2.4 Integral of motion

Deadline: 22 Nov 23:59

(1 point)

The motion of a bead on a rotating hoop is described by the dimensionless dynamical system

$$\begin{aligned} \dot{\phi} &= \omega \\ \dot{\omega} &= \sin(\phi)[\cos(\phi) - \tau - 1] \end{aligned} .$$

Here, ϕ is the angle, ω describes the angular velocity and τ is a dimensionless parameter.

Derive an integral of motion for this dynamical system in terms of (ϕ, ω, τ). Normalise your conserved quantity so that the terms that contain ω have the prefactor $+1$ and adjust the additive integration constant so that the conserved quantity is equal to -1 when $\phi = \pi/2$ and $\omega = 0$.

(in terms of omega,phi,tau)

$$\dot{\phi} = w$$

$$\dot{w} = \sin(\phi) \cdot \left[\cos(\phi) - \Upsilon - 1\right]$$

$$\ddot{\phi} = \sin(\phi) \cdot \left[\cos(\phi) - \Upsilon - 1\right]$$

$$\phi \to x$$

$$\ddot{x} - \sin(x)(\cos(x) - \Upsilon - 1) = 0 \qquad \Big|\Big| \cdot \frac{dx}{dt} \qquad (\text{Trick})$$

$$\frac{1}{2}\frac{dx}{dt} \cdot \frac{d^2x}{dt^2} - \frac{dx}{dt}\left(\sin(x)(\cos(x) - \Upsilon - 1)\right)$$

$$\frac{1}{2}\underbrace{\dot{x} \cdot \ddot{x}}_{\int} - \underbrace{\dot{x}\left(\sin(x)(\cos(x) - \Upsilon - 1)\right)}_{\int}$$

$$\Big\downarrow \text{Wolfram Alpha}$$

$$\frac{1}{2}\frac{dx}{dt} \cdot w^2 + \frac{1}{2}\frac{dx}{dt}\cos(x)(\cos(x) - 2(\Upsilon + 1)) + C$$

$$E(w, \phi) = w^2 + \cos(\phi)(\cos(\phi) - 2(\Upsilon + 1)) + C$$

$$E\left(0, \frac{\pi}{2}\right) = 0 + 0 + C \overset{!}{=} -1 \qquad \to C = -1$$

$$\underline{\underline{E(w, \phi) = w^2 + \cos(\phi)(\cos(\phi) - 2(\Upsilon + 1)) - 1}}$$