A.I.2) The Qubit ("quantum bit")

= indivisible unit of quantum information
(analogous of the bit (0 or 1) in classical information)

= 2-level quantum system
represented by a 2-dimensional Hilbert space
$\mathcal{H} = \mathbb{C}^2$ (smallest non trivial Hilb. space)

* States of a qubit

Can denote $\{|0\rangle, |1\rangle\}$ with $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ an orthonormal basis for this Hilbert space $\Rightarrow$ Any state $|\psi\rangle$ of the qubit can be expressed as

$$|\psi\rangle = a|0\rangle + b|1\rangle \quad \text{with } |a|^2 + |b|^2 = 1$$

$$= \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2$$

$\Rightarrow$ Contrary to classical bit which takes either value 1 or 0, a qubit can be in a superposition of $|0\rangle$ & $|1\rangle$ at the same time.

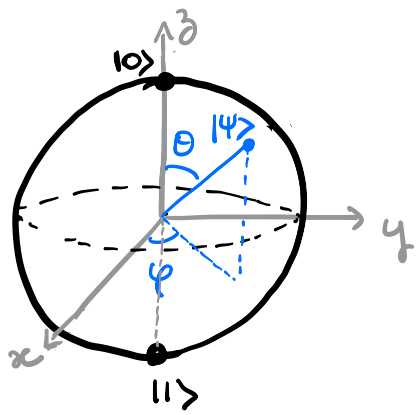
Spin-1/2 representation of the qubit:

Easier to grasp the properties of the qubit with a geometrical interpretation of its state.

* The qubit can be interpreted as a spin-1/2 object

$$\Rightarrow \begin{cases} |0\rangle \equiv |\uparrow\rangle_z = \begin{pmatrix}1\\0\end{pmatrix} \\ |1\rangle = |\downarrow\rangle_z = \begin{pmatrix}0\\1\end{pmatrix} \end{cases} \Bigg\} \text{ eigenvectors of } \sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

↳ "computational basis".



Bloch sphere

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle$$
$$= \cos(\theta/2)\,|0\rangle + e^{i\varphi}\sin(\theta/2)\,|1\rangle$$

$\theta \in [0, \pi]$ $\quad \varphi \in [0, 2\pi[$

$\Rightarrow$ The state of a qubit can be represented by a unit vector in 3d space, with orientation param. (θ, φ).

⚠ This Bloch sphere maps two-dimensional vectors $\in \mathbb{C}^2$ onto 3-d real space.

$|0\rangle$ and $|1\rangle$ are orthogonal in Hilbert space but here they appear on the same axis on the Bloch sphere.

Note: There is no generalization of the Bloch sphere for multiple qubits.

* Measurement

- Along z axis :

Can measure the qubit state along the z-axis

$\Rightarrow$ Observable is the Pauli Matrix :

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Proba of obtaining state $|0\rangle$ as outcome (eigval +1) is obtained by projecting with $\hat{P}_0 = |0\rangle\langle 0|$

$$p(|0\rangle) \equiv p(+1) = \langle\psi|\hat{P}_0|\psi\rangle$$
$$= |\langle\psi|0\rangle|^2 = |a|^2 = \cos^2(\theta/2)$$

Similarly $p(|1\rangle) \equiv p(-1) = |b|^2 = \sin^2(\theta/2)$

- Can also measure along any other axis

For instance along the x axis :

$\Rightarrow$ observable $= \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

with eigenvalues $+1, -1$

& eigenvectors $\begin{cases} |+\rangle = 1/\sqrt{2}\,(|0\rangle + |1\rangle) \\ |-\rangle = 1/\sqrt{2}\,(|0\rangle - |1\rangle) \end{cases}$

$\Rightarrow$ projectors $\hat{P}_+ = |+\rangle\langle+|$, $\hat{P}_- = |-\rangle\langle-|$

$$\Rightarrow \quad p(+1) \equiv p(|+\rangle) = \langle\psi|\hat{P}_+|\psi\rangle = \left|\frac{a+b}{\sqrt{2}}\right|^2$$

$$= \frac{1}{2}\left[1 + \sin\theta\cos\varphi\right]$$

$$p(-1) \equiv p(|-\rangle) = \langle\psi|\hat{P}_-|\psi\rangle = \left|\frac{a-b}{\sqrt{2}}\right|^2$$

$$= \frac{1}{2}\left[1 - \sin\theta\cos\varphi\right]$$

This can be easily obtained by rewriting $|\psi\rangle$ in the $|\pm\rangle$ basis :

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

$$= \frac{a}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) + \frac{b}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)$$

$$= \left(\frac{a+b}{\sqrt{2}}\right)|+\rangle + \left(\frac{a-b}{\sqrt{2}}\right)|-\rangle$$

1 single measurement cannot determine θ and φ unambiguously but if we prepare many identical copies we can obtain the expectation values:

$$\langle \sigma_x \rangle = \sin\theta \cos\varphi \qquad \langle \sigma_y \rangle = \sin\theta \sin\varphi \qquad \langle \sigma_z \rangle = \cos\theta$$

For instance:

$$\langle \sigma_x \rangle = \langle \psi | \hat{\sigma}_x | \psi \rangle$$

$$= (a^* \quad b^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= (a^* \quad b^*) \begin{pmatrix} b \\ a \end{pmatrix} = a^* b + b^* a$$

$$= \underbrace{\cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right)}_{(1/2)\sin\theta} e^{i\varphi} + \underbrace{\cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right)}_{(1/2)\sin\theta} e^{-i\varphi}$$

$$= \frac{\sin\theta}{2} \underbrace{\left(e^{i\varphi} + e^{-i\varphi}\right)}_{2\cos\varphi} = \sin\theta \cos\varphi$$

$\Rightarrow$ with enough measurements in all 3 directions can then completely characterize the state of a qubit. (with small error)

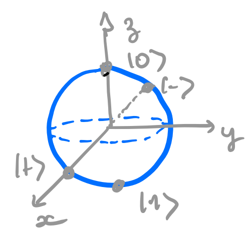
Quantum Interferences: A qubit is not the same as a probabilistic classical bit (classical bit with definite value unknown to us)

This is because of their relative phase which cause the phenomenon of quantum interference making probabilities add up in unexpected ways

<u>Example</u> • consider the states

$$\begin{cases} |+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \\ |-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \end{cases}$$

$|+\rangle = |\uparrow\rangle_x$, $|0\rangle = |\uparrow\rangle_z$, $|1\rangle = |\downarrow\rangle_z$, $|-\rangle = |\downarrow\rangle_x$



For both of them, if we measure spin along z axis
$\Rightarrow$ obtain $\begin{cases} |0\rangle \text{ with proba } 1/2 \\ |1\rangle \text{ ——— } 1/2 \end{cases}$ $\Rightarrow$ it's like generating a random bit.

• Now consider the state $\frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) \equiv |\psi\rangle$ and measure spin along z.
If we naively think of a superposition of eigenstates as an ensemble of possible states, each occurring with some proba $\Rightarrow$ would conclude that when measuring $|\psi\rangle$ would get $|0\rangle$ & $|1\rangle$ with same proba $1/2$.
But because of destructive interferences $|\psi\rangle = |0\rangle$
$\Rightarrow$ will always find $|0\rangle$ as outcome.

3) One-qubit "gates"

= (non-measuring) Operations on one qubit

- they are unitary (see last lecture) and correspond to rotation of the spin on the 3d Bloch sphere.

- they are generated by $\mathbb{1}$ and the Pauli Matrices

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

They form the simplest non-trivial irreducible representation of the rotation group (in 3d).

=> A rotation in 3d space of angle θ about the unit axis $\vec{n} = (n_x, n_y, n_z)$ $(n_x^2 + n_y^2 + n_z^2 = 1)$ is represented in 2d Hilbert space as a 2x2 unitary matrix:

$$\hat{U}(\vec{n}, \theta) = e^{-i\frac{\theta}{2}\vec{n}.\hat{\vec{\sigma}}} = \hat{\mathbb{1}} \cos\frac{\theta}{2} - i\,\vec{n}.\hat{\vec{\sigma}} \sin\frac{\theta}{2}$$

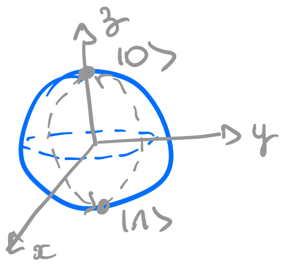
Note: it can also be convenient to use the Euler representation which tells us that any single-qubit gate $\hat{U}$ can be decomposed into products of rotations about z & y axes as:

$$\hat{U}(\alpha, \beta, \gamma) = \underset{(\times\ \text{phase})}{\hat{U}(\vec{e_z}, \gamma)}\ \hat{U}(\vec{e_y}, \beta)\ \hat{U}(\vec{e_z}, \alpha)$$

* Example

Rotation of angle φ about x-axis corresponds to

$$U(\vec{e}_x, \varphi) = \mathbb{1}\cos(\varphi/2) - i\,\sigma_x\left(\sin\varphi/2\right)$$



$$= \begin{pmatrix}\cos(\varphi/2) & 0\\ 0 & \cos(\varphi/2)\end{pmatrix} + \begin{pmatrix} 0 & -i(\sin\varphi/2)\\ -i\sin(\varphi/2) & 0\end{pmatrix}$$

Take $\varphi = \pi$

$$\Rightarrow U(\vec{e}_x, \pi) = -i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = (-i)\,\sigma_x$$

↳ gives global phase

⇒ The Pauli matrices themselves represent particular rotations.

Rmq : Pauli matrices are special because they are both Hermitian and Unitary ($\Rightarrow \sigma_{(i)}^{-1} = \sigma_{(i)}^{\dagger} = \sigma_{(i)}$ (involutory)) ⇒ they are both observables and can be used to transform the state of the system. (while preserving norm & proba dist°)

* Common 1-qubit gates :

1) $\sigma_x \equiv X$:

also called NOT (in analogy to classical)

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{|0\rangle} = \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{|1\rangle}$$

"bit flip"

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{|1\rangle} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{= |0\rangle}$$

2) $\sigma_z \equiv Z$ ("phase flip")

$$\left.\begin{aligned} |0\rangle &\mapsto |0\rangle \\ |1\rangle &\mapsto -|1\rangle \end{aligned}\right\} \Rightarrow$$

will introduce a relative phase when acting on superposition of $|0\rangle$ & $|1\rangle$

3) $\sigma_y \equiv Y$:

$$\left\{\begin{aligned} |0\rangle &\mapsto i\,|1\rangle \\ |1\rangle &\mapsto -i\,|0\rangle \end{aligned}\right.$$

If we start from qubit initialized in state $|0\rangle$ or $|1\rangle$ and only apply Pauli matrices

$\rightarrow$ can never move away from these basis states, i.e. cannot generate superposition.

4) Hadamard gate "H"

↳ (not to be confused with the Hamiltonian) ⚠

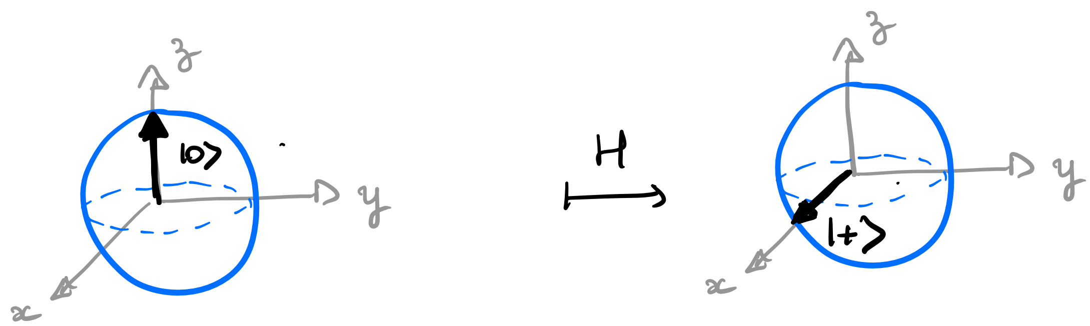
$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(rotation about $\frac{\hat{x}+\hat{z}}{\sqrt{2}}$ of angle π)

Transforms the computational basis states (eigenbasis of σ_z) into eigenbasis of σ_x:

$$\{|0\rangle, |1\rangle\} \xmapsto{H} \{|+\rangle, |-\rangle\}$$

↳ $\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$



We have $H = H^{-1}$ so that applying H again will transform back to computational basis:

$$\{|+\rangle, |-\rangle\} \xmapsto{H} \{|0\rangle, |1\rangle\}.$$