# Density operators of qubits

1

$$|\psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi} \quad \sin\frac{\theta}{2} \end{pmatrix}$$

$$|\psi\rangle\langle\psi| = \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi} \quad \cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\phi} \quad \cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{2}\cos\frac{\theta}{2} & \frac{1}{2}(\cos\phi - i\sin\phi)\sin\theta \\ \frac{1}{2}\ (\cos\phi + i\sin\phi)\sin\theta & \frac{1}{2} - \frac{1}{2}\cos\theta \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 1 \quad \cos\phi\ \sin\theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2}\sin\phi\sin\theta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$+ \frac{1}{2}\cos\theta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\left( \mathbb{1} + \cos\phi\ \sin\theta\ \sigma_x + \sin\phi\ \sin\theta\ \sigma_y + \cos\theta\ \sigma_z \right)$$

$$= \frac{1}{2}\left(\mathbb{1} + \vec{r}\cdot\hat{\vec{\sigma}}\right)$$

with $r_x = \cos\phi\ \sin\theta$

$r_y = \sin\phi\ \sin\theta$

$r_z = \cos\theta$

you all know $\quad |\vec{r}| = \sqrt{r_x^2 + r_y^2 + r_z^2} = 1$

2

$$\hat{\rho}_{mixed} = \sum_i p_i \hat{\rho}_i = \sum_i p_i \frac{1}{2}\left(\mathbb{1} + \vec{r}_i \cdot \hat{\vec{\sigma}}\right)$$

$$= \frac{1}{2}\left( \sum_i p_i\ \mathbb{1} + \sum_i p_i \vec{r}_i \cdot \hat{\vec{\sigma}} \right)$$

$$= \frac{1}{2}\left(\mathbb{1} + \vec{r}\cdot\hat{\vec{\sigma}}\right)$$

with $\vec{r} = \sum_i p_i \vec{r}_i$ and using $|\vec{r}_i| = 1$

$$|\vec{r}| = \left|\sum_i p_i \vec{r}_i\right| < \sum_i |p_i \vec{r}_i| = \sum_i p_i |\vec{r}_i| = 1$$

3a

$$\rho_{AB} = |\Psi^+\rangle\langle\Psi^+| = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)\frac{1}{\sqrt{2}}\left(\langle 01| + \langle 10|\right)$$

$$= \frac{1}{2}\left(|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|\right)$$

Using $|01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $|10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$\Rightarrow \quad \rho_{AB} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3b

$$\rho_{AB}^2 = \rho_{AB} \cdot \rho_{AB}$$

$$= \frac{1}{2}\left(|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|\right)$$

and with $\mathrm{Tr}\left(|x\rangle\langle y|\right) = \delta_{xy}$

$$\mathrm{Tr}\,\rho_{AB}^2 = \frac{1}{2}\left(1 + 0 + 0 + 1\right) = 1$$

$\Rightarrow$ pure state

3c

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{1}{2}\mathbb{1}$$

$\mathrm{Tr}\left(\rho_A^2\right) = \mathrm{Tr}\left(\frac{1}{4}\mathbb{1}\right) = \frac{1}{2} < 1 \quad \Rightarrow \quad \rho_A$ is mixed

3d

$$S(\rho_A) = -\sum_\alpha \lambda_\alpha \log \lambda_\alpha = \log 2 = 1$$

3e

$$\rho_{AB} = |\phi_{AB}\rangle\langle\phi_{AB}| = |\phi\rangle\langle\phi|_A \otimes |\chi\rangle\langle\chi|_B$$

and $\rho_A = Tr_B(\rho_{AB}) = |\phi\rangle\langle\phi|_A$ is pure

$\Rightarrow$ $S(\rho_A) = -Tr(\rho_A \log \rho_A) = 0$ for a pure state

4a

$$|\rho - \rho'| = \left|\frac{1}{12}\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)\right|$$
$$= \sqrt{\frac{1}{12^2}\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)}$$
$$= \frac{1}{12}\sqrt{|0\rangle\langle 0| + |1\rangle\langle 1|}$$
$$= \frac{1}{12}\sqrt{\mathbb{1}} = \frac{1}{12}\mathbb{1}$$

$$\Rightarrow D(\rho,\rho') = \frac{1}{2}Tr\left(\frac{1}{12}\mathbb{1}\right) = \frac{1}{12}$$

4b Using $\hat{\vec{\sigma}}^{\dagger} = \hat{\vec{\sigma}}$

$$|\rho - \rho'| = \frac{1}{2}\left|(\vec{r} - \vec{r}')\hat{\vec{\sigma}}\right|$$
$$= \frac{1}{2}\sqrt{(\vec{r} - \vec{r}')\hat{\vec{\sigma}}(\vec{r} - \vec{r}')\hat{\vec{\sigma}}}$$
$$= \frac{1}{2}\sqrt{(\vec{r} - \vec{r}')^2\mathbb{1}}$$
$$= \frac{1}{2}|\vec{r} - \vec{r}'|\mathbb{1}$$

✓