3.3 Homoclinic bifurcation

Deadline: 29 Nov 23:59

$\mu < \mu_c$ verschwinde FP

$\mu > \mu_c$ geboren FP

(4 points)

See exercise 8.4.3 and 8.4.12 in Strogatz

Consider the dynamical system

$\mu = 2x$

$$\begin{aligned} \dot{x} &= \mu x + y - x^2 = 0 \\ \dot{y} &= -x + \mu y + 2x^2 , = 0 \end{aligned} \quad (1)$$

where μ is a dimensionless parameter.

$\mu x + y - x^2 = -x + \mu y + 2x^2$

a) Using a numerical method of your choice, find the value $\mu = \mu_c$ where the system undergoes a homoclinic bifurcation. Give your result with two significant digits.

μ

A A A

b) Plot the phase portraits that occur for the cases $\mu < 0$, $\mu = 0$, $0 < \mu < \mu_c$, $\mu = \mu_c$ and $\mu > \mu_c$, marking and labelling fixed points, closed orbits, limit cycles and homoclinic orbits. Use a numerical solver, for example NDSolve[] in Mathematica (using StreamPlot[] will not give enough resolution for this task). *Upload the portraits as .png images or as .pdf.*

In order to find out how the period of a closed orbit scales as a homoclinic bifurcation is approached, it is useful to first estimate the time it takes for an orbit to pass a saddle point. To estimate the time to pass a generic saddle, consider the linearised dynamics

$$\begin{aligned} \dot{x} &= u\,x \\ \dot{y} &= s\,y \end{aligned} ,$$

where u and s are the eigenvalues of the unstable and stable directions respectively, i.e. $u > 0$ and $s < 0$.

Now, let a trajectory start from the point

$$(x(0), y(0)) = (\gamma, 1),$$

$\gamma = 0{,}01$

where $\gamma > 0$ is small.

c) Find an analytical expression for the time t_1 to escape from the saddle to $x(t_1) = 1$. *Answer in term of γ and u, even if OpenTA does not specify that.*

(in terms of gamma,u)

A A A

d) Find an analytical expression for u suitable for the system in Eq. (1).

(in terms of mu)

A A A

The time of the periodic orbit, T_μ, just below the homoclinic bifurcation where $\mu < \mu_c$ and $|\mu - \mu_c| \ll 1$ is well approximated by the time it takes to pass the saddle point, i.e. it depends only on u and γ. Hence, to find T_μ we first need to investigate how γ depends on $|\mu - \mu_c|$ for $|\mu - \mu_c| \ll 1$.

Numerically find this dependence. It should be given by the scaling law

$$\gamma \sim A|\mu - \mu_c|^a,$$

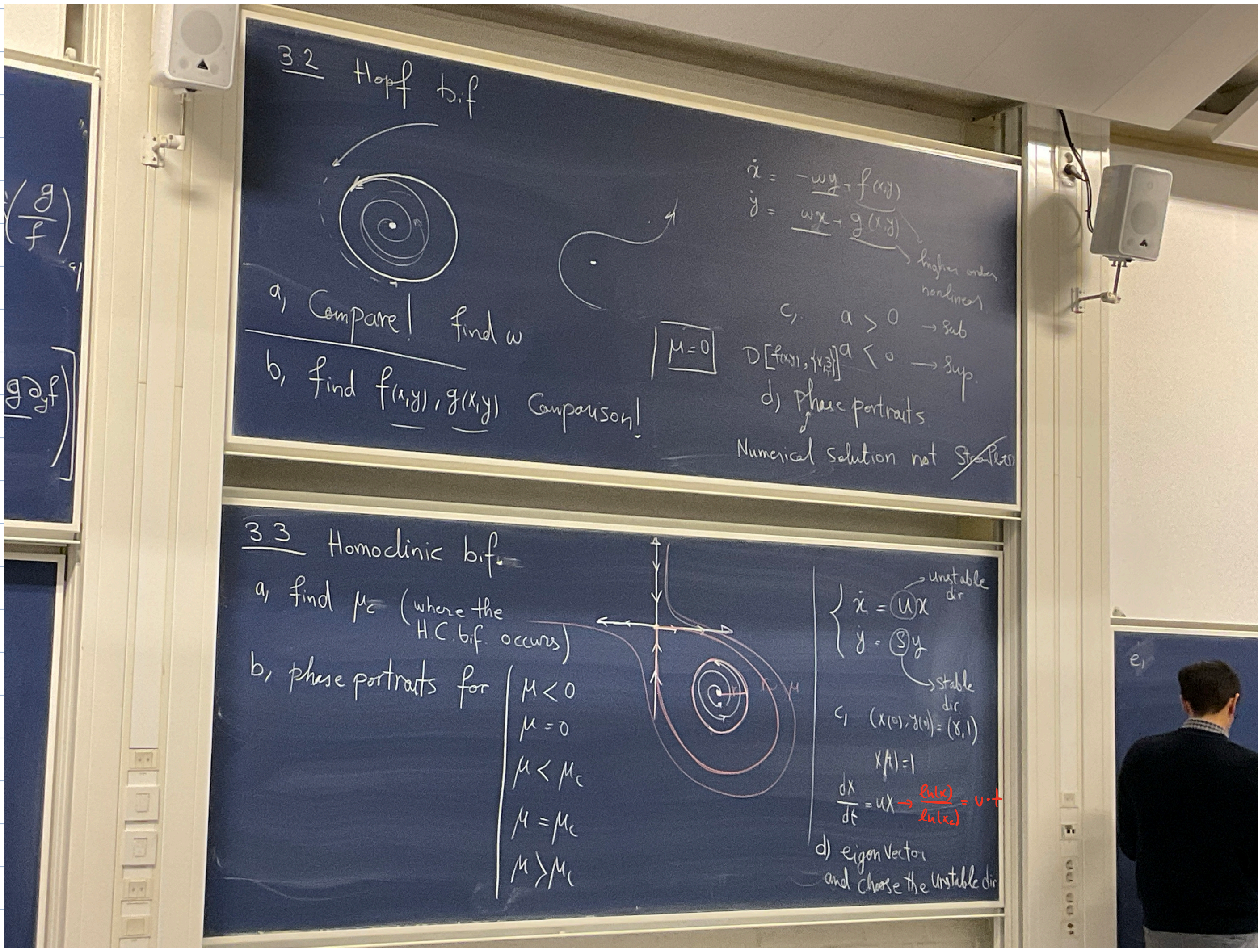
for some constants a and A. Here $\sim$ denotes asymptotic equivalence, meaning that the expression $A \sim |\mu - \mu_c|^a$ is approached in the limit of $\mu \to \mu_c$.

e) Give your estimate for a with one digit accuracy.

precision = .1

A A A

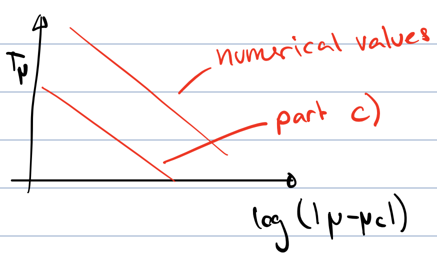
f) First, numerically evaluate the period time of the periodic orbit and plot it against $\mu - \mu_c$. Second, from the estimate in e), together with the results of c) and d), you should be able to find an estimate of the period time, T_μ, as a function of $|\mu - \mu_c|$. Plot this estimate in the same figure as the numerically evaluated period time. To show that the scaling for small $|\mu - \mu_c|$ comes out correctly, you may need to multiply T_μ by a constant to make the curves overlap for small $|\mu - \mu_c|$. *Upload the figure as a .png image of a .pdf.*

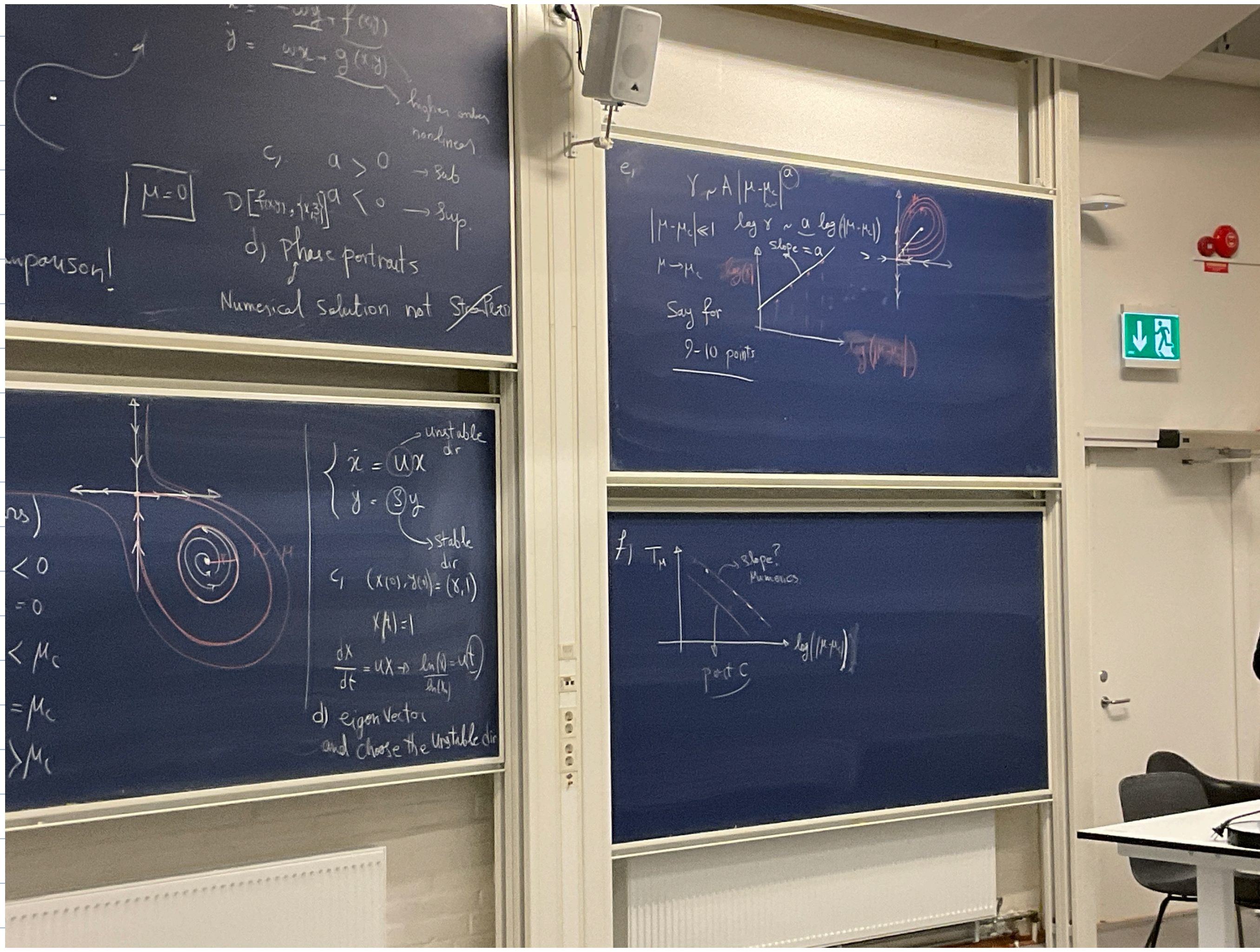


e) $\gamma \sim A\cdot|\mu-\mu_c|^a$ $\qquad \mu \to \mu_c \qquad |\mu-\mu_c| \ll 1$

$$\log(\gamma) = a\cdot \log(A|\mu-\mu_c|)$$

f)





Compute A:

$$A = \begin{pmatrix} \frac{d\dot{x}}{dx} & \frac{d\dot{x}}{dy} \\ \frac{d\dot{y}}{dx} & \frac{d\dot{y}}{dx} \end{pmatrix} = \begin{pmatrix} \mu - 2x & 1 \\ -1+4x & \mu \end{pmatrix}$$

Check FP's :

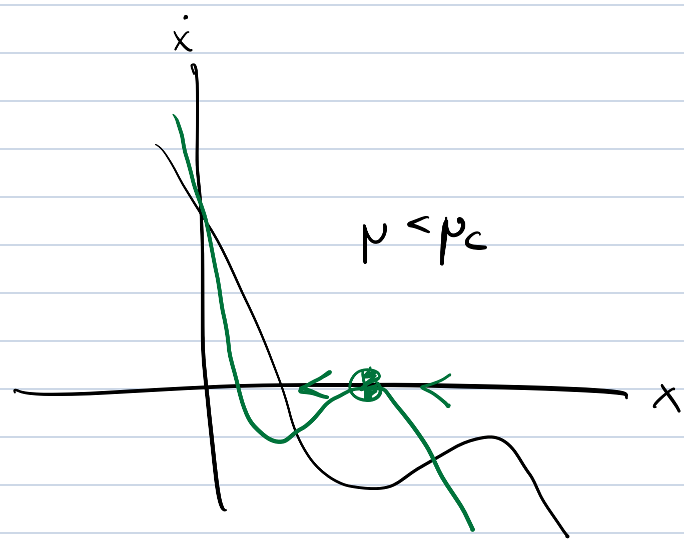
$$0 = \mu x + y - x^2$$

$$x^2 = \mu x + y$$

$$0 = -x + \mu y + 2x^2$$

$$-2x^2 + x = \mu y$$

$$-2(\mu x + y)^2 + x = \mu y$$



$\mu(x,y)$ $\qquad\qquad$ $\mu = \mu_c$

$(\mu - 2x)\,\mu$

$\mu^2 - 2\mu x = 0$ $\qquad\qquad$ $\mu\,(\mu - 2x) = 0$

$\mu = 0$

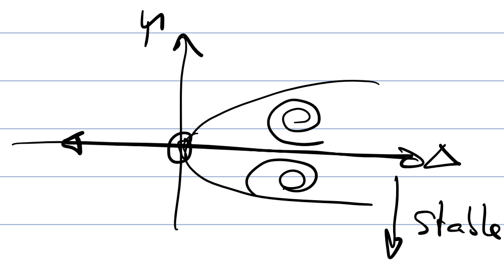
$a = 1 \quad b = -2x \quad c = 0$

$\mu_c = 2x$

$b^2 - 4ac = 0$

$4x^2 = 0$

$\dot{x} =$



put μ_c in $\det(A)$ and check if $= 0$

$$A = \begin{pmatrix} \mu - 2x & 1 \\ -1+4x & \mu \end{pmatrix} \to \begin{pmatrix} 2x - 2x & 1 \\ -1+4x & 2x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1+4x & 2x \end{pmatrix}$$

$\mu = 2x$

$\Delta = ad - bc = 0 - ((-1+4x)1)$

$= 1 - 4x$

$1 = 4x$

$x = 1/4$

$\mu = 2x \to \frac{1}{2}$

FP $[0,0]$

$x^* \Rightarrow \quad \mu = \frac{x^2 - y}{x}$

$y^* \Rightarrow \quad \mu = \frac{x - 2x^2}{y}$

$$\frac{x^2 - y}{x} = \frac{x - 2x^2}{y} \qquad \| \cdot x, \cdot y$$

$$y(x^2 - y) = x(x - 2x^2)$$

$$yx^2 - y^2 = x^2 - 2x^3$$

$$x^2\left(y - \frac{y^2}{x^2}\right) = x^2(1 - 2x)$$

$$y - \frac{y^2}{x^2} = 1 - 2x$$

a.) Find FP:

$$\dot{x} = \mu x + y - x^2 \stackrel{!}{=} 0 \qquad (I)$$

$$\dot{y} = -x + \mu y + 2x^2 \stackrel{!}{=} 0 \qquad (II)$$

$FP_1 = [0, 0]$

for FP_2:

$(II) \to \quad y = \dfrac{x - 2x^2}{\mu}$

put in (I)

$$0 = x \cdot \mu + \frac{x - 2x^2}{\mu} - x^2 \qquad \| \cdot \mu$$

$$0 = x \cdot \mu^2 + x - 2x^2 - x^2\mu$$

$$x = -x\mu^2 + 2x^2 + x^2\mu$$

$$x - 2x^2 = -x\mu^2 + x^2\mu$$

$$x(1 - 2x) = x(x\mu - \mu^2)$$

$$1 - 2x = x\mu - \mu^2$$

$$1 + \mu^2 = x\mu + 2x$$

$$1 + \mu^2 = x(\mu + 2)$$

$$x^* = \frac{1 + \mu^2}{\mu + 2} \qquad y^* = \frac{x - 2x^2}{\mu} = \frac{x(1 - 2x)}{\mu}$$

$$= \frac{1}{\mu} \cdot \left(\frac{1 + \mu^2}{\mu - 2}\right)\left(1 - \frac{2 + 2\mu^2}{\mu - 2}\right)$$

$FP_2 = [x^*, y^*]$

$\int \dot{x} = \int ux$ with initial cond

$$\int_{x^*}^{1} ux \, dx = \int_{0}^{t_1} ux \, dt$$

c)

$$\dot{x} = \mu x + y - x^2$$
$$\dot{y} = -x + \mu y + 2x^2$$

$\downarrow$ linearise

$$\dot{x} = ux \qquad \frac{dx}{dt} = ux$$

$$\dot{y} = sy$$

find the time t_1 to escape from the saddle to $x(t_1) = 1$!

Init conditions:

$$x(0) = \mu$$
$$y(0) = 1$$

Seperation of variable

$$\frac{dx}{dt} = u \cdot x$$

$$\frac{1}{x}\, dx = u \cdot dt \qquad \| \int$$

$$\int \frac{1}{x}\, dx = \int u \cdot dt$$

$$\ln|x| = u \cdot t + C$$

Put in init conditions: $x(0) = g \to \ln|g| = C$

$$\ln|x| = v \cdot t + \ln|g|$$

$$x = g \cdot e^{(v \cdot t)}$$

init cond $x(t_1) = 1$

$$1 = g \cdot e^{(v \cdot t_1)}$$

solve for t_1:

$$t_1 = \frac{\ln(1/g)}{v}$$

d) $\dot{x} = \mu x + y - x^2$ $\quad\rightarrow A = \begin{matrix} \mu - 2x & 1 \\ -1+4x & \mu \end{matrix}$

$\dot{y} = -x + \mu y + 2x^2$

$$A = \begin{pmatrix} \mu - 2x & 1 \\ -1+4x & \mu \end{pmatrix}$$

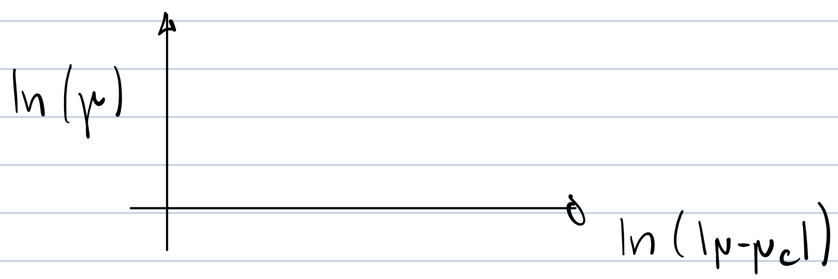
now $x^* = \dfrac{1+\mu^2}{\mu+2}$

Substitute EigVal [A] /. $x \rightarrow \dfrac{1+\mu^2}{\mu+2}$

e) To find T_ν first find $\gamma(|\nu-\nu_c|)$ for $|\nu-\nu_c| \ll 1$

$$\gamma \sim A \cdot |\nu-\nu_c|^a$$

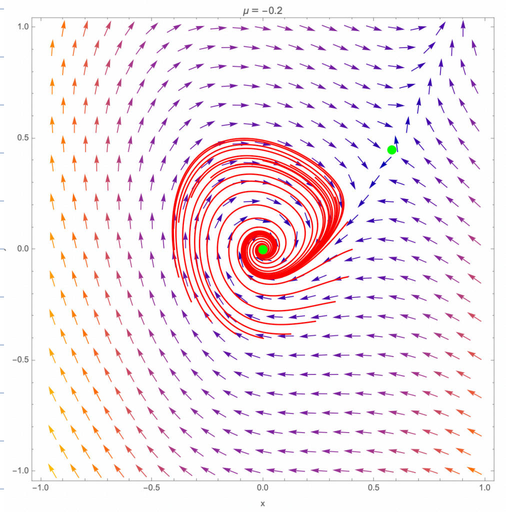
$$\gamma = \frac{1+\nu^2}{\nu+2}$$



$$\log(\gamma(\nu)) = a \cdot \log(|\nu-\nu_c|) + \log(A)$$

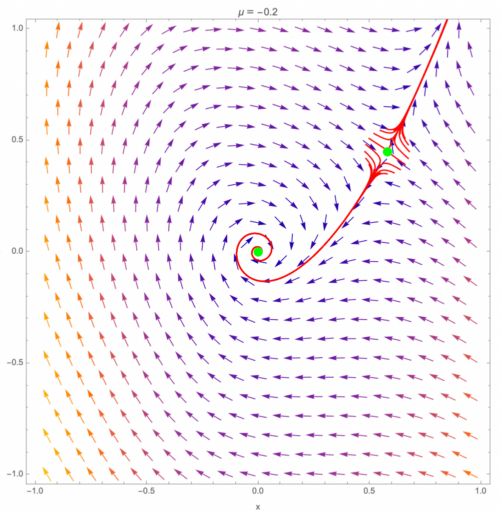
$$\log(\gamma(\nu)) \approx a \cdot \log(\text{mabs})$$

$\mu = -0{,}2$



Starting very close to FP_1.

The trajectory spirals into the stable FP_1 @ $\{0,0\}$.

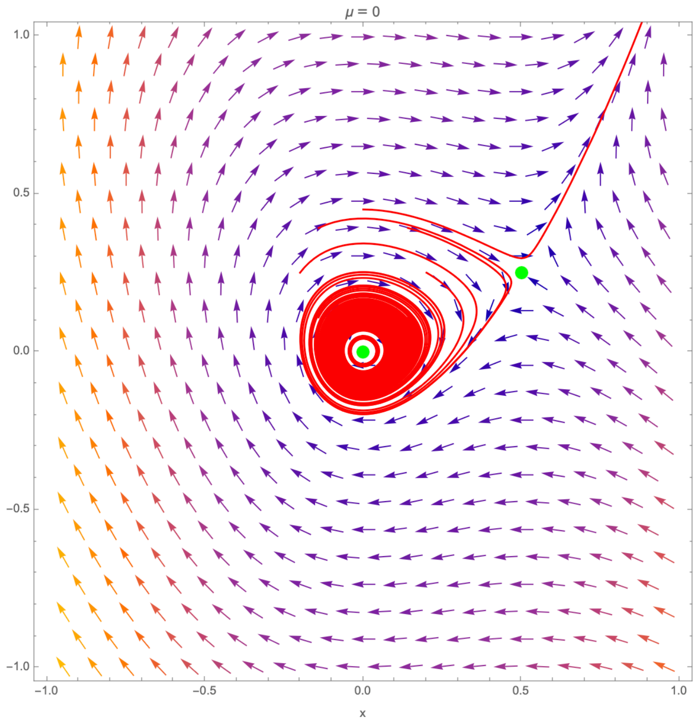


When starting the trajectories around FP_2 they either spiral into the stable FP_1 or escape towards NorthEast, with a way higher velocity than the ones going into FP_1.

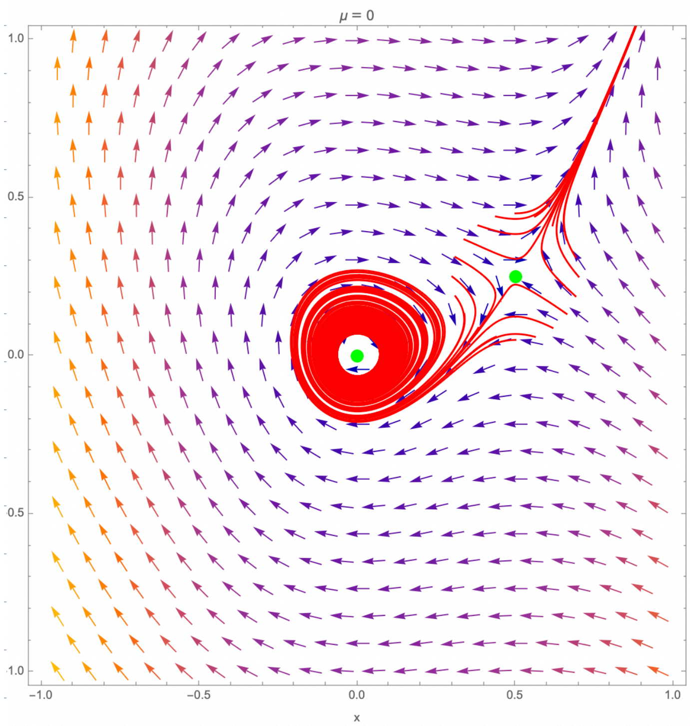
FP_2 seems to be a saddle point, since attracting in one direction, and repelling in the other (Eigenvectors)

So stable FP_1 at $\{0,0\}$ and a saddle node $FP_2(\mu)$.

$\mu = 0$

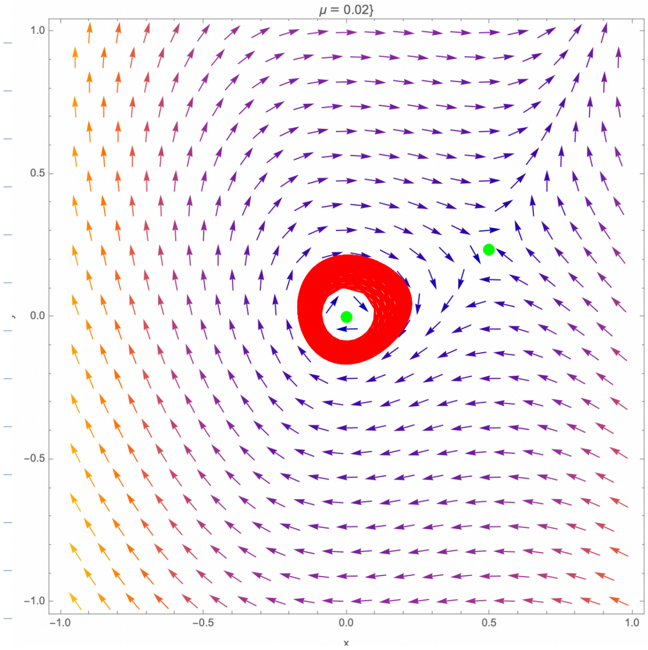


Trajectories starting close around FP_1 spiral into FP_1. FP_1 stable spiral. Into FP_1 the trajectories are very slow.

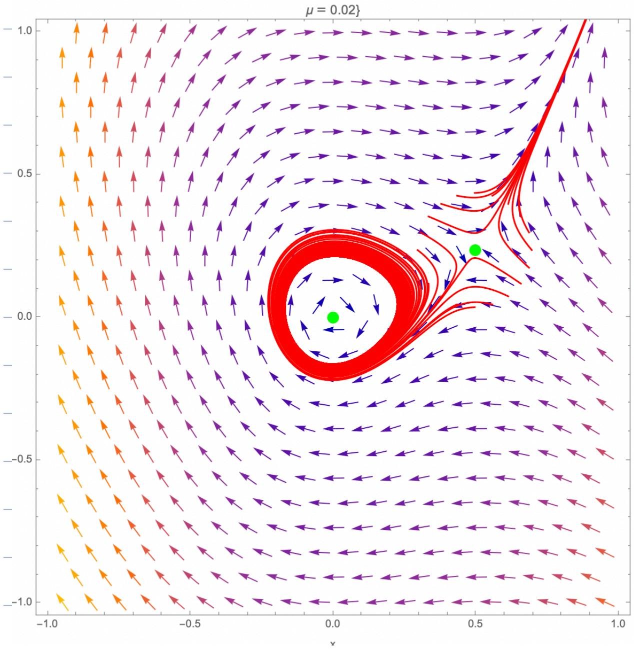


FP_2 is again as in $\mu = -0{,}2$ a saddle point

$\mu = 0{,}02$

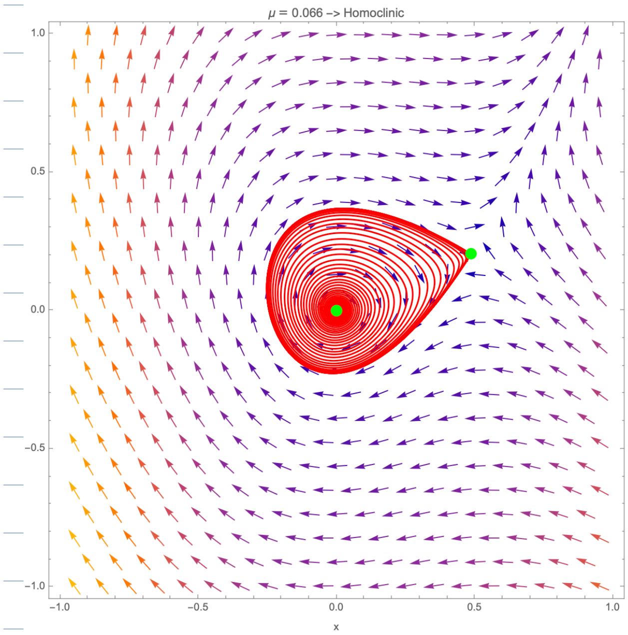


Trj. starting close around FP_1 are repelled from the FP_1, hence FP_1 seems unstable spiral. They are trapped inside the "egg" shape, limit cycle.



When starting around FP_2, the traj are attracted either to limit cycle or escape towards NorthEast. FP_2 saddle point.

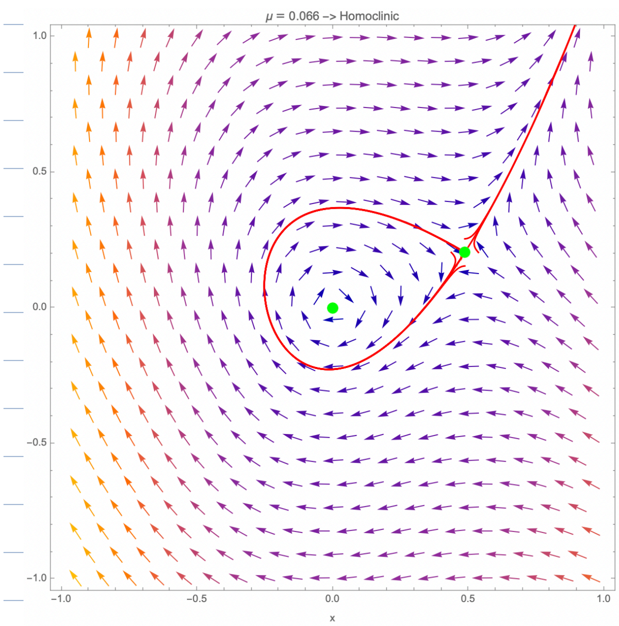
$\mu = \mu_c$



Starting trj. close around FP_1

the get trapped by Homoclinic orbit.

FP_1 unstable spiral

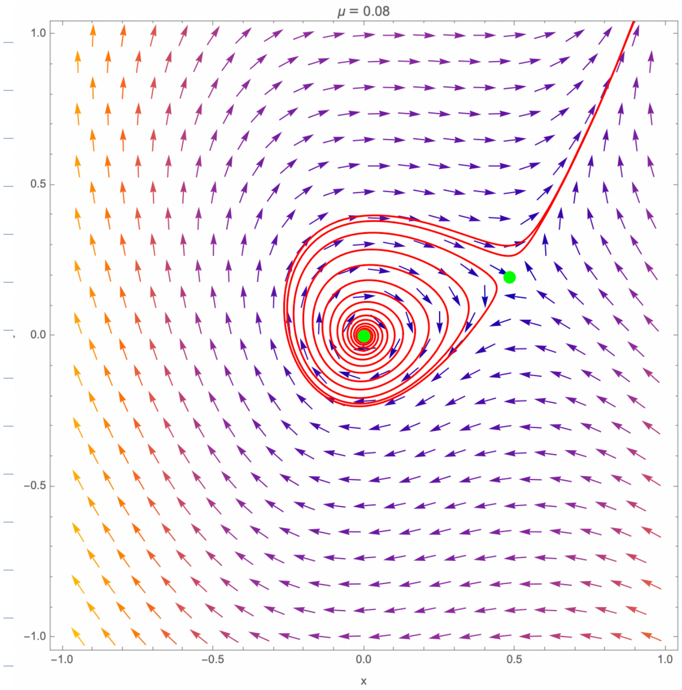


Starting just at the FP_2 one sees that

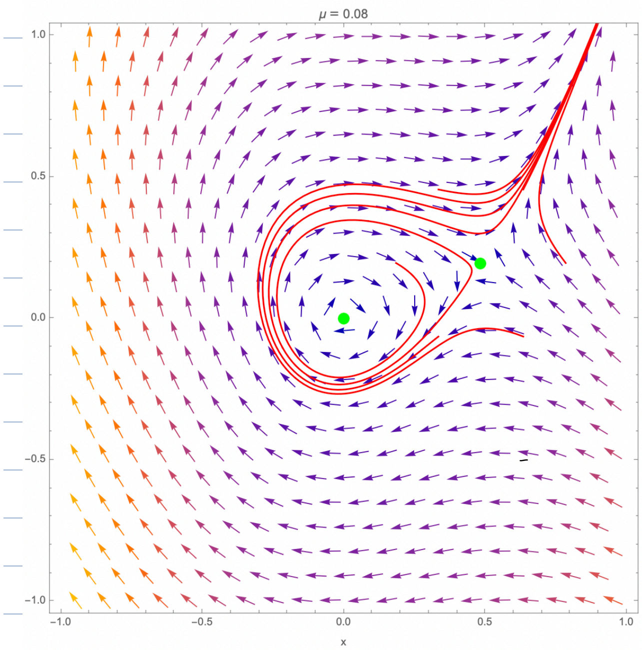
the trajectories escape the FP and

come into it again.

$\mu = 0{,}08$



Starting trj. close to FP_1 the escape from FP_1. So around FP_1 unstable spiral.



$FP2$ saddle point.