EX 5 p1

$$\underline{z} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \underline{v} \sim p_{\underline{v}} \quad \underline{\underline{\text{zero mean}}} \quad \mathbb{E}_{\underline{v}}(\underline{v}) = \underline{0}$$

$$\underline{z}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \hat{\underline{\omega}} = \underline{\omega} + \underline{v} = f(\underline{z}) + \underline{v}$$

$$\hat{\underline{\omega}} = \begin{pmatrix} \hat{r} \\ \hat{\phi} \end{pmatrix} \qquad f(\underline{z}) = \begin{pmatrix} \sqrt{x^2+y^2} \\ \arctan\left(\frac{y}{x}\right) \end{pmatrix}$$

$$\hat{\underline{z}} = f^{-1}(\hat{\underline{\omega}}) = \begin{pmatrix} \hat{r} \cdot \cos\hat{\phi} \\ \hat{r} \cdot \sin\hat{\phi} \end{pmatrix}$$

(a) find $p_{\hat{\underline{\omega}}|\underline{z}} = \int p(\hat{\underline{\omega}} | \underline{z}, \underline{v})\, p(\underline{v} | \underline{z})\, d\underline{v}$

$$p(\hat{\underline{\omega}} | \underbrace{\underline{z}, \underline{v}}) = \delta(\hat{\underline{\omega}} - f(\underline{z}) - \underline{v})$$

both are known

$\Rightarrow$ then cond. prob is deterministic

$$\underline{z} \longrightarrow \boxed{f(\cdot)} \xrightarrow{\underline{\omega}} \underset{\uparrow \underline{v}}{\oplus} \xrightarrow{\hat{\underline{\omega}}}$$

what is $p_{\hat{\underline{\omega}} | \underline{\omega}, \underline{v}} - \dfrac{p(\hat{\underline{\omega}}, \underline{\omega}, \underline{v})}{p(\underline{\omega}, \underline{v})}\Bigg|_{\underline{z} = f(\underline{\omega})} = \dfrac{p(\hat{\underline{\omega}}, \underline{v} | \underline{z}) \cdot p(\underline{z})}{p(\underline{v} | \underline{z})\, p(\underline{z})}$

if $p(\underline{z} > 0)$

$$\hat{\underline{\omega}} = f(\underline{z}) + \underline{v}$$

$\rightarrow$ we know pdf of $\underline{v}$: $p_{\underline{v}}(\underline{v})$, $\mathbb{E}_{\underline{v}}(\underline{v}) = \emptyset$

$\Rightarrow$ change of variable formula:

$$\underline{v} = \hat{\underline{\omega}} - f(\underline{z})$$

$$p_{\hat{\underline{\omega}}|\underline{z}}(\hat{\underline{\omega}} | \underline{z}) = \int_{\underline{v} \in \mathcal{V}} p(\hat{\underline{\omega}} | \underline{z}, \underline{v})\, p(\underline{v} | \underline{z})\, d\underline{v}$$

$$= \int_{\underline{v} \in \mathcal{V}} \delta(\overbrace{\hat{\underline{\omega}} - f(\underline{z}) - \underline{v}}^{\underline{v} = \hat{\underline{\omega}} - f(\underline{z})})\, p(\underline{v} | \underline{z})\, d\underline{v}$$

$$= p_{\underline{v}|\underline{z}=z}(\hat{\underline{\omega}} - f(\underline{z}) | \underline{z})$$

noise can depend on $\underline{z}$

(b) $\mathbb{E}_{\underline{\hat{\omega}}|\underline{z}}(\underline{\hat{\omega}}) = ?$ unbiased ?

EX 5

p2

$$\mathbb{E}_{\underline{\hat{\omega}}|\underline{z}}(\underline{\hat{\omega}}) = \int_{\underline{\hat{\omega}} \in \underline{\hat{W}}} \underline{\hat{\omega}} \cdot p_{\underline{\hat{\omega}}|\underline{z}}(\underline{\hat{\omega}}|\underline{z})\, d\underline{\hat{\omega}}$$

$$= \int_{\underline{\hat{\omega}}} \underline{\hat{\omega}}\, p_{\underline{V}}(\underline{\hat{\omega}} - f(\underline{z})|\underline{z})\, d\underline{\hat{\omega}}$$

$\underline{\hat{\omega}} = f(\underline{z}) + \underline{v}$

$$= \int_{\underline{v}} (f(\underline{z}) + \underline{v})\, p_{\underline{V}}(\underline{v}|\underline{z})\, d\underline{v}$$

$$= \int_{\underline{v}} f(\underline{z}) \underbrace{p_{\underline{V}}(\underline{v}|\underline{z})}\, d\underline{v} + \overbrace{\int_{\underline{v}} \underline{v}\, p_{\underline{V}}(\underline{v}|\underline{z})\, d\underline{v}}^{=0 \text{ zero mean}}$$

?

$$\int_{\underline{v}} \frac{p(\underline{v}, \underline{z})}{p(\underline{z})}\, d\underline{v} = \frac{p(\underline{z})}{p(\underline{z})} = 1$$

$$= f(\underline{z})$$

if $\underline{v}$ and $\underline{z}$ indep.

then $p(\underline{v}, \underline{z}) = p(\underline{v})\, p(\underline{z})$

and $\int_{\underline{v}} \underline{v}\, p(\underline{z}) p(\underline{v})\, d\underline{v}$ vanishes

otherwise not sure...

if $\underline{v}$ and $\underline{z}$ independent

$\Rightarrow p(\underline{v}|\underline{z}) = p(\underline{v})$

EX 5 p 3

$p_{\underline{v}} = p_{v_r} \cdot p_{v_\varphi}$ independent

assume $\underline{z}$ and $\underline{v}$ independent

$$\Rightarrow p_{\hat{\underline{\omega}}|\underline{z}} = p_{\underline{v}}\left(\hat{\underline{\omega}} - f(\underline{z})\right)$$

$$\mathbb{E}_{\hat{\underline{\omega}}|\underline{z}}\left(f^{-1}(\hat{\underline{\omega}})\right) = (*)$$

$$f^{-1}(\hat{\underline{\omega}}) = \begin{pmatrix} \hat{r}\cos\hat{\varphi} \\ \hat{r}\sin\hat{\varphi} \end{pmatrix}$$

$$p_{\underline{v}} = p_{vr} \cdot p_{v\varphi} \qquad p_{\underline{v}}(\hat{\underline{\omega}} - f(\underline{z})$$

$$(*) := \int_{v_r}\int_{v_\varphi} \begin{pmatrix} \hat{r}\cos\hat{\varphi} \\ \hat{r}\sin\hat{\varphi} \end{pmatrix} p_{v_r} \cdot p_{v_\varphi} \, dv_r \, dv_\varphi$$

$$\mathbb{E}_{\hat{\underline{\omega}}|\underline{z}}\left(f^{-1}(\hat{\underline{\omega}})\right) = \begin{pmatrix} \mathbb{E}_{\hat{\underline{\omega}}|\underline{z}}(\hat{r}\cos\hat{\varphi}) \\ \mathbb{E}_{\hat{\underline{\omega}}|\underline{z}}(\hat{r}\sin\hat{\varphi}) \end{pmatrix}$$

$$\mathbb{E}_{\hat{\underline{\omega}}|\underline{z}}(\hat{r}\cos\hat{\varphi}) = \int_{v_r}\int_{v_\varphi} \hat{r}\cos\hat{\varphi} \, p_{v_r} p_{v_\varphi} \, dv_r \, dv_\varphi$$

$$= \int_{v_r}\int_{v_\varphi} \hat{r}\cos\hat{\varphi} \, p_{v_r}(v_r) \, p_{v_\varphi}(v_\varphi) \, dv_r \, dv_\varphi$$

$$= \int_{v_r}\int_{v_\varphi} (r + v_r)\cos(\varphi + v_\varphi) \, p_{v_r}(v_r) \, p_{v_\varphi}(v_\varphi) \, dv_r \, dv_\varphi$$

$$= (*)$$

$$(*) = \int_{v_r} (r+v_r)\, p_{v_r}(v_r)\, dv_r \cdot \int_{v_\varphi} \cos(\varphi+v_\varphi)\, p_{v_\varphi}(v_\varphi)\, dv_\varphi$$

$$= \Big(r \underbrace{\int_{v_r} p_{v_r}(v_r)\, dv_r}_{=1} + \underbrace{\int_{v_r} v_r\, p_{v_r}(v_r)\, dv_r}_{=0} \Big) \cdot \int_{v_\varphi} \cos(\varphi+v_\varphi)\, p_{v_\varphi}(v_\varphi)\, dv_\varphi$$

$$= r \cdot \underbrace{\int_{v_\varphi} \cos(\varphi+v_\varphi)\, p_{v_\varphi}(v_\varphi)\, dv_\varphi}$$

$$\overset{?}{=} \cos\varphi$$ is it unbiased ?

not in general : counterexample ...

$$p_{v_\varphi}(v_\varphi) = \begin{cases} \frac{1}{2}, & v_\varphi \in [-1, 1] \\ 0, & \text{else} \end{cases}$$

$$\Rightarrow \int_{-1}^{1} \cos(\varphi+v_\varphi)\tfrac{1}{2}\, dv_\varphi = \tfrac{1}{2} \sin(\varphi+v_\varphi)\Big|_{-1}^{1} = \tfrac{1}{2}\sin(\varphi+1) - \tfrac{1}{2}\sin(\varphi-1)$$

$$= \cos\varphi \cdot \sin(1)$$

$$\neq \cos\varphi$$

$$\tfrac{1}{2}\big(\sin(x) - \sin(y)\big) = \cos\left(\frac{x+y}{2}\right) \cdot \sin\left(\frac{x-y}{2}\right)$$

$$\tfrac{1}{2}\big(\sin(x+1) - \sin(x-1)\big) = \cos\left(\frac{x+1+x-1}{2}\right) \sin\left(\frac{x+1-x+1}{2}\right)$$

$$= \cos(x) \cdot \sin(1) \neq \cos(x)$$

same for $\mathbb{E}_{\underline{\hat{\omega}}|\underline{z}}(\hat{r} \cdot \sin\hat{\varphi})$ $\hookrightarrow$ not unbiased

EX 5 p5

d) $\{\hat{\omega}_n\}_{n=1}^{N}$

$$\hat{\underline{z}}_A = f^{-1}\Big(\underbrace{\tfrac{1}{N}\sum_n \hat{\underline{\omega}}_n}\Big)$$

$$\frac{1}{N}\sum_n \hat{\underline{\omega}}_n \approx \mathbb{E}(\hat{\underline{\omega}}) = \mathbb{E}(\underline{\omega} + \underline{v})$$

$$= \mathbb{E}(\underline{\omega}) + \mathbb{E}(\underline{v})$$

$$= \underline{\omega} + \underline{0}$$

$\Rightarrow$ $\underline{z}_A$ asymptotically unbiased

if $\mathbb{E}(\underline{v}) = \underline{0}$ zero mean

$$\hat{\underline{z}}_B = \frac{1}{N}\sum_n f^{-1}(\hat{\underline{\omega}}_n) \approx \mathbb{E}(f^{-1}(\hat{\underline{\omega}}))$$

from c BIASED