4.2 Stability exponents for a toy model

Deadline: 13 Dec 23:59

(5 points)

We define a simple flow in polar coordinates on which to test that the Lyapunov exponent calculation works. Define the simple dynamics

$$\begin{aligned} \dot{r} &= \mu r - r^3 \\ \dot{\theta} &= \omega + \nu r^2 \end{aligned} , \qquad (1)$$

which has a stable fixed point and a limit cycle if $\mu > 0$.

(a) Calculate the radius r_0 and the period T of the limit cycle for $\mu > 0$. Give your result on the form $[r_0, T]$. *(0.75 points)*

(in terms of mu,omega,nu)

Transform the dynamical system (1) into the Cartesian coordinates X_1 and X_2, where $X_1 = r\cos\theta$ and $X_2 = r\sin\theta$. Compare your result to the dynamical system $\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X})$ with

$$\begin{aligned} \dot{X}_1 = F_1(\mathbf{X}) &= \frac{1}{10}X_1 - X_2^3 - X_1X_2^2 - X_1^2X_2 - X_2 - X_1^3 \\ \dot{X}_2 = F_2(\mathbf{X}) &= X_1 + \frac{1}{10}X_2 + X_1X_2^2 + X_1^3 - X_2^3 - X_1^2X_2 \end{aligned} . \qquad (2)$$

(b) Make a phase portrait of the dynamical system (2), showing a few representative trajectories. In the same figure, plot the limit cycle using a suitable representative trajectory. *Upload your figure as .pdf or .png. Using StreamPlot[] is not acceptable. (0.5 points)*

(c) For which values of μ, ω and ν is the system (1) written in Cartesian coordinates identical to (2). Write your result as the vector $[\mu, \omega, \nu]$. *(0.5 points)*

From now on, we consider only the dynamical system (2). The deformation matrix $\mathbb{M}$ corresponding to (2) satisfies the differential equation

$$\dot{\mathbb{M}}(t) = \mathbb{J}(t)\mathbb{M}(t),$$

with $\mathbb{M}(0) = I$ (the identity matrix) and $J_{ij} = \frac{\partial F_i(\mathbf{X})}{\partial X_j}$.

Set up a computer program to numerically solve the differential equation in the six variables X_1, X_2 and M_{11}, M_{12}, M_{21} and M_{22}.

(d) Starting on the limit cycle with $X_1(0) > 0$ and $X_2(0) = 0$, plot all six quantities as functions of t for one period T of the limit cycle, $t \in [0, T]$. Put all the curves in one plot using a different colour for each quantity. *Upload the figure as .pdf or .png. (0.5 points)*

(e) Give your numerical result for $\mathbb{M}(T)$ obtained in (d) to 4 relevant digits accuracy. Write it as a matrix of the form $[[M_{11}(T), M_{12}(T)], [M_{21}(T), M_{22}(T)]]$. *(0.75 points)*

precision = 0.0001

(f) Calculate the stability exponents of separations $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ of the limit cycle from the eigenvalues of $\mathbb{M}(T)$ to 4 relevant digits accuracy. Write your result as the ordered vector $[\tilde{\sigma}_1, \tilde{\sigma}_2]$ with $\tilde{\sigma}_1 \leq \tilde{\sigma}_2$. *(0.5 points)*

precision = 0.0001

(g) Using what you know from all parts of this problem, calculate the deformation matrix $\mathbb{M}(T)$ analytically. Write your exact result (in Cartesian coordinates) in the form $[[M_{11}, M_{12}], [M_{21}, M_{22}]]$. Write exponentials as $\exp()$. *(1 point)*

(h) Compute the stability exponents of separations $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ of the limit cycle analytically. Write your result on the ordered form $[\tilde{\sigma}_1, \tilde{\sigma}_2]$ with $\tilde{\sigma}_1 \leq \tilde{\sigma}_2$. *(0.5 points)*

a) find the radius r_0 and period T of the limit cycle

$$\frac{dr}{dt} = 0 \rightarrow r_1, r_{2,3} \qquad \text{for } \mu > 0 \qquad \text{should give only one } r$$

$$T = \frac{2\pi}{\left.\frac{d\theta}{dr}\right|_{r^*}}$$

b) plot phaseportrait of the limit cycle.

c)
$$\dot{x}_1 = F_1(x_1, x_2)$$
$$\dot{x}_2 = F_2(x_1, x_2)$$

find the parameters to make the 2 systems equal.

$$\frac{dx_1}{dt} = \frac{dx_1}{dr} \cdot \frac{dr}{dt} + \frac{dx_1}{d\theta} \cdot \frac{d\theta}{dt}$$

d)
$$\underline{\underline{\dot{M}}}(t) = \overset{\text{Jacob}}{\underline{\underline{J}}(t)} \cdot \underline{\underline{M}}(t) \qquad \text{(Numerical)} \rightarrow dt \sim 10^{-4}$$

$$x_1(0) = r^*, \quad x_2(0) = 0, \quad \underline{\underline{M}}(0) = \mathbb{1}$$

$$\begin{pmatrix} \dot{M}_{11} & \dot{M}_{12} \\ \dot{M}_{21} & \dot{M}_{22} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \cdot \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

plot $x_1(t)$, $x_2(t)$ and $M_1^{(t)}, M_2, M_3, M_4$

legends / colors.

e) give final $\underline{\underline{M}}(T)$ values.

f) find

$$\sigma_i = \frac{1}{T} \cdot \log\left(\text{Eigenvalue}\left(\underline{\underline{M}}(t)\right)\right)$$

Should give 2 stability exponents

with $\sigma_1 < \sigma_2$ order

g) $$\frac{dM}{dt} = \underline{\underline{J}}(t) \cdot \underline{\underline{M}} \quad \rightarrow \quad \frac{1}{M(t)}\, dM = J(t)\, dt$$

$$\int_{M(0)}^{M(t)} \frac{dM}{M} = \int_0^t \underline{\underline{J}}(t)\, dt$$

$$M(t) = M_0 \cdot \exp\left[\int J \right.$$

$\underline{\underline{J}}$ in polar coordinates:

$$J = \begin{pmatrix} \mu - 3r^2 & 0 \\ 2\mu r & 0 \end{pmatrix}$$

$$\underline{\underline{M}}_{polar}(t) = \exp\left[\underline{\underline{J}} \cdot t\right] \rightarrow \text{Back to cartesian}$$

$\underline{\underline{J}}_G$ = operator from polar to cartesian

$$\underline{\underline{J}}_G = \begin{pmatrix} \frac{\partial r}{\partial x_1} & \frac{\partial r}{\partial x_2} \\ \frac{\partial \theta}{\partial x_1} & \frac{\partial \theta}{\partial x_2} \end{pmatrix}$$

$$\underline{\underline{M}}_{\text{CARTESIAN}} = \underline{\underline{J}}_G^{-1} \cdot \underline{\underline{M}}_{\text{POLAR}} \cdot \underline{\underline{J}}_G$$

h) find stability exponents from analytical $\underline{\underline{M}}(t)$

eigenvalue[M]

a) find the radius r_0 and period T of the limit cycle

$\frac{dr}{dt} = 0 \rightarrow r_1, r_{2|3}$ for $\mu > 0$ should give only one r

$T = \frac{2\pi}{\frac{d\theta}{dr}\Big|_{r^*}}$

$\dot{r} = \mu r - r^3$

$0 = \mu r - r^3$

$r^2 = \mu$

$r_1 = 0$

$r_{2|3} = \pm\sqrt{\mu} \rightarrow r_2 = \sqrt{\mu} \rightarrow r^* = r_2$

$T = \frac{2\pi}{\frac{d\theta}{dr}\Big|_{r^*}}$ $\quad T = \frac{2\pi}{\omega + 2\nu\mu}$

$\frac{d\theta}{dr} = 2\nu r \xrightarrow{r=r^*} 2\nu \cdot \sqrt{\mu}$

$T = \frac{2\pi}{2 \cdot \nu \cdot \sqrt{\mu}}$ $\quad T = \frac{2\pi}{\dot{\theta}\big|_{r^*}} = \frac{2\pi}{\omega + \nu \cdot r^2} = \underline{\underline{\frac{2\pi}{\omega + \nu \cdot \mu}}}$

c) for which parameters μ, ω, ν is the system in cartesian coords the same as:

$$\begin{aligned} \dot{X}_1 = F_1(\mathbf{X}) &= \frac{1}{10}X_1 - X_2^3 - X_1X_2^2 - X_1^2X_2 - X_2 - X_1^3 \\ \dot{X}_2 = F_2(\mathbf{X}) &= X_1 + \frac{1}{10}X_2 + X_1X_2^2 + X_1^3 - X_2^3 - X_1^2X_2 \end{aligned} \quad . \qquad (2)$$

transform system to cartesian:

$$\dot{X}_1 = F_1(\underline{X}) = \frac{1}{10}X_1 - X_2^3 - X_1X_2^2 - X_1^2X_2 - X_2 - X_1^3$$

$$\dot{X}_2 = F_2(\underline{X}) = X_1 + \frac{1}{10}X_2 + X_1X_2^2 + X_1^3 - X_2 - X_1^2X_2$$

$$X_1 = r \cdot \cos(\theta), \; X_2 = r \cdot \sin(\theta)$$

$$\dot{r} = \mu \cdot r - r^3$$

$$\dot{\theta} = \omega + \nu \cdot r^2$$

derivate X_1 & X_2 w.r.t. time

$\dot{u}v + u\dot{v}$

$\dot{u} = \dot{r} = \mu r - r^3$

$\dot{v} = \frac{\cos(\theta)}{dt} = -\sin(\theta) \cdot \dot{\theta}$

$$\dot{X}_1 = \dot{r}\cos(\theta) - r \cdot \sin(\theta)$$

$$\dot{X}_2 = \dot{r}\sin(\theta) - r \cdot \cos(\theta)$$

$$\dot{X}_1 = (\mu r - r^3)\cos(\theta) - r \cdot \sin(\theta) \cdot (\omega + \nu \cdot r^2)$$

$$\dot{X}_2 = (\mu r - r^3)\sin(\theta) - r \cdot \cos(\theta) \cdot (\omega + \nu r^2)$$

$\dot{u}v + u\dot{v}$

$$X_1 = \underbrace{r(t)}_{u} \cdot \underbrace{\cos(\theta(t))}_{v}$$

$\dot{u} = \dot{r}$
$\dot{v} = -\sin(\theta) \cdot \dot{\theta}$

$$\frac{dX_1}{dt} = \dot{r} \cdot \cos(\theta) + r \cdot (-\sin(\theta)\,\dot{\theta})$$

$$X_2 = \underbrace{r(t)}_{u} \cdot \underbrace{\sin(\theta(t))}_{v}$$

$\dot{u} = \dot{r}$

$\dot{v} = \cos(\theta)\,\dot{\theta}$

$$\frac{dX_2}{dt} = \dot{r} \cdot \sin(\theta) + r\,(\cos(\theta)\,\dot{\theta})$$

$$\frac{dX_1}{dt} = (\mu r - r^3)\cos(\theta) - r\sin(\theta) \cdot (\omega + \nu r^2)$$

$$\frac{dX_2}{dt} = (\mu r - r^3)\sin(\theta) + r \cdot \cos(\theta) \cdot (\omega + \nu r^2)$$

$$\mu \underbrace{r \cdot \cos(\theta)}_{X_1} - r^3 \cdot \underbrace{\cos(\theta)}_{X_1} - \underbrace{r \cdot \sin(\theta)}_{X_2} \cdot \omega + \underbrace{r \cdot \sin(\theta)}_{X_2} \nu r^2$$

$$\dot{X}_1 = \mu X_1 - r^2 \cdot X_1 - \omega \cdot X_2 + \nu r^2 \cdot X_2$$

$$\dot{X}_2 = \mu \underbrace{r \cdot \sin(\theta)}_{X_2} - r^3 \cdot \underbrace{\sin(\theta)}_{X_2} + \omega \cdot \underbrace{r \cdot \cos(\theta)}_{X_1} + \nu r^3 \underbrace{\cos(\theta)}_{X_1}$$

$$\dot{X}_2 = \mu \cdot X_2 - r^2 \cdot X_2 + \omega X_1 + \nu r^2 X_1$$

$$r = \sqrt{X_1^2 + X_2^2}$$

$$\dot{X}_1 = \mu X_1 - (X_1^2 + X_2^2) X_1 - \omega \cdot X_2 + \nu (X_1^2 + X_2^2) X_2$$

$$\dot{X}_2 = \mu X_2 - (X_1^2 + X_2^2) X_2 + \omega X_1 + \nu (X_1^2 + X_2^2) X_1$$

$$\dot{X}_1 = \mu X_1 - X_1^3 - X_1 \cdot X_2^2 - \omega \cdot X_2 + \nu (X_1^2 X_2 + X_2^3)$$

$$\dot{X}_2 = \mu X_2 - X_1^2 X_2 - X_2^3 + \omega X_1 + \nu (X_1^3 + X_2^2 X_1)$$

$$\dot{X}_1 = \mu X_1 - X_1^3 - X_1 X_2^2 - \omega X_2 + \nu X_1^2 X_2 + \nu X_2^3$$

$$\dot{X}_2 = \mu X_2 - X_1^2 X_2 - X_2^3 + \omega X_1 + \nu X_1^3 + \nu X_2^2 X_1$$

$$\begin{aligned} \dot{X}_1 = F_1(\mathbf{X}) &= \frac{1}{10} X_1 - X_2^3 - X_1 X_2^2 - X_1^2 X_2 - X_2 - X_1^3 \\ \dot{X}_2 = F_2(\mathbf{X}) &= X_1 + \frac{1}{10} X_2 + X_1 X_2^2 + X_1^3 - X_2^3 - X_1^2 X_2 \end{aligned} \quad (2)$$

$\mu = 1/10 \quad \omega = 1 \quad \nu = 1$

$$J = \begin{pmatrix} \frac{\partial \dot{X}_1}{\partial x_1} & \frac{\partial \dot{X}_1}{\partial x_2} \\ \frac{\partial \dot{X}_2}{\partial x_1} & \frac{\partial \dot{X}_2}{\partial x_2} \end{pmatrix}$$

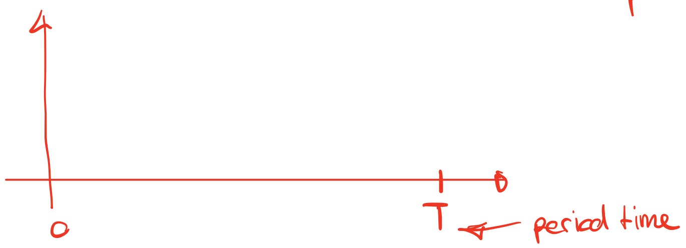
d) d) $\underline{\underline{\dot{M}}}(t) = \overset{\text{Jacob}}{\underline{\underline{J}}}(t) \cdot \underline{\underline{M}}(t)$ (Numerical) $\rightarrow dt \sim 10^{-4}$

$x_1(0) = r^*, \; x_2(0) = 0, \; \underline{\underline{M}}(0) = \mathbb{1}$

$$\begin{pmatrix} \dot{M}_{11} & \dot{M}_{12} \\ \dot{M}_{21} & \dot{M}_{22} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \cdot \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

plot $x_1(t), x_2(t)$ and $M_1, M_2, M_3, M_4^{(t)}$

legends / colors.



1.) find $\underline{\underline{J}}$

$$\underline{\underline{\dot{M}}} = \underline{\underline{M}} \cdot \underline{\underline{J}}$$

2x2 2x2 2x2

$$\begin{pmatrix} \overset{a}{M_{11}} & \overset{b}{M_{12}} \\ \overset{c}{M_{21}} & \overset{d}{M_{22}} \end{pmatrix} \cdot \begin{pmatrix} \overset{e}{J_{11}} & \overset{f}{J_{12}} \\ \overset{g}{J_{21}} & \overset{h}{J_{22}} \end{pmatrix}$$

$M(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\dot{M}_{11} = a \cdot e$

$\dot{M}_{12} = b \cdot f$

$M_{21} = cg$

$M_{22} = d \cdot h$

$M_{11} =$

$M_{22} =$

M_{11}

M_{12}

M_{21}

M_{22}

$$T = 2\pi / (1 + 1/10) = \frac{2\pi}{\frac{11}{10}} = \frac{20\pi}{11}$$

$$J = \begin{pmatrix} 0{,}1 - 3x_1^2 + 2x_1 \cdot x_2 - x_2^2 & -1 + x_1^2 - 2x_1 x_2 + 3x_2^2 \\ 1 + 3x_1^2 - 2x_1 x_2 + x_2^2 & 0{,}1 - x_1^2 + 2x_1 x_2 - 3x_2^2 \end{pmatrix}$$

for starting position $r = \sqrt{\mu}$

$$r = \sqrt{x_1^2 + x_2^2}$$

$$x_2 = 0$$

$$r = \sqrt{x_1^2}$$

$$r = x_1$$

$$x_1 = \sqrt{\mu}$$

Period $T = \dfrac{2 \cdot \pi}{\omega + \nu \cdot \mu}$ $\quad \omega = 1$, $\nu = 1$, $\mu = 0{,}1$ $\quad \rightarrow \dfrac{2\pi}{1{,}1}$

$$\begin{pmatrix} \dot{M}_{11} & \dot{M}_{12} \\ \dot{M}_{21} & \dot{M}_{22} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \cdot \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{matrix} J_{11} \cdot M_{11} + J_{12} \cdot M_{21} & J_{11} M_{12} + J_{12} \cdot M_{22} \\ J_{21} \cdot M_{11} + J_{22} \cdot M_{21} & J_{21} \cdot M_{12} + J_{22} M_{22} \end{matrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{matrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{matrix}$$

$J_{11} \cdot M_{11} + J_{12} M_{21}$ $\quad$ $J_{11} \cdot M_{12} + J_{12} \cdot M_{22}$

$J_{21} \cdot M_{11} + J_{22} M_{21}$ $\quad$ $J_{21} M_{12} + J_{22} M_{22}$

$$M(T) = M_0 \cdot \int J \, dt$$

$$2\times 2 = (2\times 2) \int (2\times 2) = \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix} \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

$$J_{CARTESIAN} = \begin{pmatrix} \frac{1}{10} - 3x_1^2 - 2x_1x_2 - x_2^2 & -1 - x_1^2 - 2x_1x_2 - 3x_2^2 \\ 1 + 3x_1^2 - 2x_1x_2 + x_2^2 & \frac{1}{10} - x_1^2 + 2x_1x_2 - 3x_2^2 \end{pmatrix}$$

$$\int_0^{TP} J_{11} \, dt = \int_0^{TP} \left(\frac{1}{10} - 3x_1^2 - 2x_1x_2 - x_2^2 \right) dt$$

$$\frac{2\pi}{1.1}$$

$$J = \begin{pmatrix} \mu - 3r^2 & 0 \\ 2\mu r & 0 \end{pmatrix}$$

$$M_{POLAR} = \exp(J \cdot t) = \begin{pmatrix} e^{(m-3r^2)\cdot t} & \\ 0 & 1 \end{pmatrix}$$

$$J_B = \begin{pmatrix} \frac{\partial r}{\partial x_1} & \frac{\partial r}{\partial x_2} \\ \frac{\partial \theta}{\partial x_1} & \frac{\partial \theta}{\partial x_2} \end{pmatrix}$$

$$M_{cart_1} = \begin{pmatrix} Ji_{11} & Ji_{12} \\ Ji_{21} & Ji_{22} \end{pmatrix} \cdot \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$M_{cart_1} = \begin{pmatrix} Ji_{11} \cdot M_{11} + Ji_{22} \cdot M_{21} & Ji_{11} \cdot M_{12} + Ji_{12} \cdot M_{22} \\ Ji_{21} \cdot M_{11} + Ji_{22} \cdot M_{21} & Ji_{21} \cdot M_{12} + Ji_{22} \cdot M_{22} \end{pmatrix}$$

633]= $\left\{\left\{\frac{e^{t\left(m-3\left(X1^2+X2^2\right)\right)}\, X1^2}{X1^2+X2^2}, 0\right\}, \left\{0, \frac{X1^2}{X1^2+X2^2}\right\}\right\}$

Out[1648]= $\{\{1.\, e^{-0.2\, t}, 0\}, \{0, 1.\}\}$