4.1 Introduction to the Lorenz model

Deadline: 13 Dec 23:59

(2 points)

The three-dimensional Lorenz flow is given by

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \; . \\ \dot{z} &= xy - bz \end{aligned} \qquad (1)$$

The Lorenz system is named after the meteorologist Edward Norton Lorenz who studied it extensively. He found that the system (1) exhibits a fractal attractor for the parameter values $\sigma = 10$, $b = 8/3$ and $r = 28$. This attractor is nowadays called Lorenz attractor.

(a) How many fixed points does the Lorenz system have, and how many of them are stable for the parameter values given above? Give your answer as the vector [number of fixed points,number of stable fixed points].

(b) Solve the equations (1) numerically using the parameters stated above for some initial condition close to the origin. Plot an approximation of Lorenz attractor obtained by discarding the initial part of the solution.
Upload your figure as .pdf or .png.

(c) Compute the stability matrix $\mathbb{J}_{ij} = \partial F_i / \partial x_j$ of the flow (1). Give your result as the matrix $[[J_{11}, J_{12}, J_{13}], [J_{21}, J_{22}, J_{23}], [J_{31}, J_{32}, J_{33}]]$.
(in terms of sigma,r,z,x,y,b)

(d) Confirm that the trace of the stability matrix, $\mathrm{tr}\mathbb{J}$, is independent of the coordinates (x, y, z). From what you have learned in the lectures and read in the course book, you should now be able to compute the sum of Lyapunov exponents for the Lorenz system (and thus the Lorenz attractor). Give your result for $\lambda_1 + \lambda_2 + \lambda_3$ for general parameter values.
(in terms of b,sigma)

a) 3 FP $\{0,0,0\}$ ...

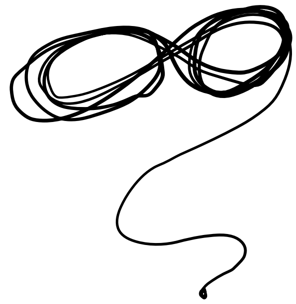
Check stability of FP

$J \to$ Jacobian and evaluate eigVal

Mathematica D[{f,g,h}, {{x,y,z}}]

b) NDSolve and plot trajectory

Plot at attractor.



c) Result of a) but not evaluated at FP

d) $\lambda_1 + \lambda_2 + \lambda_3 = \lim_{t \to \infty} \frac{1}{t} \int dt \cdot \mathrm{trace}(J(t))$

compute eigenvalues and trace $\rightarrow$ useful for 4.3

a)

$$\dot{x} = \sigma(y-x) \qquad \sigma = 10$$
$$\dot{y} = rx - y - xz \qquad b = 8/3$$
$$\dot{z} = xy - bz \qquad r = 28$$

find FP $\to FP_1 = \{0, 0, 0\}$

$$0 = 10(y-x) \quad \to \quad x = y$$
$$0 = 28x - y - xz$$
$$0 = xy - \frac{8}{3}z$$

$$0 = x(28-z) - y$$
$$y = x(28-z) \quad \to \quad (28-z) = 1$$
$$z = 27$$

$$0 = x^2 - \frac{8}{3} \cdot 27$$
$$x^2 = 8 \cdot 9 = 72$$
$$x = \pm\sqrt{72}$$

$$FP_{2|3} = \{\pm\sqrt{72}, \pm\sqrt{72}, 27\}$$

$$\dot{x} = \sigma y - \sigma x$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

Check stability of FP's.

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} \end{pmatrix} = \begin{vmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{vmatrix}$$

Check eigVl of the Jacobian :

@ FP_1 $\{0,0,0\}$:

$\lambda_1 = -22{,}82$ $\quad \lambda_2 = 11{,}82$ $\quad , \lambda_3 = -2{,}66$

@ $FP_2 = \{\sqrt{72}, \sqrt{72}, 27\}$

$\lambda_1 = -13{,}85$, $\lambda_2 = 0{,}09 + 10{,}19i$, $\lambda_3 = 0{,}09 - 10{,}19i$

@ $FP_3 = \{-\sqrt{72}, -\sqrt{72}, 27\}$

$\lambda_1 = -13{,}85$, $\lambda_2 = 0{,}09 + 10{,}19i$, $\lambda_3 = 0{,}09 - 10{,}19i$

Check stability:

$FP_1 \to$ unstable

$FP_2 \to$ unstable spiral

$FP_3 \to$ unstable spiral

number FP

Number of stable FP

$\underline{\underline{[3, 0]}}$

b)

c)

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$$

d)

$$\lambda_1 + \lambda_2 + \lambda_3 = \lim_{t \to \infty} \frac{1}{t} \int dt \cdot \text{trace}(J(t))$$

compute eigenvalues and trace → useful for 4.3

$$\text{tr}(J) = -\sigma + (-1) + (-b) = \underline{\underline{-\sigma - 1 - b}}$$