# HW3 - 2 Hopf Bifurcation

For each of the following two-dimensional flows

$$\begin{aligned} \dot{x} &= \mu x - 5y - 1 \\ \dot{y} &= 5x + \mu y + 3 \end{aligned}$$

and

$$\begin{aligned} \dot{x} &= \mu x + y - x \\ \dot{y} &= -x + \mu y + 2 \end{aligned}$$

a Hopf bifurcation occurs at the origin for $\mu = 0$.

A system with these properties can, at the bifurcation, be brought int

$$\begin{aligned} \dot{x} &= -\omega y + f \\ \dot{y} &= \omega x + g \end{aligned}$$

by a suitable change of coordinates. The functions $f$ and $g$ contain c
origin.

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## a.) What is $\omega$ for the two systems (1) & (2), respectively? Write your answer as the vector [w_1, w_2]

$\dot{x}_1 = \mu x - 5y - 1x^3 \quad = -wy + f(x,y) \Longrightarrow \quad w = 5$

In[ ]:=

$\dot{x}_2 = \mu x + y - x^2 \quad = -wy + f(x,y) \Rightarrow \quad w = -1$

Out[ ]=

$$\dot{x}_1 = \mu x - 5y - 1x^3 \quad = -\omega y + f(x,y) \implies \omega = 5$$

$$\dot{x}_2 = \mu x + y - x^2 \quad = -\omega y + f(x,y) \Rightarrow \omega = -1$$

## b.) Determine f and g for the systems (1) and (2). Write your solution as the matrix [[f_1,g_1],[f_2,g_2]]

b)

$$\dot{x}_1 = \mu x - 5y - x^3 \quad = -\omega y + f_1(x,y)$$

$$\dot{y}_1 = 5x + \mu y + 3y^3 \quad = \omega x + g_1(x,y)$$

$$f_1(x,y) = \mu x - x^3 \qquad \text{only HOT} \to -x^3$$

$$g_1(x,y) = \mu y + 3y^3 \qquad \to +3y^3$$

In[ ]:=

$$\dot{x}_2 = \mu x + y - x^2 \quad = -\omega y + f_2(x,y)$$

$$\dot{y}_2 = -x + \mu y + 2x^2 \quad = \omega x + g_2(x,y)$$

$$f_2(x,y) = \mu x - x^2 \qquad \text{only HOT} \Rightarrow -x^2$$

$$g_2(x,y) = \mu y + 2x^2 \qquad \Rightarrow 2x^2$$

$$\underline{\underline{[[-x^3, 3y^3], [-x^2, 2x^2]]}}$$

b)

$$\dot{x}_1 = \mu x - 5y - x^3 = -wy + f_1(x,y)$$
$$\dot{y}_1 = 5x + \mu y + 3y^3 = wx + g_1(x,y)$$

$$f_1(x,y) = \mu x - x^3 \quad \text{only HOT} \rightarrow -x^3$$
$$g_1(x,y) = \mu y + 3y^3 \quad \rightarrow +3y^3$$

Out[ ]=

$$\dot{x}_2 = \mu x + y - x^2 = -wy + f_2(x,y)$$
$$\dot{y}_2 = -x + \mu y + 2x^2 = wx + g_2(x,y)$$

$$f_2(x,y) = \mu x - x^2 \quad \text{only HOT} \Rightarrow -x^2$$
$$g_2(x,y) = \mu y + 2x^2 \quad \Rightarrow 2x^2$$

$$\underline{\underline{[[-x^3, 3y^3], [-x^2, 2x^2]]}}$$

# c.) Determine a for the two systems (1) and (2). Write you solution as the vector [a_1,a_2].

It can be shown that whether the bifurcation is subcritical or supercritical depends solely on the sign of the quantity $a$ defined by

$$\begin{aligned} 16a &= f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} \\ &+ \frac{1}{\omega}[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}] \end{aligned},$$

where the subscripts denote partial derivatives evaluated at the fixed point $(0, 0)$. According to this criterion, the bifurcation is supercritical if $a < 0$ and subcritical if $a > 0$.

In[1]:=
```
(* Define [f1,g1] and [f2,g2] *)
f1 = -x^3;
g1 = 3*y^3;
f2 = -x^2;
g2 = 2*x^2;

(* Define w1 and w2 from the results of a) *)
w1 = 5;
w2 = -1;

(* Define the partial derivatives for system (1) *)
fxx1 = D[f1,{x,2}];
gxx1 = D[g1,{x,2}];
fyy1 = D[f1,{y,2}];
gyy1 = D[g1,{y,2}];
fxy1 = D[D[f1,{x,1}], {y,1}];
gxy1 = D[D[g1,{x,1}], {y,1}];
fxxx1 = D[f1,{x,3}];
gyyy1 = D[g1,{y,3}];
fxyy1 = D[D[f1,{x,1}], {y,2}];
gxxy1 = D[D[g1,{x,2}], {y,1}];

a1 = (fxxx1 + fxyy1 + gxxy1 + gyyy1 + (fxy1 * (fxx1 + fyy1) - gxy1 * (gxx1 + gyy1) - f

(* Define the partial derivatives for system (2) *)
fxx2 = D[f2, {x, 2}];
gxx2 = D[g2, {x, 2}];
fyy2 = D[f2, {y, 2}];
gyy2 = D[g2, {y, 2}];
fxy2 = D[D[f2, {x, 1}], {y, 1}];
gxy2 = D[D[g2, {x, 1}], {y, 1}];
fxxx2 = D[f2, {x, 3}];
gyyy2 = D[g2, {y, 3}];
fxyy2 = D[D[f2, {x, 1}], {y, 2}];
gxxy2 = D[D[g2, {x, 2}], {y, 1}];

a2 = (fxxx2 + fxyy2 + gxxy2 + gyyy2 + (fxy2 * (fxx2 + fyy2) - gxy2 * (gxx2 + gyy2) - f
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Out[17]= $\frac{3}{4}$

Out[28]= $-\frac{1}{2}$

# d.) Draw phase portraits of the global dynamics for

positive and negative $\mu$ for each of the systems (1) and (2). Make sure that
these phase portraits verify the criteria you found in subtask c). Use a numerical solver, for example NDSolve] in Mathematica
(using StreamPlot] will not give enough resolution for this task).

In[2840]:=
```
ClearAll["Global'*"];

m1 = 0.02;
m2 = -0.02;

(*Define system 11*)
eq11 = x'[t] == m1 * x[t] - 5 * y[t] - x[t]^3;
eq12 = y'[t] == 5 * x[t] + m1 * y[t] + 3 * y[t]^3;

(* Define system 12 *)
eq13 = x'[t] == m1 * x[t] + y[t] - x[t]^2;
eq14 = y'[t] == -x[t] + m1 * y[t] + 2 * x[t]^2;

(*Define system 21 *)
eq21 = x'[t] == m2 * x[t] - 5 * y[t] - x[t]^3;
eq22 = y'[t] == 5 * x[t] + m2 * y[t] + 3 * y[t]^3;

(* Define system 22 *)
eq23 = x'[t] == m2 * x[t] + y[t] - x[t]^2;
eq24 = y'[t] == -x[t] + m2 * y[t] + 2 * x[t]^2;

system11 = {eq11, eq12};
system12 = {eq13, eq14};

system21 = {eq21, eq22};
system22 = {eq23, eq24};

(* Define the time range for the solutions *)
t0 = 0;
tMax = 100;

(* Define initial conditions *)
FP = {0,0};
radius = 0.1;
numInitialCond = 4;
initPoints = Table[{x[0] == FP[[1]] + radius * Cos[i * 2 * Pi / numInitialCond], y[0] == 
```

```
(* Trajectories *)
sol11 = Table[NDSolve[{system11, initPoints[[i]]}, {x, y}, {t, t0, tMax}], {i, Length[in
sol12 = Table[NDSolve[{system12, initPoints[[i]]}, {x, y}, {t, t0, tMax}], {i, Length[in
sol21 = Table[NDSolve[{system21, initPoints[[i]]}, {x, y}, {t, t0, tMax}], {i, Length[in
sol22 = Table[NDSolve[{system22, initPoints[[i]]}, {x, y}, {t, t0, tMax}], {i, Length[in

(* Create parametric plots with arrows *)
parametricPlotWithArrows[sol_, style_] :=
  ParametricPlot[Evaluate[{x[t], y[t]} /. sol], {t, t0, tMax},
    PlotStyle → style] /. Line[x_] :> {Arrowheads[{0., 0.04, 0.04, 0.04, 0.}], Arrow[x]

TP11 = parametricPlotWithArrows[#, Red] & /@ sol11;
TP12 = parametricPlotWithArrows[#, Blue] & /@ sol12;
TP21 = parametricPlotWithArrows[#, Green] & /@ sol21;
TP22 = parametricPlotWithArrows[#, Magenta] & /@ sol22;

xMin = -0.5;
xMax = 0.5;
yMin = -0.5;
yMax = 0.5;

sSystem11 = {m1 * x - 5 * y - x^3,5 * x + m1 * y + 3 * y^3};
sSytsem12 = {m1 * x + y - x^2,-x + m1 * y + 2 * x^2};
sSystem21 = {m2 * x - 5 * y - x^3,5 * x + m2 * y + 3 * y^3};
sSystem22 = {m2 * x + y - x^2, -x + m2 * y + 2 * x^2};

mag = 0.5;
(* Create stream plots for each system *)
SP11 = StreamPlot[{sSystem11}, {x, xMin*mag, xMax*mag}, {y, yMin*mag, yMax*mag}, Strea
SP12 = StreamPlot[{sSytsem12}, {x, xMin*mag, xMax*mag}, {y, yMin*mag, yMax*mag}, Strea
SP21 = StreamPlot[{sSystem21}, {x, xMin*mag, xMax*mag}, {y, yMin*mag, yMax*mag}, Strea
SP22 = StreamPlot[{sSystem22}, {x, xMin*mag, xMax*mag}, {y, yMin*mag, yMax*mag}, Strea

Show[SP21, TP21, PlotLabel→"m=-0.2 system 1"]
Show[SP11, TP11, PlotLabel→"m=0.2 system 1"]
Show[SP22, TP22, PlotLabel→"m=-0.2 system 2"]
Show[SP12, TP12, PlotLabel→"m=0.2 system 2"]


Show[TP11, TP12, TP21, TP22, PlotRange → All];
```

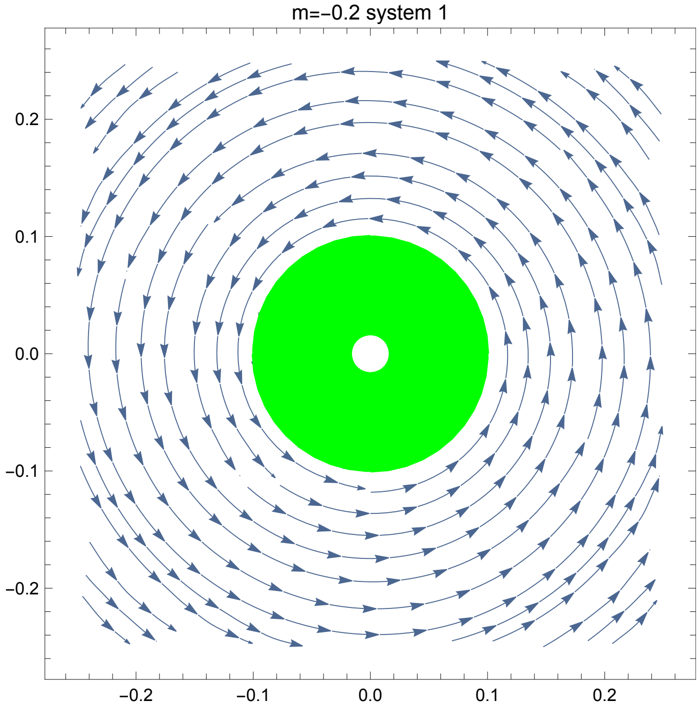
NDSolve : At t == 32.091076640184724`, step size is effectively zero; singularity or stiff system suspected.

NDSolve : At t == 32.186726815656726`, step size is effectively zero; singularity or stiff system suspected.

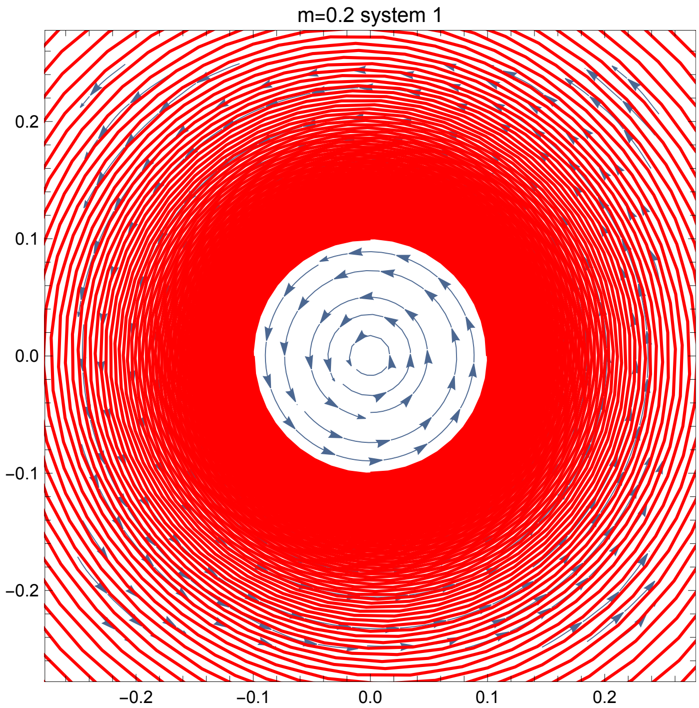
NDSolve : At t == 32.091076640184724`, step size is effectively zero; singularity or stiff system suspected.

General : Further output of NDSolve::ndsz will be suppressed during this calculation.

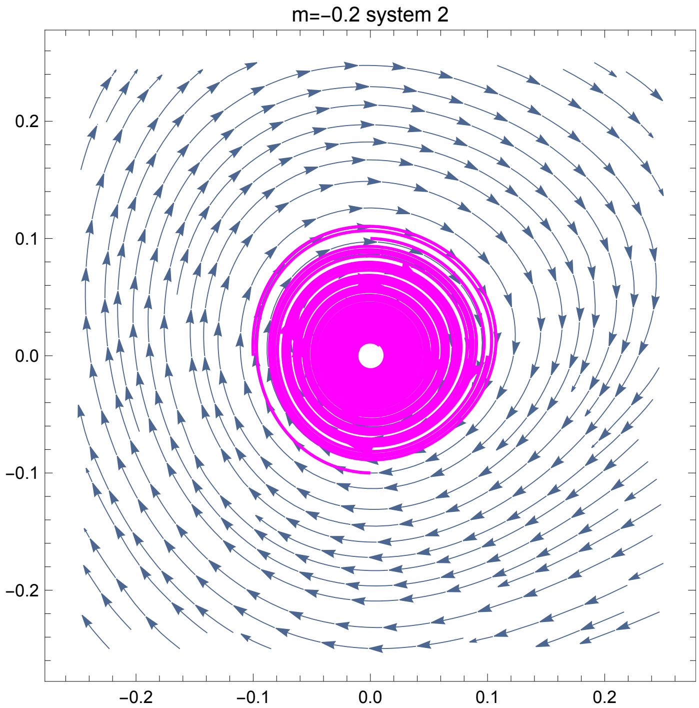
Out[2883]=



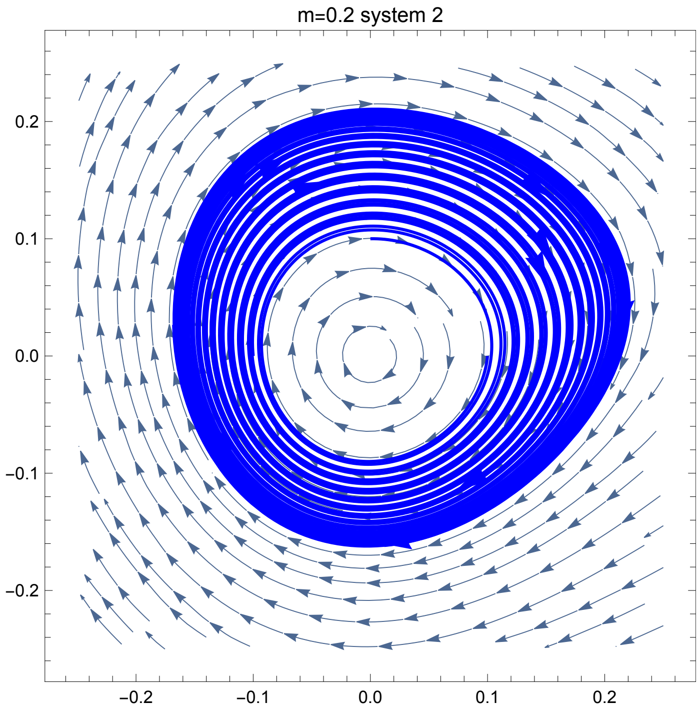

Out[2884]=



Out[2885]=




Out[2886]=




It can be seen, that the system 1 undergoes a subcritical bifurcation. After the bifurcation the trajectories quickly escape.
For system 2 a supercritical behaviour is observed, the last plot approaches a limit cycle which can nicely be seen with the almost filled outer contour of the “egg” shape.

In[ ]:=