4.1 Introduction to the Lorenz model

The three-dimensional Lorenz flow is given by

$$\dot{x} = \sigma\,(y - x)$$
$$\dot{y} = r\,x - y - x\,z$$
$$\dot{z} = x\,y - b\,z$$

The Lorenz system is named after the meteorologist Edward Norton Lorenz who studied it extensively. He found that the system (1) exhibits a fractal attractor for the parameter values $\sigma = 10$, $b = 8/3$ and $r = 28$. This attractor is nowadays called Lorenz attractor.

a) How many fixed points does the Lorenz system have, and how many of them are stable for the parameter values given above? Give your answer as the vector [number of fixed points, number of stable fixed points].

There are three fixed points. Fixed point 1 = (0,0)
Also check the stability, (Linear stability analysis) calculate the Jacobian (=stability matrix) and look at the eigenvalues. Look at the sign of the eigenvalues (positive unstable, negative stable).
Jacobian can be calculated by Mathematica with D[{f,g,h},{x,y,z}]

To find the fixed points start with checking the trivial solution, which happens to be fixed point 1 = (0,0,0). From the first equation one sees that $x = y$, using this in the second equation one finds $z = r - 1$ and applying this information to the third equation leads to $x = \pm\sqrt{b(r-1)}$. Thus fixed point 2 = $\left(\sqrt{b(r-1)},\ \sqrt{b(r-1)},\ r-1\right)$ and fixed point 3 = $\left(-\sqrt{b(r-1)},\ -\sqrt{b(r-1)},\ r-1\right)$. To check the stability of the fixed points, the Jacobian can be calculated and then evaluated at the fixed points.

The Jacobian is [[$-\sigma$, σ, 0], [$r - z$, -1, $-x$], [y, x, $-b$]].

For $\sigma = 10$, $b = 8/3$ and $r = 28$ the Jacobian evaluated at each fixed point.

Fixed point 1: [[−10, 10, 0], [28, −1, 0], [0, 0, −8/3]]. The eigenvalues are
$\lambda_1 = \frac{1}{2}\left(-11 - \sqrt{1201}\right)$, $\lambda_2 = \frac{1}{2}\left(-11 + \sqrt{1201}\right)$ and $\lambda_3 = -\frac{8}{3}$.

Fixed point 2: $\left[[-10, 10, 0], \left[1, -1, -6\sqrt{2}\right], \left[6\sqrt{2}, 6\sqrt{2}, -8/3\right]\right]$. The eigenvalues are
$\lambda_1 = -11, 185$, $\lambda_2 = -1, 241 + i * (1, 427)$ and $\lambda_3 = -1, 241 - i * (1, 427)$.

Fixed point 3: $\left[[-10, 10, 0], \left[1, -1, -6 i\sqrt{2}\right], \left[6 i\sqrt{2}, 6 i\sqrt{2}, -8/3\right]\right]$. The eigenvalues are
$\lambda_1 = -11.319 + i * 5.687$ $\lambda_2 = -11.319 + i * 5.687$ and $\lambda_3 = 8.973$

To determine the stability of the fixed points the eigenvalues need to be analysed.

Complex Eigenvalues:

Positive real part: The fixed point is unstable and the system behaves like an unstable oscillator a.k.a unstable spiral.

Zero real part: The fixed point is stable and the system behaves like an undamped oscillator a.k.a center.

Negative real part: The fixed point is stable and the system behaves like a damped oscillator a.k.a stable spiral.

Real Eigenvalues:

Zero Eigenvalues: System is unstable (This is just a trivial case of the complex eigenvalue that has a zero part).

Positive distinct Eigenvalues: Unstable fixed point.

Negative distinct Eigenvalues: Stable fixed point.

Positive and negative Eigenvalues: Saddle node (unstable).

This allow us to classify the fixed point 1 to be unstable, fixed point 2 is

In[ ]:=
```
ClearAll["Global`*"]

(*Fixed points for the system*)
FixedPoint1 = {0,0,0};
FixedPoint2 = {Sqrt[b(r-1)],Sqrt[b(r-1)],r-1};
FixedPoint2 = {-Sqrt[b(r-1)],-Sqrt[b(r-1)],r-1};

f1 = s(y-x);
f2 = r*x-y-x*z;
f3 = x*y-b*z;

Jacobi = D[{f1,f2,f3},{{x,y,z}}];

(*Parameters*)
s = 10;
b = 8/3;
r = 28;
x = Sqrt[-b(r-1)];
y = Sqrt[-b(r-1)];
z = r-1;

Eig = N[Eigenvalues[Jacobi]]
```

Out[ ]= {-11.3199 + 5.68695 i, -11.3199 - 5.68695 i, 8.97305}

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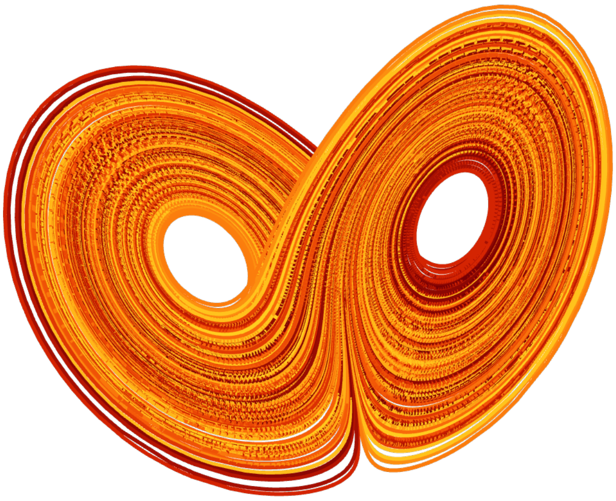
b) Solve the equations (1) numerically using the parameters stated above for some initial condition close to the origin. Plot an approximation of Lorenz attractor obtained by discarding the initial part of the solution.

Upload your figure as .pdf or .png.

Plot (y,z),(xy) and (xz)

```
In[•]:= ClearAll["Global`*"]

(* Lorenz variables *)
s = 10;
r = 28;
b = 8/3;

(* Differential equation system *)
Equation1 = x'[t] == s*(y[t] - x[t]);
Equation2 = y'[t] == r*x[t] - y[t] - x[t]*z[t];
Equation3 = z'[t] == x[t]*y[t] - b*z[t];
System = {Equation1, Equation2, Equation3};

(* Fixed points of the differential equation system *)
FixedPoint1 = {0, 0, 0};
FixedPoint2 = {Sqrt[b (r - 1)], Sqrt[b (r - 1)], r - 1};
FixedPoint3 = {-Sqrt[b (r - 1)], -Sqrt[b (r - 1)], r - 1};

(* Parameters for the plot *)
eta = 0.00001;
t0 = 0;
tMax = 10000;
t0Plot = 20;

(* Starting point of the trajectory (distance from fixed points handled via eta) *)
StartingPoint = {x[0] == FixedPoint1[[1]] - eta, y[0] == FixedPoint1[[2]] - eta, z[0] == Fix

Solution = NDSolve[{System, StartingPoint}, {x, y, z}, {t, t0, tMax}, MaxSteps → ∞];

(* Plotting *)
Show[ParametricPlot3D[Evaluate[{x[t], y[t], z[t]} /. Solution], {t, t0Plot, 400},
  PlotPoints → 1000,
  PlotStyle → Directive[Thick, RGBColor[.8, 0, 0]],
  ColorFunction → (ColorData["SolarColors", #4] &),
  PlotRange → All],
 RotationAction → "Clip", Boxed → False, (*SphericalRegion → False,*)
 Axes → False]
```

Out[∘]=


c) Compute the stability matrix $J_{ij} = \partial F_i / \partial x_j$ of the flow (1). Give your result as the matrix $[(J_{11}, J_{12}, J_{13}], [J_{21}, J_{22}, J_{23}], [J_{31}, J_{32}, J_{33}]]$.

In[∘]:=
```
f1 = s(y-x);
f2 = r*x-y-x*z;
f3 = x*y-b*z;

Jacobi = D[{f1,f2,f3},{{x,y,z}}]
```

Out[∘]= $\left\{\{-10, 10, 0\}, \{28 - z, -1, -x\}, \left\{y, x, -\frac{8}{3}\right\}\right\}$

d) Confirm that the trace of the stability matrix, traceJ, is independent of the coordinates (x, y, z). From what you have learned in the lectures and read in the course book, you should now be able to compute the sum of Lyapunov exponents for the Lorenz system (and thus the Lorenz attractor). Give your result for $\lambda_1 + \lambda_2 + \lambda_3$ for

general parameter values.

Calculate the Eigenvalues and then build the trace

lim_{t->\infty}\frac{1}{t}\int dt trace{J(t)} useful for 4.3