Problem 3:

Laplace $(0, \frac{1}{\lambda})$

RV X with pdf $p_X(x) = \frac{\lambda}{2} e^{-\lambda|x|} \quad \forall \; x \in [-\infty, \infty]$

(a) Char. function $\Phi(\omega)$!

$$\Phi_X\{\omega\} = \int_{\mathbb{R}} f_X(\xi) e^{+j\xi\omega} d\xi$$

$$= \int_{-\infty}^{\infty} \frac{\lambda}{2} e^{-\lambda|\xi|} e^{+j\xi\omega} d\xi$$

$$|\xi| = \begin{cases} -\xi & , \xi < 0 \\ \xi & , \xi \geq 0 \end{cases}$$

$$= \int_{-\infty}^{0} \frac{\lambda}{2} e^{-\lambda(-\xi)} e^{j\xi\omega} d\xi + \int_{0}^{\infty} \frac{\lambda}{2} e^{-\lambda\xi} e^{j\xi\omega} d\xi$$

$$= \int_{-\infty}^{0} \frac{\lambda}{2} e^{\xi(\lambda+j\omega)} d\xi + \int_{0}^{\infty} \frac{\lambda}{2} e^{\xi(-\lambda+j\omega)} d\xi$$

$$= \frac{\lambda}{2} \frac{1}{\lambda+j\omega} e^{\xi(\lambda+j\omega)} \Big|_{-\infty}^{0} + \frac{\lambda}{2} \frac{1}{-\lambda+j\omega} \underbrace{e^{\xi(-\lambda+j\omega)}}_{e^{-\xi(\lambda-j\omega)}} \Big|_{0}^{\infty}$$

$$= \frac{\lambda}{2} \frac{1}{\lambda+j\omega} [1-0] + \underbrace{\frac{\lambda}{2} \frac{1}{-\lambda+j\omega} (0-1)}_{\text{WOLFRAM} \checkmark}$$

$$= \frac{\lambda}{2} \left[\frac{1}{\lambda+j\omega} + \underbrace{\frac{1}{\lambda-j\omega}}_{} \right]$$

$$= \frac{\lambda}{2} \left[\frac{(\lambda - j\omega + \lambda + j\omega)}{(\lambda+j\omega)(\lambda-j\omega)} \right]$$

$$= \frac{\lambda}{2} \left[\frac{2\lambda}{\lambda^2 + \omega^2} \right] = \frac{\lambda^2}{\lambda^2+\omega^2} = \frac{1}{1 + \frac{\omega^2}{\lambda^2}} = \frac{1}{1+b^2\omega^2}$$

$$\Phi_X(\omega) = \frac{\lambda^2}{\lambda^2+\omega^2}$$

$\Phi_X(0) = 1 \checkmark \qquad |\Phi_X(\omega)| \leq 1 \checkmark$

$$\Phi_X(\omega) = \frac{\lambda^2}{\lambda^2 + \omega^2}$$

$$\left.\frac{d\Phi_X(\omega)}{d\omega}\right|_{\omega=0} = -\left.\frac{2\lambda^2\omega}{(\lambda^2+\omega^2)^2}\right|_{\omega=0} = 0 \quad \checkmark \qquad \mu = 0$$

$$\mathbb{E}(X) = 0 \quad \checkmark$$

$$\left.\frac{d^2\Phi_X(\omega)}{d\omega^2}\right|_{\omega=0} = \lambda^2 \left.\left(\frac{8\omega^2}{(\lambda^2+\omega^2)^3} - \frac{2}{(\lambda^2+\omega^2)^2} \right)\right|_{\omega=0}$$

$$= \lambda^2 \cdot \left(-\frac{2}{\lambda^4} \right) = -\frac{2}{\lambda^2} = j^2\, \mathbb{E}(X^2) = -\mathbb{E}(X^2)$$

$$\Rightarrow \mathbb{E}(X^2) = \frac{2}{\lambda^2} = \sigma^2 \checkmark$$

$$\left.\frac{d^3\Phi_X(\omega)}{d\omega^3}\right|_{\omega=0} = \lambda^2 \left.\left(\frac{24\omega}{(\lambda^2+\omega^2)^3} - \frac{48\omega^3}{(\lambda^2+\omega^2)^4} \right)\right|_{\omega=0} = 0 \quad \checkmark$$

$$\left.\frac{d^4\Phi_X(\omega)}{d\omega^4}\right|_{\omega=0} = \lambda^2 \left.\left(-\frac{288\omega^2}{(\lambda^2+\omega^2)^4} + \frac{24}{(\lambda^2+\omega^2)^3} + \frac{384\omega^2}{(\lambda^2+\omega^2)^5} \right)\right|_{\omega=0}$$

$$= \lambda^2 \cdot \frac{24}{\lambda^{2\cdot 3}} = \frac{24}{\lambda^4} = \underbrace{j^4}_{1}\, \mathbb{E}(X^4)$$

(b) n^{th} moment: $\mathbb{E}(X^n) = \int_{-\infty}^{\infty} \xi^n f_X(\xi)\, d\xi$

$f_X(\xi)$ exists

$$= \int_{-\infty}^{0} \xi^n f_X(\xi)\, d\xi + \int_{0}^{\infty} \xi^n f_X(\xi)\, d\xi$$

and integrals converge

$$\mathbb{E}(X^n) = \int_{0}^{-\infty} -\xi^n f_X(\xi)\, d\xi + \int_{0}^{\infty} \xi^n f_X(\xi)\, d\xi$$

$$= \int_{0}^{\infty} -(-u)^n f_X(-u)\,(-du) + \int_{0}^{\infty} \xi^n f_X(\xi)\, d\xi$$

let $u = -\xi$

$\Rightarrow \frac{d\xi}{du} = -1$

$\Rightarrow d\xi = -du$

even symmetry: $f_X(u) = f_X(-u)$

$$= \int_0^\infty (-u)^n f_X(u)\, du + \int_0^\infty \xi^n f_X(\xi)\, d\xi$$

$$= \int_0^\infty (-\xi)^n f_X(\xi)\, d\xi + \int_0^\infty \xi^n f_X(\xi)\, d\xi$$

If n is odd:

$$\mathbb{E}(X^n) = -\int_0^\infty \xi^n f_X(\xi)\, d\xi + \int_0^\infty \xi^n f_X(\xi)\, d\xi$$

$$= 0 \quad \checkmark$$