3.1 Index of fixed point

Deadline: 29 Nov 23:59

(2 points)

Each of the following systems has a fixed point at $(x^*, y^*) = (0, 0)$. For each system, either draw a (rough) phase plot (for example using StreamPlot[]) and obtain the index, or evaluate the index using an analytical integral formula.
You do not need to prepare and upload your phase plots in the parts a)-d) of this exercise, it is enough to upload the code you used to solve the problem (or StreamPlots if that was how you obtained the index). Moreover, you will not recieve any feedback on the correctness of your answer for this exercise from OpenTA.

a) Give the index for the fixed point of $\dot{x} = y - x$, $\dot{y} = x^2$.

A A A

b) Give the index for the fixed point of the Cartesian system $\dot{x}$ and $\dot{y}$ corresponding to $\dot{r} = h(r)$, $\dot{\theta} = 0$, where r and θ are polar coordinates. Let $h(r)$ be a smooth function with $h(r) \sim ar + O(r^2)$ for small values of r and $a \neq 0$.

A A A

c) Give the index for the fixed point of $\dot{x} = y^3$, $\dot{y} = x$.

A A A

d) Give the index for the fixed point of $\dot{x} = (x^2 + y^2)^{|n|/2} \cos[n \arctan(y/x)]$,
$\dot{y} = (x^2 + y^2)^{|n|/2} \sin[n \arctan(y/x)]$, where n is a non-zero integer number and $\arctan(y/x)$ is evaluated on the suitable branch.

(in terms of n)

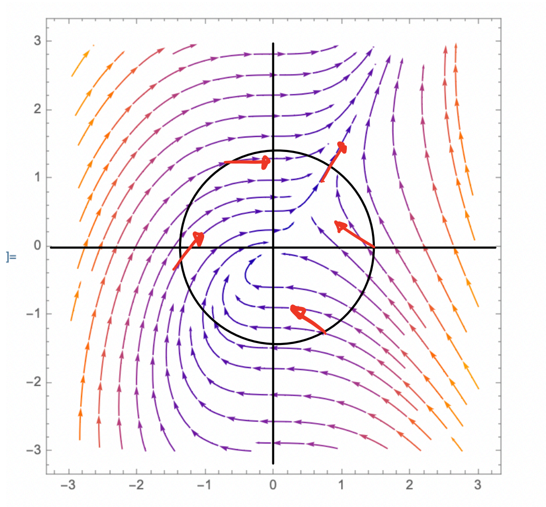
A A A

a) give the index for the fixed point:

$\dot{x} = y - x$

$\dot{y} = x^2$

$FP = (0,0) = (x^*, y^*)$



$\phi = \operatorname{atan}\left(\frac{\dot{y}}{\dot{x}}\right)$

$\dot{x} = f(x,y)$

$\dot{y} = g(x,y)$

$d\phi = \frac{\partial \phi}{\partial f} df + \frac{\partial \phi}{\partial g} dg$

$\underline{\underline{I = 0}}$

b)

b) Give the index for the fixed point of the Cartesian system $\dot{x}$ and $\dot{y}$ corresponding to $\dot{r} = h(r)$, $\dot{\theta} = 0$, where r and θ are polar coordinates. Let $h(r)$ be a smooth function with $h(r) \sim ar + O(r^2)$ for small values of r and $a \neq 0$.

$\dot{r} = h(r)$ $\qquad h(r) = a \cdot r$ $\qquad a \neq 0$ $\qquad r$ small

$\dot{\theta} = 0$ $\qquad \dot{r} = a \cdot r$

Polar to cartesian:

$$x = r \cdot \cos(\theta) \qquad (I)$$
$$y = r \cdot \sin(\theta) \qquad (II)$$

$\dot{u}v + u\dot{v}$

derive I & II with respect to t

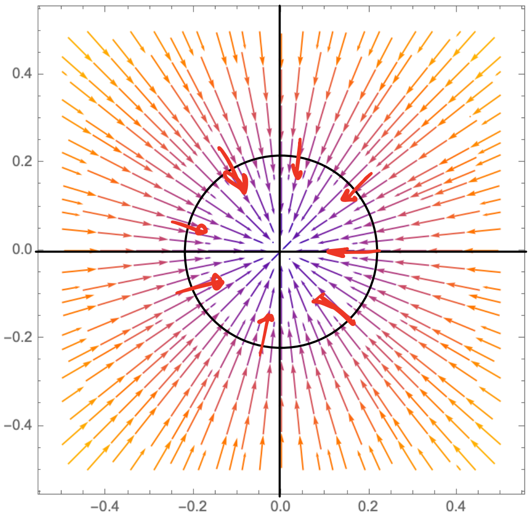
$$\dot{x} = a \cdot r \cdot \cos(\theta) - r \cdot \sin(\theta) \cdot \dot{\theta}$$
$$\dot{y} = a \cdot r \cdot \sin(\theta) + r \cdot \cos(\theta) \cdot \dot{\theta}$$

with $\dot{\theta} = 0$:

$$\dot{x} = a \cdot r \cdot \cos(\theta) \qquad (III)$$
$$\dot{y} = a \cdot r \cdot \sin(\theta) \qquad (IV)$$

substitute (III) & (IV) with (I) & (II)

$$\dot{x} = a \cdot x$$
$$\dot{y} = a \cdot y$$

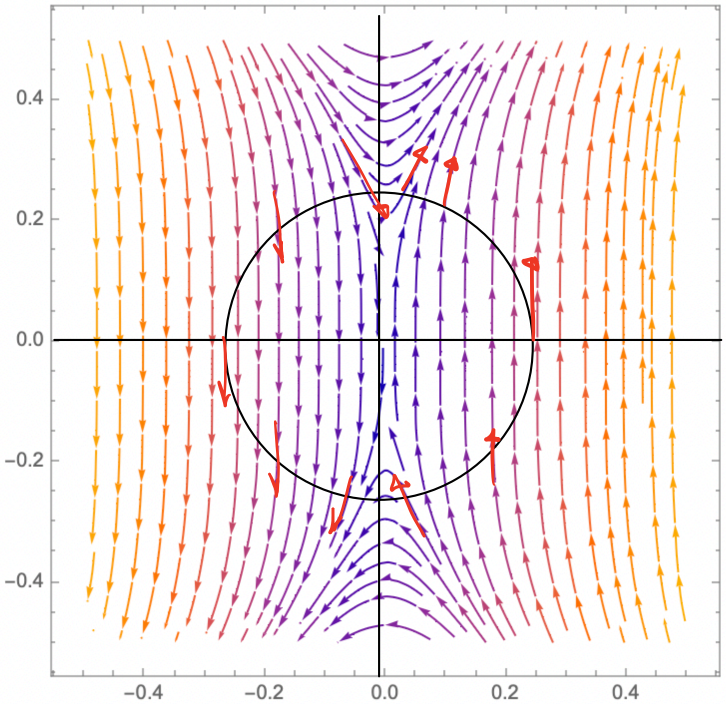


$I = 1$

c) Give the index for the fixed point of, x'=y^3 ; y' = x

In[86]:=
```
ClearAll["Global`*"];
eq1 = y^3;
eq2 = x;

StreamPlot[{eq1, eq2}, {x, -0.5, 0.5}, {y, -0.5, 0.5}, StreamPoints → Fine]
```

Out[89]=



$I = -1$

d) Give the index for the fixed point of $\dot{x} = (x^2+y^2)^{|n|/2}\cos[n\arctan(y/x)]$, $\dot{y} = (x^2+y^2)^{|n|/2}\sin[n\arctan(y/x)]$, where n is a non-zero integer number and $\arctan(y/x)$ is evaluated on the suitable branch.

(in terms of n)

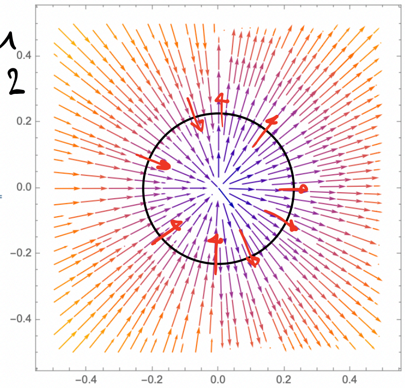
d) Give the index for the fixed point of

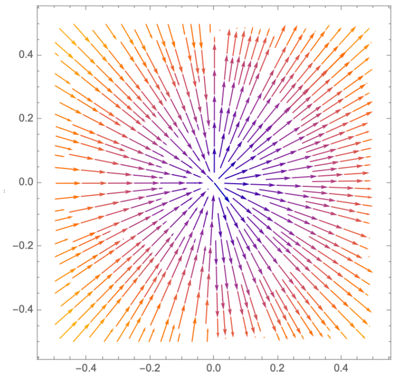
```
ClearAll["Global'.*"];
n = 1;
eq1 = (x^2 + y^2)^(Abs[n]/2)*Cos[n*ArcTan[y/x]];
eq2 = (x^2 + y^2)^(Abs[n]/2)*Sin[n*ArcTan[y/x]];

n = -1;
eq3 = (x^2 + y^2)^(Abs[n]/2)*Cos[n*ArcTan[y/x]];
eq4 = (x^2 + y^2)^(Abs[n]/2)*Sin[n*ArcTan[y/x]];

StreamPlot[{eq1, eq2}, {x, -0.5, 0.5}, {y, -0.5, 0.5}, StreamPoi
StreamPlot[{eq3, eq4}, {x, -0.5, 0.5}, {y, -0.5, 0.5}, StreamPoi
```

```
ClearAll["Global'.*"];
n = 1;
eq1 = (x^2 + y^2)^(Abs[n]/2)*Cos[n*ArcTan[y/x]];
eq2 = (x^2 + y^2)^(Abs[n]/2)*Sin[n*ArcTan[y/x]];

n = 3;
eq3 = (x^2 + y^2)^(Abs[n]/2)*Cos[n*ArcTan[y/x]];
eq4 = (x^2 + y^2)^(Abs[n]/2)*Sin[n*ArcTan[y/x]];

StreamPlot[{eq1, eq2}, {x, -0.5, 0.5}, {y, -0.5, 0.5}
StreamPlot[{eq3, eq4}, {x, -0.5, 0.5}, {y, -0.5, 0.5}
```

$n=1$
$I=2$

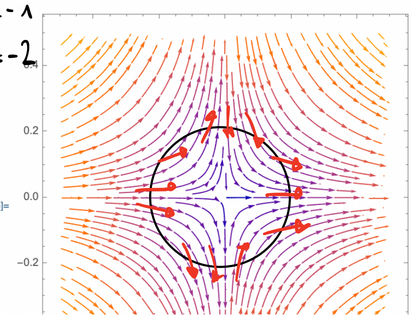
Out[75]=

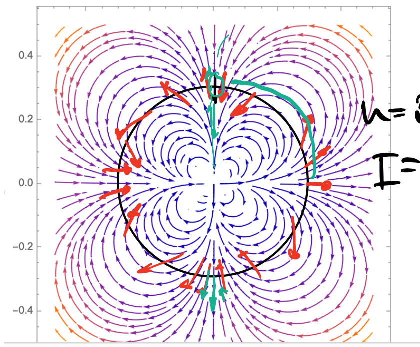




$n=-1$
$I=-2$

Out[76]=



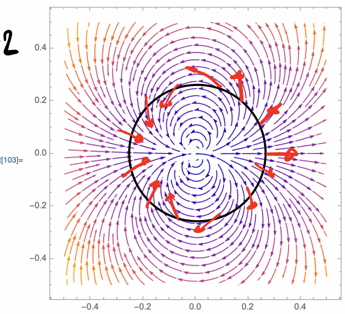


$n=3$
$I=4$

$I=2$

$n=2$

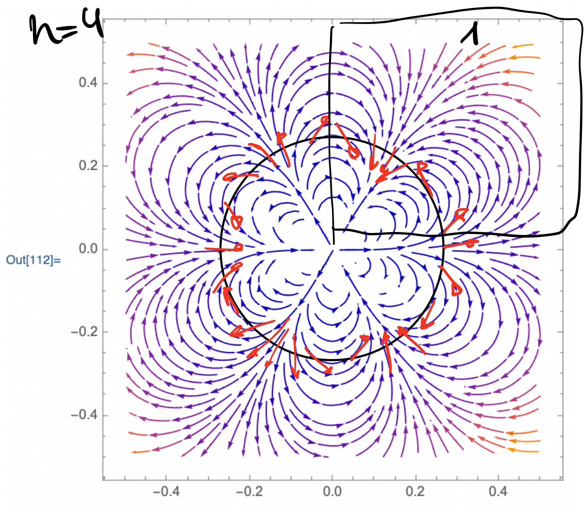
Out[103]=



$n=-1 \to I = -2 \to 2n$

$n=3 \to I = 4 \to n+1$

$I=n$



I = 4

$$I = \frac{\Delta \Phi}{2\pi} \qquad \Phi = \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right) \qquad \frac{d}{dx}\tan^{-1}(z) = \frac{1}{1+z^2}\frac{d}{dz}$$

$$I = \frac{1}{2\pi}\oint d\Phi$$

$$d\Phi = \partial_x \Phi + \partial_y \Phi$$

$$= \frac{f^2}{f^2+g^2}\left[\left(\frac{f\cdot\partial_x g - f\partial_x g}{f^2}\right) + \left(\frac{f\,\partial_y g - g\partial_y f}{f^2}\right)\right]$$

$$= \frac{1}{f^2+g^2}\left(f\underbrace{(\partial_x+\partial_y)g}_{\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y}} - g(\partial_x+\partial_y)f\right)$$

$$I = \frac{1}{2\pi}\oint d\Phi = \frac{1}{2\pi}\left[\int_{x_1}^{x_2}\underbrace{d\Phi\,dx}_{y=y_1} + \int_{y_1}^{y_2}\underbrace{d\Phi\,dy}_{x=x_2} + \int_{x_2}^{x_1}\underbrace{d\Phi\,dx}_{y=y_2} + \int_{y_2}^{y_1}\underbrace{d\Phi\,dy}_{x=x_1}\right]$$

Sometimes transform to polar coord.

$$x = r\cdot\cos(\theta)$$

$$y = r\cdot\sin(\theta)$$

$$r^2 = x^2+y^2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$