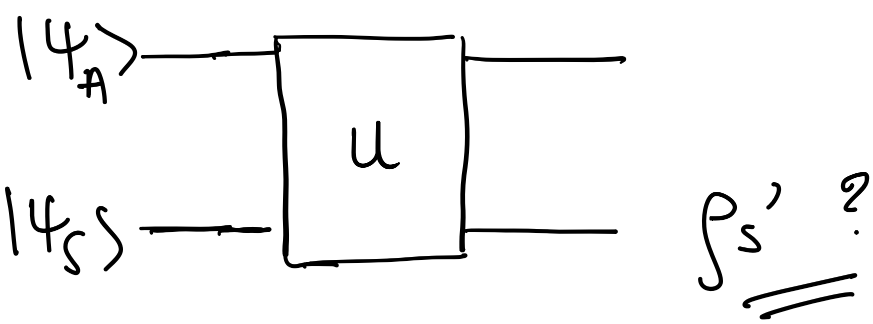
② Quantum Channels

We have seen that a pure state of a bipartite system SA may behave like a mixed state when we observe subsystem S alone, and that an orthogonal measurement of the bipartite system can realize a non orthogonal generalized measurement on S alone.

Now we ask: if a state of SA undergoes unitary evolution, how do we describe the evolution of S alone?



Generally, U will entangle A & S.

For example consider

$$\hat{U}: \underbrace{(\alpha|0\rangle + \beta|1\rangle)_S}_{|\psi_S\rangle} \otimes |0\rangle_A \longmapsto \alpha|00\rangle + \beta|11\rangle$$

which can be written as

$$\hat{M}_0 |\psi_S\rangle |0\rangle_A + \hat{M}_1 |\psi_S\rangle |1\rangle_A$$

where $\hat{M}_0 = \frac{1+Z}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\hat{M}_1 = \frac{1-Z}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

or in other basis for A, e.g. $|\pm\rangle_A$:

$$\frac{1}{\sqrt{2}} \left(\alpha |0\rangle_S + \beta |1\rangle_S \right) \otimes |+\rangle_A$$
$$+ \frac{1}{\sqrt{2}} \left(\alpha |0\rangle_S - \beta |1\rangle_S \right) \otimes |-\rangle_A .$$

$$= \hat{M}_+ |\psi_S\rangle |+\rangle_A + \hat{M}_- |\psi_S\rangle |-\rangle_A$$

where $\hat{M}_+ = \frac{1}{\sqrt{2}} \mathbb{1}$, $\hat{M}_- = \frac{1}{\sqrt{2}} Z$

This can be generalized:

$$\left[|\psi_S\rangle \otimes |\psi_A\rangle \underset{\text{unitary evol}}{\longmapsto} \sum_\mu \left(\hat{M}_\mu |\psi_S\rangle \right) \otimes |\mu_A\rangle \right]$$
$$\equiv |\Phi_{SA}\rangle$$

where $\{|\mu\rangle_A\}$ denotes an orthonormal basis for A.

(This is the Schmidt form for the entangled state)

The normalization of $|\Phi\rangle_{SA}$ leads to

$$\langle \Phi_{SA} | \Phi_{SA} \rangle = 1$$

$$= \sum_{\mu\nu} \underbrace{\langle \mu_A | \nu_A \rangle}_{\delta_{\mu\nu}} \langle \psi_S | M_\mu^\dagger M_\nu | \psi_S \rangle$$

$$= \sum_\mu \langle \psi_S | M_\mu^\dagger M_\mu | \psi_S \rangle$$

$$\Rightarrow \boxed{\sum_\mu M_\mu^\dagger M_\mu = \mathbb{1}.}$$

Now the state of S alone after unitary evolution of SA is obtained by tracing over A:

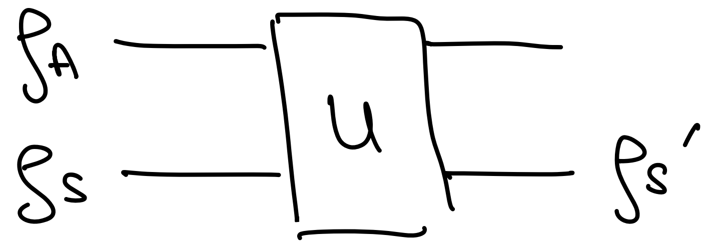
$$\rho_S' = \mathrm{Tr}_A \left(|\Phi_{SA}\rangle \langle \Phi_{SA}| \right)$$

$$= \sum_\mu \langle \mu_A | \Phi_{SA} \rangle \langle \Phi_{SA} | \mu_A \rangle$$

$$= \sum_\mu \sum_{\nu\nu'} \langle \mu | \nu \rangle M_\nu | \psi_S \rangle \langle \nu' | \mu \rangle \langle \psi_S | M_{\nu'}^\dagger$$

$$\Rightarrow \left[\rho_S' = \sum_\mu M_\mu \underbrace{|\psi_S\rangle\langle\psi_S|}_{\rho_S} M_\mu^\dagger \right]$$

This generalizes to the case where A & S are both initially in (unentangled) mixed states:



$$\hat{U}: \quad \rho_S \otimes \rho_A \longmapsto \rho'_{SA} = \sum_{\mu\nu} M_\mu \, \rho_S \, M_\nu^\dagger \; |\mu\rangle_{A\,A}\langle\nu|$$

ρ_{SA}''

and as above

$$\rho_S' = \mathrm{Tr}_A \left(\rho'_{SA} \right)$$

$$= \sum_\mu \langle \mu | \rho'_{SA} | \mu \rangle$$

$$= \sum_\mu M_\mu \, \rho_S \, M_\mu^\dagger$$

$\Rightarrow$ when the total (closed) system SA evolves via the unitary transformation above, the density operator ρ_S (of S alone) is subjected to a linear map $\mathcal{E}$ which acts as:

$$\boxed{\mathcal{E} : \rho_S \longmapsto \mathcal{E}(\rho_S) = \rho_S' = \sum_\mu M_\mu \rho M_\mu^\dagger} \quad (*)$$

and is not (necessarily) unitary.

This is how we would describe noise acting on S: S interacts with envt (A) and the state of S is modified according to the map $\mathcal{E}$.

Such a linear map $\mathcal{E}$, with $\sum_\mu M_\mu^\dagger M_\mu = \mathbb{1}$, is called a quantum channel.

Note:

The word "channel" comes from communication theory. We can imagine a sender who transmits the state ρ_s through a communication link to another party. This party receives the modified state $\mathcal{E}(\rho_s)$ (due to interaction with environment during transmission).

A quantum channel maps density op. to density op, and thus must have the following properties (easily verifiable):

* linearity: $\mathcal{E}(\alpha\rho_1 + \beta\rho_2) = \alpha\,\mathcal{E}(\rho_1) + \beta\,\mathcal{E}(\rho_2)$
* preserves Hermiticity: $\rho = \rho^\dagger \Rightarrow \mathcal{E}(\rho) = \mathcal{E}(\rho)^\dagger$
* —— positivity: $\rho \geq 0 \Rightarrow \mathcal{E}(\rho) \geq 0$
* —— trace: $\mathrm{Tr}(\rho) = \mathrm{Tr}(\mathcal{E}(\rho))$ $(=1)$

Other names for "quantum channels"
- "superoperators"
- "trace-preserving completely positive map" (TPCP map)

Eq. (*) above is said to be an "operator-sum representation" of the quantum channel, and the operators $\{M_\mu\}$ are called the "Kraus operators", or "operation elements", of the channel.

<u>Note</u>: The operator-sum rep. of the quant. channel is not unique because we can perform the partial trace on A in any basis we want. For ex, use $|i\rangle$ basis instead of $|\mu\rangle$ with

$$|\mu\rangle_A = \sum_i |i\rangle_A V_{i\mu} . \qquad (V \text{ unitary})$$

$$\rho'_{SA} = \sum_{\mu\nu} M_\mu \rho_S M_\nu^\dagger \sum_{ij} \overbrace{|i\rangle V_{i\mu}}^{|\mu\rangle_A} \overbrace{\langle j| (V^\dagger)_{\nu j}}^{{}_A\langle\nu|}$$

$$= \sum_{ij} \underbrace{\left(\sum_\mu V_{i\mu} M_\mu\right)}_{N_i} \rho_S \underbrace{\sum_\nu M_\nu^\dagger (V^\dagger)_{\nu j}}_{N_j^\dagger} |i\rangle\langle j|$$

$\Rightarrow$ The $\{N_i\}$ are another set of Kraus operators.

- The special case where there is only one term in the operator - sum rep corresp. to the unitary evolution of S :

$$\rho_S \mapsto \mathcal{E}(\rho_S) = M_\mu \rho_S M_\mu^\dagger$$

- If there are at least two terms, then the states of S become entangled with the envt (A) under evolution governed by the unitary transfo U acting jointly on S and A. Therefore state of S becomes mixed when we trace out A.

=> Quantum Channels are imptt because they provide us with a formalism for discussing decoherence, the evolution of pure states into mixed states.

Then one important question is the question of reversibility :

when evolution is unitary, we know that it can always be reversed (because unitary op are invertible). Thus if today's state was obtained by applying

U (unitary) on yesterday's state, we can in principle recover yesterday's state by applying $U^{-1} = U^\dagger$ on today's state.

→ is the same thing true for general (non-unitary) quantum channels?
(in which case decoherence could be reversed).

Consider a quantum channel $\mathcal{E}_1$ with Kraus operators $\{M_\mu\}$ - Can $\mathcal{E}_1$ be inverted by another channel $\mathcal{E}_2$ with Kraus op. $\{N_i\}$?

This would mean:

$$(\mathcal{E}_2 \circ \mathcal{E}_1)(\rho) = \mathcal{E}_2(\mathcal{E}_1(\rho))$$

composition

$$= \mathcal{E}_2\left(\sum_\mu M_\mu \rho M_\mu^\dagger\right)$$

$$= \sum_{\mu i} N_i M_\mu \rho M_\mu^\dagger N_i^\dagger$$

$$= \rho$$

sum of positive terms

$$\Rightarrow \quad N_i M_\mu \rho M_\mu^\dagger N_i^\dagger \propto \rho$$

$$\Rightarrow \quad N_i M_\mu \propto \mathbb{1} \qquad \forall i, \mu.$$

$$\Rightarrow \quad N_i M_\mu = \lambda_{i\mu} \mathbb{1}.$$

Now $$M_\nu^\dagger M_\mu = M_\nu^\dagger \underbrace{\left(\sum_i N_i^\dagger N_i\right)}_{\mathbb{1}} M_\mu$$

$$= \sum_i \underbrace{M_\nu^\dagger N_i^\dagger}_{\lambda_{i\nu}^* \mathbb{1}} \underbrace{N_i M_\mu}_{\lambda_{i\mu} \mathbb{1}}$$

$$= \underbrace{\sum_i \lambda_{i\nu}^* \lambda_{i\mu}}_{\equiv \beta_{\nu\mu}} \mathbb{1}$$

M_μ being a square matrix (because maps ρ of a syst into ρ' of same syst) it has a polar decomposition

$$M_\mu = U_\mu \sqrt{M_\mu^\dagger M_\mu} \qquad U_\mu \text{ unitary}$$

$$= \sqrt{\beta_{\mu\mu}}\, U_\mu \qquad (1)$$

$$\Rightarrow \quad M_\nu^\dagger M_\mu = \sqrt{\beta_{\mu\mu}} \sqrt{\beta_{\nu\nu}} \quad U_\nu^\dagger U_\mu$$

$$\text{and} = \beta_{\nu\mu} \mathbb{1}$$

$$\Rightarrow \quad U_\nu^\dagger U_\mu = \frac{\beta_{\nu\mu}}{\sqrt{\beta_{\mu\mu}}\sqrt{\beta_{\nu\nu}}} \mathbb{1}$$

$$\text{or} \quad U_\mu = \frac{\beta_{\nu\mu}}{\sqrt{\beta_{\mu\mu}}\sqrt{\beta_{\nu\nu}}} U_\nu \quad (2) \quad \forall \mu, \nu.$$

($\times$ U_ν on the left)

$$\Rightarrow \quad M_\mu \overset{(1),(2)}{=} \frac{\beta_{\nu\mu}}{\sqrt{\beta_{\nu\nu}}} U_\nu = \frac{\beta_{\nu\mu}}{\beta_{\nu\nu}} M_\nu . \quad (\forall \mu, \nu)$$

$\Rightarrow$ We have shown that, if the quantum channel $\mathcal{E}_1$ is invertible (by another channel $\mathcal{E}_2$), then the Kraus operators $\{M_\mu\}$ of $\mathcal{E}_1$ are all proportional to the same unitary: $\hat{M}_\mu = \underbrace{c(\mu)}_{\text{constant}} \hat{U}_{\mathcal{E}_1}$

$$\Rightarrow \quad \mathcal{E}_1(\rho) = \sum_\mu M_\mu^\dagger \rho M_\mu$$

$$= \left(\sum_\mu c^*(\mu) c(\mu) \right) \times \hat{U}_{\mathcal{E}_1}^\dagger \rho U_{\mathcal{E}_1}$$

i.e. $\mathcal{E}_1$ is a unitary map.

=> A quantum channel can only be inverted if it is unitary.

=> Thus, decoherence is irreversible.

Once the system S becomes entangled with the environment (A), we cannot undo the damage if we don't have access to A.

Decoherence causes quantum info to leak into a system's envt, and because we cannot control the envt, this info cannot be recovered.

This argument applies to a channel which maps S to itself ($\rho_S \mapsto \mathcal{E}(\rho_S) = \rho_S'$) or to another system S' with same dimension (in that case the Polar decomposition applies)

<u>But</u>, the conclusion can be evaded if S' has a dimension larger than S.

This exception is used in Quantum Error Correction.

* Single-qubit channels

Our discussion of quantum channels has been quite general and abstract so far.
Now we will look at some particular examples of quantum channels, corresponding to certain noise processes, acting on a single qubit.

→ Bit-flip channel

Suppose we have a qubit (q) initially in state $|\psi\rangle$ which interacts with some "environment" (E) -

The qubit will remain intact with proba $(1-p)$, while with proba p the qubit undergoes a "bit-flip" error process:

$$|\psi\rangle \mapsto X|\psi\rangle \qquad \begin{pmatrix} |0\rangle \mapsto |1\rangle \\ |1\rangle \mapsto |0\rangle \end{pmatrix}$$

This means the qubit and envt interact as:

$$\hat{U}: |\psi\rangle \otimes |0\rangle_E \longmapsto \Big(\sqrt{1-p}\, |\psi\rangle \otimes |0\rangle_E + \sqrt{p}\, X|\psi\rangle \otimes |1\rangle_E \Big) \equiv |\Phi\rangle_{qE}$$

In terms of density matrices :

$$\hat{U} : \quad \underbrace{\rho_{qE} = \rho_q \otimes \rho_E}_{= |\psi\rangle\langle\psi| \otimes |0\rangle_E {}_E\langle 0|} \longmapsto \rho'_{qE} = |\Phi\rangle_{qE\,qE}\langle\Phi|$$

What is the operator-sum representation (i.e. what are the Kraus operators M_μ) for this channel ?

$$\rho_q = |\psi\rangle\langle\psi| \longmapsto \mathcal{E}(\rho_q) = \sum_\mu M_\mu \rho_q M_\mu^\dagger$$

$$=$$

$$\mathcal{E}(\rho_q) = \mathrm{Tr}_E \left(\rho'_{qE}\right)$$

$$= {}_E\langle 0|\Phi_{qE}\rangle\langle\Phi_{qE}|0\rangle_E + {}_E\langle 1|\Phi_{qE}\rangle\langle\Phi_{qE}|1\rangle_E$$

$$= (1-p) \underbrace{|\psi\rangle\langle\psi|}_{\rho_q} + p \; X \underbrace{|\psi\rangle\langle\psi|}_{\rho_q} X$$

$$= \sum_{\mu=0,1} M_\mu \, \rho_q \, M_\mu^\dagger$$

with $\begin{cases} M_0 = \sqrt{1-p} \; \mathbb{1} \\ M_1 = \sqrt{p} \; X \end{cases}$

We easily check the completeness relation:

$$\sum_{\mu=0,1} M_\mu^\dagger M_\mu = (1-p)\,\mathbb{1} + p\,\underbrace{X^2}_{\mathbb{1}} = \mathbb{1}$$

(Note: this is one possible op-sum representation. Remember that the Kraus op. are not unique)

$\Rightarrow$ Generally, if the qubit (q) starts in a mixed state ρ_q, it evolves as

$$\rho_q \mapsto \mathcal{E}(\rho_q) = (1-p)\,\rho_q + p\,X\rho_q X$$

Geometric picture (Bloch Ball representation)

We know that the density op of a single qubit can be expressed as:

$$\rho_q = \frac{\mathbb{1} + \vec{r}\cdot\vec{\sigma}}{2}$$

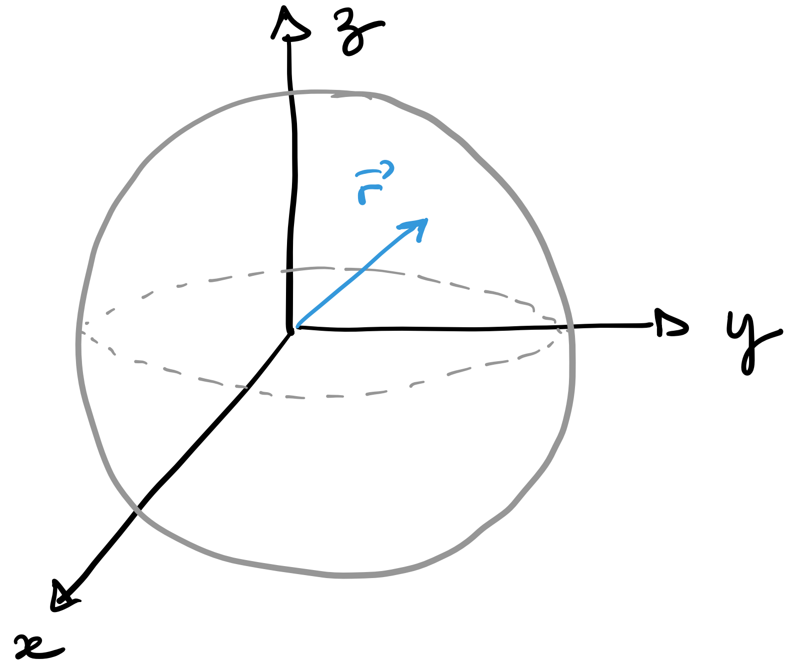
where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) = (X, Y, Z)$

(Pauli operators).

and $\vec{r}$ is a 3-dimensional real vector

$$\vec{r} = (r_x, r_y, r_z)^{\in \mathbb{R}^3} \text{ with } |\vec{r}| \leq 1.$$

$\vec{r}$ = "spin polarization" of the qubit.



(pure states have $|\vec{r}|=1$ $\rightarrow$ on the surface
mixed states have $|\vec{r}|<1$ $\rightarrow$ in the interior)

The bit-flip channel thus maps

$$\rho_q = \frac{\mathbb{1}+\vec{r}\cdot\vec{\sigma}}{2} = \frac{1}{2}\left(\mathbb{1} + r_x X + r_y Y + r_z Z\right)$$

to

$$\mathcal{E}(\rho_q) = (1-p)\,\rho_q + p\,X\rho_q X$$

$$= (1-p)\left(\frac{\mathbb{1}+\vec{r}\cdot\vec{\sigma}}{2}\right) + p\,\underbrace{X\left(\frac{\mathbb{1}+\vec{r}\cdot\vec{\sigma}}{2}\right)X}_{=}$$

$$\frac{1}{2}\Big(\underset{\substack{\| \\ \mathbb{1}}}{X^2} + r_x \underset{\substack{\| \\ X}}{X^3} + r_y \underbrace{XYX}_{\substack{-X^2Y \\ \| \\ -Y}} + r_z \underbrace{XZX}_{\substack{-X^2Z \\ \| \\ -Z}}\Big)$$

where we used

$$\sigma_i^2 = \mathbb{1} \quad \text{and} \quad \underbrace{\{\sigma_j, \sigma_k\}}_{\sigma_j \sigma_k + \sigma_k \sigma_j} = 2\,\delta_{jk}\,\mathbb{1}$$

$(\sigma_x = X,\ \sigma_y = Y,\ \sigma_z = Z)$

$$\Rightarrow \mathcal{E}(\rho_q) = (1-p)\left(\frac{\mathbb{1} + \vec{r}\cdot\vec{\sigma}}{2}\right)$$

$$+ \frac{p}{2}\left(\mathbb{1} + r_x X - r_y Y - r_z Z\right)$$

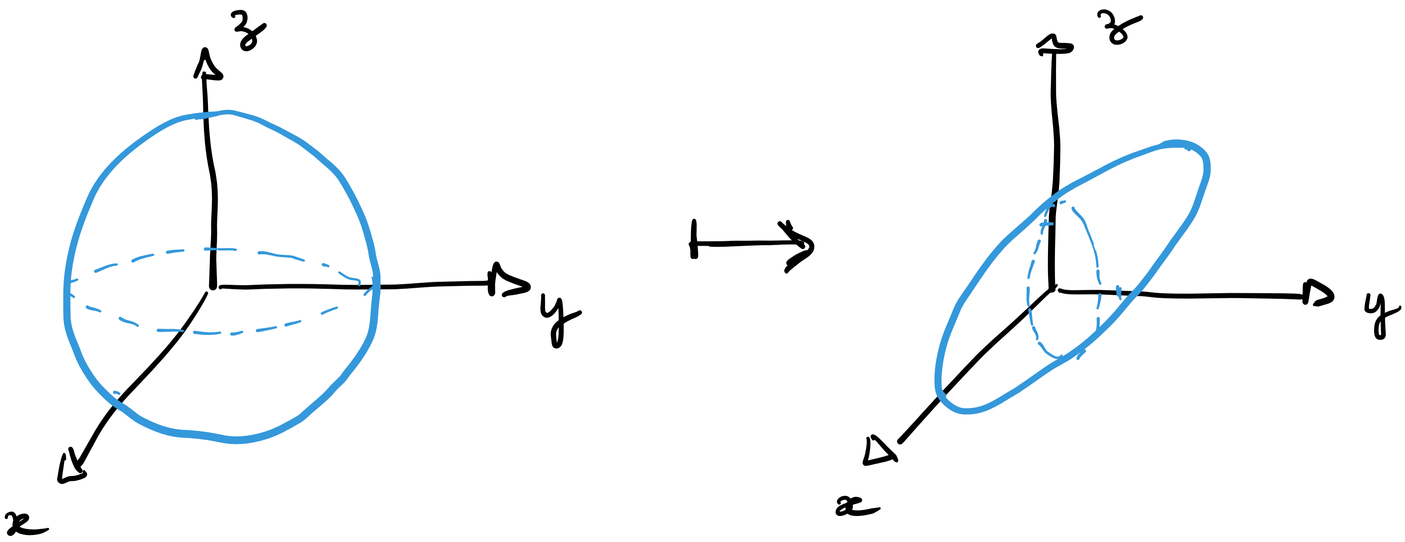
$$= \frac{1}{2}\left(\mathbb{1} + r_x X + (1-2p)\, r_y Y + (1-2p)\, r_z Z\right)$$

The component r_x is unchanged while the r_y and r_z components are changed according to

$$r_y \longmapsto (1-2p)\, r_y < r_y$$

$$r_z \longmapsto (1-2p)\, r_z < r_z$$

i.e. the (y,z) plane of the Bloch Ball is uniformly contracted by a factor $(1-2p)$:



"cigar" shape (prolate)

We know that the $tr(\rho_q^2)$ tells us about the "purity" of the qubit state

$(tr(\rho^2) = 1 \Leftrightarrow \rho$ is pure $)$

Here $$tr(\rho_q^2) = tr\left(\left[\frac{\mathbb{1} + (\vec{r}\cdot\vec{\sigma})}{2}\right]^2\right)$$

$$= [\ldots] = \left(\frac{1 + |\vec{r}|^2}{2}\right)$$

The Bit-flip channel contracts r_y, r_z and thus, decreases $|\vec{r}|^2$.

This means that the "purity" of the qubit state decreases $\rightarrow$ becomes more mixed.

* "Phase-flip" channel

Another possible error can be modeled by the phase-flip channel characterized by

$$\begin{cases} |0\rangle \longmapsto |0\rangle \\ |1\rangle \longmapsto -|1\rangle \end{cases}$$

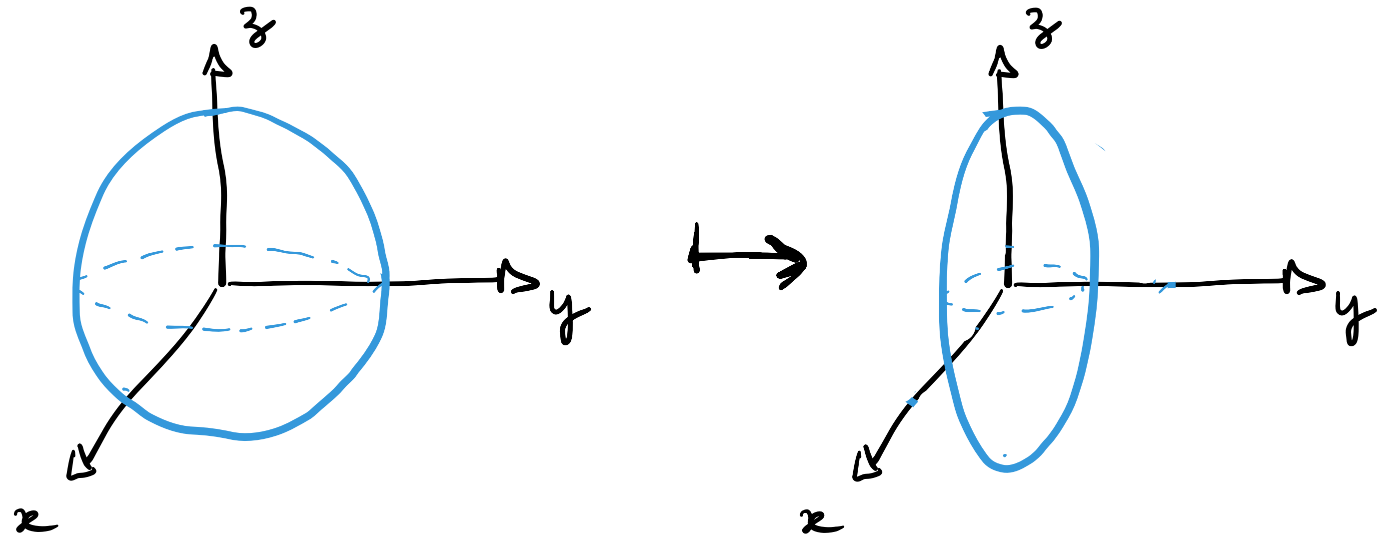
In that case, a possible set of Kraus operators is

$$\begin{cases} M_0 = \sqrt{1-p}\ \hat{\mathbb{1}} \\ M_1 = \sqrt{p}\ \hat{Z} \end{cases}$$

and the corresponding operator-sum represent° of the channel is:

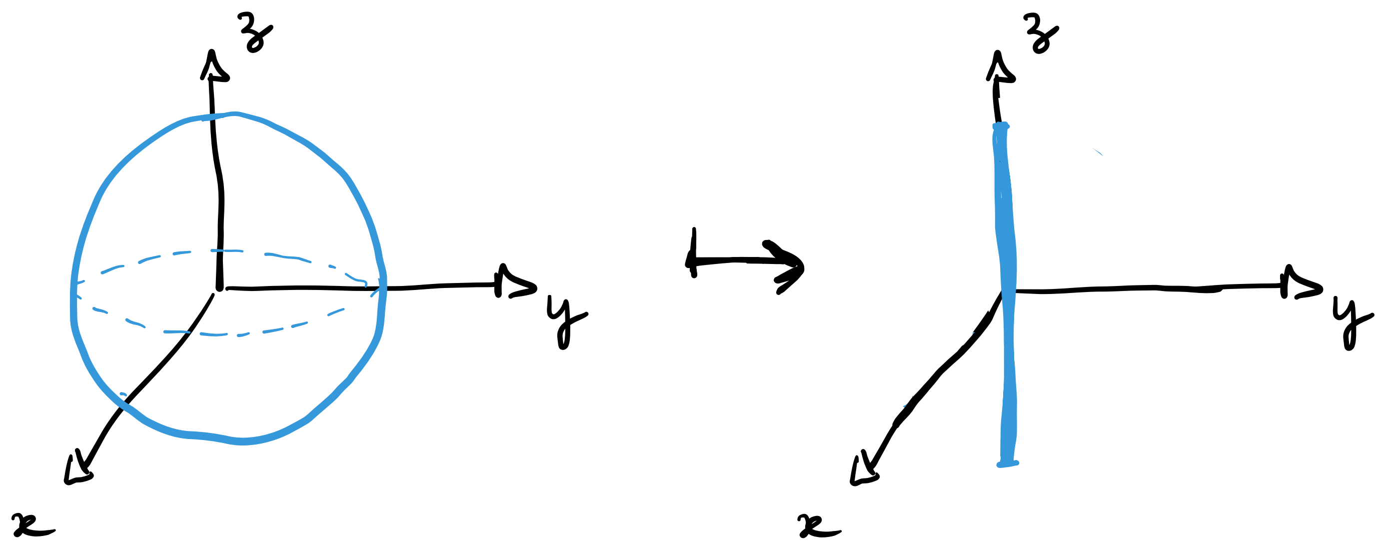
$$\rho_q \longmapsto \mathcal{E}(\rho_q) = (1-p)\,\rho_q + p\, Z \rho_q Z$$

In the Bloch-ball representation, the r_z component is now unchanged while the r_x and r_y components are contracted by a factor $(1-2p)$.



Note : Special case $p = 1/2$

In the case $p = 1/2$, $(1 - 2p) = 0 \Rightarrow$ the corresponding map on the Bloch sphere is $(r_x, r_y, r_z) \longmapsto (0, 0, r_z)$



* "Bit-phase flip" channel

This is the channel with Kraus operators

$$\begin{cases} M_0 = \sqrt{1-p}\ \mathbb{1} \\ M_1 = \sqrt{p}\ Y \end{cases}$$

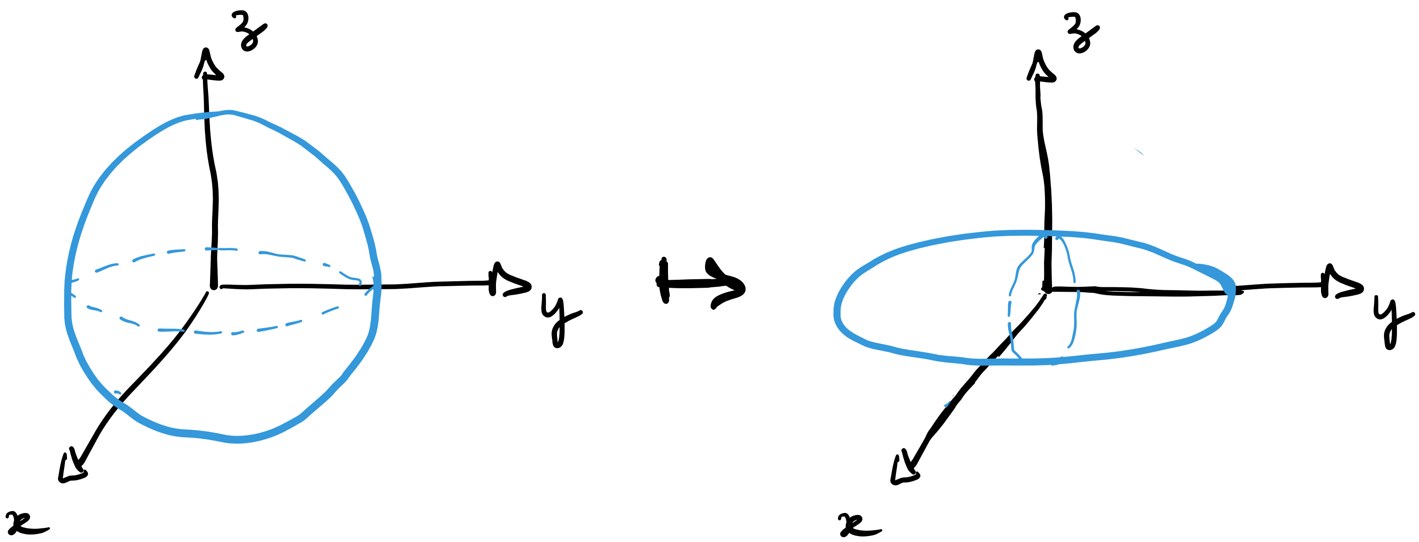
Since $Y = iXZ$ it has the action of flipping both the phase and the bit :

$$\begin{cases} |0\rangle \mapsto i|1\rangle \\ |1\rangle \mapsto -i|0\rangle \end{cases}$$

Corresponding sum-op rep :

$$\rho_q \mapsto \mathcal{E}(\rho_q) = (1-p)\,\rho_q + p\, Y \rho_q Y$$

In the Bloch-ball representation, the r_y component is now unchanged while the r_x and r_z components are contracted by a factor $(1-2p)$.



In general, error on qubits will corresp to a combination of all three preambles channels.

An arbitrary qubit channel may be written in the operator-sum representation as:

$$\mathcal{E}(\rho_q) = \sum_\mu M_\mu \rho_q M_\mu^\dagger$$

where $M_\mu = \alpha_\mu \mathbb{1} + \beta_\mu X + \gamma_\mu Y + \delta_\mu Z$

In the Bloch-ball representation this is equivalent to the mapping:

$$\vec{r} \longmapsto A\vec{r} + \vec{b}$$

where A and $\vec{b}$ can be related to $\alpha_\mu, \beta_\mu, \gamma_\mu, \delta_\mu$.

Note :

- maps $\vec{r} \mapsto A\vec{r}$ with no displacement of the center of the Bloch Ball are called "unital maps".

They take $\rho \propto \mathbb{1}$ to itself.

the 3 maps studied above are unital.

One important property of unital maps is they can never compress only one component of the vector $\vec{r}$. Thus they can never lead to "pancake" shapes (oblate) $\rightarrow$ "no-pancake" theorem.

Because this would lead to a map that is not completely positive.

$\Rightarrow$ unital maps can only shrink 2 (cigar, prolate) or 3 components (smaller ball).

- maps $\vec{r} \mapsto A\vec{r} + \vec{b}$ also displace the center of the Bloch ball and are called non-unital.

They do not map $\rho \propto \mathbb{1}$ into itself.

Ex : More realistic noise channel

* Depolarizing Channel

= important model of decohering qubit where with proba $(1-p)$ the qubit is unchanged, and with proba p it is "depolarized", meaning that it becomes completely mixed (maximally) : $\rho_q' = \frac{\mathbb{1}}{2}$.

i.e. $$\mathcal{E}(\rho_q) = (1-p)\rho_q + p\,\frac{\mathbb{1}}{2}$$

This is not an operator-sum representation but we can use the property :

$$X\rho X + Y\rho Y + Z\rho Z = \frac{3}{2}\mathbb{1}$$

Thus,

$$\mathcal{E}(\rho_q) = (1-p)\rho_q + \frac{p}{3}\left(X\rho X + Y\rho Y + Z\rho Z\right)$$

$$= \sum_{\mu=0}^{3} M_\mu\, \rho_q\, M_\mu^\dagger$$

with $\begin{cases} M_0 = \sqrt{1-p}\ \mathbb{1} \\ M_1 = \sqrt{\frac{p}{3}}\, X, \quad M_2 = \sqrt{\frac{p}{3}}\, Y, \quad M_3 = \sqrt{\frac{p}{3}}\, Z \end{cases}$

$\Rightarrow$ interpretation: the state of the qubit is untouched with proba $(1-p)$, and the operators X, Y, Z are applied with proba $\frac{p}{3}$.

Bloch ball representation:

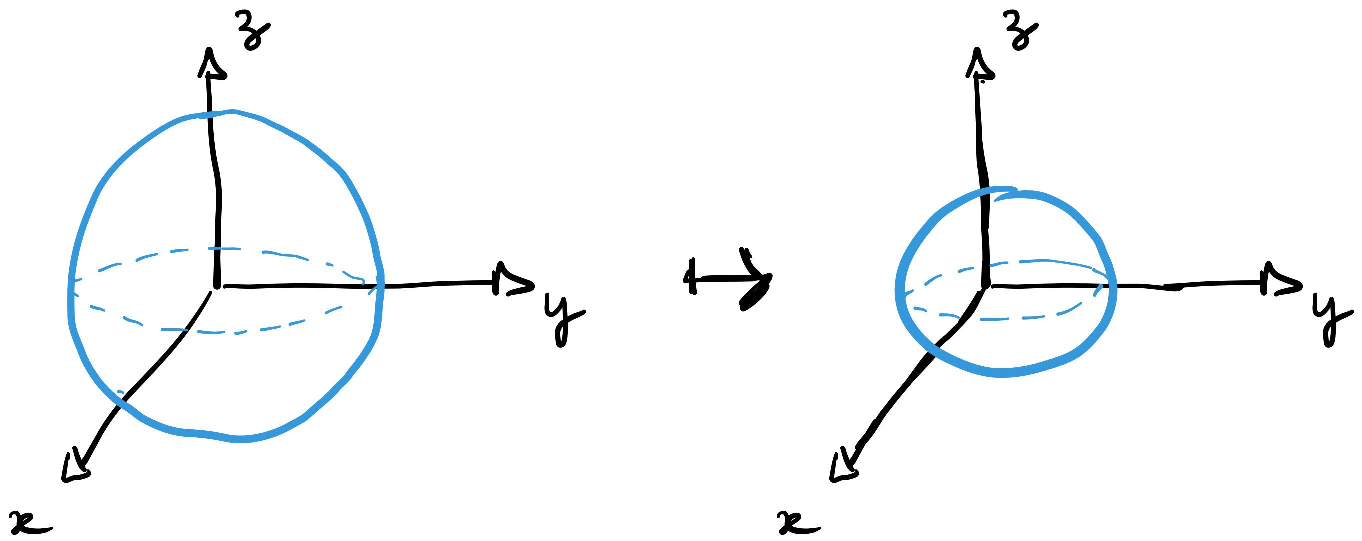
$$\rho_q = \frac{\mathbb{1} + \vec{r}\cdot\vec{\sigma}}{2} \mapsto \mathcal{E}(\rho_q) = (1-p)\rho_q + p\frac{\mathbb{1}}{2}$$

$$= \left(\frac{1-p}{2} + \frac{p}{2}\right)\mathbb{1} + \left(\frac{1-p}{2}\right)\vec{r}\cdot\vec{\sigma}$$

$$= \frac{1}{2}\left(\mathbb{1} + (1-p)\,\vec{r}\cdot\vec{\sigma}\right)$$

$\Rightarrow$ the depolarizing channel contracts uniformly the whole sphere as:

$$\vec{r} \mapsto (1-p)\,\vec{r}$$



The larger p is $\rightarrow$ the more the Ball shrinks.

The density matrix $\rho^* = \frac{\mathbb{1}}{2} \quad (\vec{r} = \vec{0})$

is a "fixed point" of the channel $\mathcal{E}(\rho^*) = \rho^*$.

<u>Note</u>: Reversibility ?

we could imagine performing a uniform inflation to reverse the uniform contraction of the Bloch ball.

But the trouble is that the inflation is not positive, and thus not a channel.

(would take ρ into ρ' with negative eigenvalues.)

We see again that decoherence is not reversible : it can shrink the ball but no physical process can inflate it again.

(a channel running backward in time is not a channel).