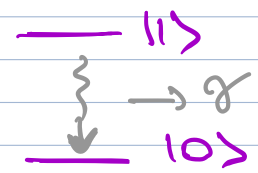
* Amplitude - damping channel

↳ spontaneous decay of a qubit

Ex: when the qubit is experimentally realized with ground + excited state of an ion
→ decay from the excited state to the grand state via photon (γ) emission.



Suppose environment is initially in state $|0\rangle_E$ (representing the vacuum : no photon)

If the system (qubit) is in its g.s $|0\rangle$ then nothing happens. But if the qubit is in excited state $|1\rangle$, there is a proba p for the qubit to relax into its g.s $|0\rangle$.

The unitary description for the process is:

$$\hat{U} : \begin{cases} |0\rangle \otimes |0\rangle_E \longmapsto |0\rangle \otimes |0\rangle_E \\ |1\rangle \otimes |0\rangle_E \longmapsto \sqrt{1-p}\, |1\rangle \otimes |0\rangle_E \\ \qquad\qquad\qquad\qquad + \sqrt{p}\, |0\rangle \otimes \underbrace{|1\rangle_E}_{\text{state with 1 photon}} \end{cases}$$

The action of $\hat{U}$ can be expressed as:

$$U |\psi\rangle |0\rangle_E = \sum_{\mu=0,1} (M_\mu |\psi\rangle) \otimes |\mu\rangle_E$$

with

$$\begin{cases} M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \\ M_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \end{cases}$$

Kraus operators.

Note: we can easily check the completeness relation

$$\sum_{\mu=0,1} M_\mu^\dagger M_\mu = \begin{pmatrix} 1 & 0 \\ 0 & 1-p \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix} = \mathbb{1}$$

The operator M_1 induces a "quantum jump" i.e. it changes $|1\rangle$ into $|0\rangle$, corresp to the physical decay process of losing a quantum of energy into the envt.

M_0 describes what happens if no jump occurs : leaves $|0\rangle$ unchanged but reduces the amplitude of a $|1\rangle$ state (physically this happens because a qu. of energy was not lost to the envt and thus, the envt now perceives it to be more likely that the syst. is in state $|0\rangle$, rather than $|1\rangle$).

A general density matrix ρ_q $= \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}$ evolves as :

$$\rho_q \longmapsto \mathcal{E}(\rho_q) = \sum_{\mu=0,1} M_\mu \, \rho_q \, M_\mu^\dagger$$

$$= \begin{pmatrix} 1 & \\ & \sqrt{1-p} \end{pmatrix} \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 1 & \\ & \sqrt{1-p} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \rho_{00} & \rho_{01}\sqrt{1-p} \\ \rho_{10}\sqrt{1-p} & \rho_{11}(1-p) \end{pmatrix} + \begin{pmatrix} p\rho_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \rho_{00} + p\,\rho_{11} & \rho_{01}\sqrt{1-p} \\ \rho_{10}\sqrt{1-p} & \rho_{11}\,(1-p) \end{pmatrix}$$

The spontaneous decay affects both the diagonal and off-diagonal components.

There is a population transfer btw the $|0\rangle$ and $|1\rangle$ state : proba. of being in state $|1\rangle$ is decreased $(\mathcal{E}(\rho))_{11} = \rho_{11}(1-p)$

while proba. of being in state $|0\rangle$ is increased by the same amount :
$(\mathcal{E}(\rho))_{00} = \rho_{00} + p\,\rho_{11}$.

(which describe coherence)

The off-diago terms are both suppressed by $\sqrt{1-p}$.

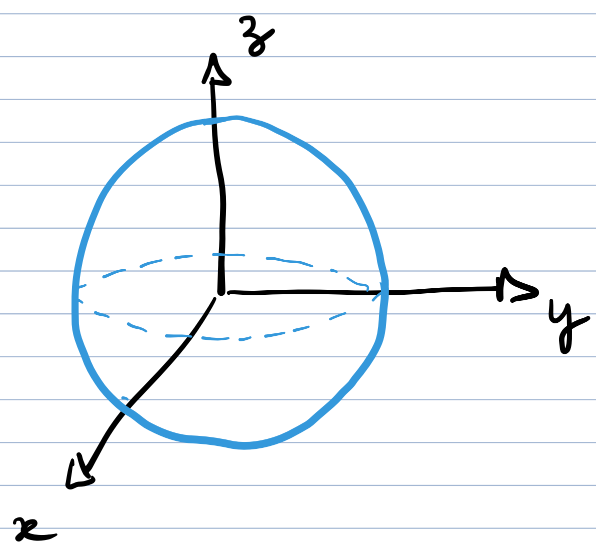
Bloch - Ball representation :

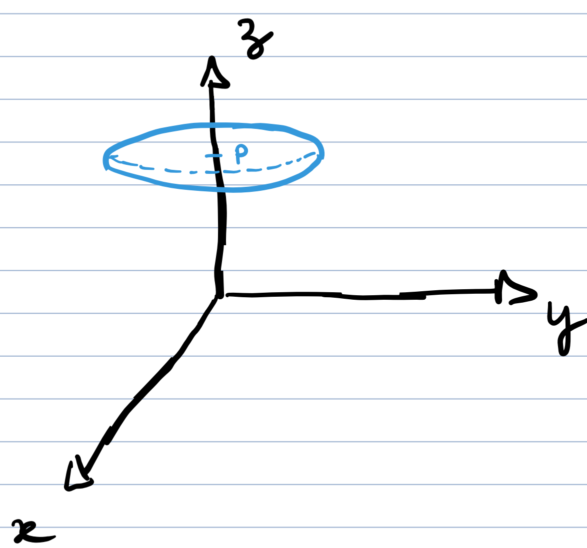
$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \frac{\mathbb{1} + \vec{r}\cdot\vec{\sigma}}{2} = \frac{1}{2}\begin{pmatrix} 1+r_z & r_x + i r_y \\ r_x - i r_y & 1 - r_z \end{pmatrix}$$

$$\mathcal{E}(\rho) = \begin{pmatrix} \rho_{00} + p\,\rho_{11} & \rho_{01}\sqrt{1-p} \\ \rho_{10}\sqrt{1-p} & \rho_{11}(1-p) \end{pmatrix} \equiv \frac{1}{2}\begin{pmatrix} 1+r_z' & r_x' + i r_y' \\ r_x' - i r_y' & 1 - r_z' \end{pmatrix}$$

leads to :

$$\begin{cases} r_x' = r_x\sqrt{1-p} \\ r_y' = r_y\sqrt{1-p} \\ r_z' = r_z + (1-r_z)\,p \\ \quad\;\; = r_z(1-p) + p \end{cases}$$





This channel compresses the Bloch Ball and also shifts the center.

$\Downarrow$

It is said to be non unital

(it does not map $\rho = \frac{\mathbb{1}}{2}$ to $\rho = \frac{\mathbb{1}}{2}$)

- Note: Because it is non unital, the pancake (oblate) shape is allowed.

- Note: Time - dependence:

If one thinks as $p = \Gamma \Delta t$ where $\Gamma =$ spontaneous decay rate per unit time, we find the density operator after time $t = n \Delta t$ by applying the channel n times in succession.

The ρ_{11} matrix element then decays as

$$\rho_{11} \longmapsto (1-p)^n \rho_{11} = \left(1 - \frac{\Gamma t}{n}\right)^n \rho_{11}$$

$$\xrightarrow[n \to \infty]{} e^{-\Gamma t} \rho_{11}$$

which is the usual exponential decay law.

$$\Rightarrow \rho(t) = \begin{pmatrix} \rho_{00} + (1 - e^{-\Gamma t}) \rho_{11} & e^{-\Gamma t/2} \rho_{01} \\ e^{-\Gamma t/2} \rho_{10} & e^{-\Gamma t} \rho_{11} \end{pmatrix}$$

Then on the Bloch ball we have

$$\begin{cases} r_x(t) = r_x(0)\, e^{-\Gamma t/2} \\ r_y(t) = r_y(0)\, e^{-\Gamma t/2} \\ r_z(t) = r_z(0)\, e^{-\Gamma t} + \left(1 - e^{-\Gamma t}\right) \end{cases}$$

$\Rightarrow$ we can visualize the effects of the amplitude-damping (spontaneous decay) channel as a flow on the Bloch ball, which moves every point in the unit ball towards a fixed point, where $|0\rangle$ is located.

* Phase - damping channel (see Tutorial #13)

One last channel which can model a noise process is the "phase-damping" channel. This process is purely quantum mechanical as it describes the loss of quantum info without the loss of energy.
It also provides a simple picture of decoherence in real physical situations, without any inessential mathematical detail.

Here the qubit actually does not make any transition in the $\{|0\rangle, |1\rangle\}$ basis.

Instead, the environment (initially in state $|0\rangle_E$) "scatters" off of the qubit occasionally (with proba p), and is "kicked" into state $|1\rangle_E$ if the qubit is in state $|0\rangle$, or in state $|2\rangle_E$ if the qubit is in state $|1\rangle$.

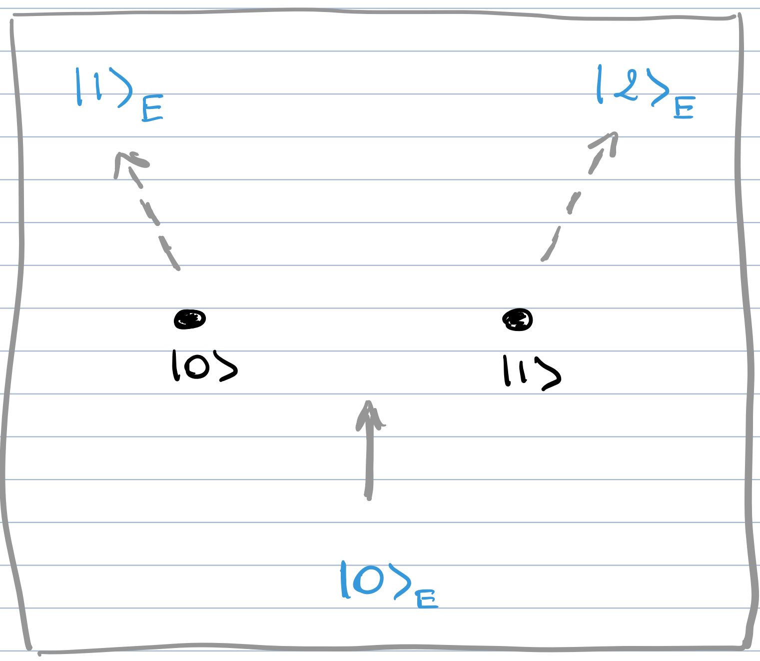
The unitary representation for this channel is:

$$\hat{U}: \begin{cases} |0\rangle \otimes |0\rangle_E \longmapsto \sqrt{1-p}\, |0\rangle \otimes |0\rangle_E + \sqrt{p}\, |0\rangle \otimes |1\rangle_E \\ |1\rangle \otimes |0\rangle_E \longmapsto \sqrt{1-p}\, |1\rangle \otimes |0\rangle_E + \sqrt{p}\, |1\rangle \otimes |2\rangle_E \end{cases}$$

Physically, this channel can describe a heavy particle (e.g. an interstellar dust grain) interacting a background gas of light particles (e.g. photons).

The dust grain does not have a well defined position, it can be in a superposition of $\neq$ positions, labeled 0 and 1.

It can happen that a photon encounters the dust grain and gets scattered by it. And the way the γ gets scattered depends on the position of the dust grain:



($|0\rangle_E$, $|1\rangle_E$, $|2\rangle_E$ would describe $\neq$ orthogonal states of the photon).

$\Rightarrow$ The state of the photon becomes entangled with the dust grain.

In a quantum computer, the environment can be some light particles interacting with the qubits

for instance particles from cosmic rays

see: Nature **584**, 551 (2020)
e.g.

This channel can be rewritten in terms of Kraus operators M_μ as:

$$U|\psi\rangle \otimes |0\rangle_E = \sum_{\mu=0}^{2} M_\mu |\psi\rangle \otimes |\mu\rangle_E$$

$$\Rightarrow \quad M_0 = \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} = \sqrt{1-p}\ \mathbb{1}.$$

$$M_1 = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix}$$

<u>Note</u> : These 3 Kraus operators are in fact not linearly independent $\Rightarrow$ can reduce to a set of two Kraus operators (see exercise).

$\Rightarrow$ An initial density matrix ρ_q for the qubit will thus evolve as

$$\rho_q \longmapsto \mathcal{E}(\rho_q) = \sum_{\mu=0}^{2} M_\mu \, \rho_q \, M_\mu^\dagger$$

$$\rho_q = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \qquad = \begin{pmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{pmatrix}$$

$\Rightarrow$ the diagonal terms remain invariant while the off-diago terms are suppressed by a factor $(1-p)$.

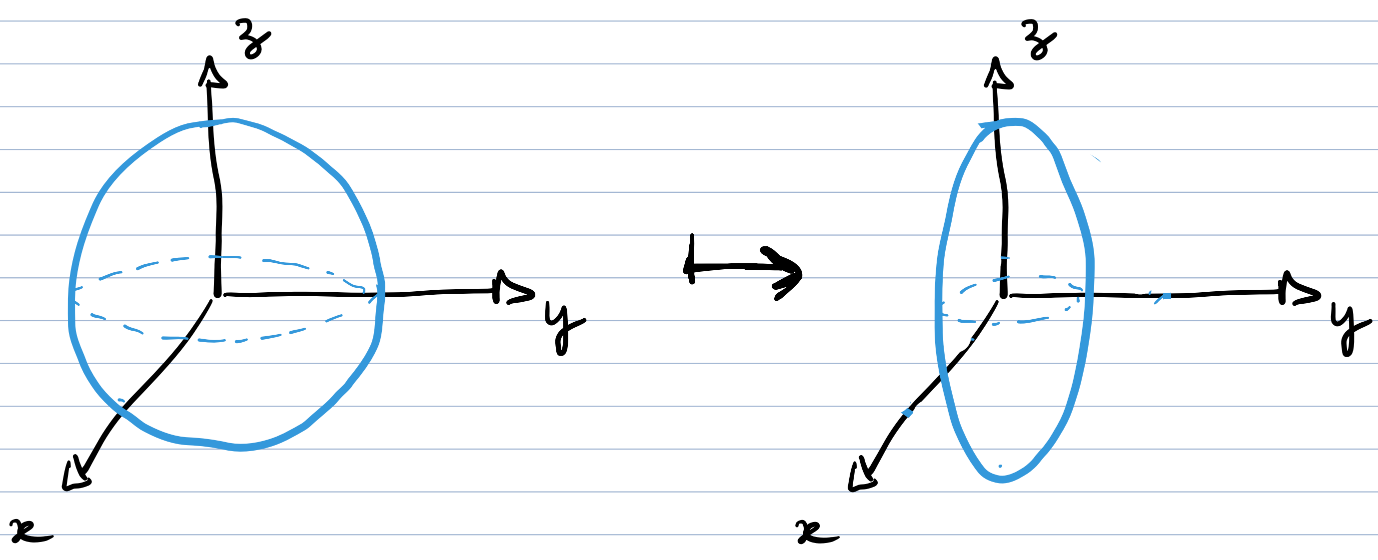
Bloch - ball representation:

$$\rho_q = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+r_z & r_x + i r_y \\ r_x - i r_y & 1 - r_z \end{pmatrix}$$

$$\mapsto \mathcal{E}(\rho_q) = \begin{pmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+r_z' & r_x' + i r_y' \\ r_x' - i r_y' & 1 - r_z' \end{pmatrix}$$

$$\Rightarrow \begin{cases} r_x' = (1-p)\, r_x \\ r_y' = (1-p)\, r_y \\ r_z' = r_z \end{cases}$$

$\Rightarrow$ Similar to the "phase - flip" channel: the Bloch ball shrinks to a prolate ellipsoid aligned with the z axis.



Continuous dephasing :

If we consider that there are many photons in the envt $\Rightarrow$ dephasing occurs continuously in time.

Suppose the proba of a scattering event per unit time is Γ $\Rightarrow$ $p = \Gamma \Delta t \ll 1$ (for very small time interval Δt).

The evolution over time $t = n \Delta t$ is governed by the repeated action of the channel $\mathcal{E}$ (repeated n times).

$\Rightarrow$ the off-diago elements of the initial density become suppressed by a factor

$$(1-p)^n = (1 - \Gamma \Delta t)^n$$

$$= \left(1 - \frac{\Gamma t}{n}\right)^n$$

$$\xrightarrow[n \to \infty]{} e^{-\Gamma t}$$

$$\Rightarrow \underbrace{(\mathcal{E} \circ \dots \circ \mathcal{E})}_{\mathcal{E}^n} (\rho_q) = \begin{pmatrix} \rho_{00} & e^{-\Gamma t} \rho_{01} \\ e^{-\Gamma t} \rho_{10} & \rho_{11} \end{pmatrix}$$

(n times)

$\Rightarrow$ the off-diago terms decay exponentially with time.

After a long time

$$t \gg \frac{1}{\Gamma} \Rightarrow e^{-\Gamma t} \to 0.$$

i.e. the off-diago terms are negligible

This means that if we started with a qubit (dust grain) in a coherent superposition state $|\psi\rangle = a|0\rangle + b|1\rangle$

i.e. with $\rho_q = |\psi\rangle\langle\psi|$

$$= |a|^2 |0\rangle\langle 0| + |b|^2 |1\rangle\langle 1|$$

$$+ ab^* |0\rangle\langle 1| + a^* b |1\rangle\langle 0|$$

$$= \begin{pmatrix} |a|^2 & ab^* \\ a^* b & |b|^2 \end{pmatrix}$$

then the density op evolves into the mixed state :

$$\begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} = |a|^2 |0\rangle\langle 0| + |b|^2 |1\rangle\langle 1|$$

$\Rightarrow$ the state decoheres, in the basis $\{|0\rangle, |1\rangle\}$.

Under continuous dephasing in time, the Ball deflates exponentially as

$$\begin{cases} r_x' = e^{-\Gamma t} r_x \\ r_y' = e^{-\Gamma t} r_y \end{cases}$$

$\Rightarrow$ in the limit of large time $t \gg 1/\Gamma$, the Ball deflates completely to

$$r_x' = r_y' = 0$$

i.e. it degenerates to the z axis.