* Quantum Phase Estimation (QPE)

The QFT is the key to a general procedure known as phase estimation which in turn is the main ingredient in many quantum algorithms.

Suppose we have a unitary operator $\hat{U}$ with eigenvectors $|\psi\rangle$ and eigenvalue λ.

$$\hat{U}|\psi\rangle = \lambda |\psi\rangle$$

The goal is to determine λ (exactly or approximately).

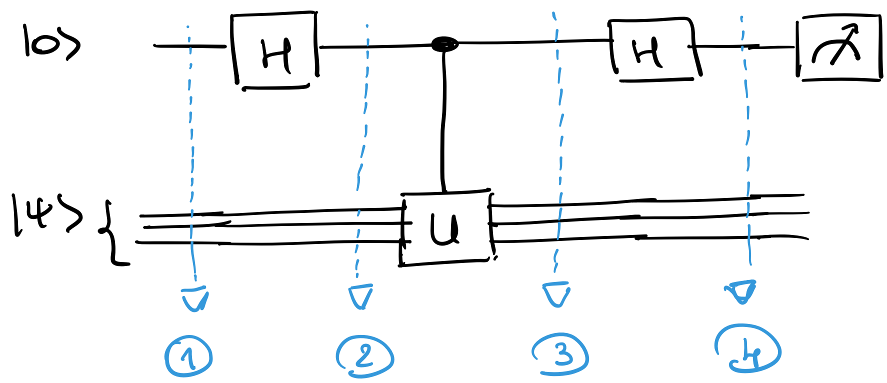
Since $\hat{U}$ is unitary, $|\lambda| = 1 \Rightarrow$ we can write $\lambda = e^{(2\pi i)\theta}$ with $\theta \in [0,1)$

Thus the pb is to determine θ.

Here we will assume that we can prepare the state $|\psi\rangle$.

How can we determine θ ?

One possibility is to consider the following circuit:



①: $|\psi\rangle|0\rangle$

②: $|\psi\rangle \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

③: $\frac{1}{\sqrt{2}}\left[|\psi\rangle|0\rangle + \underbrace{U|\psi\rangle}_{e^{2i\pi\theta}|\psi\rangle} |1\rangle \right]$

Phase kickback

④: $\frac{1}{\sqrt{2}}\left[|\psi\rangle \frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right) + e^{2i\pi\theta} |\psi\rangle \frac{1}{\sqrt{2}}\left(|0\rangle-|1\rangle\right)\right]$

$= \frac{1}{2} |\psi\rangle \left[\left(1+e^{2i\pi\theta}\right)|0\rangle + \left(1-e^{2i\pi\theta}\right)|1\rangle \right]$

Then measure the top qubit:

$$\begin{cases} \text{proba of measuring } |0\rangle = \left|\dfrac{1+e^{2i\pi\theta}}{2}\right|^2 = \cos^2(\pi\theta) \\ \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ } |1\rangle = \left|\dfrac{1-e^{2i\pi\theta}}{2}\right|^2 = \sin^2(\pi\theta) \end{cases}$$

$\Rightarrow$ info about θ is encoded in the measurement probabilities.

if $\theta = 0$ or $\theta = 1/2$ $\Rightarrow$ the proba are 0 and 1 and one can determine θ with certainty with only one measurement.

other values of θ can also be determined (with less statistical confidence)
with repeated measurements can obtain θ
(from ≠ce in the statistics btw $|0\rangle$ & $|1\rangle$ outcomes)

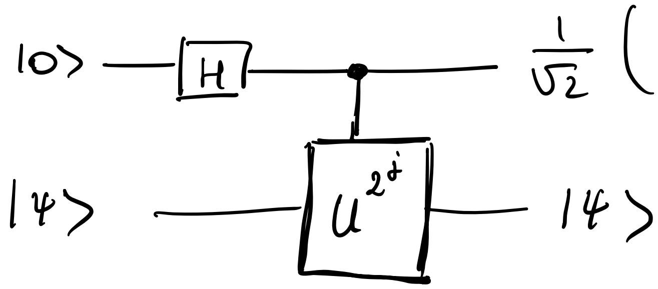
However with this method, measuring θ to exponential accuracy would require expo number of trials.

which will be needed for the factoring pb.

There is another (better) method:
if one can (efficiently) compute high powers of U such as

$$U^{2^j}$$

$\Rightarrow$ can produce states:



$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{(2\pi i)\,2^j\theta}\,|1\rangle\right)$$

if $\theta = \frac{\theta_{n-1}}{2} + \dots + \frac{\theta_{n-j}}{2^j} \dots + \frac{\theta_0}{2^n}$

$$2^j\theta = \theta_{n-1}\,2^{j-1} + \dots + \theta_{n-j} + \underbrace{\frac{\theta_{n-j-1}}{2} + \dots \frac{\theta_0}{2^{n-j}}}_{\text{fractional}}$$

$\Rightarrow$ measuring the fractional part to one bit of accuracy is like measuring θ_{n-j-1}

Using several qubits for the top register, one can make use of the QFT to extract θ.

$\rightarrow$ one for each power j up to the precisi° we want

General setup of the Quantum Phase Estimation:

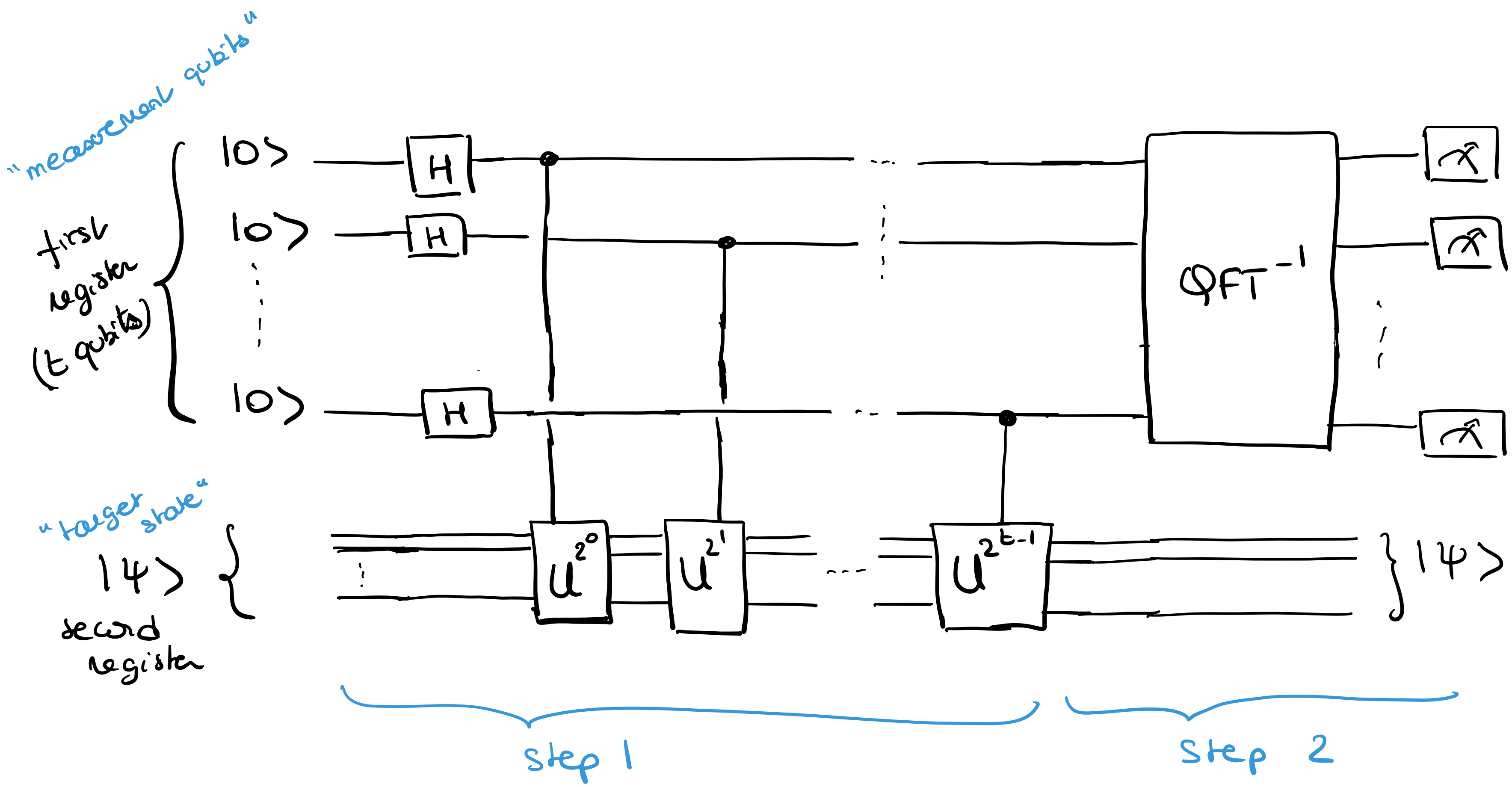
The QPE procedure uses two qubit registers

* A register initialized to $|\psi\rangle$ (if $|\psi\rangle$ is a n-qubit state this register contains n qubits).

* A second register with t qubits initialized to $|0\rangle$.

How we choose t will depend on the accuracy we want for our estimate of θ, and with what proba we wish the QPE procedure to be successful.

The QPE circuit is the following:


"measurement qubits"
first register (t qubits)
$|0\rangle$
$|0\rangle$
$|0\rangle$
H
H
H
U^{2^0}
U^{2^1}
$U^{2^{t-1}}$
QFT^{-1}
"target state"
$|\psi\rangle$
second register
$|\psi\rangle$
step 1
step 2

• <u>Step 1</u> : $H^{\otimes t}$ is applied to the top register and controlled-$\left(U^{2^j}\right)$ operations are applied to $|\psi\rangle$.

This produces a phase kick back into the top register while the bottom register is unchanged.

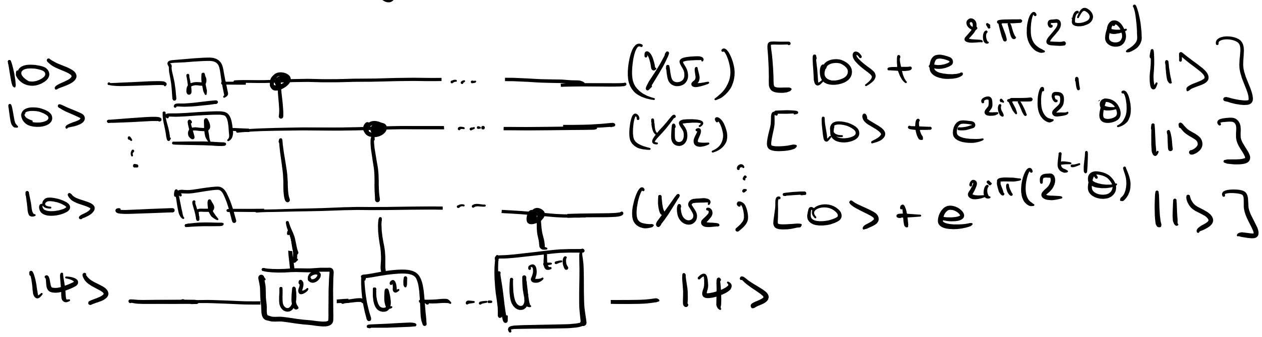
For instance consider the combined state of $|\psi\rangle$ and one qubit of the top register:

$$|\psi\rangle \otimes |0\rangle \overset{H}{\longmapsto} |\psi\rangle \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$\overset{C(U^{2^j})}{\longmapsto} \frac{1}{\sqrt{2}}\Big[|\psi\rangle|0\rangle + \underbrace{U^{2^j}|\psi\rangle}_{\left(e^{2i\pi\theta}\right)^{2^j}|\psi\rangle} |1\rangle$$

$$= |\psi\rangle \otimes \frac{1}{\sqrt{2}}\left[|0\rangle + \left(e^{2i\pi\theta}\right)^{2^j} |1\rangle \right]$$

$\Rightarrow$ after the first step we have



so the total state is :

$$|\psi\rangle \otimes \frac{1}{\sqrt{2}}\left[|0\rangle + e^{2i\pi(2^{t-1}\theta)}|1\rangle\right] \otimes \cdots$$

$$\otimes \cdots \frac{1}{\sqrt{2}}\left[|0\rangle + e^{2i\pi(2^{0}\theta)}|1\rangle\right]$$

$$= \frac{1}{2^{t/2}}\Big(|\overbrace{00\ldots 0}^{|0\rangle}\rangle + e^{2i\pi(2^0\theta)}|\overbrace{00\ldots 1}^{|1\rangle}\rangle$$

$$+ \cdots + e^{2i\pi(\underbrace{2^{t-1}+2^{t-2}+\ldots+2^0}_{2^t-1})\theta}|\underbrace{11\ldots 1}_{|2^t-1\rangle}\rangle\Big)$$

$$= \frac{1}{2^{t/2}} \sum_{y=0}^{2^t-1} e^{(2i\pi)y\theta}|y\rangle$$

$$= |\widetilde{2^t\theta}\rangle$$

We recognize the QFT $|\tilde{x}\rangle$ of $|x = 2^t\theta\rangle$.

Suppose θ can be expressed exactly in t bits :

$$\theta = \frac{\theta_{t-1}}{2} + \frac{\theta_{t-2}}{2^2} + \cdots \frac{\theta_1}{2^{t-1}} + \frac{\theta_0}{2^t}$$

$$= (.\theta_{t-1}\,\theta_{t-2}\cdots\theta_1\,\theta_0)$$

$$\Rightarrow 2^t\theta = \theta_{t-1}2^{t-1} + \theta_{t-2}2^{t-2} + \cdots + \theta_1 2 + \theta_0$$

$$\Rightarrow |2^t\theta\rangle \equiv |\theta_{t-1}\,\theta_{t-2}\cdots\theta_1\,\theta_0\rangle .$$

Thus, applying the inverse QFT will lead θ exactly $\Rightarrow$

- Step 2 : Apply the inverse QFT to the top register

$$(U_{QFT})^{-1} = (U_{QFT})^{\dagger}$$

$\rightarrow$ simply need to apply the inverse circuit of the QFT.

and measure in the computational basis.

What if θ cannot be expressed exactly with t bits ?

$\rightarrow$ will obtain θ with approximation :

Let us go back to state $|\widetilde{2^t\theta}\rangle$:

$$|\widetilde{2^t\theta}\rangle = \frac{1}{2^{t/2}} \sum_{y=0}^{2^t-1} e^{(2i\pi)y\theta} |y\rangle$$

Applying the QFT^{-1} to this state will produce :

$$|2^t\theta\rangle = \frac{1}{2^{t/2}} \sum_{y=0}^{2^t-1} e^{(2i\pi)y\theta} \left[U_{QFT}^{-1} |y\rangle\right]$$

$$= \frac{1}{2^{t/2}} \sum_{y=0}^{2^t-1} e^{(2i\pi)y\theta}$$

$$\times \left[\frac{1}{2^{t/2}} \sum_{x=0}^{2^t-1} e^{-(2i\pi)xy/2^t} |x\rangle\right]$$

$$= \frac{1}{2^t} \sum_{x,y=0}^{2^t-1} e^{(2i\pi)y\left[\theta - \frac{x}{2^t}\right]} |x\rangle$$

Note :

$$U_{QFT}|x\rangle = \sum_{y=0}^{2^t-1} e^{2i\pi x \frac{y}{2^t}} |y\rangle \cdot \frac{1}{2^{n/2}}$$

$$\underbrace{U_{QFT}^{-1}}_{= (U_{QFT})^\dagger} |y\rangle = \sum_{x=0}^{2^t-1} e^{-2i\pi x \frac{y}{2^t}} |x\rangle \quad \frac{1}{2^{n/2}}$$

If we now measure in the computational basis, the proba of outcome $|x\rangle$ is

$$\text{proba}(x) = \Bigg| \underbrace{\frac{1}{2^t} \sum_{y=0}^{2^t-1} e^{(2\pi i) y \left[\theta - \frac{x}{2^t}\right]}}_{\equiv \alpha(x)} \Bigg|^2$$

This distribution peaks at $x = 2^t\theta$.

Note: the following mathematical derivation is only provided for your information, as the steps are not all provided in the literature. However it is not necessary to go through it in detail for the understanding of the course.

How do we see this ?

Let b be the integer $\in \{0, \dots 2^t - 1\}$ such that $\frac{b}{2^t} = .b_{t-1} \dots b_0$ is the best t-bit approximation to θ which is less than θ

this means: the difference $\delta \equiv \theta - \frac{b}{2^t}$ satisfies

$$0 \leq \delta \leq 2^{-t}.$$

We want to bound the proba such that $|x - b| > k_\varepsilon$

↳ small integer (desired tolerance to error)

⇒ we want to evaluate

$$P \equiv \sum_x \text{proba}(x, |x-b| > k_\varepsilon) = \sum_{\substack{x=0 \\ \text{such that} \\ |x-b| > k_\varepsilon}}^{2^t-1} |\alpha(x)|^2$$

We have $x, b \in \{0, 1, \dots 2^t - 1\}$.

But $x - b$ can be negative

$\Rightarrow \quad -(2^{t-1}-1) \leq x - b \leq 2^{t-1}$

(range of integers (including negative) that can be stored on t bits (1 bit for the sign))

and $|x-b| > k_\varepsilon \Rightarrow \begin{cases} x - b \geq k_\varepsilon + 1 \\ \text{or} \\ x - b \leq -(k_\varepsilon + 1) \end{cases}$

$$\Rightarrow P = \sum_{\substack{x \\ \text{with} \\ \underbrace{x-b}_{x'} = -(2^{t-1}-1) \to -(k_\varepsilon+1)}} |\alpha(x)|^2 + \sum_{\substack{x \\ \text{with} \\ \underbrace{x-b}_{x'} = k_\varepsilon+1 \to 2^{t-1}}} |\alpha(x)|^2$$

$$= \sum_{x'=-(2^{t-1}-1)}^{-(k_\varepsilon+1)} |\alpha(x'+b)|^2 + \sum_{x'=k_\varepsilon+1}^{2^{t-1}} |\alpha(x'+b)|^2$$

Let us write:

$$\alpha(x'+b) = \frac{1}{2^t} \sum_{y=0}^{2^t-1} e^{(2i\pi)y\left[\theta - \frac{b+x'}{2^t}\right]}$$

$\hookrightarrow$ amplitude of $|(x'+b) \pmod{2^t}\rangle$

This is a geometric series $\sum_{k=0}^{A-1} e^{ikX} = \frac{1-e^{iAX}}{1-e^{iX}}$

$$\Rightarrow \alpha(x'+b) = \frac{1}{2^t} \frac{1-e^{(2i\pi)\left[\theta-\frac{b+x'}{2^t}\right]2^t}}{1-e^{(2i\pi)\left[\theta-\frac{b+x'}{2^t}\right]}}$$

$$= \frac{1}{2^t} \frac{1-e^{(2i\pi)\left[2^t\theta-(b+x')\right]}}{1-e^{(2i\pi)\left[\theta-(b+x')/2^t\right]}}$$

$$= \frac{1}{2^t} \left(\frac{1-e^{(2\pi i)\left[2^t\delta-x'\right]}}{1-e^{(2\pi i)\left[\delta-x'/2^t\right]}} \right)$$

$$\Rightarrow |\alpha(x'+b)| = \frac{1}{2^t} \left| \frac{1-e^{(2\pi i)\left[2^t\delta-x'\right]}}{1-e^{(2\pi i)\left[\delta-x'/2^t\right]}} \right|$$

$$\leq \frac{1}{2^t} \frac{2}{\left|1-e^{(2\pi i)\left[\delta-x'/2^t\right]}\right|}$$

where we used $|1-e^{i\varphi}| \leq 2$ for φ real

and we also have $|1-e^{i\varphi}| \geq 2|\varphi|/\pi$

for $-\pi \leq \varphi \leq \pi$

here we have: $-\pi \leq 2\pi\left(\delta - \frac{x'}{2^t}\right) \leq +\pi$

Because $-2^{t-1}+1 \leq x' \leq 2^{t-1}$

$-2^{t-1}+1 \leq x' \leq 2^{t-1} \rightarrow \frac{1}{2}-\frac{1}{2^t} \geq -\frac{x'}{2^t} \geq -\frac{1}{2}$

$0 \leq \delta \leq 2^{-t}$ $\rightarrow \underbrace{\underbrace{\delta}_{\leq \frac{1}{2^t}} + \frac{1}{2} - \frac{1}{2^t}}_{\frac{1}{2} \geq} \geq \delta - \frac{x'}{2^t} \geq -\frac{1}{2} + \underbrace{\delta}_{\geq 0} \geq -\frac{1}{2}$

$\Rightarrow 2\pi \times \frac{1}{2} \geq 2\pi\left(\delta - \frac{x'}{2^t}\right) \geq -\frac{1}{2} \times 2\pi$ ✓

$$\Rightarrow |\alpha(x'+b)| \leq \frac{2}{2^t \times 2 \times 2\pi\left[\delta - x'/2^t\right]/\pi}$$

$$= \frac{1}{2^{t+1}\left[\delta - x'/2^t\right]}$$

$$= \frac{1}{2\left[2^t\delta - x'\right]}$$

$$\Rightarrow P \leq \frac{1}{4}\left(\sum_{x'=-(2^{t-1}-1)}^{-(k_\varepsilon+1)} \frac{1}{(\underbrace{2^t\delta}_{\in[0,1]} - \underbrace{x'}_{<0})^2} + \sum_{y=k_\varepsilon+1}^{2^{t-1}} \frac{1}{(\underbrace{2^t\delta}_{\in[0,1]} - \underbrace{x'}_{>0})^2} \right)$$

$$\leq \frac{1}{4}\left(\sum_{x=-(2^{t-1}-1)}^{-(k_\varepsilon+1)} \frac{1}{x^2} + \sum_{x=k_\varepsilon+1}^{2^{t-1}} \frac{1}{(x-1)^2} \right)$$

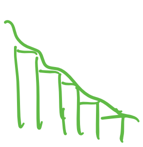
$$\underbrace{\sum_{x=k_\varepsilon+1}^{2^{t-1}-1} \frac{1}{x^2} \qquad\qquad \sum_{x=k_\varepsilon}^{2^{t-1}-1} \frac{1}{x^2}}$$

$$= 2\sum_{x=k_\varepsilon}^{2^{t-1}-1} \frac{1}{x^2} - \left(\frac{1}{k_\varepsilon^2}\right)$$

↳ to remove dble counting

$$\leq 2\sum_{x=\varepsilon}^{2^{t-1}-1} \left(1/x^2\right)$$

$$\leq \frac{1}{2}\sum_{x=k_\varepsilon}^{2^{t-1}-1} \frac{1}{x^2} \leq \frac{1}{2}\int_{k_\varepsilon-1}^{2^{t-1}-1} \frac{dx}{x^2} = \frac{1}{2}\left[-\frac{1}{x}\right]_{k_\varepsilon-1}^{2^{t-1}-1}$$



$$= \frac{1}{2}\left(\underbrace{\frac{-1}{(2^{t-1}-1)}}_{<0} + \frac{1}{k_\varepsilon-1} \right) \leq \frac{1}{2(k_\varepsilon-1)}$$

$\Rightarrow$ the probability of obtaining an outcome $|x\rangle$ with $|x-b| > k_\varepsilon$ is smaller than $\dfrac{1}{2(k_\varepsilon-1)}$.

or equivalently the proba of obtaining an outcome $|x\rangle$ with

$$|x-b| \leq k_\varepsilon$$

is greater than $1 - \dfrac{1}{2(k_\varepsilon-1)}$.

if we want to approximate θ to an accuracy 2^{-n}

meaning: $\theta \approx \theta^{approx} = \frac{\theta_{n-1}}{2} + \dots + \frac{\theta_0}{2^n}$

$$\Rightarrow 2^t \theta^{approx} = \theta_{n-1} 2^{t-1} + \dots + \theta_0 2^{t-n}$$

while we have $b = b_{t-1} 2^{t-1} + \dots b_1 2 + b_0$.

we want $|2^t \theta^{approx} - b| \leq k_\varepsilon$

$$\Rightarrow \left| \underbrace{\left(\theta_{n-1} 2^{n-1} + \dots + \theta_0 2^{t-n}\right)}_{} - \left(\underbrace{b_{t-1} 2^{t-1} + \dots + b_{t-n} 2^{t-n}}_{} \right.\right.$$
$$\left.\left. + \, b_{t-n-1} 2^{t-n-1} + \dots + b_0 \right) \right| \leq k_\varepsilon$$

$$\Rightarrow |b_{t-n-1} 2^{t-n-1} + \dots + b_0| \leq k_\varepsilon$$

$\Rightarrow$ choose k_ε = larger integer that can be represented on $t-n$ bits:

$$k_\varepsilon = 2^{t-n-1} + 2^{t-n-2} + \dots 2 + 1$$

$$= \sum_{j=0}^{t-n-1} 2^j = \frac{1 - 2^{t-n}}{1-2} = 2^{t-n} - 1$$

$\Rightarrow$ from what we have derived above the proba of obtaining an approximation of θ correct to this accuracy is greater than

$$1 - \frac{1}{2(u_{\varepsilon}-1)} = 1 - \frac{1}{2(2^{t-n}-2)}$$

for $(t-n \geq 2)$

Therefore if we want to successfully obtain θ accurate to n bits with proba of success at least $1-\epsilon$, we choose t such that

$$1 - \frac{1}{2(2^{t-n}-2)} = 1 - \epsilon$$

$$\frac{1}{2^{t-n}-2} = 2\epsilon$$

$$2^{t-n} - 2 = \frac{1}{2\epsilon}$$

$$2^{t-n} = 2 + \frac{1}{2\epsilon}$$

$$\Rightarrow \quad t - n = \log_2\left(2 + \frac{1}{2\epsilon}\right)$$

$$\Rightarrow \quad \boxed{t = n + \log_2\left(2 + \frac{1}{2\epsilon}\right)}$$

$\Rightarrow$ there is a trade-off btw precision (accuracy) & proba of success.

Summary of QPE

Inputs: (1) "Black box" which can perform $C(U^{2^j})$ operations.

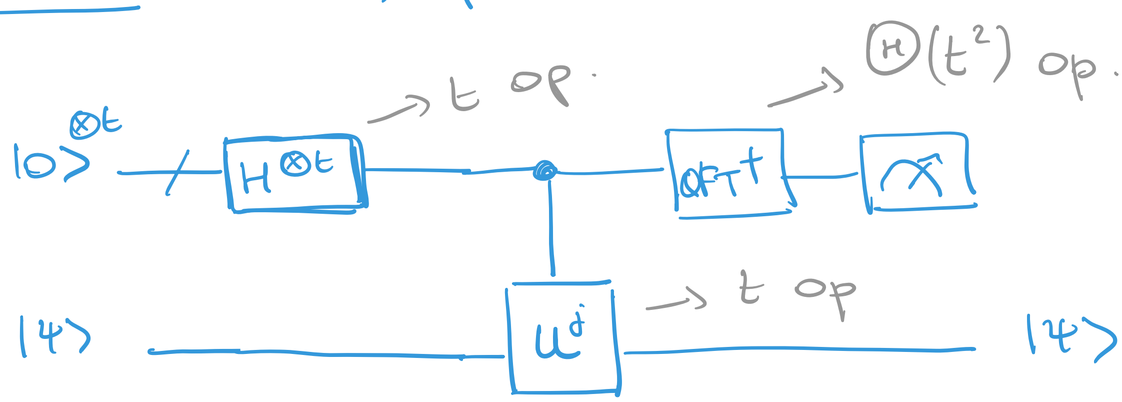
(2) eigenstate $|\psi\rangle$ of $\hat{U}$ with eigenvalue $e^{(2\pi i)\theta}$

(3) $t = n + \log(2 + 1/2\epsilon)$ qubits initialized to $|0\rangle$

where $\begin{cases} n = \text{accuracy on } \theta \text{ we want} \\ 1-\epsilon = \text{proba of success} \end{cases}$

Output: $|2^t \theta\rangle$ = n-bit approximation to θ.

Runtime: $\Theta(t^2)$ operations



<u>Note:</u> in the beginning we have assumed that we are able to build $|\psi\rangle$ eigenstate of U.

What if we don't know how to prepare it?

Suppose instead we prepare (bottom register)

$$|\Phi\rangle = \sum_{(\mu)} c^{(\mu)} |\psi_\mu\rangle$$

↳ eigenstates of $\hat{U}$ with eigenvalue $\lambda_\mu = e^{2\pi i \theta^{(\mu)}}$

Intuitively the result of running QPE will give an output state close to

$$\sum_\mu c^{(\mu)} \; |\psi_\mu\rangle \otimes |2^t \theta^{(\mu)}\rangle$$

When we meas. the top register, the proba of getting state $|2^t \theta^{(\mu)}\rangle$ is $|c^{(\mu)}|^2$

$\Rightarrow$ proba of measuring $\theta^{(\mu)}$ accurate to n bits is:

$$|c^{(\mu)}|^2 \times \underbrace{(1-\epsilon)}_{\text{success proba of QPE.}}$$

While QPE is in principle efficient, the implementation of QPE discussed above is not yet suitable for implementation on current quantum computers (QC).

One issue is that QPE requires many qubits to get θ to high accuracy,

but this is becoming less of a problem as many QC now have ~ 100 qubits.

The main issue is the noise, due to the interaction of the QC with the environment, which leads to error. This error can be propagated and even enhanced via 2-qubit controlled gates (will be discussed later in the semester). Since the QPE circuit has many controlled gates, the error can rapidly explode, making the computation useless.

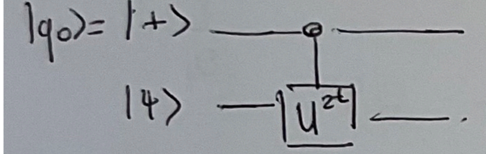
One possible alternative is the <u>iterative quantum phase estimation</u> which uses only one ancilla qubit and uses mid-circuit measurements to determine the bit values representing θ one after the other.

(see below and tutorial 6)

Iterative phase estimation. $\to$ 1 ancilla qubit only.

$$U|\psi\rangle = e^{2\pi i \theta}|\psi\rangle \qquad \theta = .(\theta_{n-1}\cdots\theta_0) = \frac{\theta_{n-1}}{2} + \cdots + \frac{\theta_0}{2^n}$$

ex $|\psi\rangle$ = 1 qubit. state



$|0\rangle + e^{2\pi i 2^t \theta}|1\rangle$ (phase kickback)

$|\psi\rangle$

$t=0 \Rightarrow$ phase $e^{2\pi i\theta} = e^{2\pi i(.\theta_{n-1}\theta_{n-2}\cdots\theta_0)}$

$t=1 \Rightarrow$ phase $e^{4\pi i\theta} = e^{2\pi i(.\theta_{n-2}\cdots\theta_0)}$ $\rightsquigarrow$ eliminates θ_{n-1}

etc

$t=n-1 \Rightarrow e^{i\pi\theta_0} \to$ state $|q_0\rangle = |\pm\rangle$.

Step 1 : apply $U^{2^{n-1}} \Rightarrow$ measure $|q_0\rangle$ in $|\pm\rangle$ basis (ie apply H + measure in comp basis)

$\Rightarrow$ know $\theta_0 = 0$ or 1.

Step 2 : apply U 2^{n-2} times

$$|q_0\rangle = |0\rangle + e^{2\pi i\left(\frac{\theta_1}{2}+\frac{\theta_0}{4}\right)}|1\rangle$$

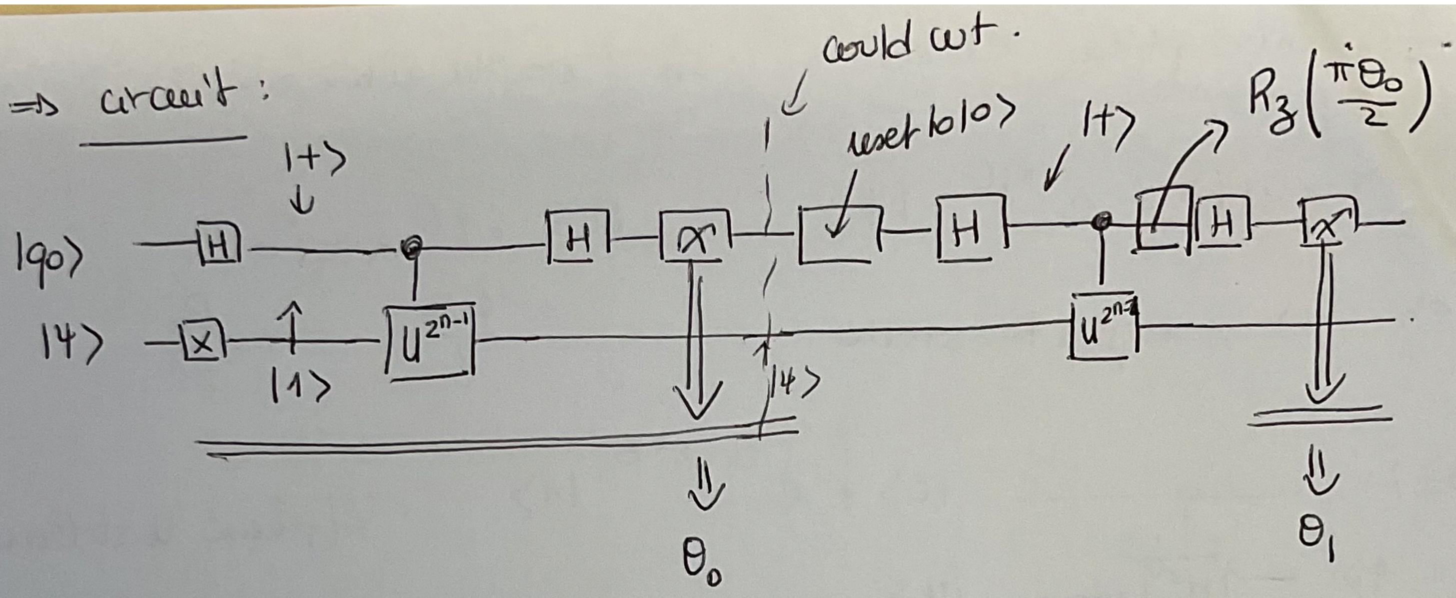
now apply $\underbrace{U\left(\vec{e}_z, -\frac{\pi\theta_0}{2}\right)}_{\begin{pmatrix}1 & \\ & e^{\frac{-i\pi\theta_0}{2}}\end{pmatrix}}|q_0\rangle = |0\rangle + e^{\pi i\theta_1}\underbrace{e^{\frac{\pi i\theta_0}{2}}e^{\frac{-i\pi\theta_0}{2}}}_{1}|1\rangle$

$= |\pm\rangle$ depending on θ_1

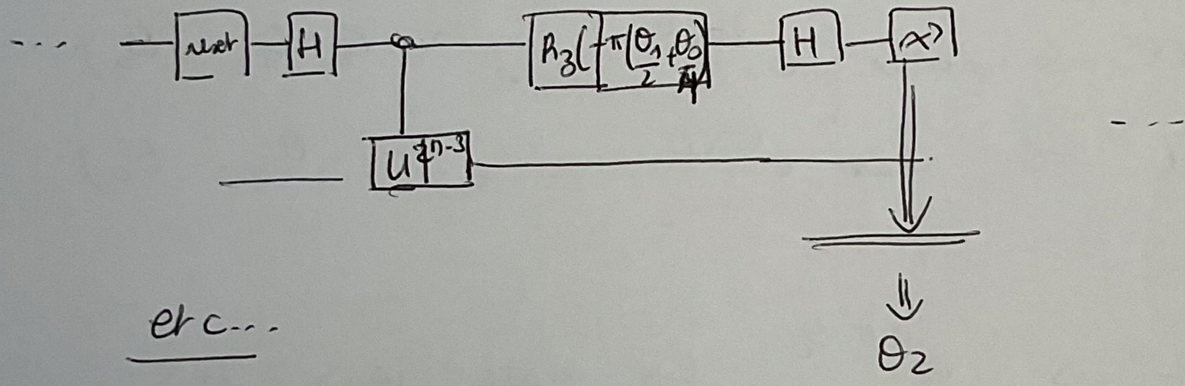
$\Rightarrow$ apply H + measure $\Rightarrow \theta_1$

Step k : apply U 2^{n-k} times, rotation around z angle $-\pi.(\theta_{k-2}\cdots\theta_0)$, Hadamard $\Rightarrow \theta_{k-1}$

⇒ circuit :



could cut.
reset to |0>
|+>
$R_z\left(\frac{\pi\theta_0}{2}\right)$
|+>
$|q_0\rangle$
H
H
$|\psi\rangle$
X
$U^{2^{n-1}}$
$U^{2^{n-2}}$
|1>
|ψ>
θ_0
θ_1



reset
H
$R_z\left(\pi\left(\frac{\theta_1}{2}+\frac{\theta_0}{4}\right)\right)$
H
$U^{2^{n-3}}$
θ_2

etc...