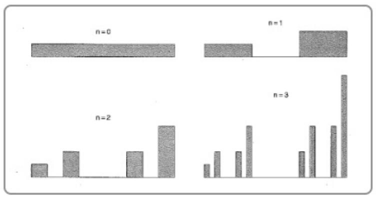
4.2 Renyi dimension of weighted Cantor set

Deadline: 20 Dec 23:59



(3 points)

In this exercise, your are going to calculate and analyze the Renyi dimension spectrum of the a weighted Cantor set. The symmetric Cantor set was discussed in the lecture and you know that the box counting dimension of this set is $D_0 = \frac{\ln 2}{\ln 3} \approx 0.63093$. Recall that the Cantor set was generated by removing the middle third from the unit interval, then the remaining two subintervals had their middle thirds removed and so on.

Now extend this construction by allocating a probability (or "mass") to each subinterval at each division. Allocate $\frac{2}{3}$ of the existing probability in an interval being divided to the right-hand side of the subinterval, and $\frac{1}{3}$ to the left as is shown in the figure above.

(a) Calculate analytically the Renyi dimension spectrum D_q of the weighted Cantor set. Make sure that for $q = 0$, you recover the box counting dimension of the Cantor set. Give your result as a function of q.

(in terms of q)

(b) Using the expression derived in (a), make a plot of D_q as a function of q for $q \in [-20, 20]$.

Upload your plot as a .pdf of .png.

(c) Using the expression derived in (a), compute explicitly D_1 (information dimension) and D_2 (correlation dimension) of the weighted Cantor set. Give your result as the vector $[D_1, D_2]$

(d) Using the expression derived in (a), compute explicitly $D_{-\infty} = \lim_{q \to -\infty} D_q$ and $D_\infty = \lim_{q \to \infty} D_q$ of the weighted Cantor set. Give your result as the vector $[D_{-\infty}, D_\infty]$

Navid

$$D_q = \frac{1}{1-q} \lim_{\varepsilon \to 0} \frac{\ln\left(I_q(\varepsilon)\right)}{\ln(1/3)}$$

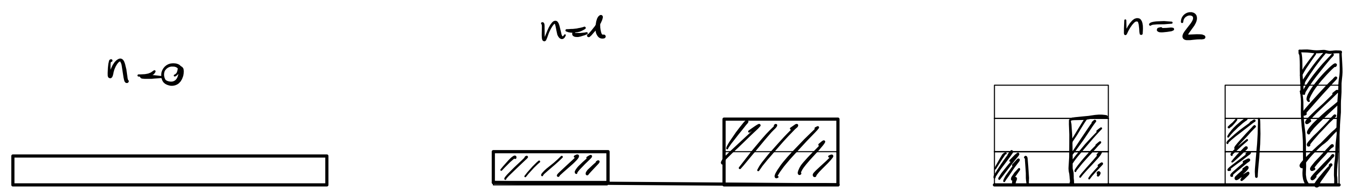
$q = 0 \rightarrow$ box counting

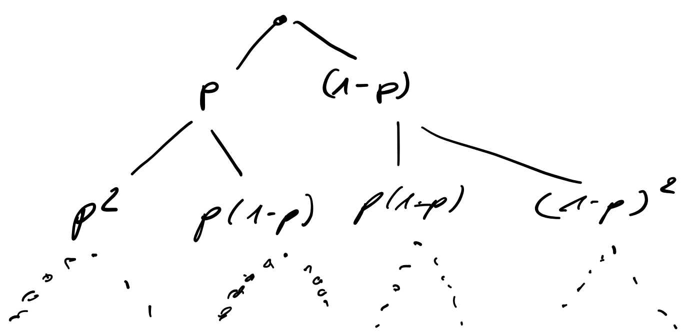
$q = 1 \rightarrow$ Information dimension (take limit $q \rightarrow 1$)

$q = 2 \rightarrow$ Correlation dimension

$$I_q(\varepsilon) \rightarrow \sum_{j=0}^{N_{box}} p_j^q(\varepsilon) \qquad p_j(\varepsilon) = \frac{N_j}{N_{POINTS}}$$

$$\sum^{N_{box}} 1 = N_{box}$$





$$\binom{n}{k} p^k (1-p)^{n-k}$$

$$I_q(\varepsilon) = \sum_{j=0}^{N_{box}} p_j^{\,q}(\varepsilon) = \sum_{k=0}^{N_{STEPS}} \binom{n}{k} \left[\left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k} \right]^{\cancel{2}\,q}$$

To compute I_q:

a) $$\sum_k \binom{n}{k} r^k = (1+r)^n$$

b) D_q is always decreasing

c) D_1 and D_2 $(q=2)$

$\downarrow$

$\lim_{q \to 1}$

d) $D_{-\infty}$ should be bigger

$D_{+\infty}$ should be the smaller

$\Big\}$ always decreasing!

a) $$D_q = \frac{1}{1-q} \lim_{\varepsilon \to 0} \frac{\ln(I(q,\varepsilon))}{\ln(\frac{1}{\varepsilon})} \qquad (\mathrm{I})$$

$$I(q,\varepsilon) = \sum_{j=0}^{N_{box}} p_j^{\,q}(\varepsilon)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left[p^k \cdot (1-p)^{n-k} \right]^q$$

$$= \sum_{k=0}^{n} \binom{n}{k} (p^q)^k ((1-p)^q)^{n-k}$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^{n-k} \cdot y^k$$

so $x = (1-p)^q$

$y = p^q$

$$I(q) = (x+y)^n = ((1-p)^q + p^q)^n \qquad (\mathrm{II})$$

put II in I

$$D_q = \frac{1}{1-q} \cdot \lim_{\varepsilon \to 0} \frac{\ln\left(\left((1-p)^q + p^q\right)^n\right)}{\ln(1/\varepsilon)}$$

$$= \frac{1}{1-q} \cdot \lim_{\varepsilon \to 0} \frac{n \cdot \ln\left(\left((1-p)^q + p^q\right)\right)}{\ln(\frac{1}{\varepsilon})}$$

$$\varepsilon = (1/3)^n \qquad \ln(\varepsilon) = n \cdot \ln(\tfrac{1}{3}) \quad n = \frac{\ln(\varepsilon)}{\ln(3)}$$

$$= \frac{1}{1-q} \cdot (-) \frac{\ln(\varepsilon) \cdot \ln\left(\left(\frac{2}{3}\right)^q + \left(\frac{1}{3}\right)^q\right)}{\ln(1/\varepsilon) \cdot \ln(3)}$$

$$= (-1) \cdot \frac{1}{1-q} \cdot \frac{\cancel{\ln(\varepsilon)} \cdot \ln\left(\left(\frac{2}{3}\right)^q + \left(\frac{1}{3}\right)^q\right)}{\cancel{\ln(\varepsilon)} \cdot \ln(3)}$$

$$= \frac{1}{1-q} \cdot \frac{\ln\left(\left(\frac{2}{3}\right)^q + \left(\frac{1}{3}\right)^q\right)}{\ln(3)}$$

$$= \frac{1}{1-q} \cdot \frac{\ln\left(\frac{2^q+1}{3^q}\right)}{\ln(3)}$$

$$= \frac{\cdot\left(\ln(2^q+1) - (\ln(3^q)\right)}{(1-q) \cdot \ln(3)}$$

a) $$= \frac{\ln(2^9+1) - 9\cdot\ln(3)}{(1-9)\cdot\ln(3)}$$