1.2 Imperfect transcritical bifurcation

Deadline: 15 Nov 23:59

P. 70

$h + xr - x^2 = 0$

(4 points)

Consider the dynamical system $\dot{x} = f(x, h, r)$ with

$$f(x, h, r) = h + x(r - x)$$

for different values of the parameters h and r. For $h = 0$, the system undergoes a transcritical bifurcation at $r = 0$ along the r-direction. Define the fixed points $x^*(h, r)$ of the system by $f(x^*(h, r), h, r) = 0$.

a) Make a plot of the (h, r) plane. Label different regions according to the number and the types (stable or unstable) of fixed points for (h, r) in that region. The different regions are separated by a bifurcation curve that you are supposed to draw into your plot. Upload the plot as .pdf or .png.

b) Make a three-dimensional (x^*, h, r)-plot of the surface of fixed points, where $f(x^*, h, r) = 0$. Upload the plot (.pdf or .png).

c) Find an analytical expression for the bifurcation curve $[h_c(r), r]$. Write your result as a vector that depends on r.

(in terms of r)

A A A

d) At the bifurcation point $h = r = 0$, a transcritical bifurcation occurs along the r-direction. Similarly, for each point, $[h_c(r), r]$, on the bifurcation curve, a transcritical bifurcation occurs in one direction that depends on r. Find this direction analytically. Write your result as a vector in the $[h, r]$-plane that depends on r. For definiteness, normalise the vector to unity and make sure your solution is equal to $[0, 1]$ for $h = 0$.

(in terms of r)

A A A

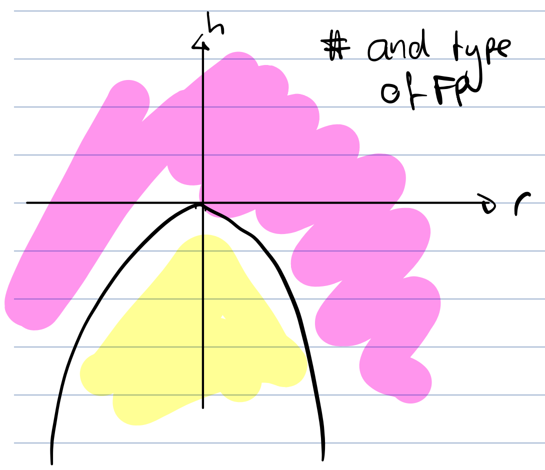
a)

$f(x,r,h) = h + x(r-x) = h + rx - x^2$

Solve for f(x)==0, r

fps = Solve[f(x,h,r) = = 0, r]

$f'(x_1^*)$ & $f'(x_2^*)$

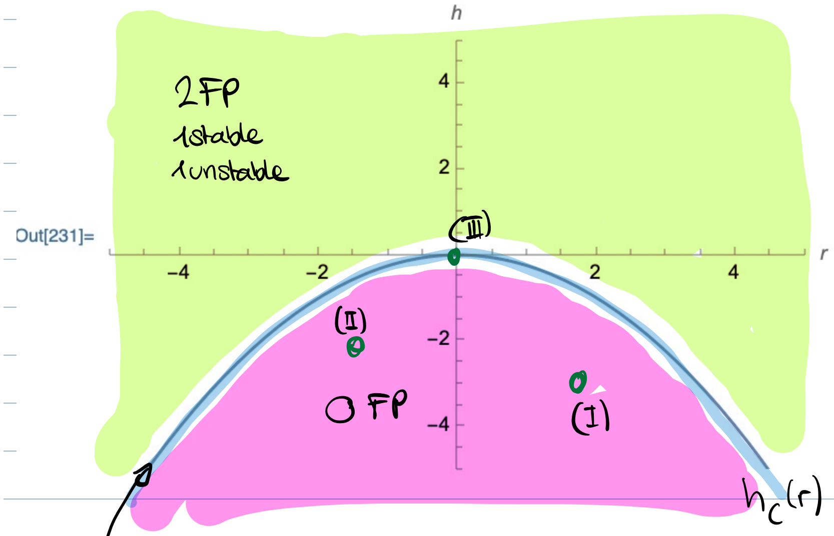


$x_1^*(h,r) = \frac{1}{2}\left(r - \sqrt{4h + r^2}\right)$

$x_2^*(h,r) = \frac{1}{2}\left(r + \sqrt{4h + r^2}\right)$

$\dot{x} = \frac{-r^2}{4} + x(r-x)$

$0 = \frac{d}{dx}(rx - x^2) = r - 2x \rightarrow x^* = \frac{r}{2}$



$h_c(r) = -\frac{r^2}{4}$

1 FP on the line
→ halfstable

eval $\dot{x}$ at some points:

$h=-2 \quad , r=-1$

$\dot{x} = -1 - x - x^2 \overset{!}{=} 0$ ← only $\mathbb{C}$ solutions so 0 FPs.

$h = 0 \qquad r = 0$ ← on critical curve $h_c(r)$

$\dot{x} = x^2 \quad \Rightarrow \quad 0 = x$

no Flow → half stable

$h=1 \qquad r=1:$ $\qquad\qquad$ $h=1 \quad r=-1:$

$\dot{x} = 1 + x - x^2$ $\qquad\qquad$ $\dot{x} = 1 - x - x^2$

$\boxed{x_1^* = \frac{1}{2} \pm \frac{\sqrt{5}}{2} \qquad\qquad x_2^* = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}}$ → 1 unstable FP, 1 stable FP

Type of FP was also determined with manipulate in Mathematica.

Intersections with $\dot{x}=0$ only occur if in pink area. Tangentially with $\dot{x}=0$ line if on critical $h_c(r)$ curve $\dot{x} = h_c(r) + rx - x^2$. If in green area no fixpoints at all.

c)

$$0 = \dot{x}(x^*) = h + \frac{r}{2}\left(r - \frac{r}{2}\right)$$

$$-h = \frac{r}{2}\left(r - \frac{r}{2}\right) = \frac{r^2}{4} \rightarrow h(r) = \frac{-r^2}{4}$$

Bifurcation curve:

$$\begin{bmatrix} \\ r \end{bmatrix} = \begin{bmatrix} -\left(\frac{r}{2}\right)^2 \\ r \end{bmatrix}$$

d) how does h change if r increases?

$$\left[\frac{h_c(r)}{dr}, 1\right]$$ normalise to be unit vector

ex Class

$$\frac{-\frac{r^2}{4}}{dr} = \left[-\frac{r}{2}, 1\right]$$

$$\text{norm}: \sqrt{\left(\frac{h(r)}{dr}\right)^2 + 1^2} : \sqrt{r^2 \cdot \frac{1}{4} + 1}$$

$$\left[\frac{-\frac{r}{2}}{\text{norm}}, \frac{1}{\text{norm}}\right]$$

$$\left[\frac{-r}{\sqrt{r^2+4}}, \frac{1}{\sqrt{\frac{r^2}{4}+1}}\right]$$