FFR110 - Homework Problem 3 - Part 1

Computational Biology

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Contents

1 Problem 1, Stochastic dynamics in large but finite populations

The SIS-model

$$\frac{dI}{dt} = \frac{\alpha}{S+I} * S * I - \beta * I$$
$$\frac{dS}{dt} = -\frac{\alpha}{S+I} * S * I + \beta * I$$

with α and β positive constants. Also S and I only take values $>= 0$, summed $S+I = N$ give the entire population which in this problem is constant.

1.1 a) Deterministic model

Problem 1a

$S = N - I$

$$\frac{dI}{dt} = \frac{\alpha}{S+I} S \cdot I - \beta I$$

$$\frac{dI}{dt} = \frac{\alpha}{N} SI - \beta I$$

$$\frac{dI}{dt} = I \cdot \left(\frac{\alpha}{N} S - \beta \right) \qquad \| \; S = N - I \qquad (1)$$

$$\frac{dI}{dt} = I \cdot \left(\frac{\alpha}{N} (N - I) - \beta \right)$$

$$\frac{dI}{dt} \overset{!}{=} 0 \qquad \text{(Steady state)}$$

$$0 = I \left(\frac{\alpha}{N} (N - I) - \beta \right)$$

$$I_1^* = 0$$

$$\frac{\alpha}{N} (N - I) = \beta$$

$$I_2^* = N \left(1 - \frac{\beta}{\alpha} \right)$$

Steady states at $I^* = \left(0, \; N \left(1 - \frac{\beta}{\alpha}\right)\right)$

Stability analysis

expand around steady states:

$$f(I^* + \eta) = f(I^*) + \eta f'(I^*)$$

with $f(I) = (1) = -\frac{\alpha I^2}{N} + \alpha I - \beta I$

now around I_1^*:

$$\dot{\eta} = \eta \, \underbrace{(\alpha - \beta)}_{\tau}$$

if $\tau < 0$ stable $\Rightarrow \alpha < \beta$

if $\tau > 0$ unstable $\Rightarrow \alpha > \beta$

around I_2^*:

$$\dot{\eta} = -\eta \, (\alpha - \beta)$$

due to the sign change, conditions flip

stable $\Rightarrow \alpha > \beta$

unstable $\Rightarrow \alpha < \beta$.

1.2 b) Stochastic model for finite population size

$n \to n+1$ $\qquad b_n = \alpha \cdot n \left(1 - \frac{n}{N}\right)$ ← infection

(2)

$n \to n-1$ $\qquad d_n = \beta n$ ← recovery

from Lecture notes:

$$\frac{d}{dt} \rho_n(t) = \lambda_{n-1} \rho_{n-1} + \mu_{n+1} \cdot \rho_{n+1} - (\lambda_n + \mu_n) \rho_n$$

describes probability to have n individuals at time t.

now lets write this with (2):

$$\frac{\partial}{\partial t} p_n = b_{n-1} p_{n-1} + d_{n+1} p_{n+1} - (b_n + d_n) p_n \qquad (3)$$

Now introduce the new operator as in the Lecture.

$$E^+_{g_n} = g_{n+1} \qquad \text{and} \qquad E^-_{g_n} = g_{n-1}$$

and also $I' = \frac{n}{N}$. $\qquad$ (4)

Rewrite (3) with the new operator:

$$\frac{\partial}{\partial t} p_n = (E^- - 1) b_n p_n + (E^+ + 1) d_n p_n \qquad (5)$$

with (4) we find:

$$p(t) = \mathcal{N} \, p(I', t)$$ (assume that p_n smooth)

$b_n = N\, b(I')$

$d_n = N\, d(I')$

$b(I') = \alpha I'(1 - I')$

$d(I') = \beta I'$

this leads to

$$E^{\pm} g(I') = \sum_{k=0}^{\infty} \frac{(\pm \frac{1}{N})^k}{k!} \frac{\partial^k g}{\partial I'^k} = e^{\pm \frac{1}{N}\frac{\partial}{\partial I'}} g(I')$$

$$e^{\pm \frac{1}{N}\frac{\partial}{\partial I'}} \overset{N \to \infty}{\cong} 1 + \frac{1}{N}\frac{\partial}{\partial I'} \qquad (6)$$

put into (5):

$$\frac{\partial}{\partial t} p_n(I', t) = \left[(e^{-\frac{1}{N}\frac{\partial}{\partial I'}} - 1) N b(I') + (e^{\frac{1}{N}\frac{\partial}{\partial I'}} - 1) N d(I') \right] \rho(I', t)$$

inserting the limit (6):

$$\frac{\partial}{\partial t} \rho(I', t) = -\frac{\partial}{\partial I'} (b(I') \rho(I', t)) + \frac{\partial}{\partial I'} (d(I') \rho(I', t))$$

Now define $V(I') = b(I') - d(I')$

giving:

$$\frac{dI}{dt} = V(I') = b(I') - d(I') = \alpha I'(1 - I') - \beta I'$$

In the stochastical model the steady state
$I_1^* = 0$ is an attracting FP. It can be seen as a quasi steady state.

1.3 c) Towards efficient simulation of the stochastic model

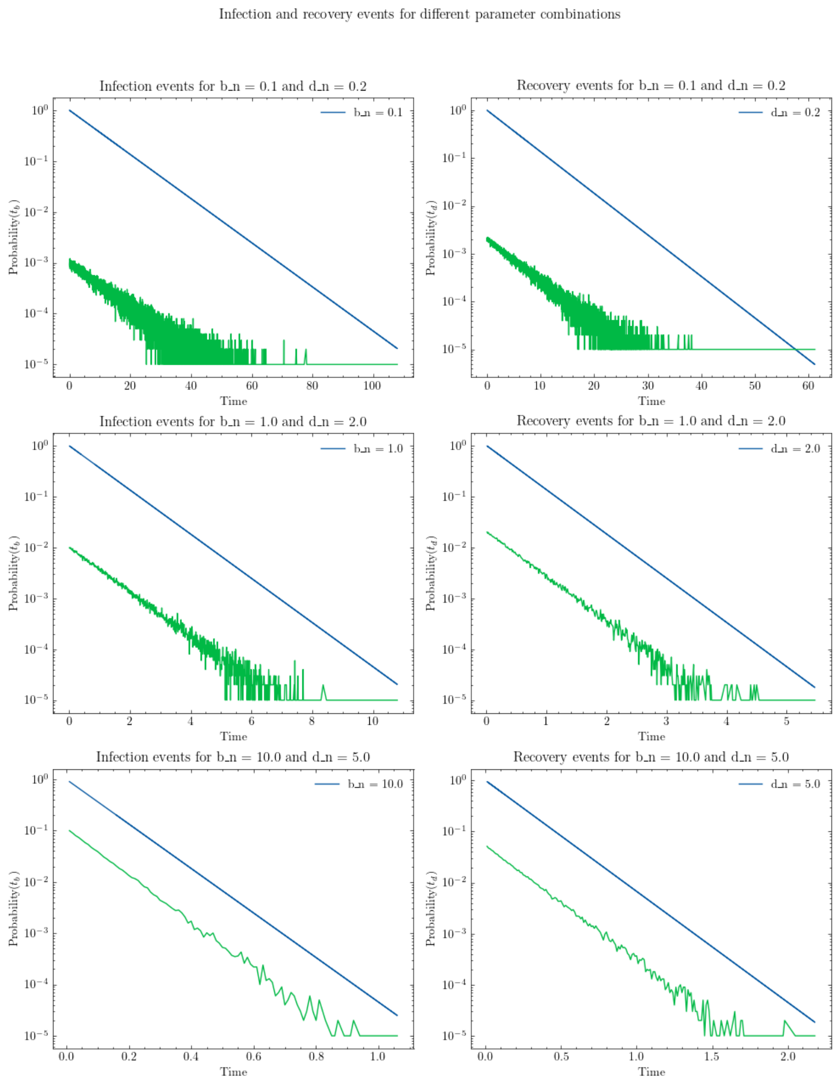

Figure 1: Typical recovery and infections times for the given parameter combinations in the exercise.

The blue lines in the plots are the step (iv) from the exercise to verify that the distributions decay exponentially with $-\lambda * t$. One finds in deed that, both the recover and the infections, do follow an exponential behaviour described by $e^{b_n * t}$ and analogue $e^{d_n * t}$ (trendline of green is parallel to blue). This leads to the following characteristic times it takes for a infection and recovery as display in Table 1.

Parameters	**Infection time [s]**	**Recover time [s]**
$b_n = 0.1$,$d_n = 0.2$	10	4.98
$b_n = 1$,$d_n = 2$	1	0.5
$b_n = 10$,$d_n = 5$	0.1	0.2

Table 1: Average times for a infection / recovery in the SIS model

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