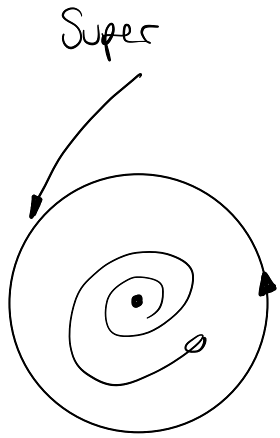
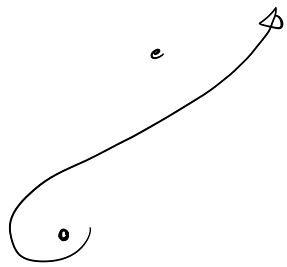
Navid





$$\dot{x} = -wy + f(x,y)$$
$$\dot{y} = wx + g(x,y)$$

a) Compare → find w?

b) Find f & g, by comparison

$\mu = 0$

c)

$a > 0$ sub critical

$a < 0$ Super critical

3rd order derivative

$D[f(x,y), \{x, \underset{n}{3}\}]$

d) Phase portraits. (NDSolve

3.2 Hopf bifurcation

Deadline: 29 Nov 23:59

(2 points)

See exercises 8.2.14-15 in Strogatz

For each of the following two-dimensional flows

$$\begin{aligned} \dot{x} &= \mu x - 5y - 1x^3 \\ \dot{y} &= 5x + \mu y + 3y^3 \end{aligned} \qquad (1)$$

W = 5

and

$$\begin{aligned} \dot{x} &= \mu x + y - x^2 \\ \dot{y} &= -x + \mu y + 2x^2 \end{aligned} \qquad (2)$$

W = -1

a Hopf bifurcation occurs at the origin for $\mu = 0$.

A system with these properties can, at the bifurcation, be brought into the form

$$\begin{aligned} \dot{x} &= -\omega y + f(x,y) \\ \dot{y} &= \omega x + g(x,y) \end{aligned},$$

by a suitable change of coordinates. The functions f and g contain only higher-order (non-linear) terms that vanish at the origin.

a) What is ω for the two systems (1) and (2), respectively? Write your answer as the vector $[\omega_{(1)}, \omega_{(2)}]$.

b) Determine f and g for the systems (1) and (2). Write your solution as the matrix $[[f_{(1)}, g_{(1)}], [f_{(2)}, g_{(2)}]]$.

(in terms of x,y)

It can be shown that whether the bifurcation is subcritical or supercritical depends solely on the sign of the quantity a defined by

$$\begin{aligned} 16a &= f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} \\ &+ \frac{1}{\omega}\left[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\right] \end{aligned},$$

where the subscripts denote partial derivatives evaluated at the fixed point $(0,0)$. According to this criterion, the bifurcation is supercritical if $a < 0$ and subcritical if $a > 0$.

c) Determine a for the two systems (1) and (2). Write you solution as the vector $[a_{(1)}, a_{(2)}]$.

Using this result you should be able to determine what kind of bifurcations the systems (1) and (2) undergo at $\mu = 0$.

d) Draw phase portraits of the global dynamics for positive and negative μ for each of the systems (1) and (2). Make sure that these phase portraits verify the criteria you found in subtask c). Use a numerical solver, for example NDSolve[] in Mathematica (using StreamPlot[] will not give enough resolution for this task).
Upload the phase portraits as .png images or .pdf.

a)

$$\dot{x}_1 = \mu x - 5y - 1x^3 \quad = -wy + f(x,y) \implies \quad w_1 = 5$$

$$\dot{x}_2 = \mu x + y - x^2 \quad = -wy + f(x,y) \Rightarrow \quad w_2 = -1$$

$$\underline{\underline{w = [5, -1]}}$$

b)

$$\dot{x}_1 = \mu x - 5y - x^3 \qquad = -wy + f_1(x,y)$$
$$\dot{y}_1 = 5x + \mu y + 3y^3 \qquad = wx + g_1(x,y)$$

$$f_1(x,y) = \mu x - x^3 \qquad \text{only HOT} \rightarrow -x^3$$
$$g_1(x,y) = \mu y + 3y^3 \qquad \rightarrow +3y^3$$

$$\dot{x}_2 = \mu x + y - x^2 \qquad = -wy + f_2(x,y)$$
$$\dot{y}_2 = -x + \mu y + 2x^2 \qquad = wx + g_2(x,y)$$

$$f_2(x,y) = \mu x - x^2 \qquad \text{only HOT} \Rightarrow -x^2$$
$$g_2(x,y) = \mu y + 2x^2 \qquad \Rightarrow 2x^2$$

$$\underline{\underline{[[-x^3, 3y^3], [-x^2, 2x^2]]}}$$

c) $a_1 = 3/4$

$a_2 = -1/2$

$[3/4, -1/2]$

d) for $S(1)$ $a = \frac{3}{4}$

so $S(1)$ undergoes a subcritical bif.

for $S(2)$ $a = -\frac{1}{2}$

so $S(2)$ undergoes a supercritical bif.