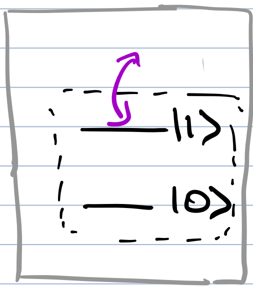
Part B

I. Quantum Mechanics of open systems.

In the first lecture we have formulated the rules of quantum mechanics for closed systems.

In practice, however, closed quantum systems do not exist and the observations we make are always limited to a small part of a much larger system.

In particular, a quantum computer will inevitably interact with its surroundings, and qubits will interacts with each others, which will result in decoherence and in the decay of the quantum information stored in the device. Unless we can successfully combat decoherence, our computer is sure to fail.



→ This is the subject of quantum-error correction (fault-tolerant QC

that we will study.

But before that, we need to understand the rules of quantum mechanics that apply to <u>open quantum systems</u>.

The main differences with closed syst are:

* states are <u>not</u> represented by rays ($e^{i\alpha}|\psi\rangle$) in the Hilbert space
  → but by <u>density operators</u>.

* measurements are <u>not</u> orthogonal projectors
  → but <u>generalized measurements</u>.

* time-evolution is <u>not</u> described by a unitary operator
  → but by <u>quantum channels</u>.

① Density operators

→ density op. formalism is necessary to characterize the state of an open syst (ex: one qubit coupled to an environment, one qubit coupled to another qubit)

Note: this formalism can also be used to characterize states of closed syst, as an alternative to the state-vector formulation.

Let's start with an example:

Consider one qubit A interacting with another qubit B, such that the total system $A \cup B$ is closed.

Suppose that we only have access to A (we can only "observe" A).

Take $|\Psi\rangle_{AB} = a\,|0\rangle_A|0\rangle_B + b\,|1\rangle_A|1\rangle_B$

We want to characterize the measurement outcomes for A alone (irrespective of the outcomes of any measurements of the inaccessible qubit B)

→ The corresponding observable is of the form:

$$\hat{\Omega}_A \otimes \hat{1}_B$$

Hermitian op acting on A

identity acting on B.

The expectation value of the observable in state $|\psi\rangle_{AB}$ is:

$$\langle \Omega_A \rangle \equiv {}_{AB}\langle \psi | \hat{\Omega}_A \otimes \hat{1}_B | \psi \rangle_{AB}$$

$$= \left(a^* \langle 00| + b^* \langle 11|\right) \hat{\Omega}_A \otimes \hat{1}_B \left(a|00\rangle + b|11\rangle\right)$$

$$= |a|^2 \langle 0 | \hat{\Omega}_A | 0 \rangle \underbrace{\langle 0 | 0 \rangle}_{1} + |b|^2 \langle 1 | \hat{\Omega}_A | 1 \rangle \underbrace{\langle 1 | 1 \rangle}_{1}$$

(where we have used ${}_B\langle 0 | 1 \rangle_B = 0$)

This can be re-written as:

$$\langle \hat{\sigma}_A \rangle = tr\left(\hat{\rho}_A \, \hat{\sigma}_A\right)$$

where $tr() = trace(\ )$

and $\hat{\rho}_A = |a|^2 \, |0\rangle\langle 0| + |b|^2 \, |1\rangle\langle 1|$

$=$ <u>density operator for qubit A</u> (density "matrix")

indeed:

$$tr\left(\hat{\rho}_A \hat{\sigma}_A\right) = \sum_{i=0,1} \langle i| \hat{\rho}_A \hat{\sigma}_A |i\rangle$$

$$= \langle 0| \left(|a|^2 |0\rangle\langle 0| + |b|^2 |1\rangle\langle 1|\right) \hat{\sigma}_A |0\rangle$$

$$+ \langle 1| \left(|a|^2 |0\rangle\langle 0| + |b|^2 |1\rangle\langle 1|\right) \hat{\sigma}_A |1\rangle$$

$$= |a|^2 \langle 0| \hat{\sigma}_A |0\rangle + |b|^2 \langle 1| \hat{\sigma}_A |1\rangle$$

$\hat{\rho}_A$ is a "reduced" density. It can be obtained by tracing the density op of the full system AB over B:

$\hat{\rho}_A = Tr_B\left(\hat{\rho}_{AB}\right)$ where $\hat{\rho}_{AB} = |\psi\rangle_{AB} \, {}_{AB}\langle\psi|$

check:

$$\mathrm{Tr}_B \left(|\psi\rangle_{AB\,AB}\langle\psi| \right)$$

$$= \sum_{i=0,1} {}_B\langle i|\psi\rangle_{AB\;AB}\langle\psi|i\rangle_B$$

$$= {}_B\langle 0| \left(a|00\rangle + b|11\rangle \right)\left(a^*\langle 00| + b^*\langle 11| \right)|0\rangle_B$$

$$+ {}_B\langle 1| \left(\qquad \right)\left(\qquad \right)|1\rangle_B$$

$$= |a|^2 \, |0\rangle\langle 0| + |b|^2 \, |1\rangle\langle 1|$$

$$= \hat{\rho}_A$$

The density operator has a natural "ensemble interpretation": $\hat{\rho}_A$ represents an ensemble of quantum states ($|0\rangle$ and $|1\rangle$), each occuring with a specified probability ($|a|^2$ and $|b|^2$).

For any observable $\hat{\Gamma}_A$ that we measure on system A, we cannot tell the difference btw measuring in the joint state $|\psi\rangle_{AB}$ of the composite system AB, and measuring in the corresponding ensemble of possible states for A alone.

Note: Take the case $a = b = 1/\sqrt{2}$

$$\Rightarrow \hat{\rho}_A = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1| \right) = \frac{1}{2}\hat{\mathbb{1}}$$

Thus if we measure the spin of A along any axis:

$$\langle \vec{\sigma}.\hat{n} \rangle = tr\left(\hat{\rho}_A \, \vec{\sigma}.\hat{n} \right)$$

$$= \frac{1}{2} tr\left(\vec{\sigma}.\hat{n} \right)$$

$$= 0$$

($\vec{\sigma}.\vec{n}$ are all traceless matrices)

$\Rightarrow$ no matter what axis we measure along, we get a uniform distribution of outcomes, (generate a random bit $\rightarrow$ spin "up" or "down" with proba 1/2)

This is different from the case when

$$|\psi\rangle_A = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$\begin{cases} \langle \sigma_z \rangle = \cos\theta = 0 \\ \langle \sigma_x \rangle = \sin\theta \cos\varphi = \underline{\underline{1}} \\ \langle \sigma_y \rangle = \sin\theta \quad \sin\varphi = 0 . \end{cases}$$

$\Rightarrow$ if one prepares an equally-weighted superposition of $|00\rangle$ and $|11\rangle$ in the 2-qubit world, the state of qubit A behaves incoherently (along any axis it is an equiprobable mixture of spin up & spin down).

When A and B interact, they can become entangled

The entanglement destroys the coherence of a superposition of states of A, so that some of the phases become inaccessible if we look at A alone

Generally :

if we have two subsystems A and B represented by Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ with orthonormal bases $\{|i\rangle_A\}$ and $\{|\mu\rangle_B\}$ and forming a total closed system AB.

A general state for AB is

$$\boxed{|\psi\rangle_{AB} = \sum_{i\mu} a_{i\mu} \, |i\rangle_A \otimes |\mu\rangle_B}$$

"bipartite" state

(with $\sum_{i\mu} |a_{i\mu}|^2 = 1$)

The expectation value of an observable

$\hat{\Omega}_A \otimes \mathbb{1}_B$ is

$$\langle \hat{\Omega}_A \rangle \equiv {}_{AB}\langle \psi | \hat{\Omega}_A \otimes \mathbb{1}_B |\psi\rangle_{AB}$$

$$= \sum_{i\mu} \sum_{j\nu} a_{j\nu}^* a_{i\mu} \langle j | \hat{\Omega}_A | i \rangle \underbrace{\langle \nu | \mathbb{1} | \mu \rangle}_{\delta_{\mu\nu}}$$

$$= \sum_{ij\mu} a_{j\mu}^* a_{i\mu} \langle j | \hat{\Omega}_A | i \rangle$$

$$= \mathrm{tr}\left(\hat{\rho}_A \hat{\Omega}_A \right)$$

where $\boxed{\begin{aligned}\hat{\rho}_A &= \mathrm{tr}_B \left(\overbrace{|\psi\rangle_{AB\,AB}\langle\psi|}^{\rho_{AB}} \right) \\ &= \sum_\mu {}_B\langle\mu|\psi\rangle_{AB}\;{}_{AB}\langle\psi|\mu\rangle_B \\ &= \sum_{ij\mu} a_{j\mu}^* \, a_{i\mu} \, |i\rangle_A \, {}_A\langle j|\end{aligned}}$

partial trace of $\hat{\rho}_{AB}$ over subsystem B.

$=$ "reduced" density operator of subsystem A

General Properties of density operators $\hat{\rho}$

* Hermiticity : $\hat{\rho}^\dagger = \hat{\rho}$
* Non-negativity : $\langle\phi|\hat{\rho}|\phi\rangle \geq 0 \quad \forall\, \phi.$

can be seen with $\hat{\rho}_A = \sum_{ij} a_{j\mu}^* a_{i\mu} |i\rangle\langle j|$

$$\langle\phi|\hat{\rho}_A|\phi\rangle = \sum_{ij\mu} a_{j\mu}^* \, a_{i\mu} \quad \langle\phi|i\rangle\langle j|\phi\rangle$$

$$= \sum_\mu \left| \sum_i a_{i\mu} \langle\phi|i\rangle \right|^2$$

$$\geq 0$$

* Unit trace : $tr(\hat{\rho}) = 1$

for ex: $tr(\hat{\rho}_A) = tr\left(\sum_{ij} a^*_{j\mu} a_{i\mu} |i\rangle\langle j|\right)$

$$= \sum_{i'} \sum_{ij} a^*_{j\mu} a_{i\mu} \underbrace{\langle i'|i\rangle}_{\delta_{ii'}} \underbrace{\langle j|i'\rangle}_{\delta_{ji'}}$$

$$= \sum_{i'} |a_{i'\mu}|^2$$

$= 1$ as $|\psi\rangle_{AB}$ is normalized.

$\Rightarrow$ Due to these properties, any density operator $\hat{\rho}$ can be diagonalized in an orthonormal basis $\{|a\rangle\}$. And the eigenvalues of $\hat{\rho}$ are non-negative real numbers which sum up to one:

$$\boxed{\hat{\rho} = \sum_a p_a |a\rangle\langle a| \quad \text{where} \quad p_a \geq 0 \quad \text{and} \quad \sum_a p_a = 1}$$

$\Rightarrow$ can think of the p_a's as a proba distribution.

In the ensemble interpretation, the eigenvalue p_a is the probability that the subsystem A has been prepared in state $|a\rangle$.

Cannot distinguish btw $\hat{\rho}$ and a situation where we have A closed and would flip a coin to sample from a proba. distribution such that with proba p_a we prepare state $|a\rangle$

Pure versus mixed states :

* if there is only one non-zero eigenvalue $p_a = 1$

$$\Rightarrow \hat{\rho} = |a\rangle\langle a|$$

$\Rightarrow$ The state is "pure"

and can be represented by a ray $e^{i\alpha}|a\rangle$ (just up to a phase)

ex: . state of a closed system.

. state of a subsystem A part of $A \cup B$ if A and B are not entangled.

* otherwise $\hat{\rho}$ is an incoherent mixture of the states $\{|a\rangle\}$.

↳ the relative phases of the $|a\rangle$'s are not accessible experimentally (⇒ cannot write a ket for the state of A).

⇒ the state is "mixed"

ex: • state of a subsystem A entangled with B when one has no access to B.

Even if the state of the global system is pure (like AB here), the state of the subsystem A (or B) is not necessarily.

It can be mixed or pure

$$\hat{\rho}_A = \sum p_a |a\rangle\langle a| \qquad\qquad \hat{\rho}_A = |a\rangle\langle a|.$$

A few more general properties of density matrices:

* The density operator of a pure state $|\psi\rangle$: $\hat{\rho} = |\psi\rangle\langle\psi|$

satisfies $\hat{\rho}^2 = \hat{\rho}$ $\qquad |\psi\rangle\underbrace{\langle\psi|\psi\rangle}_{1}\langle\psi| = |\psi\rangle\langle\psi|$

* For any $\hat{\rho}$: $\mathrm{tr}(\hat{\rho}^2) \leq 1$

* and $\mathrm{tr}(\hat{\rho}^2) = 1 \iff \hat{\rho}$ is pure

Example : density operator of a single qubit.

A qubit in a pure state can be rep. by

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

$$\Rightarrow \rho = |\psi\rangle\langle\psi|$$

$$= \cos^2\left(\frac{\theta}{2}\right)|0\rangle\langle 0| + \sin^2\left(\frac{\theta}{2}\right)|1\rangle\langle 1|$$

$$+ \cos\left(\frac{\theta}{2}\right)e^{-i\varphi}\sin\left(\frac{\theta}{2}\right)|0\rangle\langle 1|$$

$$+ \cos\left(\frac{\theta}{2}\right)e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\langle 0|$$

$$= \begin{pmatrix} \cos^2(\theta/2) & e^{-i\varphi}\cos(\theta/2)\sin(\theta/2) \\ e^{i\varphi}\cos\theta/2\sin\theta/2 & \sin^2(\theta/2) \end{pmatrix}$$

[...]

$$= \frac{1}{2}\begin{pmatrix} 1+r_3 & r_1 - ir_2 \\ r_1 + ir_2 & 1-r_3 \end{pmatrix}$$

$$= \frac{1}{2}\left(\hat{\mathbb{1}} + \vec{r}\cdot\vec{\sigma}\right)$$

where the r_1, r_2, r_3 are real and $|\vec{r}| = 1$.

Now for a mixed state

$$\hat{\rho} = \sum_k p_k \, |\psi_k\rangle\langle\psi_k|$$

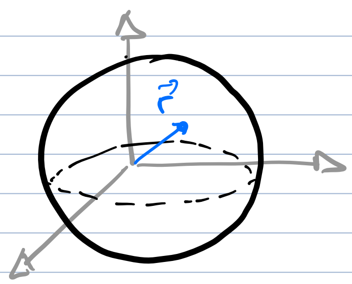
$$= \sum_k p_k \, \frac{1}{2} \left(\mathbb{1} + \vec{r}_k \cdot \vec{\sigma} \right)$$

$$= \frac{1}{2} \underbrace{\sum_k p_k}_{1} \mathbb{1} + \frac{1}{2} \sum_k p_k \, \vec{r}_k \cdot \vec{\sigma}$$

$$= \frac{1}{2} \left[\mathbb{1} + \underbrace{\left(\sum_k p_k \, \vec{r}_k \right)}_{\equiv \vec{r}} \cdot \vec{\sigma} \right]$$

now $|\vec{r}| = \left| \sum_k p_k \vec{r}_k \right| \leq \sum_k p_k \underbrace{|\vec{r}_k|}_{1} = 1$

Can think of this in term of a "Bloch Ball"



with vectors $\vec{r}$ representing the state ρ.

→ if ρ is pure : $|\vec{r}| = 1$

⇒ pure states live on the surface of the Ball (Bloch sphere)

→ if ρ is mixed : $|\vec{r}| < 1$

⇒ mixed states live in the interior of the Ball.

Example of the Bell states:

$$|\Phi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$$

$$|\Psi^{\pm}\rangle = \frac{|10\rangle \pm |01\rangle}{\sqrt{2}}$$

They are pure states. But if we look at one subsystem only:

ex: $\hat{\rho}_A = \mathrm{Tr}_B \left(|\Phi^+\rangle\langle\Phi^+| \right) = \frac{1}{2} \mathbb{1}_A$

$\Rightarrow \hat{\rho}_A^{\,2} = \frac{1}{4} \mathbb{1}_A \Rightarrow \mathrm{Tr}(\rho_A^{\,2}) = 1/2 \Rightarrow \hat{\rho}_A$ mixed.

The Bell states are also "maximally" entangled $\rightarrow$ meaning that when we trace over one subsystem (e.g. qubit B), the density operator $\hat{\rho}_A$ is a multiple of the identity. (see more next lecture)

In general, there is a way of determining whether a bipartite pure state is entangled:

it is called the Schmidt decompo.

- Schmidt decomposition of a bipartite pure state

$$|\psi\rangle_{AB} = \sum_{i\mu} C_{i\mu} \, |i\rangle_A \otimes |\mu\rangle_B$$

Using the basis in which the density operator of A is diagonal, we can put the bipartite pure state in a standard form.

$$|\psi\rangle_{AB} = \sum_{a\mu} D_{a\mu} \, |a\rangle_A \otimes |\mu\rangle_B \longrightarrow \text{orthonormal bases for } \mathcal{H}_A \,\&\, \mathcal{H}_B$$

$$= \sum_a |a\rangle_A \otimes \underbrace{\sum_\mu D_{a\mu} |\mu\rangle_B}_{\equiv |\tilde{a}\rangle_B}$$

not necessarily orthonormal.

Since $\{|a\rangle_A\}$ is eigenbasis of $\hat{\rho}_A \Rightarrow$

$$\hat{\rho}_A = \sum_a p_a \, |a\rangle_A {}_A\langle a| = \sum_{aa'} p_a \, |a\rangle_A {}_A\langle a'| \, \delta_{aa'} \quad (1)$$

On the other hand:

$$\hat{\rho}_A = \mathrm{tr}_B\left(|\psi\rangle_{AB} {}_{AB}\langle\psi|\right) = \mathrm{tr}_B\left(\sum_{aa'} |a\rangle_A {}_A\langle a'| \otimes |\tilde{a}\rangle_B {}_B\langle\tilde{a}'|\right)$$

$$= \sum_{aa'} |a\rangle_A {}_A\langle a'| \otimes \underbrace{\sum_\mu {}_B\langle\mu|\tilde{a}\rangle_B {}_B\langle\tilde{a}'|\mu\rangle_B}_{= \; {}_B\langle\tilde{a}'| \underbrace{\left(\sum_\mu |\mu\rangle_B {}_B\langle\mu|\right)}_{\hat{1}_B} |\tilde{a}\rangle_B}$$

$$= \sum_{aa'} {}_B\langle \tilde{a}' | \tilde{a} \rangle_B \left(|a\rangle_A {}_A\langle a'| \right) \qquad (2)$$

identify (1) = (2) $\Rightarrow$

$${}_B\langle \tilde{a}' | \tilde{a} \rangle_B = p_a \delta_{aa'} \Rightarrow \{ |\tilde{a}\rangle_B \} \text{ are orthogonal}$$

They can also be normalized by rescaling:

$$|\tilde{\tilde{a}}\rangle_B \equiv \frac{1}{\sqrt{p_a}} |\tilde{a}\rangle_B$$

Then

$$\boxed{|\psi\rangle_{AB} = \sum_a \sqrt{p_a}\, |a\rangle_A \otimes |\tilde{\tilde{a}}\rangle_B}$$

Schmidt decomposition for $|\psi\rangle_{AB}$.

Note : Any bipartite pure state can be expressed in this form but the orthonormal bases $\{|a\rangle_A\}$ and $\{|\tilde{\tilde{a}}\rangle_B\}$ are $\neq$ for every state.

Note * $\rho_B = \mathrm{Tr}_A \left(|\psi\rangle_{AB} {}_{AB}\langle \psi| \right)$

$$= \sum_a p_a |\tilde{\tilde{a}}\rangle_B {}_B\langle \tilde{\tilde{a}}|$$

$\Rightarrow$ ρ_A & ρ_B have the same non-zero eigenvalues.

* We have

$$|\psi\rangle_{AB} = \sum_{i\mu} C_{i\mu} \, |i\rangle_A \, |\mu\rangle_B$$
$$= \sum_a \sqrt{p_a} \, |a\rangle_A \, |\tilde{a}\rangle_B$$

The orthon. bases are related via a unitary transfo

$$|a\rangle = \sum_i |i\rangle \, U_{ia}$$
$$|\tilde{a}\rangle = \sum_\mu |\mu\rangle \, V^*_{\mu a}$$

$$\Rightarrow C_{i\mu} = \sum_a U_{ia} \sqrt{p_a} \, \underbrace{V^*_{\mu a}}_{(V^\dagger)_{a\mu}}$$

$$\Rightarrow \underline{C = U D V^\dagger} \quad \text{where } D = \begin{pmatrix} \sqrt{p_1} & & \\ & \ddots & \\ & & \sqrt{p_n} \end{pmatrix}$$

"singular value decompo of C"

* <u>Schmidt number and entanglement</u>

The Schmidt number (SN) of a bipartite pure state $|\psi\rangle_{AB}$
= # of non-zero eigenvalues of $\hat{\rho}_A$ (or $\hat{\rho}_B$)
= # of terms in the Schmidt decompo :

SN — Schmidt number

$$|\psi_{AB}\rangle = \sum_{a=1}^{SN} \sqrt{p_a} \, |a\rangle_A \otimes |\tilde{a}\rangle_B .$$

- if SN > 1, $|\psi\rangle_{AB}$ is entangled (A & B have quantum correlations).

- if SN = 1, $|\psi\rangle_{AB}$ is unentangled (separable):

  $|\psi\rangle_{AB} = |\varphi\rangle_A \otimes |\chi\rangle_B$ is a direct product of pure states ($\hat{\rho}_A = |\varphi\rangle\langle\varphi|$, $\hat{\rho}_B = |\chi\rangle\langle\chi|$)

$\boxed{\text{Note}}$: if $|\psi\rangle_{AB}$ is separable, A & B can still be correlated.

ex: $|\uparrow\rangle_A \otimes |\uparrow\rangle_B$ is separable but the 2 spins are correlated because point in the same direction.

<u>However</u> this state can be prepared while keeping A & B far away from each other.
(only need to send a classical message, like a phone call, to the preparers).

Now the state $\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A |\uparrow_B\rangle + |\downarrow\rangle_A |\downarrow\rangle_B\right)$ can only be prepared by applying a <u>collective</u> unitary transfo to $|\uparrow\rangle_A |\uparrow\rangle_B$.

$\Rightarrow$ To entangle 2 systems we must bring them together and allow them to interact.

‖ Local unitary transformations cannot increase the Schmidt number.

Density operators (continued)

Note: The ensemble representation $\rho = \sum_a p_a |a\rangle\langle a|$ of density operators is ambiguous (not uniquely defined)

Take $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

$$\rho_A = \mathrm{Tr}_B\left(|\psi\rangle_{AB\,AB}\langle\psi|\right) = \sum_{i=0,1} {}_B\langle i|\psi_{AB}\rangle\langle\psi_{AB}|i\rangle_B$$

$$= \frac{1}{2}\left(|0\rangle_{A\,A}\langle 0| + |1\rangle_{A\,A}\langle 1|\right) = \frac{1}{2}\mathbb{1} = \begin{pmatrix} 1/2 & \\ & 1/2 \end{pmatrix}$$

Now take the trace in another basis

$$\rho_A = \sum_{\mu=+,-} {}_B\langle\mu|\psi_{AB}\rangle\langle\psi_{AB}|\mu\rangle_B$$

we have

$${}_B\langle +|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\Big(\underbrace{\langle +|0\rangle}_{1/\sqrt{2}}|0\rangle_A + \underbrace{\langle +|1\rangle}_{1/\sqrt{2}}|1\rangle_A\Big)$$

$$= \frac{1}{2}\left(|0\rangle_A + |1\rangle_A\right) = \frac{1}{\sqrt{2}}|+\rangle_A$$

$${}_B\langle -|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\Big(\underbrace{\langle -|0\rangle}_{1/\sqrt{2}}|0\rangle_A + \underbrace{\langle -|1\rangle}_{-1/\sqrt{2}}|1\rangle_A\Big)$$

$$= \frac{1}{\sqrt{2}}|-\rangle_A$$

$$\Rightarrow \quad \rho_A = \frac{1}{2}\left(|+\rangle\langle+| + |-\rangle\langle-|\right)$$

$$= \frac{1}{2}\mathbb{1} = \begin{pmatrix} 1/2 & \\ & 1/2 \end{pmatrix}$$

The ensemble representation is ambiguous!

Another example:

- $|\psi\rangle_{AB} = \frac{7}{10}|00\rangle + \frac{1}{10}|01\rangle + \frac{1}{10}|10\rangle + \frac{7}{10}|11\rangle$

$$\rho_A = \mathrm{Tr}_B\left(|\psi\rangle\langle\psi|_{ABAB}\right) = {}_B\langle 0|\psi\rangle_{AB}\ {}_{AB}\langle\psi|0\rangle_B + {}_B\langle 1|\psi\rangle\langle\psi|1\rangle_B$$

$$= \left(\frac{7}{10}|0\rangle + \frac{1}{10}|1\rangle\right)\left(\frac{7}{10}\langle 0| + \frac{1}{10}\langle 1|\right)$$

$$+ \left(\frac{1}{10}|0\rangle + \frac{7}{10}|1\rangle\right)\left(\frac{1}{10}\langle 0| + \frac{7}{10}\langle 1|\right)$$

$$= \frac{50}{100}\underbrace{\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)}_{\mathbb{1}}$$

$$+ \frac{14}{100}\underbrace{\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)}_{X}$$

$$= \frac{1}{2} \mathbb{1} + \frac{7}{50} X$$

which the DM assoc. with e.g. :

$$\rho_A = \sum_{i=1}^{2} p_i \, |\phi_i\rangle\langle\phi_i|$$

where

$$p_1 = p_2 = 1/2 \quad \text{and}$$

$$\begin{cases} |\phi_1\rangle = \frac{1}{\sqrt{50}} \left(7|0\rangle + |1\rangle \right) \\ |\phi_2\rangle = \frac{1}{\sqrt{50}} \left(|0\rangle + 7|1\rangle \right). \end{cases}$$

Or $\rho_A = \sum_{\alpha=1,2} \lambda_\alpha \, |\psi_\alpha\rangle\langle\psi_\alpha|$

where $\lambda_1 = \frac{16}{25} = 0.64$ and $\lambda_2 = \frac{9}{25} = 0.36$

and :

$$\begin{cases} |\psi_1\rangle = |+\rangle \\ |\psi_2\rangle = |-\rangle \end{cases}$$

In the second represent° the λ_α are the eigenvalues of $\hat{\rho}$.

$\Rightarrow$ the ensemble rep. of mixed quantum state is ambiguous.

So far we have talked about criteria to tell whether a state is entangled or not, but we have not discussed how to quantify the entanglement.

One of the most widely used measure is the Von Neumann entanglement entropy (VN).

The VN entang. entropy is based upon the the Shannon entropy for classical info.

* Classical Shannon entropy

If one has a discrete random variable X taking values $x \in \{x_1, x_2 \ldots x_n\}$ with probability $p(x_i)$, the Shannon entropy is defined as:

$$H(X) = \sum_{i=1}^{n} p(x_i) \underbrace{\log\left(1/p(x_i)\right)}_{\text{info. meas. for each event.}}$$

where log is usually taken to be $\log_2$ in information theory. (and $0 \log 0 = 0$).

H(X) quantifies the average amount of information gained with each outcome.

Let's understand the formula above.

- More information is associated with small proba. of event.

  if we read one bit from a bit-string containing only 0's → no info can be gained. However if we have 0's & 1's we can encode and gain info.

  ⇒ the info measure for each event $I(x_i)$ should decrease monotonically with $p(x_i)$

  ie $I(x_i) < I(x_j)$ if $p(x_i) > p(x_j)$

- info measure should be positive

- If there are multiple <u>independent</u> events the joint informativeness should equal the sum of individual informativeness:

$$\text{if } z_{ij} \equiv (X = x_i,\ Y = y_j)$$

$$\text{then } I(z)_{ij} = I(x_i) + I(y_j)$$

$$\Rightarrow I(x_i) = \log\left(1/p(x_i)\right) \text{ fulfills these criteria.}$$

Then $H(X)$ represents the average informativeness over all events (by weighting with proba for each event)

$$H(X) = \sum_{i=1}^{n} p(x_i) \log(1/p(x_i))$$
$$= -\sum_{i=1}^{n} p(x_i) \log(p(x_i))$$

$H(X)$ is maximal $\overset{= \log_2 n}{}$ when $p(x_i) = 1/n$ $\forall i = 1, \dots n$.

and minimal $(= 0)$ when $p(x_i) = 0$ or 1.

<u>ex</u>: • coin flip with 50% chance of getting heads or tail

$$H = \frac{1}{2} \log_2 2 + \frac{1}{2} \log_2 2$$
$$= 1.$$

• if have 100% of getting heads

$$H = 0 \log 0 + 1 \log 1 = 0.$$

* <u>Quantum Von Neumann entanglement entropy</u>

The VN entropy is defined analogously to the Shannon entropy.

For a density matrix $\rho = \sum_\alpha \lambda_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| = \sum_i p_i |\phi_i\rangle\langle\phi_i|$

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho)$$
$$= -\sum_\alpha \lambda_\alpha \log \lambda_\alpha$$

where λ_α = eigenvalues of ρ.

$S(\rho)$ Quantifies the mixedness of ρ -

- $S(\rho)$ cancels when $\lambda_\alpha = 0, 1$ ie when ρ is pure.
- $S(\rho)$ is maximal $= (\log_2 d)$ when $\lambda_\alpha = \frac{1}{d}$ $\forall \alpha$ (where d = dimension of ρ).

How much a bipartite pure state $|\psi\rangle_{AB}$ is entangled (ie how much A & B are entangled) is given by $S(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A) = S(\rho_B)$

where $\rho_A = \mathrm{Tr}_B \left(|\psi\rangle_{AB} \, {}_{AB}\langle\psi| \right)$

$S(\rho_A)$ tells us how much info we are missing by looking at A alone.

Example: • Bell states have RDM $\rho_A = \frac{\mathbb{1}}{2}$ which is maximally mixed -

$$S(\rho_A) = \log_2 2 = 1.$$

$\Rightarrow$ The Bell states are maximally entangled.

• $|\psi\rangle_{AB} = \frac{7}{10}|00\rangle + \frac{1}{10}|01\rangle + \frac{1}{10}|10\rangle + \frac{7}{10}|11\rangle$

$$\rho_A = \mathrm{Tr}_B \left(|\psi\rangle\langle\psi|_{ABAB} \right)$$

$= \frac{1}{2}\mathbb{1} + \frac{7}{50} X$ with eigenvalues

$\lambda_1 = \frac{16}{25} = 0.64$ and $\lambda_2 = \frac{9}{25} = 0.36$

$$\Rightarrow \quad S(\rho_A) = -0.64 \log_2 0.64 - 0.36 \log_2 0.36$$

$$\simeq 0.943$$

$S(\rho_A) < 1$ but is close to the maximal value of 1.

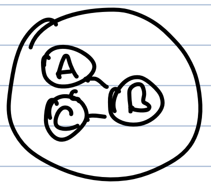
* Other measures exist:

ex: Rényi entropies

$$H_n(\rho) = \frac{1}{1-n} \log \left(\sum_\alpha \lambda_\alpha^n \right)$$

that are a generalization of the VN entropy (VNE) (the limit $n \to 1$ coincides with VNE).

* Entanglement for mixed states are more complicated. e.g when want to determine entang btw A & B in syst ABC -



One intuitive measure is the mutual information

$$I(A,B) = -\left[S(\rho_{AB}) - S(\rho_A) - S(\rho_B) \right]$$

However this has been shown to measure both classical & quantum correlations.

Several other measures have been developped, which we will not cover in this course.