In[∘]:= (* Find all the zero positions *)

```
f[x_] := r * x + 4 * x^3 - 9 * x^5
xstar = Solve[r * x + 4 * x^3 - 9 * x^5 == 0, x] // Simplify
```

Out[∘]= $\left\{\{x \to 0\}, \left\{x \to -\frac{1}{3}\sqrt{2-\sqrt{4+9r}}\right\}, \left\{x \to \frac{1}{3}\sqrt{2-\sqrt{4+9r}}\right\}, \left\{x \to -\frac{1}{3}\sqrt{2+\sqrt{4+9r}}\right\}, \left\{x \to \frac{1}{3}\sqrt{2+\sqrt{4+9r}}\right\}\right\}$

In[∘]:=

$$\text{xstar1[r\_]} = 0;$$

$$\text{xstar2[r\_]} = -\frac{1}{3}\sqrt{2-\sqrt{4+9r}};$$

$$\text{xstar3[r\_]} = \frac{1}{3}\sqrt{2-\sqrt{4+9r}};$$

$$\text{xstar4[r\_]} = -\frac{1}{3}\sqrt{2+\sqrt{4+9r}};$$

$$\text{xstar5[r\_]} = \frac{1}{3}\sqrt{2+\sqrt{4+9r}};$$

$$\text{xstar1[r\_]} = 0;$$

In[∘]:=

```
(* Determine which xstars are stable and which are instable *)
dir1[r_] = D[f[x], x] /. xstar[[1]]
dir2[r_] = D[f[x], x] /. xstar[[2]]
dir3[r_] = D[f[x], x] /. xstar[[3]]
dir4[r_] = D[f[x], x] /. xstar[[4]]
dir5[r_] = D[f[x], x] /. xstar[[5]]

(* eval at r = -0.2*)
r1 = -0.2
dir1[r1]
dir2[r1]
dir3[r1]
dir4[r1]
dir5[r1]
```

Out[∘]= r

Out[∘]= $r + \frac{4}{3}\left(2 - \sqrt{4 + 9\,r}\right) - \frac{5}{9}\left(2 - \sqrt{4 + 9\,r}\right)^2$

Out[∘]= $r + \frac{4}{3}\left(2 - \sqrt{4 + 9\,r}\right) - \frac{5}{9}\left(2 - \sqrt{4 + 9\,r}\right)^2$

Out[∘]= $r + \frac{4}{3}\left(2 + \sqrt{4 + 9\,r}\right) - \frac{5}{9}\left(2 + \sqrt{4 + 9\,r}\right)^2$

Out[∘]= $r + \frac{4}{3}\left(2 + \sqrt{4 + 9\,r}\right) - \frac{5}{9}\left(2 + \sqrt{4 + 9\,r}\right)^2$

Out[∘]= -0.2

Out[∘]= -0.2

Out[∘]= 0.340658

Out[∘]= 0.340658

Out[∘]= -2.29621

Out[∘]= -2.29621

In[∘]:= `0.3406575088170066` ⋮⋮⋮`

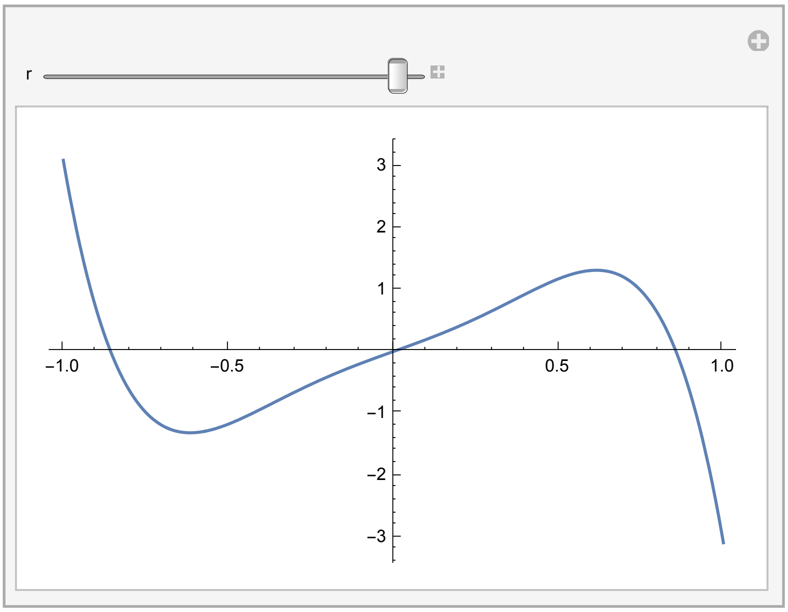
Out[∘]= $0.340658\,⋮^3$

In[∘]:= $0.3406575088170066`\,⋮^3$

Out[∘]= $0.340658\,⋮^3$

In[∘]:=
```
Manipulate[Plot[r * x + 4 * x^3 - 9 * x^5, {x, -1, 1}], {r, -2, 2}]
```

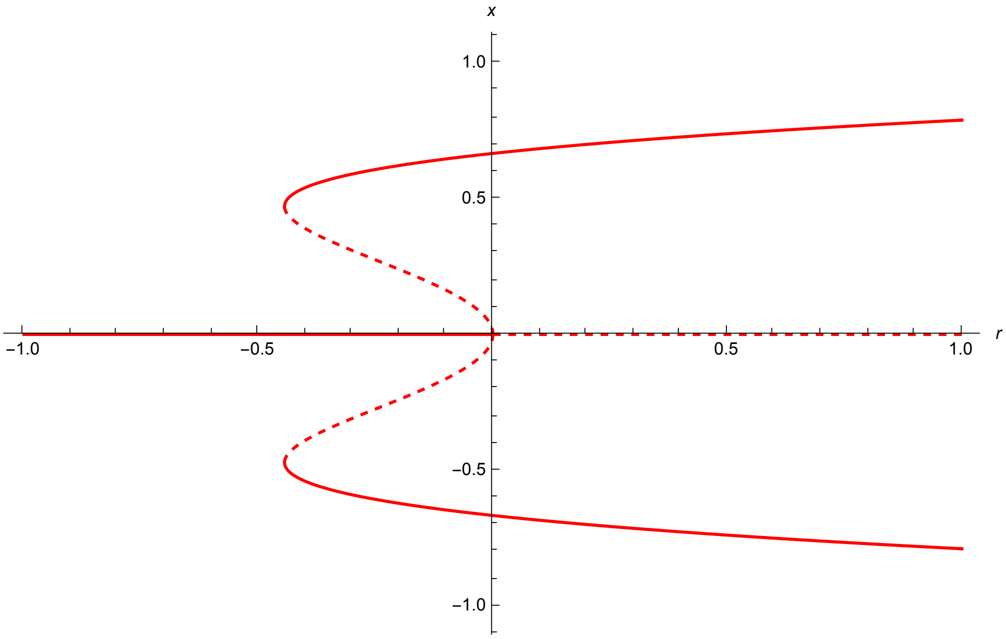
Out[∘]=

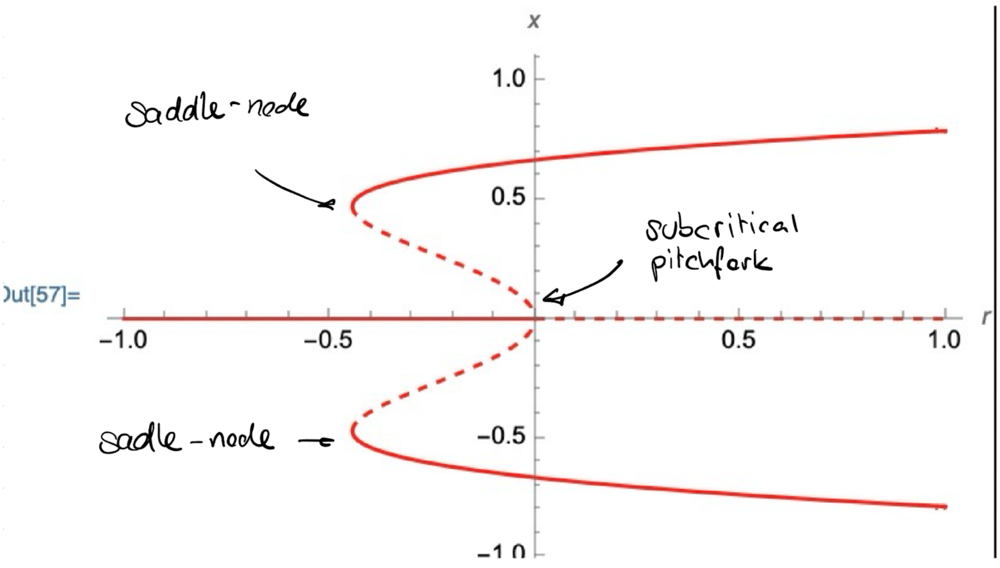


In[∘]:=
```
(* Determine a range for r*)
rmin = -1;
rmax = 1;
```

In[∘]:=
```
(* First solution is for r < 0*)
plot1 = Plot[xstar1[r], {r, rmin, 0}, PlotStyle → {Red}];
plot11 = Plot[xstar1[r], {r, 0, rmax}, PlotStyle → {Red, Dashed}];
plot2 = Plot[xstar2[r], {r, rmin, rmax}, PlotStyle → {Red, Dashed}];
plot3 = Plot[xstar3[r], {r, rmin, rmax}, PlotStyle → {Red, Dashed}];
plot4 = Plot[xstar4[r], {r, rmin, rmax}, PlotStyle → {Red}];
plot5 = Plot[xstar5[r], {r, rmin, rmax}, PlotStyle → {Red}];
```

In[∘]:=
```
(* Combine all plots together *)
Show[plot1, plot11, plot2, plot3, plot4, plot5,
 AxesLabel → {r, x}, PlotRange → {{rmin, rmax}, {rmin, rmax}}]
```





At the point - 4/9 saddle - node bifurcation occurs . If approaching from the left, so r < -4/9, one jumps from 1 to 3 FPs if r = -4/9. In the moment when r > -4/9 another two FPs are created, so a total of five exists . Three of them are stable, as can be seen in the plot with the continuous lines, and two are unstable (dashed lines) . Ar the origin 2 FPs are lost, and the stable FP on the x = 0 line become s unstable. There an subcritical pitchform bifurcation occurs.

```
(* Find the saddle-node bifurcations analytically *)
df[x_, r_] = r + 12 x^2 - 45 x^4;
```

In[∘]:=
```
b1[r_] = df[x, r] /. xstar[[3]]
```

Out[∘]= $r + \frac{4}{3}\left(2 - \sqrt{4 + 9\,r}\right) - \frac{5}{9}\left(2 - \sqrt{4 + 9\,r}\right)^2$

```
(* question b → where is r_c*)
Solve[b1[r] == 0, r]
```

$$\left\{\left\{r \to -\frac{4}{9}\right\}, \{r \to 0\}\right\}$$