider the following set of parameters: $J = 1.0$, $U = 3.0$.

Trotterized Time-Evolution:

1. What is the expression of the time-evolution operator $\hat{U}_{LO}(t)$ approximated with a leading-order (LO) Trotter expansion?

2. Using Qiskit, program a quantum circuit which implements such time-evolution operator at LO Trotter, starting from initial state $|\Psi_0\rangle$.

3. We want to measure the survival probability of the initial state $P(t) = |\langle\Psi_0|\Psi(t)\rangle|^2$ obtained from this Trotterized time-evolution. Implement the corresponding measurement of $P(t)$ on Qiskit.

4. The measurement procedure performed n_{shots} times gives us the expectation value of the measurement corresponding to $P(t)$. If we repeat this a few times we can

[2] Here we used Qiskit's ordering convention where qubit (0) is on the right of the ket while qubit $L-1$ is on the left

deduce the mean value $\bar{P}(t)$ and standard deviation $\sigma_P(t)$ for each time t, which allows for estimating the uncertainties (error bars) of the obtained results.

Plot $P(t)$ with error bars as a function of time t, for $t \in [0, 4]$, for a given number of Trotter steps N_{Trott} and shots n_{shots} (for example, $N_{Trott} = 3$ and $n_{shots} = 500$).

5. In order to estimate the effect of Trotterization, we would like now to compare the above results to those obtained with *exact* time evolution of the system. Compute the exact time evolution operator[3] $\hat{U}(t)$, and repeat the steps above. Plot on the same figure the survival probability $P(t)$ obtained with LO Trotter and exact time evolution, and study the changes as you vary the number of Trotter steps.

Charge and Spin densities:

One way to study how information gets spread onto the lattice through electron dynamics, is to study the behaviour of the charge and spin densities [4]. For a given state $|\Psi\rangle$, these quantities are defined for each site ℓ as

$$\rho_\ell^{charge} = \langle\Psi|\hat{n}_{\ell,\uparrow}|\Psi\rangle + \langle\Psi|\hat{n}_{\ell,\downarrow}|\Psi\rangle \ , \tag{5}$$

$$\rho_\ell^{spin} = \langle\Psi|\hat{n}_{\ell,\uparrow}|\Psi\rangle - \langle\Psi|\hat{n}_{\ell,\downarrow}|\Psi\rangle \ . \tag{6}$$

The charge density corresponds the average number of electrons at site ℓ, while the spin density provides the net spin polarization at ℓ.

1. Using the JW mapping of $\hat{n}_{\ell,s}$ onto qubits, express ρ_ℓ^{charge} and ρ_ℓ^{spin} in terms of probabilities of measurement outcomes, and implement the corresponding measurements onto Qiskit.

2. Compute and plot the charge and spin densities of the initial state $|\Psi_0\rangle$ as a function of the site number.

3. Compute and plot the charge and spin densities of the time-evolved state $|\Psi(t)\rangle$ at time $t = 2.5$ (as a function of the site number), obtained with both exact and Trotterized time evolution. Try different numbers of Trotter steps.

Finally, can you think of another qubit ordering than the one we have used (shown in Fig. 1) that would have minimized the number of $\hat{Z}$ gates in the JW mapping?

[3] For example, you can use python functions to diagonalize and exponentiate matrices

[4] These quantities were measured on a quantum computer simulating Fermi-Hubbard model dynamics in a study by Google AI in 2000: `https://arxiv.org/pdf/2010.07965`.