DYS HW4-2 Stability Exponents for a toy model

We define a simple flow in polar coordinates on which to test that the Lyapunov exponent calculation works. Define the simple dynamics

In[762]:=

$$\begin{aligned} \dot{r} &= \mu r - r^3 \\ \dot{\theta} &= \omega + \nu r^2 \end{aligned}, \qquad (1)$$

which has a stable fixed point and a limit cycle if $\mu > 0$.

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a.) Calculate radius r_0 and the period of the limit cycle for mu > 0

In[763]:=

(a) Calculate the radius r_0 and the period T of the limit cycle for $\mu > 0$. Give your result on the form $[r_0, T]$. *(0.75 points)*

(in terms of mu,omega,nu)

A A A

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(in terms of mu,omega,nu)

A A A

In[764]:=

b.) Make a phase portrait of the dynamical system (2), showing a few representative trajectories. In the same figure, plot the limit cycle using a suitable representative trajectory. Upload your figure as .pdf or .png. Using StreamPlot]] is

not acceptable.
(0.5 points)

In[765]:=

$$\begin{aligned} \dot{X}_1 = F_1(\mathbf{X}) &= \frac{1}{10}X_1 - X_2^3 - X_1X_2^2 - X_1^2X_2 - X_2 - X_1^3 \\ \dot{X}_2 = F_2(\mathbf{X}) &= X_1 + \frac{1}{10}X_2 + X_1X_2^2 + X_1^3 - X_2^3 - X_1^2X_2 \end{aligned} . \qquad (2)$$

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$$\dot{X}_1 = \mu X_1 - X_1^3 - X_1 X_2^2 - \omega X_2 + \nu X_1^2 X_2 + \nu X_2^3$$

$$\dot{X}_2 = \mu X_2 - X_1^2 X_2 - X_2^3 + \omega X_1 + \nu X_1^3 + \nu X_2^2 X_1$$

In[766]:=

$$\begin{aligned} \dot{X}_1 = F_1(\mathbf{X}) &= \frac{1}{10}X_1 - X_2^3 - X_1X_2^2 - X_1^2X_2 - X_2 - X_1^3 \\ \dot{X}_2 = F_2(\mathbf{X}) &= X_1 + \frac{1}{10}X_2 + X_1X_2^2 + X_1^3 - X_2^3 - X_1^2X_2 \end{aligned} . \quad (2)$$

$\mu = 1/10 \qquad \omega = 1 \qquad \nu = 1$

$$\dot{X}_1 = \mu X_1 - X_1^3 - X_1 X_2^2 - \omega X_2 + \nu X_1^2 X_2 + \nu X_2^3$$

$$\dot{X}_2 = \mu X_2 - X_1^2 X_2 - X_2^3 + \omega X_1 + \nu X_1^3 + \nu X_2^2 X_1$$

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$$\begin{aligned} \dot{X}_1 = F_1(\mathbf{X}) &= \frac{1}{10}X_1 - X_2^3 - X_1X_2^2 - X_1^2X_2 - X_2 - X_1^3 \\ \dot{X}_2 = F_2(\mathbf{X}) &= X_1 + \frac{1}{10}X_2 + X_1X_2^2 + X_1^3 - X_2^3 - X_1^2X_2 \end{aligned} . \quad (2)$$

$\mu = 1/10 \qquad \omega = 1 \qquad \nu = 1$

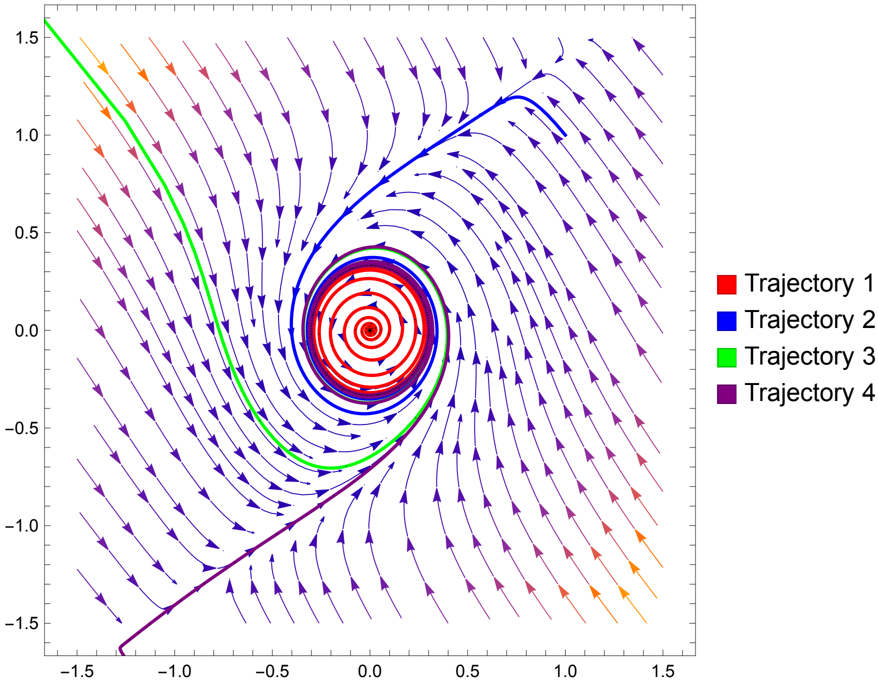
In[767]:=

```
ClearAll["Global`*"]
X1prime = m * X1 - X1^3 - X1 * X2^2 - w * X2 + n * X1^2 * X2 + n * X2^3;
X2prime = m * X2 - X1^2 * X2 - X2^3 + w * X1 + n * X1^3 + n * X2^2 * X1;

m = 0.1;
w = 1;
n = 1;

system = {
```

```
  X1'[t] == m*X1[t] - X1[t]^3 - X1[t]*X2[t]^2 - w*X2[t] + n*X1[t]^2*X2[t] + n*X2[t]^3,
  X2'[t] == m*X2[t] - X1[t]^2*X2[t] - X2[t]^3 + w*X1[t] + n*X1[t]^3 + n*X2[t]^2*X1[t]
};

FP1 = {0,0};
FP2 = {1,1};
FP3 = {-2, 2};
FP4 = {-1,-2};
delta = 0.001;
startingPoint1 = FP1 + delta;
startingPoint2 = FP2;
startingPoint3 = FP3;
startingPoint4 = FP4;

initialConditions1 = {X1[0] == startingPoint1[[1]], X2[0] == startingPoint1[[2]]};
initialConditions2 = {X1[0] == startingPoint2[[1]], X2[0] == startingPoint2[[2]]};
initialConditions3 = {X1[0] == startingPoint3[[1]], X2[0] == startingPoint3[[2]]};
initialConditions4 = {X1[0] == startingPoint4[[1]], X2[0] == startingPoint4[[2]]};


solution1 = NDSolve[{system, initialConditions1}, {X1, X2}, {t, 0, 200}];
solution2 = NDSolve[{system, initialConditions2}, {X1, X2}, {t, 0, 200}];
solution3 = NDSolve[{system, initialConditions3}, {X1, X2}, {t, 0, 200}];
solution4 = NDSolve[{system, initialConditions4}, {X1, X2}, {t, 0, 200}];


Clear[plotWithArrowsWithLegend]

plotWithArrowsWithLegend[solution_, color_, label_] :=
 Module[{trajectory, arrows},
  trajectory = ParametricPlot[Evaluate[{X1[t], X2[t]} /. solution], {t, 0, 200},
    AspectRatio → 1, PlotRange → All, AxesLabel → {"X1", "X2"},
    PlotStyle → Directive[color, Thickness[0.005]],
    PlotLabel → label];

  arrows = Table[Arrow[{{X1[t], X2[t]} /. solution, {X1[t + 1], X2[t + 1]} /. solution

  Show[trajectory, Graphics[{Arrowheads[Medium], arrows}]]
]

SP = StreamPlot[{X1prime, X2prime}, {X1,-1.5,1.5},{X2,-1.5,1.5}];

(* Example usage with legend *)
PP1 = plotWithArrowsWithLegend[solution1, Red, "Trajectory 1"];
PP2 = plotWithArrowsWithLegend[solution2, Blue, "Trajectory 2"];
PP3 = plotWithArrowsWithLegend[solution3, Green, "Trajectory 3"];
PP4 = plotWithArrowsWithLegend[solution4, Purple, "Trajectory 4"];
```

```
legend = SwatchLegend[{Red, Blue, Green, Purple}, {"Trajectory 1", "Trajectory 2", "Tr

Show[SP, Legended[Show[PP1, PP2, PP3, PP4], legend]]
```

Out[799]=



c.) Polar to Cartesian and compare with system 2

Done by hand:

In[800]:=

d.) Plot M and X1, X2 quantities

From now on, we consider only the dynamical system (2). The deformation matrix $\mathbb{M}$ corresponding differential equation

$$\dot{\mathbb{M}}(t) = \mathbb{J}(t)\mathbb{M}(t),$$

with $\mathbb{M}(0) = I$ (the identity matrix) and $J_{ij} = \frac{\partial F_i(\mathbf{X})}{\partial X_j}$.

In[801]:=

Set up a computer program to numerically solve the differential equation in the six variables X_1, X_2 M_{22}.

(d) Starting on the limit cycle with $X_1(0) > 0$ and $X_2(0) = 0$, plot all six quantities as functions of limit cycle, $t \in [0, T]$. Put all the curves in one plot using a different colour for each quantity. *Uploa* *(0.5 points)*

In[802]:=

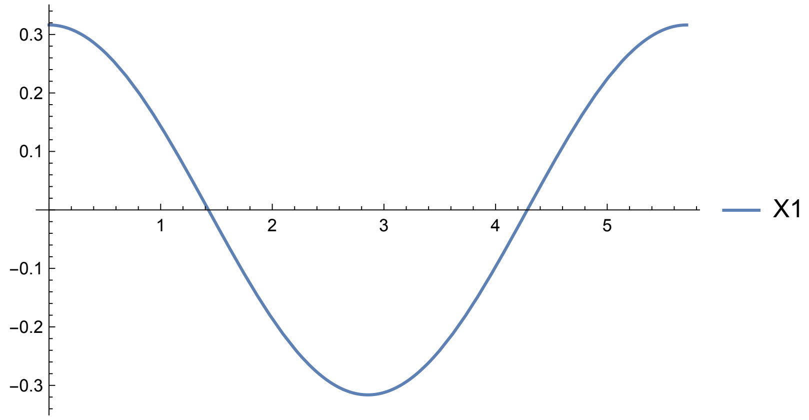
$$\begin{aligned} \dot{X}_1 = F_1(\mathbf{X}) &= \frac{1}{10}X_1 - X_2^3 - X_1 X_2^2 - X_1^2 X_2 - X_2 - X_1^3 \\ \dot{X}_2 = F_2(\mathbf{X}) &= X_1 + \frac{1}{10}X_2 + X_1 X_2^2 + X_1^3 - X_2^3 - X_1^2 X_2 \end{aligned} \quad . \qquad ($$

In[803]:=

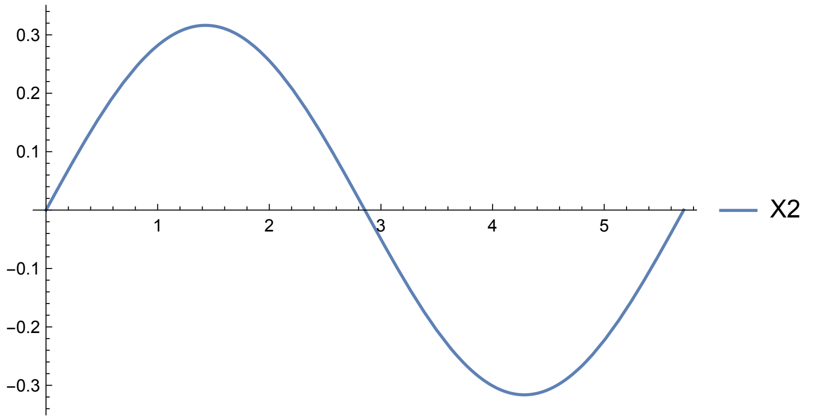
```
ClearAll["Global`*"]

(* Define the system of differential equations *)
X1prime[X1_, X2_] :=
    0.1 * X1 - X2^3 - X1*(X2^2) - (X1^2) * X2 - X2 - X1^3;
X2prime[X1_, X2_] :=
    X1 + 0.1 * X2 + X1 * (X2^2) + X1^3 - X2^3 - (X1^2) * X2;
w = 1;
nu = 1;
mu = 0.1;
PeriodTime = (2 * Pi)/(w + nu * mu);
t0 = 0;
(*tMax = 20;*)
tMax = PeriodTime;

(* Define the Jacobian matrix *)
J[X1_, X2_] := {{D[X1prime[X1, X2], X1], D[X1prime[X1, X2], X2]},
                 {D[X2prime[X1, X2], X1],D[X2prime[X1, X2], X2]}}

J[X1,X2] // MatrixForm;

(* Define the system of differential equations for M' = M * J *)
M11prime[X1_, X2_, M11_, M21_] := J[X1, X2][[1,1]] * M11 + J[X1, X2][[1,2]] * M21;
M12prime[X1_, X2_, M12_, M22_] := J[X1, X2][[1,1]] * M12 + J[X1, X2][[1,2]] * M22;
M21prime[X1_, X2_, M11_, M21_] := J[X1, X2][[2,1]] * M11 + J[X1, X2][[2,2]] * M21;
M22prime[X1_, X2_, M12_, M22_] := J[X1, X2][[2,1]] * M12 + J[X1, X2][[2,2]] * M22;

(* Set the initial conditions *)
initialConditions = {X1[t0] == Sqrt[mu], X2[t0] == 0, M11[t0] == 1, M12[t0] == 0, M21[t0]

(* Solve the system of differential equations *)
solution = NDSolve[{
   X1'[t] == X1prime[X1[t], X2[t]],
   X2'[t] == X2prime[X1[t], X2[t]],
   M11'[t] == M11prime[X1[t], X2[t], M11[t], M21[t]],
   M12'[t] == M12prime[X1[t], X2[t], M12[t], M22[t]],
   M21'[t] == M21prime[X1[t], X2[t], M11[t], M21[t]],
   M22'[t] == M22prime[X1[t], X2[t], M12[t], M22[t]],
   initialConditions
   }, {X1, X2, M11, M12, M21, M22}, {t, t0, tMax}, MaxStepSize->0.001];

(* Plot individual *)
P11 = Plot[Evaluate[{X1[t]/. solution}], {t, t0, tMax},
```

```
  PlotLegends → {"X1"}]
P12 = Plot[Evaluate[{X2[t]/. solution}], {t, t0, tMax},
  PlotLegends → {"X2"}]
PM11 = Plot[Evaluate[{M11[t]/. solution}], {t, t0, tMax},
  PlotLegends → {"M11"}]
PM12 = Plot[Evaluate[{M12[t]/. solution}], {t, t0, tMax},
  PlotLegends → {"M12"}]
PM21 = Plot[Evaluate[{M21[t]/. solution}], {t, t0, tMax},
  PlotLegends → {"M21"}]
PM22 = Plot[Evaluate[{M22[t]/. solution}], {t, t0, tMax},
  PlotLegends → {"M22"}]
(* Plot the solutions *)
Plot[Evaluate[{X1[t], X2[t], M11[t], M12[t], M21[t], M22[t]} /. solution], {t, t0, tMa
  PlotLegends → {"X1", "X2", "M11", "M12", "M21", "M22"}]

(* Print the numerical values at timestep T = tMax *)
tPeriod = PeriodTime
valuesAtTMax = {X1[tPeriod], X2[tPeriod], M11[tPeriod], M12[tPeriod], M21[tPeriod], M2

Print["Numerical values at T =", tMax];
Print["X1:", valuesAtTMax[[1]]];
Print["X2:", valuesAtTMax[[2]]];
Print["M11:", valuesAtTMax[[3]]];
Print["M12:", valuesAtTMax[[4]]];
Print["M21:", valuesAtTMax[[5]]];
Print["M22:", valuesAtTMax[[6]]];
```

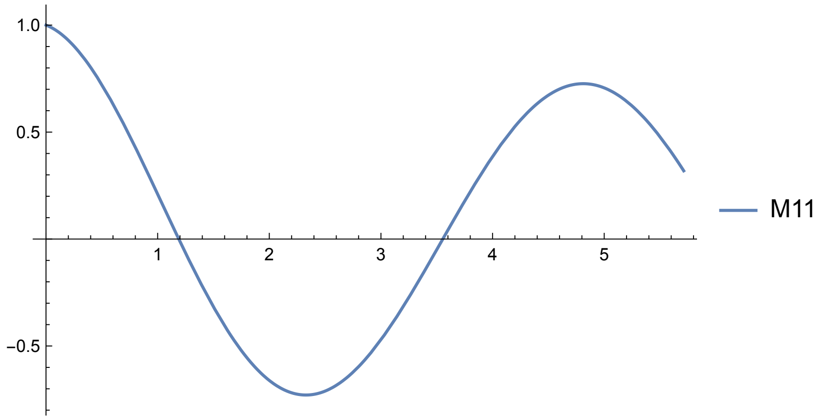
Out[820]=



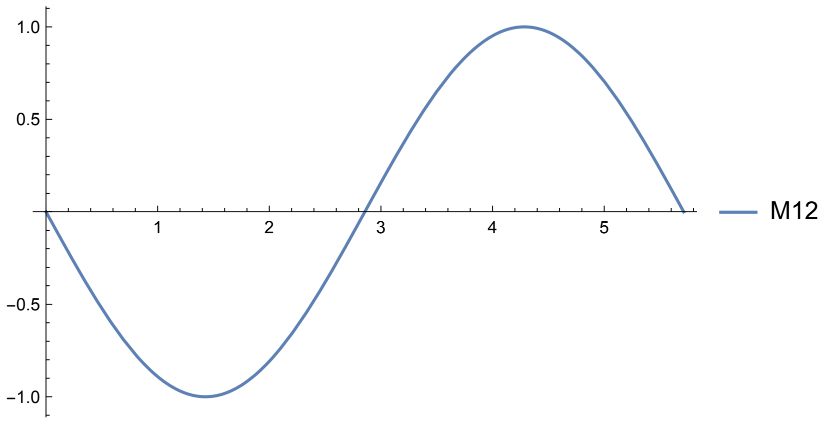
Out[821]=



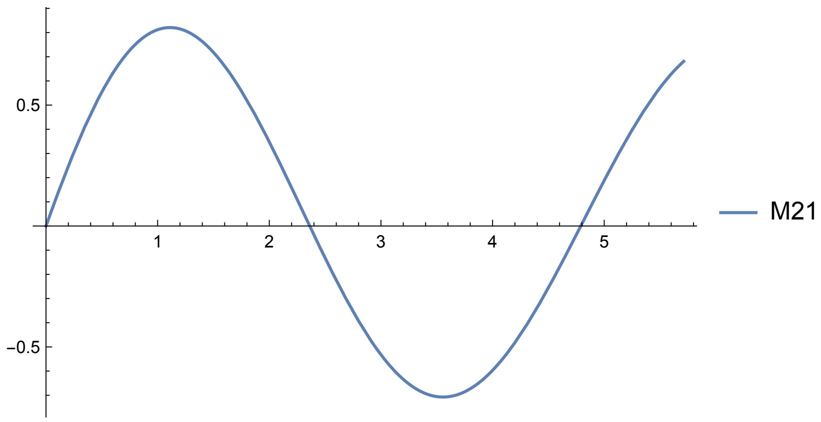
Out[822]=



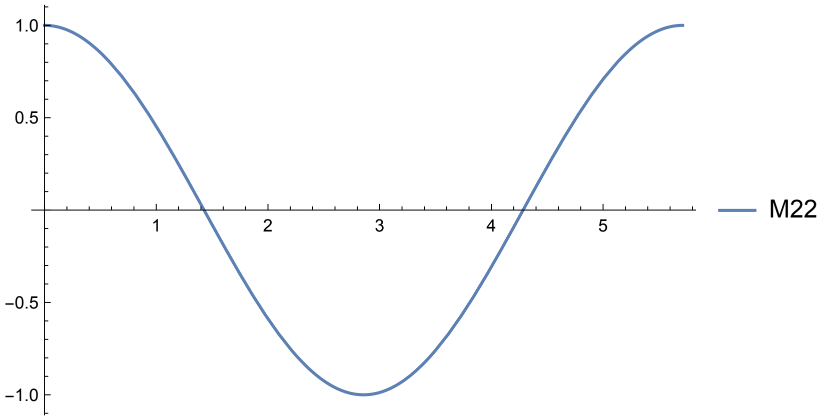
Out[823]=



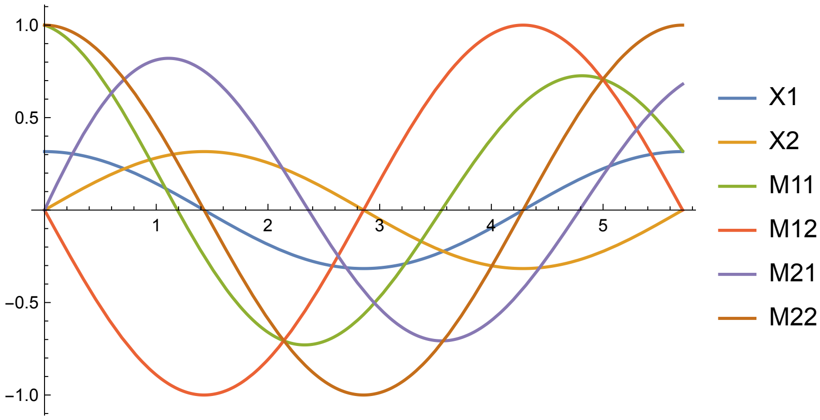
Out[824]=



Out[825]=



Out[826]=



Out[827]= 5.71199

Numerical values at T =5.71199

X1:0.316228

X2:-5.21206×10^{-9}

M11:0.319053

M12:1.6482×10^{-8}

M21:0.680947

M22:1.

f) Calculate the stability exponents of separations simga~1 and sigma~2
of the limit cycle from the eigenvalues of M(T) to 4 relevant
digits accuracy. Write your result as the ordered vector [simga~1, simga~2] with simga~1 ≤ simga~2. (0.5 points)

In[836]:=
```
M = {{M11[tPeriod], M12[tPeriod]}, {M21[tPeriod], M22[tPeriod]}} /. solution

sigmas = Log[Eigenvalues[M]]
sigmas = sigmas / tPeriod
```

Out[836]= $\{\{\{0.319053, 1.6482\times10^{-8}\}, \{0.680947, 1.\}\}\}$

Out[837]= $\{8.89036\times10^{-9}, -1.1424\}$

Out[838]= $\{1.55644\times10^{-9}, -0.2\}$

g) Using what you know from all parts of this problem, calculate the deformation matrix M(T) analytically. Write your
exact result (in Cartesian coordinates) in the form [[M11, M121, M21, M22]]. Write exponentials as exp(). (1 point)

In[839]:=
```
ClearAll["Global`*"]

tPeriod = 20*Pi / 11;

Jpol = {{m - 3 * r^2, 0}, {2 * n * r, 0}};
M0 = {{1,0},{0,1}};
Mpolar = M0.MatrixExp[Jpol*t];

r[X1_, X2_] := Sqrt[X1^2 + X2^2];
phi[X1_, X2_] := ArcTan[X1, X2];

Jpol2cart = {{D[r[X1,X2], X1], D[r[X1,X2], X2]},{D[phi[X1,X2], X1], D[phi[X1,X2], X2]}
Jpol2CartInv = Inverse[Jpol2cart];

Mcart = Jpol2CartInv . Mpolar . Jpol2cart // Simplify;
Mcart1 = Mcart /. r→(Sqrt[X1^2 + X2^2]) // Simplify;
m = 1/10;
n = 1;
t = tPeriod;
X1 = Sqrt[1/10];
X2 = 0;
Mcart1 // Simplify // MatrixForm
```

Out[855]//MatrixForm=

$$\begin{pmatrix} e^{-4\pi/11} & 0 \\ 1 - e^{-4\pi/11} & 1 \end{pmatrix}$$

h.) compute the analytical stability exponents

In[856]:=
```
N[Mcart1] // Simplify //MatrixForm
sigmas = Log[Eigenvalues[Mcart1]];
sigmas = N[sigmas / tPeriod]
```

Out[856]//MatrixForm=

$$\begin{pmatrix} 0.319053 & 0. \\ 0.680947 & 1. \end{pmatrix}$$

Out[858]= {0., -0.2}

In[859]:=