② Example: Quantum Teleportation

Quantum Teleportation = technique for moving quantum states around in space, using classical communication.

Suppose we have 2 parties A (Alice) and B (Bob) in 2 $\neq$ locations.

Alice has a qubit in her possession which is in a superposition state

$$\rightarrow |\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\varphi}\sin(\theta/2)\,|1\rangle$$
$$= a\,|0\rangle + b\,|1\rangle$$

She wants to send this qubit to Bob, but she can only send him classical info.

This seems impossible because Alice does not know the state $|\psi\rangle$
(she only has one copy so cannot measure and determine θ & φ)
and even if she knew $|\psi\rangle$, describing

it with perfect precision would take an infinite amount of classical info. because $\theta, \varphi \in \mathbb{R}$.

But now suppose that A & B share an entangled pair of qubits in a Bell state

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|\underset{A}{0}\underset{B}{0}\rangle + |\underset{A}{1}\underset{B}{1}\rangle\right).$$

(can e.g. be prepared by a 3rd observer who sends 1 qubit to Bob and the other to Alice).

Q.T. uses this Bell pair to move $|\psi\rangle$ from A to B by sending only bits of classical info.

The steps are :

① Initially the joint state of the 3 qubits is:

$$\underline{|\psi\rangle} \otimes \underline{|\Phi^+\rangle}_{AB}$$

$$= \underbrace{(a|0\rangle + b|1\rangle)}_{A} \otimes \frac{1}{\sqrt{2}}\left(|\underset{A}{0}\,\underset{B}{0}\rangle + |\underset{A}{1}\,\underset{B}{1}\rangle\right)$$

$$= \frac{a}{\sqrt{2}}\,\underbrace{|\overset{\text{ctrl}}{0}\,\overset{\text{target}}{0}\rangle}_{A} \otimes \underbrace{|0\rangle}_{B} + \frac{a}{\sqrt{2}}\,\underbrace{|01\rangle}_{A} \otimes \underbrace{|1\rangle}_{B}$$

$$+ \frac{b}{\sqrt{2}}\,\underbrace{|10\rangle}_{A} \otimes \underbrace{|0\rangle}_{B} + \frac{b}{\sqrt{2}}\,\underbrace{|11\rangle}_{A} \otimes \underbrace{|1\rangle}_{B}.$$

② Alice interacts her qubit $|\psi\rangle$ with her half of the Bell pair

with a CNOT (CX) gate (with $|\psi\rangle$ the control)

followed by a $\underline{H}$ gate on the left qubit.

$\Rightarrow$ the new 3-qubit state after CNOT :

$$\frac{a}{\sqrt{2}}|00\rangle \otimes |0\rangle + \frac{a}{\sqrt{2}}|01\rangle \otimes |1\rangle$$

$$+ \frac{b}{\sqrt{2}}|11\rangle \otimes |0\rangle + \frac{b}{\sqrt{2}}|10\rangle \otimes |1\rangle$$

after H gate.

$$\frac{a}{\sqrt{2}}|+0\rangle \otimes |0\rangle + \frac{a}{\sqrt{2}}|+1\rangle \otimes |1\rangle$$

$$+ \frac{b}{\sqrt{2}}|-1\rangle \otimes |0\rangle + \frac{b}{\sqrt{2}}|-0\rangle \otimes |1\rangle$$

$$= \frac{a}{2}\left(|00\rangle + |10\rangle\right) \otimes |0\rangle$$

$$+ \frac{a}{2}\left(|01\rangle + |11\rangle\right) \otimes |1\rangle$$

$$+ \frac{b}{2}\left(|01\rangle - |11\rangle\right) \otimes |0\rangle$$

$$+ \frac{b}{2}\left(|00\rangle - |10\rangle\right) \otimes |1\rangle$$

$$= \frac{1}{2} \underline{\underline{|00\rangle}} \otimes \left[a|0\rangle + b|1\rangle \right]$$

$$+ \frac{1}{2} \underline{\underline{|01\rangle}} \otimes \left[a|1\rangle + b|0\rangle \right]$$

$$+ \frac{1}{2} \underline{\underline{|10\rangle}} \otimes \left[a|0\rangle - b|1\rangle \right]$$

$$+ \frac{1}{2} \underline{\underline{|11\rangle}} \otimes \left[a|1\rangle - b|0\rangle \right]$$

Alice Bob.

③ Alice measures her 2 qubits in the computational basis

→ she obtains 2 classical bits:

00, 01, 10, 11.

and sends these 2 classical bits to Bob.

④ Then Bob knows what operations he has to do to recover the original qubit state

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

* if A sends (00) $\longrightarrow$ B does nothing.
* ______ (01) $\longrightarrow$ B applies X to his qubit
* ______ (10) $\longrightarrow$ B applies Z
* ______ (11) $\longrightarrow$ B applies X and Z :

$$\left(a|1\rangle - b|0\rangle \xmapsto{X} a|0\rangle - b|1\rangle \xmapsto{Z} a|0\rangle + b|1\rangle = |\psi\rangle \right)$$

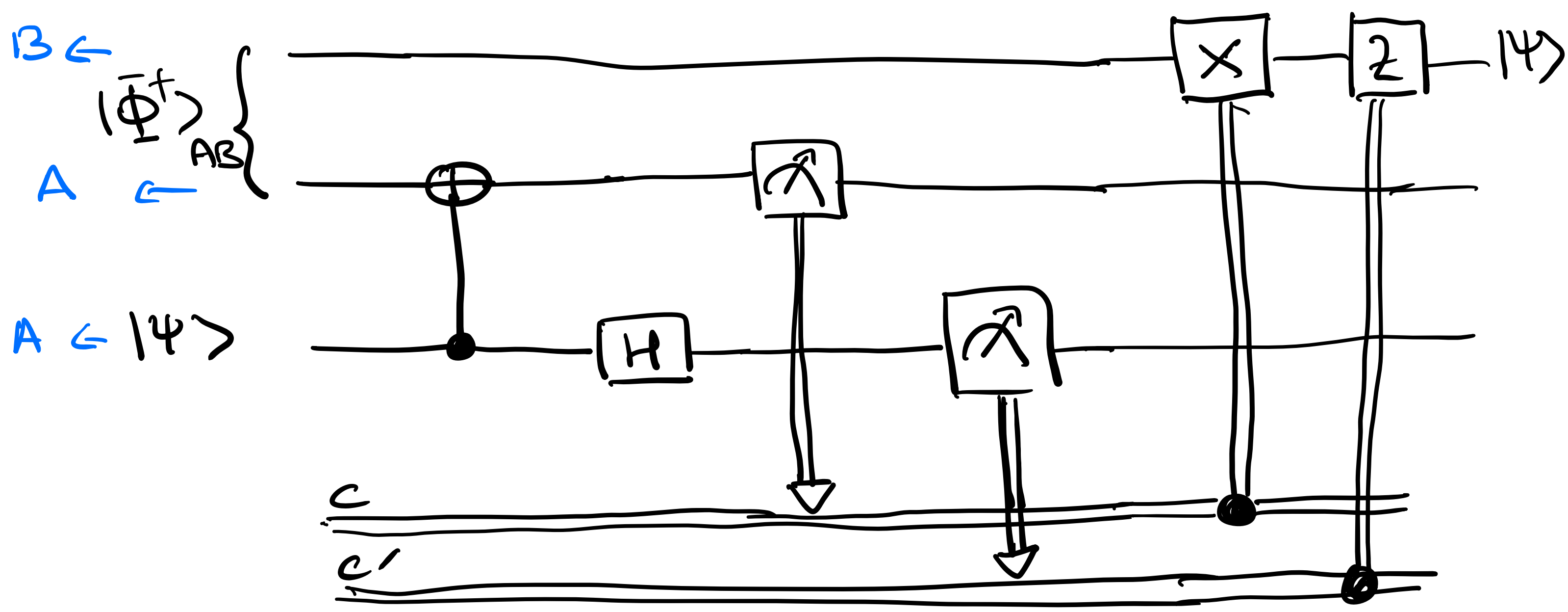
This is an amazing result.
We used entanglement in order to move a quantum state from A to B, by applying only local operations (on each qubit individually) and classical communication (LOCC).

In principle,
This is true for any distance separating Alice and Bob – as long as classical communication is possible.

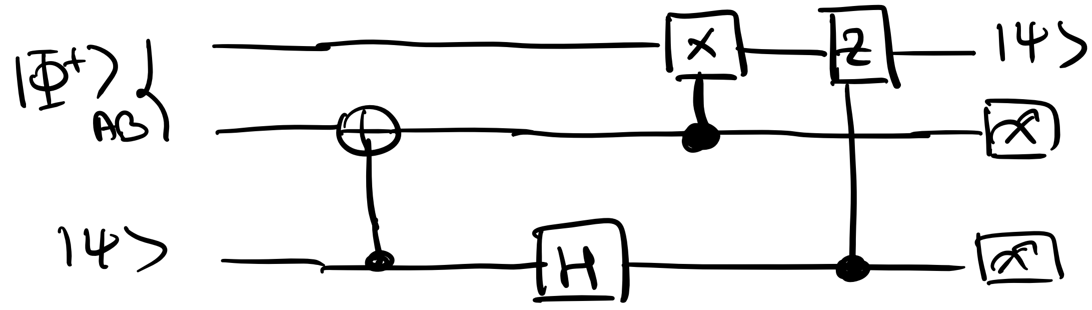
Note : * The qubit on Alice's side has been destroyed : Q. teleportation moves the qubit state from A to B but does not copy.

(copying an unknown qubit is impossible → "no-cloning thm" (see tutorial #1))

Circuit :



* Using the "principle of deferred measurement", we can also rewrite the circuit as:



can check that this gives the same result.

⚠ But here no classical info is transmitted (only quantum info)
$\Rightarrow$ cannot be achieved if A & B are far away
$\Rightarrow$ interpretation as "teleportation" is lost.