3) Multiple Qubits and Entanglement.

* 2-qubit system

Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$ spanned by

$|0\rangle \otimes |0\rangle, \quad |0\rangle \otimes |1\rangle, \quad |1\rangle \otimes |0\rangle, \quad |1\rangle \otimes |1\rangle$

$\equiv \quad |00\rangle, \quad |01\rangle, \quad |10\rangle, \quad |11\rangle$

$\Rightarrow$ Arbitrary 2-qubit state :

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$$
$$\equiv \begin{pmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{pmatrix}$$

with $\langle\psi|\psi\rangle = 1 \Rightarrow \sum_{\substack{i=0,1 \\ j=0,1}} |a_{ij}|^2 = 1$

Note : we will typically consider that qubits are distinguishable. (which is the case in a QC)

* N- qubit system

Generally, the Hilbert space for a N-qubit system is

$$\mathbb{C}^{2^N} = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{N \text{ times}}$$

with orthonormal basis states $|x_{N-1} x_{N-2} \dots x_1 x_0\rangle$

where $x_i = 0$ or 1.

$$(|00\dots00\rangle, |00\dots01\rangle, |00\dots10\rangle \dots |11\dots11\rangle)$$

Since the basis states are labeled by binary strings, we can write each of them as:

$$|x\rangle$$

where $x = x_{N-1} \cdot 2^{N-1} + x_{N-2} \cdot 2^{N-2} + \dots + x_1 \cdot 2^1 + x_0 \cdot 2^0$

is the number represented by the binary string

For example,

$x_{N-1} \dots x_1 \, x_0$		x
$00\dots00$	$\longrightarrow$	0
$00\dots01$	$\longrightarrow$	1
$00\dots11$	$\longrightarrow$	$1\times 2 + 1 = 3.$
$11\dots11$	$\longrightarrow$	$1.\,2^{N-1} + 1.\,2^{N-2} + \dots + 1.\,2 + 1 = 2^N - 1$

A general arbitrary N-qubit state can then be written as:

$$\boxed{|\psi\rangle = \sum_{x=0}^{2^N-1} a_x \, |x\rangle}$$

Note

The number of basis states is 2^N, i.e. it grows exponentially with N

for $N = 300 \rightarrow$ this is $2^{300} \sim 10^{90}$

$\Rightarrow$ would need 10^{90} classical basis states to represent a 300-qubit state which is more than the number of atoms in the entire universe!

$\Rightarrow$ In general, there is no simple/efficient classical representation
of quantum states, which is why
it is difficult to simulate a quantum system with a classical computer.

The reason behind this expo. growth is in fact the phenomenon of entanglement, which is a type of correlations that can exist between quantum systems (e.g. qubits) when they interact with each other and which has no classical analogous (purely quantum).

$\boxed{\text{Entanglement:}}$

* If we have quantum systems (here qubits) that don't interact, they can be described independently from each other, in which case the N-qubit state $|\psi\rangle$ can be written as a tensor product state:

$$|\psi\rangle = |\phi_1\rangle_{A_1} \otimes |\phi_2\rangle_{A_2} \otimes \dots \otimes |\phi_N\rangle_{A_N}$$

(A_i label qubits)

This typically occurs when the qubits are separated by large distances (in space).

This is because interactions in physics are usually local, i.e. systems interact when they are close to each others.

The tensor product state can be one computational-basis state $|x\rangle$, but it can also be a tensor product of 1-qubit superposition states, e.g.:

$$|\psi\rangle = |+\rangle \otimes |-\rangle \otimes \cdots \otimes |+\rangle$$

It is easy to prepare a particular tensor product state since it only requires to act <u>locally</u> on each qubit.
So it can be done by having N parties, each acting on their individual qubit (no need to make the qubits interact).

$\Rightarrow$ such a state is <u>not entangled</u> and has a simple classical description: only need $2N$ real parameters to describe it (2 per qubit (θ, φ)).

* But in general, if they interact, the qubits become entangled and cannot be described independently, hence the complicated representation

$$|\psi\rangle = \sum_{x=0}^{2^N-1} a_x |x\rangle$$

If a state $|\psi\rangle$ cannot be written as a product state (in any basis) then $|\psi\rangle$ is said to be entangled

(meaning that the subsystems $A_1, A_2 \ldots A_N$ are entangled with each others)

Such state cannot be created by remote parties acting locally on their own qubit (even if they communicate classically, e.g. phone calls ...)

The only way to entangle qubits is to bring them close to each other so they can interact.

Once they are entangled they can be sent far away from each others and will remain entangled.

$\Rightarrow$ "non-local correlations"

Examples

1) $|\psi\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$

Is $|\psi\rangle$ entangled ?

ie can we write $|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$?

$$|\phi_1\rangle = a|0\rangle + b|1\rangle$$
$$|\phi_2\rangle = c|0\rangle + d|1\rangle$$

$$\Rightarrow |\phi_1\rangle \otimes |\phi_2\rangle = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

$$\Rightarrow |\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$$

if $ac = ad = bc = bd = 1/2$.

$\Rightarrow a = b = c = d = 1/\sqrt{2}$

$$\Rightarrow |\psi\rangle = |+\rangle \otimes |+\rangle$$

where $|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

$\Rightarrow |\psi\rangle$ is not entangled.

2) $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$|\Phi^+\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$

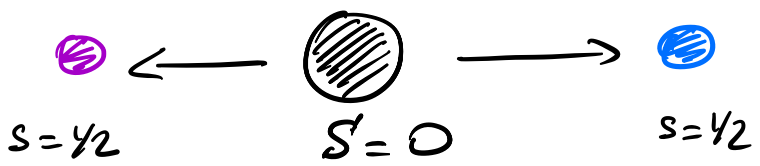
if $\begin{cases} ac = bd = 1/\sqrt{2} \\ ad = bc = 0 \end{cases}$

$\Rightarrow$ this is not possible

$\Rightarrow$ $|\Phi^+\rangle$ cannot be written as a tensor product and thus is entangled.

Physics example: Entanglement generation in particle decay.

Decay of a spin-0 particle into two spin-1/2 particles:



Because of conservation laws (angular momenta) the emitted particles remain in an entangled superposition of spin states

$$\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \equiv \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

even when separated by arbitrarily large distances.

⇒ they cannot be described independently from each other.

→ this is entanglement

⇒ <u>Consequence</u>:

if we measure the spin projection (σ_z) of one particle we will know with certainty the spin projection of the other:

if measure $|0\rangle$ $\xrightarrow[\text{(collapse)}]{}$ other is in state $|1\rangle$

and vice versa.

<u>Note</u>: The state $\dfrac{|01\rangle - |10\rangle}{\sqrt{2}}$

is called a "Bell state" or "EPR pair".

↳ J. Bell (1964)

Einstein
Podolsky
Rosen
(1935)

There are 4 Bell states:

$$\boxed{\begin{aligned} |\Psi^{\pm}\rangle &\equiv \frac{|01\rangle \pm |10\rangle}{\sqrt{2}} \\ |\Phi^{\pm}\rangle &\equiv \frac{|00\rangle \pm |11\rangle}{\sqrt{2}} \end{aligned}}$$

They are the simplest example of (maximally) entangled systems.

Now, How to entangle qubits on a quantum computer ?

* <u>Two-qubit gates</u>

= unitary op. that act collectively on 2 qubits

CX / CNOT = conditional gate that performs an X gate on the <u>target</u> qubit if the <u>control</u> qubit is in state $|1\rangle$, and does nothing otherwise

("controlled-X" or "controlled-NOT")

Action on computational basis states :

$$\Rightarrow \text{CNOT}\,(\underset{\text{control}}{|x_1\rangle} \otimes \underset{\text{target}}{|x_0\rangle}) = \begin{cases} |x_1\rangle \otimes |x_0\rangle & \text{if } |x_1\rangle = |0\rangle \\ |x_1\rangle \otimes |1-x_0\rangle & \text{if } |x_1\rangle = |1\rangle \end{cases}$$

$(|x_i\rangle = |0\rangle$ or $|1\rangle)$

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\hat{\text{CNOT}} = |0\rangle\langle 0| \otimes \hat{\mathbb{1}} + |1\rangle\langle 1| \otimes \hat{X}$$

⚠ If the target is $|x_1\rangle$ & the control is $|x_0\rangle$

$$\Rightarrow \text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\widehat{\text{CNOT}} = \hat{\mathbb{1}} \otimes |0\rangle\langle 0| + \hat{X} \otimes |1\rangle\langle 1|$$

Can easily verify that CNOT acts on 2-qubit column vectors as described above:

if $|x_1\rangle = \begin{pmatrix} a \\ 1-a \end{pmatrix}$ $(a = 0 \text{ or } 1)$

target

and * $|x_0\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (ctrl)

$$|x_1\rangle \otimes |x_0\rangle = \begin{pmatrix} a \\ 0 \\ 1-a \\ 0 \end{pmatrix} \Rightarrow \text{CNOT} \begin{pmatrix} a \\ 0 \\ 1-a \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 1-a \\ 0 \end{pmatrix}$$

→ unchanged

* if $|x_0\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\Rightarrow |x_1\rangle \otimes |x_0\rangle = \begin{pmatrix} 0 \\ a \\ 0 \\ 1-a \end{pmatrix}$

and $\text{CNOT} \begin{pmatrix} 0 \\ a \\ 0 \\ 1-a \end{pmatrix} = \begin{pmatrix} 0 \\ 1-a \\ 0 \\ a \end{pmatrix} = \begin{pmatrix} 1-a \\ a \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(Note: Here we organize qubits as $|x_{n-1}\, x_{n-2} \cdots x_1\, x_0\rangle$ – Some textbooks use the convention $|x_0\, x_1 \cdots x_{n-1}\rangle$ → then the matrices look $\neq$)

Action of CNOT on qubits in superposition?

$$\mathrm{CNOT}\Big(\underbrace{\underset{\substack{\uparrow \\ \text{ctrl}}}{|+\rangle} \otimes \underset{\substack{\uparrow \\ \text{target}}}{|0\rangle}}_{\substack{\text{tensor product state} \\ \text{(unentangled)}}}\Big) = \mathrm{CNOT}\left[\left(\frac{1}{\sqrt{2}}\,|0\rangle + |1\rangle\right)\otimes|0\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\,\mathrm{CNOT}\Big[\,|00\rangle + |10\rangle\,\Big]$$

$$= \underbrace{\frac{1}{\sqrt{2}}\Big[\,|00\rangle + |11\rangle\,\Big]}_{\substack{\text{Bell state} \\ \text{(entangled)}}}$$

⇒ the CNOT gate can generate entanglement because it represents a global action on both qubits.

(<u>Note</u> : cannot just tell via phone call if the ctrl qubit is in $|0\rangle$ & $|1\rangle$ because it is in a superpo.

The situation is different when acting on computational-basis states (as previously) because in that case one can tell if ctrl is in $|0\rangle$ or $|1\rangle$ with def. proba & call the other observer who has the target qubit

but in that case we do not create entanglement)

In general we can define a "controlled-U" gate which applies a 1-qubit U gate to the target qubit if the control qubit is in state $|1\rangle$.

$\Rightarrow$ corresponding matrix in computational basis: (acting on $|x_1 x_0\rangle$)

$$\begin{array}{cc} \uparrow & \uparrow \\ \text{ctrl} & \text{target} \end{array}$$

$$\begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & 0 \\ & & U_{11} & U_{12} \\ & 0 & U_{21} & U_{22} \end{pmatrix}$$

operator: $|0\rangle\langle 0| \otimes \hat{1} + |1\rangle\langle 1| \otimes \hat{U}$

* Measurements :

- local measurements :

can measure a single-qubit, for instance along z axis — The corresponding observable is $\mathbb{1} \otimes \sigma_z$ (if we measure the qubit on the right)

$$\equiv \mathbb{1} \otimes Z$$

2 eigenvalues ± 1 → eigenstates $|00\rangle, |10\rangle$
→ —— $|01\rangle, |11\rangle$.

Projectors :

$$\begin{cases} \hat{P}_{+1} = |00\rangle\langle 00| + |10\rangle\langle 10| = \hat{\mathbb{1}} \otimes |0\rangle\langle 0| \\ \hat{P}_{-1} = |01\rangle\langle 01| + |11\rangle\langle 11| = \hat{\mathbb{1}} \otimes |1\rangle\langle 1| \end{cases}$$

⇒ proba for the correspdg outcomes :

$$p(+1) = \langle\psi|\hat{P}_{+1}|\psi\rangle = |a_{00}|^2 + |a_{10}|^2$$

$$p(-1) = \langle\psi|\hat{P}_{-1}|\psi\rangle = |a_{01}|^2 + |a_{11}|^2$$

2-qubit state collapse to $\dfrac{(a_{00}|0\rangle + a_{10}|1\rangle)}{\sqrt{p(|0\rangle)}} \otimes |0\rangle$

⇒ destroys entanglement.

- ## Global measurements :

can also do collective measurement of the 2-qubit system:

$\Rightarrow$ Observable $= \sigma_z \otimes \sigma_z = Z \otimes Z$

Projectors

$$\hat{P}_{+1} = |00\rangle\langle 00| + |11\rangle\langle 11|$$

$$\hat{P}_{-1} = |01\rangle\langle 01| + |10\rangle\langle 10|$$

$$p(+1) = |a_{00}|^2 + |a_{11}|^2$$

$\hookrightarrow$ post - meas^t state

$$|\psi\rangle_{post} = \frac{P_{+1}|\psi\rangle}{\|P_{+1}|\psi\rangle\|} = \frac{a_{00}|00\rangle + a_{11}|11\rangle}{\sqrt{|a_{00}|^2 + |a_{11}|^2}}$$

$\Rightarrow$ can preserve (even create) entanglement.

* Three - qubit gates

Toffoli or CCNOT gate :

if the two control qubits are in state $|1\rangle$ then the X gate is applied to the target qubit.

$$\text{CCNOT}\left(|\underset{\text{ctrl}}{x_2\, x_1}\, \underset{\text{target}}{x_0}\rangle\right) = \begin{cases} |x_2\, x_1\, 1-x_0\rangle \text{ if} \\ \quad |x_2\rangle = |1\rangle \text{ and} \\ \quad |x_1\rangle = |1\rangle \\ |x_2\, x_1\, x_0\rangle \text{ otherwise} \end{cases}$$

$$\Rightarrow \underset{(8 \times 8)}{\text{Matrix}} = \begin{pmatrix} \begin{pmatrix} \mathbb{1}_{6\times 6} \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$$

entangled 3-qubit states:

$$|GHZ\rangle_{\pm} = \frac{1}{\sqrt{2}}\left(|000\rangle \pm |111\rangle\right)$$

$$|W\rangle = \frac{1}{\sqrt{2}}\left(|001\rangle + |010\rangle + |100\rangle\right)$$

these states cannot be written as

$$|\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle$$

or $|\Phi_{12}\rangle \otimes |\phi_3\rangle$

$\Rightarrow$ they have genuine 3-qubit entanglement.

can use CCNOT to prepare them, but interestingly this is not needed: H & CNOT gates are sufficient.

- Measuring one of the qubits of the GHZ state leads to the collapse of the state into

$|000\rangle$ or $|111\rangle$

which are unentangled (product states).

⇒ entanglement is completely destroyed if one of the qubit is lost.

- If we measure e.g. the right qubit of the W state, the 3-qubit state collapses to

$$|001\rangle \text{ or } \frac{1}{\sqrt{2}} |010\rangle + |100\rangle$$

$$= \frac{1}{\sqrt{2}} \underbrace{(|01\rangle + |10\rangle)}_{\substack{\text{Bell state} \\ \text{(entangled)}}} \otimes |0\rangle$$

→ thus if we measure one-qubit, the remaining 2-qubit system can retain entanglement.

This is not the case for the GHZ state.
Thus the GHZ state only contain genuine 3-qubit entanglement while the W state contains both 2-qubit and 3-qubit entanglement.

We note how "fragile" entanglement can be. If one of the qubit in the GHZ state interacts with the environment, entanglement and thus information is lost.