Week 11

In This Lecture …

- Self-balancing trees ([S] Ch.13.1-13.4)

Coming Up …

- Text processing algorithms

Nerds You Should Know

Next in a series on famous computer scientists …

She is considered to be the first computer programmer!

Nerds You Should Know (cont)

Ada Lovelace (1815-1852)

- Daughter of English poet Lord Byron
- Privately schooled in mathematics and science
- Met Charles Babbage as a teenager (1833)
- Published notes on … (1843)
  - Babbage's Analytical Engine
  - algorithm for calculating "Bernoulli numbers" on AE
    - which thus became the first computer program
- Programming language Ada named after her (1980)

Nerds You Should Know (cont)

The first published computer algorithm …

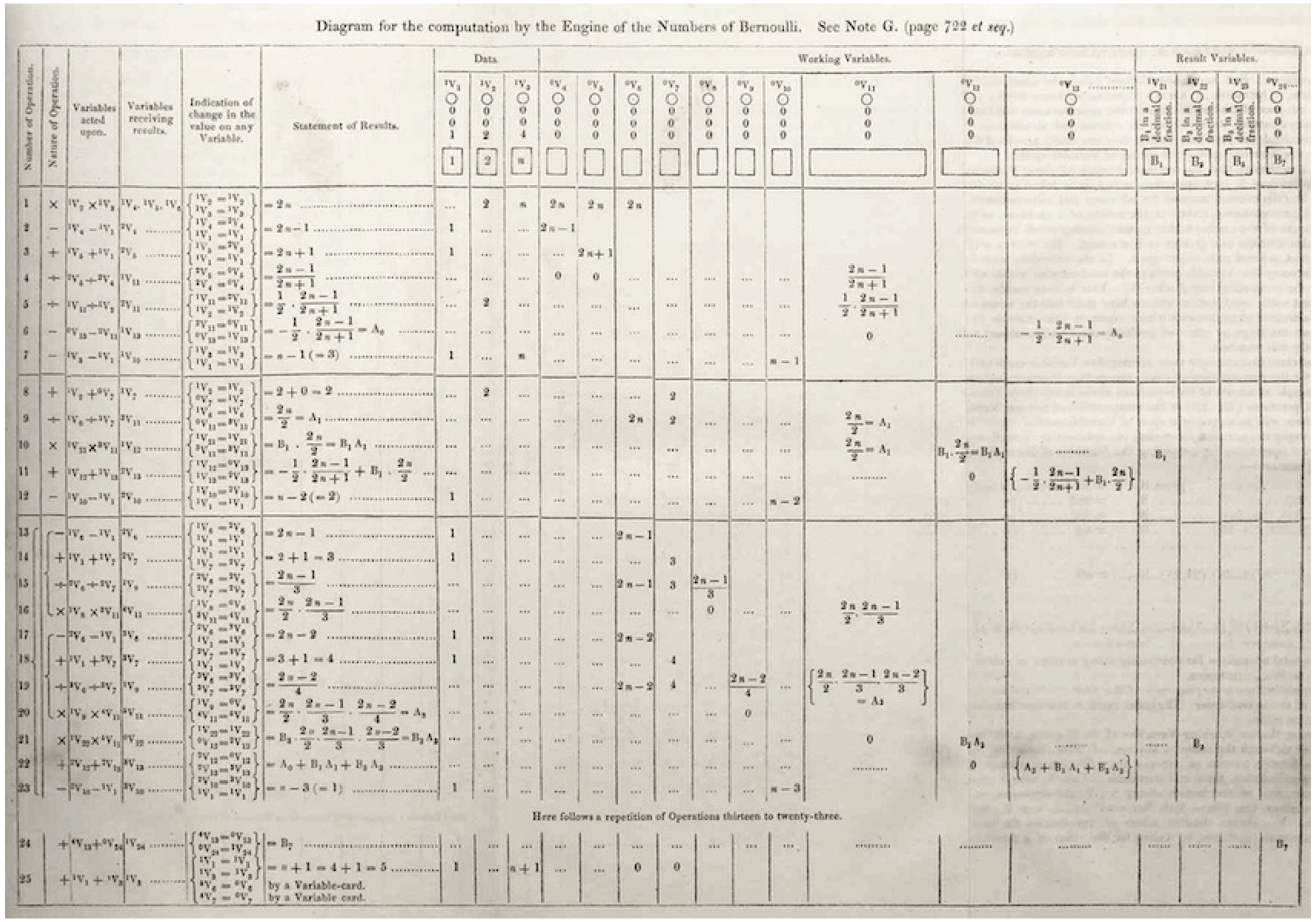


Diagram for the computation by the Engine of the Numbers of Bernoulli. See Note G. (page 722 *et seq.*)

Number of Operation.	Nature of Operation.	Variables acted upon.	Variables receiving results.	Indication of change in the value on any Variable.	Statement of Results.	Data.			Working Variables.										Result Variables.			
						$^1V_1$ ○ 0 0 0 1 $\boxed{1}$	$^1V_2$ ○ 0 0 0 2 $\boxed{2}$	$^1V_3$ ○ 0 0 0 4 $\boxed{n}$	$^0V_4$ ○ 0 0 0 0	$^0V_5$ ○ 0 0 0 0	$^0V_6$ ○ 0 0 0 0	$^0V_7$ ○ 0 0 0 0	$^0V_8$ ○ 0 0 0 0	$^0V_9$ ○ 0 0 0 0	$^0V_{10}$ ○ 0 0 0 0	$^0V_{11}$ ○ 0 0 0 0	$^0V_{12}$ ○ 0 0 0 0	$^0V_{13}$ ○ 0 0 0 0	$^1V_{21}$ ○ B_1 in a decimal fraction. $\boxed{B_1}$	$^1V_{22}$ ○ B_3 in a decimal fraction. $\boxed{B_3}$	$^1V_{23}$ ○ B_5 in a decimal fraction. $\boxed{B_5}$	$^0V_{24}$ ○ 0 0 0 ... $\boxed{B_7}$
1	×	$^1V_2 \times {}^1V_3$	$^1V_4, {}^1V_5, {}^1V_6$	$\left\{\begin{matrix} ^1V_2 = {}^1V_2 \\ ^1V_3 = {}^1V_3 \end{matrix}\right\}$	$= 2n$	...	2	n	2n	2n	2n											
2	−	$^1V_4 - {}^1V_1$	$^2V_4$	$\left\{\begin{matrix} ^1V_4 = {}^2V_4 \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$= 2n - 1$	1	...	...	2n − 1													
3	+	$^1V_5 + {}^1V_1$	$^2V_5$	$\left\{\begin{matrix} ^1V_5 = {}^2V_5 \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$= 2n + 1$	1	...	...	...	2n + 1												
4	÷	$^2V_5 \div {}^2V_4$	$^1V_{11}$	$\left\{\begin{matrix} ^2V_5 = {}^0V_5 \\ ^2V_4 = {}^0V_4 \end{matrix}\right\}$	$= \frac{2n-1}{2n+1}$	...	...	...	0	0	...	...	...	...	...	$\frac{2n-1}{2n+1}$						
5	÷	$^1V_{11} \div {}^1V_2$	$^2V_{11}$	$\left\{\begin{matrix} ^1V_{11} = {}^2V_{11} \\ ^1V_2 = {}^1V_2 \end{matrix}\right\}$	$= \frac{1}{2} \cdot \frac{2n-1}{2n+1}$	...	2	...	...	...	...	...	...	...	...	$\frac{1}{2} \cdot \frac{2n-1}{2n+1}$						
6	−	$^0V_{13} - {}^2V_{11}$	$^1V_{13}$	$\left\{\begin{matrix} ^2V_{11} = {}^0V_{11} \\ ^0V_{13} = {}^1V_{13} \end{matrix}\right\}$	$= -\frac{1}{2} \cdot \frac{2n-1}{2n+1} = A_0$	...	...	...	...	...	...	...	...	...	...	0	....	$-\frac{1}{2} \cdot \frac{2n-1}{2n+1} = A_0$				
7	−	$^1V_3 - {}^1V_1$	$^1V_{10}$	$\left\{\begin{matrix} ^1V_3 = {}^1V_3 \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$= n - 1\ (= 3)$	1	...	n	...	...	...	...	...	...	n − 1							
8	+	$^1V_2 + {}^0V_7$	$^1V_7$	$\left\{\begin{matrix} ^1V_2 = {}^1V_2 \\ ^0V_7 = {}^1V_7 \end{matrix}\right\}$	$= 2 + 0 = 2$	...	2	...	...	...	...	2										
9	÷	$^1V_6 \div {}^1V_7$	$^3V_{11}$	$\left\{\begin{matrix} ^1V_6 = {}^1V_6 \\ ^0V_{11} = {}^3V_{11} \end{matrix}\right\}$	$= \frac{2n}{2} = A_1$	...	...	...	...	...	2n	2	...	...	...	$\frac{2n}{2} = A_1$						
10	×	$^1V_{21} \times {}^3V_{11}$	$^1V_{12}$	$\left\{\begin{matrix} ^1V_{21} = {}^1V_{21} \\ ^3V_{11} = {}^3V_{11} \end{matrix}\right\}$	$= B_1 \cdot \frac{2n}{2} = B_1 A_1$	...	...	...	...	...	...	...	...	...	...	$\frac{2n}{2} = A_1$	$B_1 \cdot \frac{2n}{2} = B_1 A_1$	....	B_1			
11	+	$^1V_{12} + {}^1V_{13}$	$^2V_{13}$	$\left\{\begin{matrix} ^1V_{12} = {}^0V_{12} \\ ^1V_{13} = {}^2V_{13} \end{matrix}\right\}$	$= -\frac{1}{2} \cdot \frac{2n-1}{2n+1} + B_1 \cdot \frac{2n}{2}$ ...	...	...	...	...	...	...	...	...	...	...	....	0	$\left\{-\frac{1}{2} \cdot \frac{2n-1}{2n+1} + B_1 \cdot \frac{2n}{2}\right\}$				
12	−	$^1V_{10} - {}^1V_1$	$^2V_{10}$	$\left\{\begin{matrix} ^1V_{10} = {}^2V_{10} \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$= n - 2\ (= 2)$	1	...	...	...	...	...	...	...	...	n − 2							
13	−	$^1V_6 - {}^1V_1$	$^2V_6$	$\left\{\begin{matrix} ^1V_6 = {}^2V_6 \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$= 2n - 1$	1	...	...	...	...	2n − 1											
14	+	$^1V_1 + {}^1V_7$	$^2V_7$	$\left\{\begin{matrix} ^1V_1 = {}^1V_1 \\ ^1V_7 = {}^2V_7 \end{matrix}\right\}$	$= 2 + 1 = 3$	1	...	...	...	...	...	3										
15	÷	$^2V_6 \div {}^2V_7$	$^1V_8$	$\left\{\begin{matrix} ^2V_6 = {}^2V_6 \\ ^2V_7 = {}^2V_7 \end{matrix}\right\}$	$= \frac{2n-1}{3}$	...	...	...	...	...	2n − 1	3	$\frac{2n-1}{3}$									
16	×	$^1V_8 \times {}^3V_{11}$	$^4V_{11}$	$\left\{\begin{matrix} ^1V_8 = {}^0V_8 \\ ^3V_{11} = {}^4V_{11} \end{matrix}\right\}$	$= \frac{2n}{2} \cdot \frac{2n-1}{3}$	...	...	...	...	...	...	...	0	...	...	$\frac{2n}{2} \cdot \frac{2n-1}{3}$						
17	−	$^2V_6 - {}^1V_1$	$^3V_6$	$\left\{\begin{matrix} ^2V_6 = {}^3V_6 \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$= 2n - 2$	1	...	...	...	...	2n − 2											
18	+	$^1V_1 + {}^2V_7$	$^3V_7$	$\left\{\begin{matrix} ^2V_7 = {}^3V_7 \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$= 3 + 1 = 4$	1	...	...	...	...	...	4										
19	÷	$^3V_6 \div {}^3V_7$	$^1V_9$	$\left\{\begin{matrix} ^3V_6 = {}^3V_6 \\ ^3V_7 = {}^3V_7 \end{matrix}\right\}$	$= \frac{2n-2}{4}$	...	...	...	...	...	2n − 2	4	...	$\frac{2n-2}{4}$	...	$\left\{\frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{3} = A_3\right\}$						
20	×	$^1V_9 \times {}^4V_{11}$	$^5V_{11}$	$\left\{\begin{matrix} ^1V_9 = {}^0V_9 \\ ^4V_{11} = {}^5V_{11} \end{matrix}\right\}$	$= \frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{4} = A_3$	...	...	...	...	...	...	...	...	0	...							
21	×	$^1V_{22} \times {}^5V_{11}$	$^0V_{12}$	$\left\{\begin{matrix} ^1V_{22} = {}^1V_{22} \\ ^0V_{12} = {}^2V_{12} \end{matrix}\right\}$	$= B_3 \cdot \frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{3} = B_3 A_3$	...	...	...	...	...	...	...	...	...	...	0	$B_3 A_3$	....	......	B_3		
22	+	$^2V_{12} + {}^2V_{13}$	$^3V_{13}$	$\left\{\begin{matrix} ^2V_{12} = {}^0V_{12} \\ ^2V_{13} = {}^3V_{13} \end{matrix}\right\}$	$= A_0 + B_1 A_1 + B_3 A_3$	...	...	...	...	...	...	...	...	...	...	....	0	$\left\{A_0 + B_1 A_1 + B_3 A_3\right\}$				
23	−	$^2V_{10} - {}^1V_1$	$^3V_{10}$	$\left\{\begin{matrix} ^2V_{10} = {}^3V_{10} \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$= n - 3\ (= 1)$	1	...	...	...	...	...	...	...	...	n − 3							
Here follows a repetition of Operations thirteen to twenty-three.																						
24	+	$^4V_{13} + {}^0V_{24}$	$^1V_{24}$	$\left\{\begin{matrix} ^4V_{13} = {}^0V_{13} \\ ^0V_{24} = {}^1V_{24} \end{matrix}\right\}$	$= B_7$	...	...	...	...	...	...	...	...	...	...	....	....	....	......	......	......	B_7
25	+	$^1V_1 + {}^1V_3$	$^1V_3$	$\left\{\begin{matrix} ^1V_1 = {}^1V_1 \\ ^1V_3 = {}^1V_3 \\ ^5V_6 = {}^0V_6 \\ ^5V_7 = {}^0V_7 \end{matrix}\right\}$	$= n + 1 = 4 + 1 = 5$ .... by a Variable-card. by a Variable card.	1	...	n + 1	...	...	0	0										

Tree Review

Binary search trees …

- data structures designed for *O(log n)* search
- consist of nodes containing item (incl. key) and two links
- can be viewed as recursive data structure (subtrees)
- have overall ordering (data(Left) < root < data(Right))
- insert new nodes as leaves (or as root), delete from anywhere
- have structure determined by insertion order (worst: *O(n)*)
- operations: insert, delete, search, rotate, rebalance, …

Randomised BST Insertion

Effects of order of insertion on BST shape:

- best case (for at-leaf insertion): keys inserted in pre-order
  (median key first, then median of lower half, median of upper half, etc.)
- worst case: keys inserted in ascending/descending order
- average case: keys inserted in random order $\Rightarrow O(\log_2 n)$

Tree ADT has no control over order that keys are supplied.

Can the algorithm itself introduce some randomness?

In the hope that this randomness helps to balance the tree …

Sidetrack: Random Numbers

How can a computer pick a number at random?

- it cannot

Software can only produce pseudo random numbers.

- a pseudo random number is one that is predictable
  - (although it may appear unpredictable)

⇒ Implementation may deviate from expected theoretical behaviour

Sidetrack: Random Numbers (cont)

The most widely-used technique is called the Linear Congruential Generator (LCG)

- it uses a recurrence relation:
    - $X_{n+1} = (a \cdot X_n + c)$ *mod* m, where:
        - m is the "modulus"
        - a, $0 < a < m$ is the "multiplier"
        - c, $0 \leq c \leq m$ is the "increment"
        - X_0 is the "seed"
    - if c=0 it is called a multiplicative congruential generator

LCG is not good for applications that need extremely high-quality random numbers

- the period length is too short (length of the sequence at which point it repeats itself)
- a short period means the numbers are correlated

Sidetrack: Random Numbers (cont)

Trivial example:

- for simplicity assume c=0
- so the formula is $X_{n+1} = a \cdot X_n \; mod \; m$
- try $a=11=X_0$, m=31, which generates the sequence:

```
11, 28, 29, 9, 6, 4, 13, 19, 23, 5, 24, 16, 21, 14, 30, 20, 3, 2, 22, 25,
27, 18, 12, 8, 26, 7, 15, 10, 17, 1, 11, 28, 29, 9, 6, 4, 13, 19, 23, 5,
24, 16, 21, 14, 30, 20, 3, 2, 22, 25, 27, 18, 12, 8, 26, 7, 15, 10, 17, 1,
11, 28, 29, 9, 6, 4, 13, 19, 23, 5, 24, 16, 21, 14, 30, 20, 3, 2, 22, 25, 27,
18, 12, 8, 26, 7, 15, 10, 17, 1, 11, 28, 29, 9, 6, 4, 13, 19, 23, 5, 24, 16,
21, 14, 30, 20, 3, 2, 22, 25, 27, 18, 12, 8, 26, 7, 15, 10, 17, 1, 11, 28,
29, 9, 6, 4, 13, 19, 23, 5, 24, 16, 21, 14, 30, 20, 3, 2, 22, 25, 27, 18,
12, 8, 26, 7, 15, 10, 17, 1, ...
```

- all the integers from 1 to 30 are here

Sidetrack: Random Numbers (cont)

Another trivial example:

- again let c=0
- try a=12=X_0 and m=30
  - that is, $X_{n+1} = 12 \cdot X_n \textit{ mod } 30$
  - which generates the sequence:

```
12, 24, 18, 6, 12, 24, 18, 6, 12, 24, 18, 6, 12, 24, 18, 6, 12, 24, 18, 6,
12, 24, 18, 6, 12, 24, 18, 6, 12, 24, 18, 6, ...
```

- notice the period length … clearly a terrible sequence

Sidetrack: Random Numbers (cont)

It is a complex task to pick good numbers. A bit of history:

Lewis, Goodman and Miller (1969) suggested

- $X_{n+1} = 7^5 \cdot X_n \bmod (2^{31}-1)$
- note:
  - 7^5 is 16807
  - 2^{31}-1 is 2147483674
  - $X_0 = 0$ is not a good seed value

Most compilers use LCG-based algorithms that are slightly more involved; see www.mscs.dal.ca/~selinger/random/ for details (including a short C program that produces the exact same pseudo-random numbers as **gcc** for any given seed value)

Sidetrack: Random Numbers (cont)

- Two functions are required:

```
srandom(int seed) // sets its argument as the seed
```

```
random() // uses a LCG technique to generate random
         // numbers in the range 0 .. RAND_MAX
```

where the constant **`RAND_MAX`** is defined in **`stdlib.h`**
(depends on the computer: on the CSE network, RAND_MAX = 2147483647)

- The period length of this random number generator is very large approximately $16 \cdot ((2^{31}) - 1)$

Sidetrack: Random Numbers (cont)

To convert the return value of **`random()`** to a number between 0 .. RANGE

- compute the remainder after division by RANGE+1

Using the remainder to compute a random number is not the best way:

- can generate a 'better' random number by using a more complex division
- but good enough for most purposes

Some applications require more sophisticated, *cryptographically secure* pseudo random numbers

Exercise #1: Random Numbers

Write a program to simulate 10,000 rounds of Two-up.

- Assume a $10 bet at each round
- Compute the overall outcome and average per round

```
#include <stdlib.h>
#include <stdio.h>

#define RUNS 10000
#define BET  10

int main(void) {
   int coin1, coin2, n, sum = 0;
   for (n = 0; n < RUNS; n++) {
      do {
  coin1 = random() % 2;
  coin2 = random() % 2;
      } while (coin1 != coin2);
      if (coin1==1 && coin2==1)
  sum += BET;
      else
  sum -= BET;
   }
   printf("Final result: %d\n", sum);
   printf("Average outcome: %f\n", (float) sum / RUNS);
   return 0;
}
```

Sidetrack: Random Numbers (cont)

Seeding

There is one significant problem:

- every time you run a program with the same seed, you get exactly the same sequence of 'random' numbers (why?)

To vary the output, can give the random seeder a starting point that varies with time

- an example of such a starting point is the current time, **`time(NULL)`**
  (NB: this is different from the UNIX command **`time`**, used to measure program running time)

```
#include <time.h>
time(NULL) // returns the time as the number of seconds
           // since the Epoch, 1970-01-01 00:00:00 +0000

// time(NULL) on October 10th, 2017, 12:59pm was 1507600763
// time(NULL) about a minute later was 1507600825
```

Randomised BST Insertion

Approach: normally do leaf insert, randomly do root insert.

```
insertRandom(tree,item)
   Input  tree, item
   Output tree with item randomly inserted

   if tree is empty then
      return new node containing item
   end if
   // p/q chance of doing root insert
   if random() mod q < p then
      return insertAtRoot(tree,item)
   else
      return insertAtLeaf(tree,item)
   end if
```

E.g. 30% chance ⇒ choose *p=3, q=10*

Randomised BST Insertion (cont)

Cost analysis:

- similar to cost for inserting keys in random order: $O(\log_2 n)$
- does not rely on keys being supplied in random order

Approach can also be applied to deletion:

- standard method promotes inorder successor to root
- for the randomised method …
  - promote inorder successor from right subtree, OR
  - promote inorder predecessor from left subtree

Splay Trees

Splay Trees

A kind of "self-balancing" tree …

Splay tree insertion modifies insertion-at-root method:

- by considering parent-child-granchild (three level analysis)
- by performing double-rotations based on p-c-g orientation

The idea: appropriate double-rotations improve tree balance.

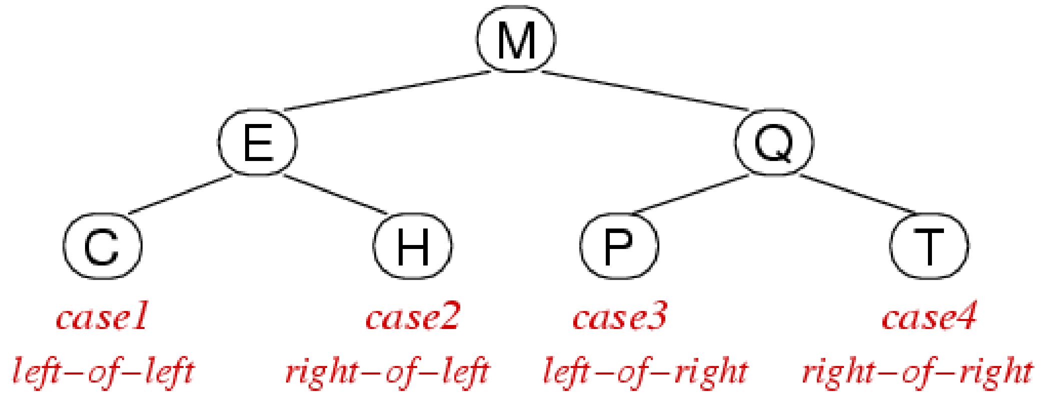
Splay tree implementations also do rotation-in-search:

- can provide similar effect to periodic rebalance
- improves balance, but makes search more expensive

Splay Trees (cont)

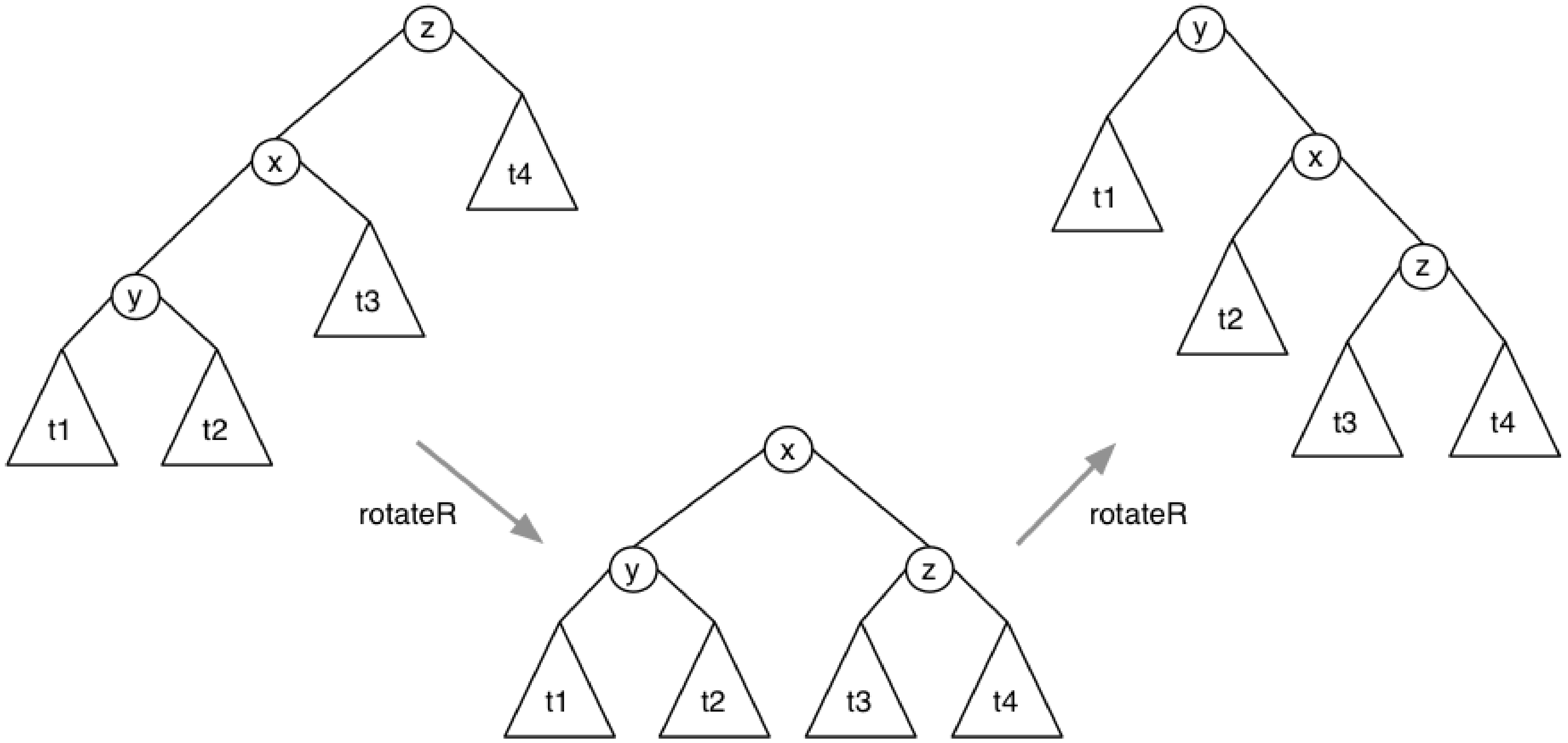
Cases for splay tree double-rotations:

- case 1: grandchild is left-child of left-child
- case 2: grandchild is right-child of left-child
- case 3: grandchild is left-child of right-child
- case 4: grandchild is right-child of right-child



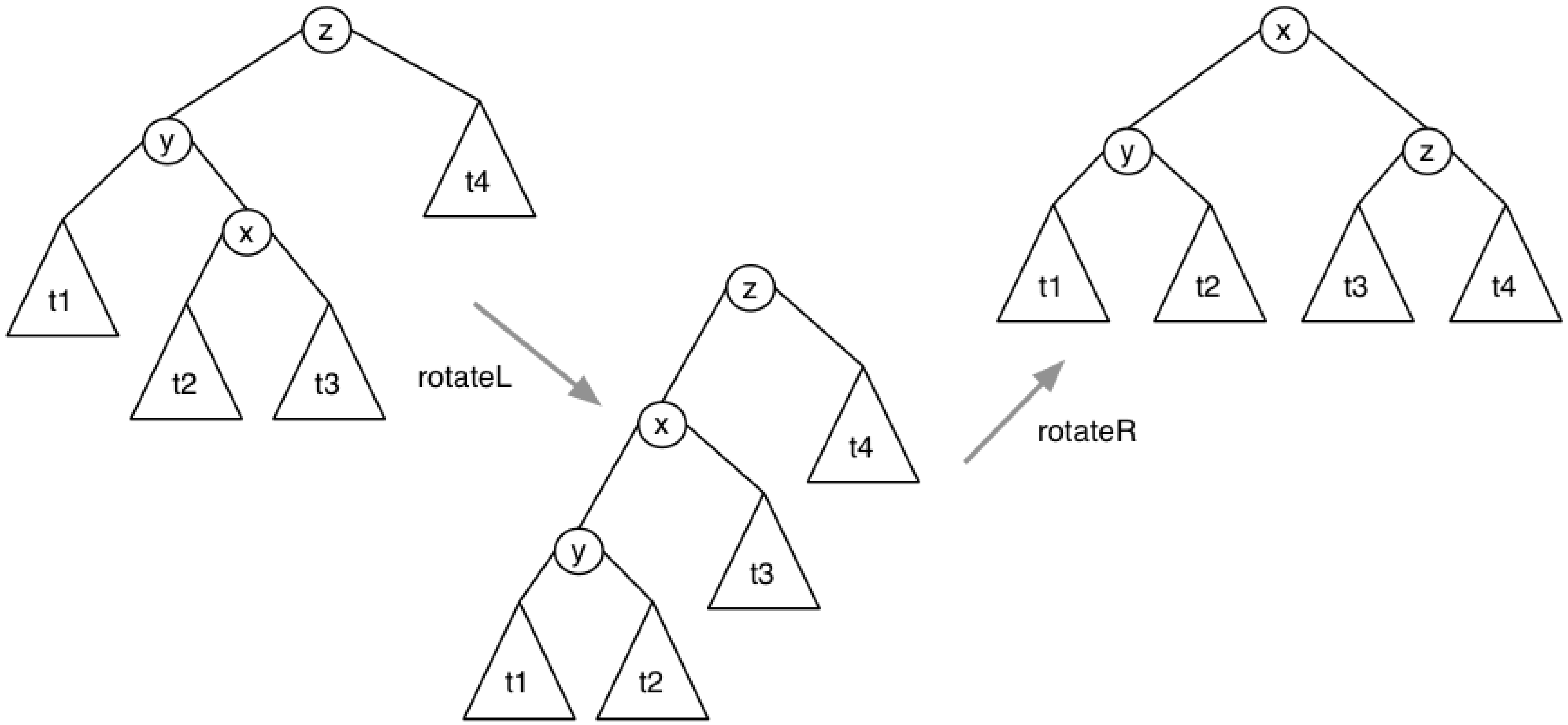
Splay Trees (cont)

Example: double-rotation case for left-child of left-child:



Splay Trees (cont)

Example: double-rotation case for right-child of left-child:



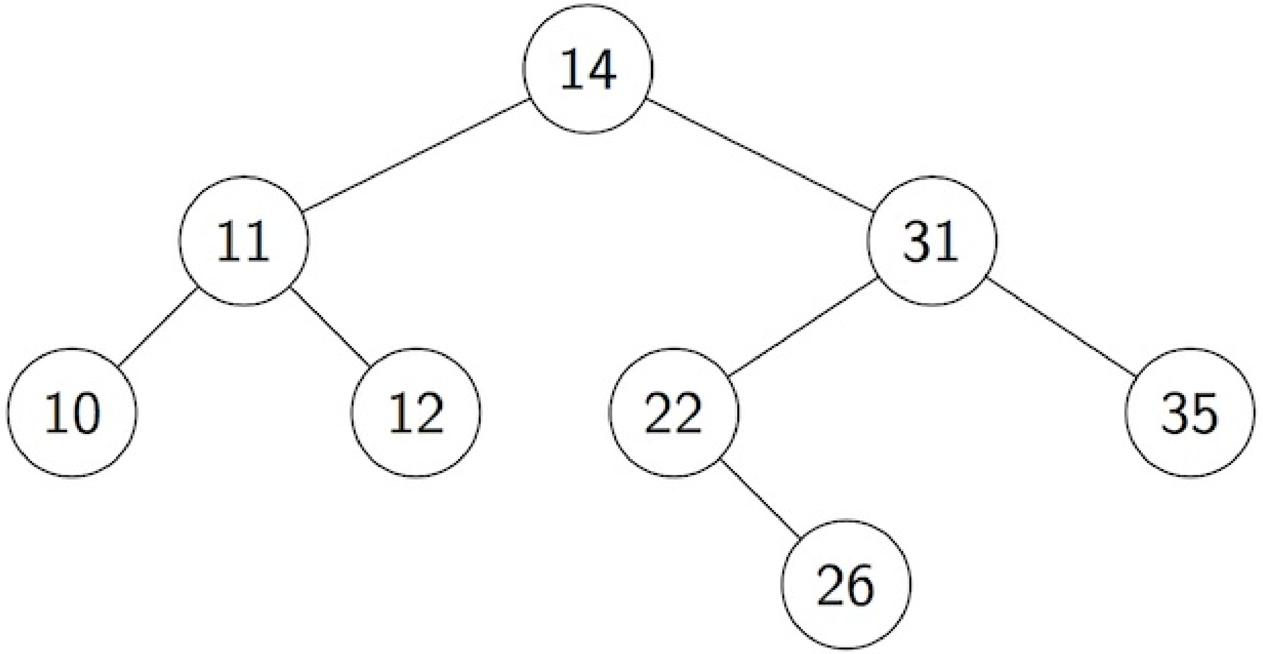
Splay Trees (cont)

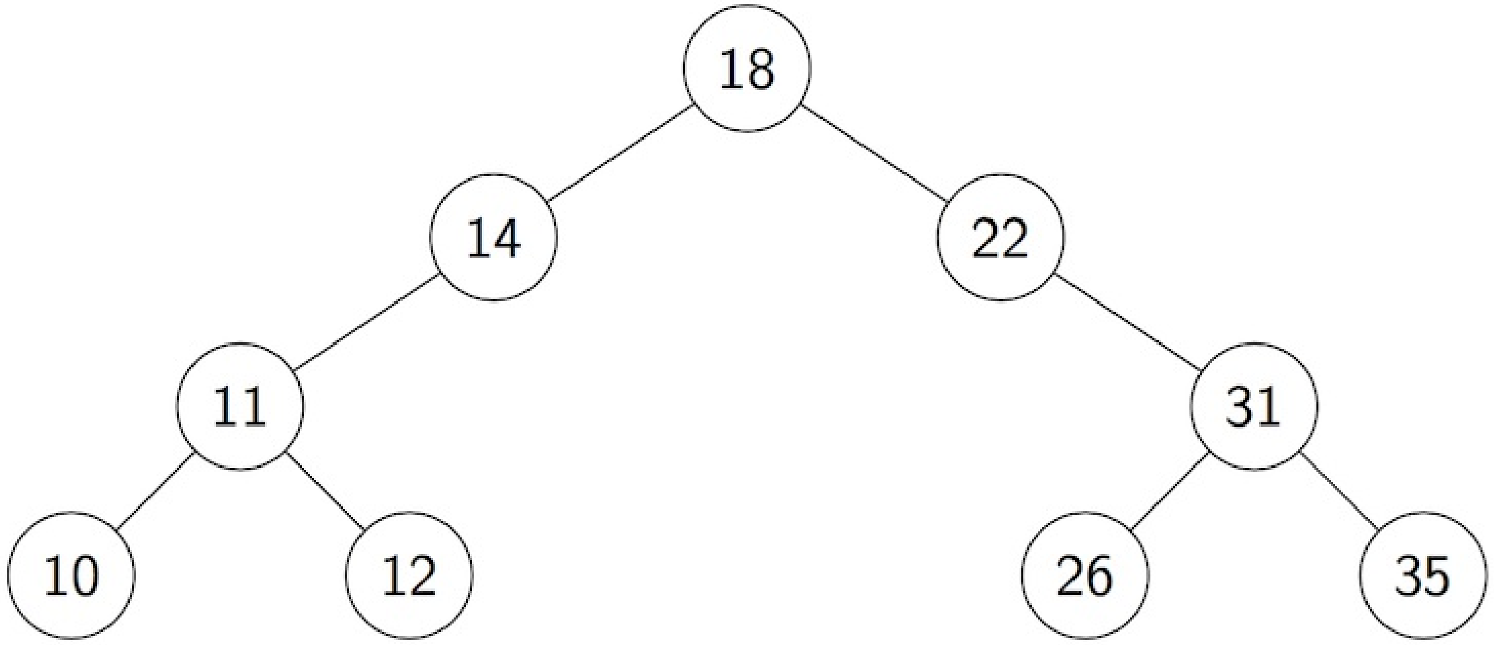
Algorithm for splay tree insertion:

```
insertSplay(tree,item):
|  Input  tree, item
|  Output tree with item splay-inserted
|
|  if tree is empty then return new node containing item
|  else if item=data(tree) then return tree
|  else if item<data(tree) then
|  |  if left(tree) is empty then
|  |     left(tree)=new node containing item
|  |  else if item<data(left(tree)) then
|  |        // Case 1: left-child of left-child
|  |     left(left(tree))=insertSplay(left(left(tree)),item)
|  |     left(tree)=rotateRight(left(tree))
|  |  else  // Case 2: right-child of left-child
|  |     right(left(tree))=insertSplay(right(left(tree)),item)
|  |     left(tree)=rotateLeft(left(tree))
|  |  end if
|  |  return rotateRight(tree)
|  else if item>data(tree) then
|  |  if right(tree) is empty then
|  |     right(tree)=new node containing item
|  |  else if item<data(right(tree)) then
|  |        // Case 3: left-child of right-child
|  |     left(right(tree))=insertSplay(left(right(tree)),item)
|  |     right(tree)=rotateRight(right(tree))
|  |  else  // Case 4: right-child of right-child
|  |     right(right(tree))=insertSplay(right(right(tree)),item)
|  |     right(tree)=rotateLeft(right(tree))
|  |  end if
|  |  return rotateLeft(tree)
|  end if
```

Exercise #2: Splay Trees

Insert **`18`** into this splay tree:




18
14
22
11
31
10
12
26
35

Splay Trees (cont)

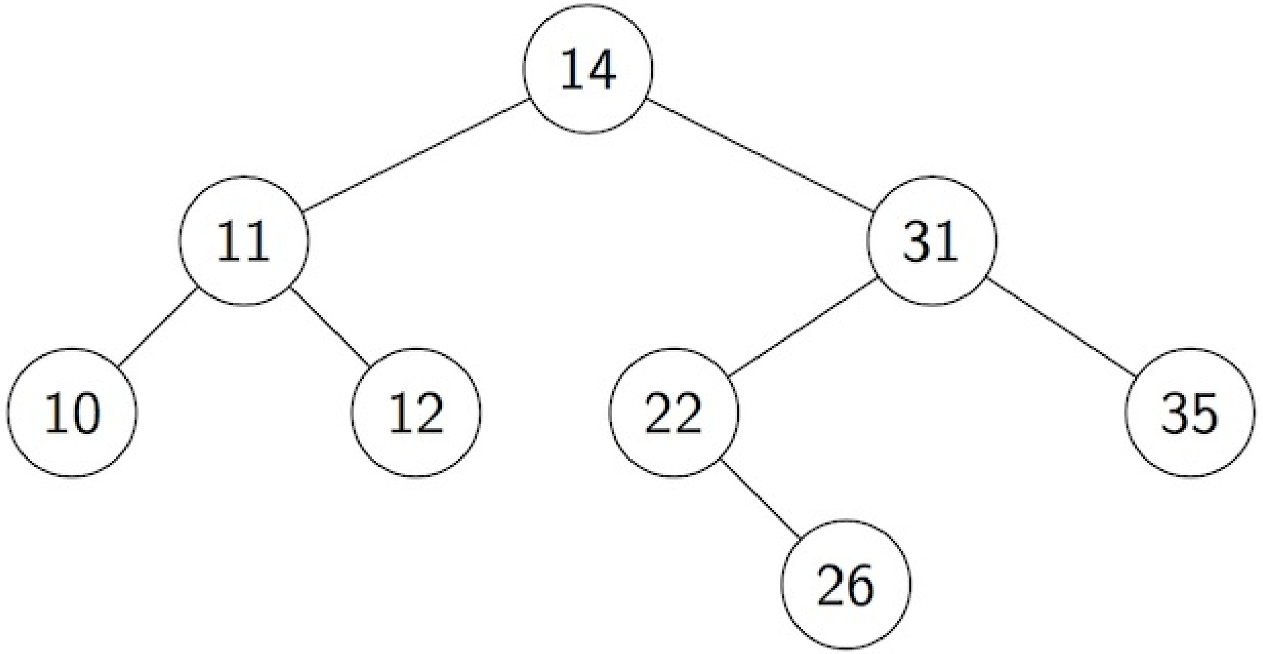
Searching in splay trees:

```
searchSplay(tree,item):
|   Input  tree, item
|   Output address of item if found in tree
|          NULL otherwise
|
|   if tree=NULL then
|      return NULL
|   else
|   |  tree=splay(tree,item)
|   |  if data(tree)=item then
|   |     return tree
|   |  else
|   |     return NULL
|   |  end if
|   end if
```

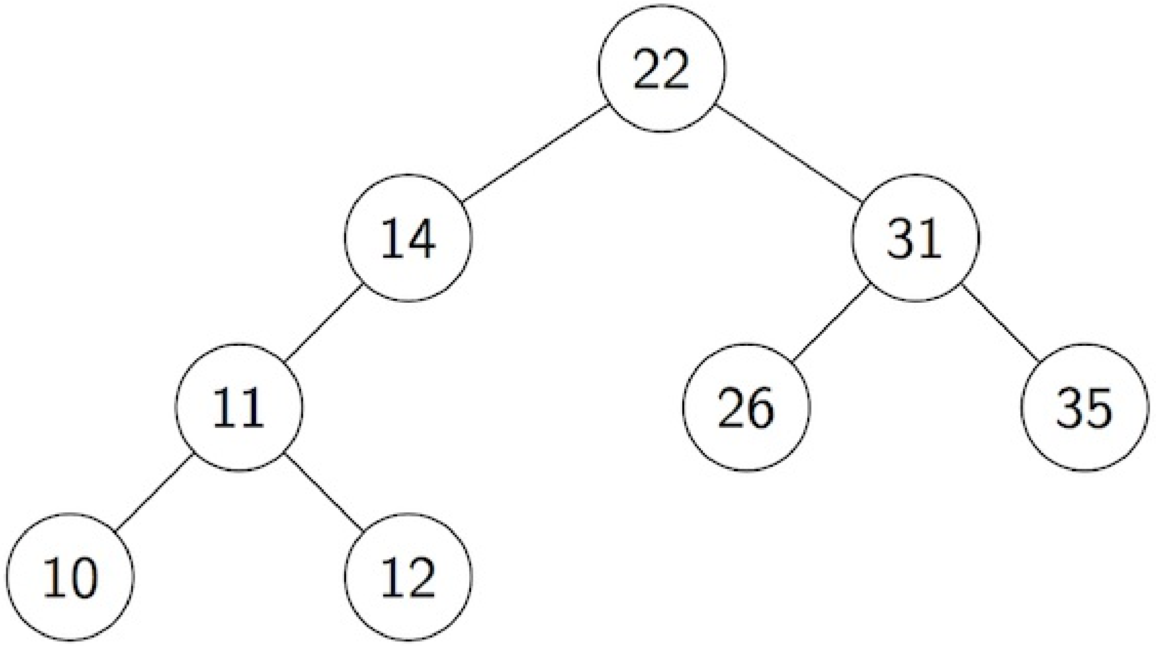
where **splay()** is similar to **insertSplay()**,
except that it doesn't add a node … simply moves **item** to root if found, or nearest node if not found

Exercise #3: Splay Trees

If we search for **22** in the splay tree



… how does this affect the tree?


22
14
31
11
26
35
10
12

Splay Trees (cont)

Analysis of splay tree performance:

- assume that we "splay" for both insert and search
- consider: *m* insert+search operations, *n* nodes
- total number of comparisons: average $O((n+m) \cdot log_2(n+m))$

Gives good overall (amortized) cost.

- insert cost not significantly different to insert-at-root
- search cost increases, but …
  - improves balance on each search
  - moves frequently accessed nodes closer to root

But … still has worst-case search cost $O(n)$

Real Balanced Trees

Better Balanced Binary Search Trees

So far, we have seen …

- randomised trees … make poor performance unlikely
- occasional rebalance … fix balance periodically
- splay trees … reasonable amortized performance
- but both types still have *O(n)* worst case

Ideally, we want both average/worst case to be *O(log n)*

- AVL trees … fix imbalances as soon as they occur
- 2-3-4 trees … use varying-sized nodes to assist balance
- red-black trees … isomorphic to 2-3-4, but binary nodes

AVL Trees

AVL Trees

Invented by Georgy Adelson-Velsky and Evgenii Landis

Approach:

- insertion (at leaves) may cause imbalance
- repair balance as soon as we notice imbalance
- repairs done locally, not by overall tree restructure

A tree is unbalanced when: abs(height(left)-height(right)) > 1

This can be repaired by a single rotation:

- if left subtree too deep, rotate right
- if right subtree too deep, rotate left

Problem: determining height/depth of subtrees may be expensive.

AVL Trees (cont)

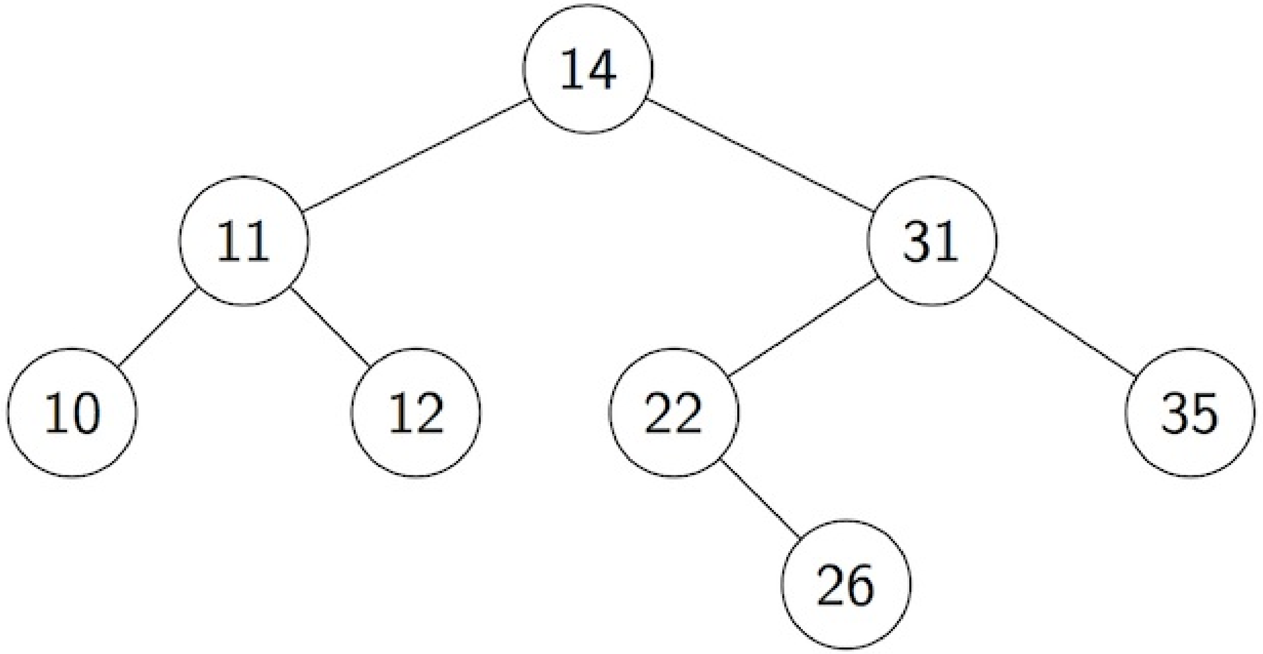
Implementation of AVL insertion

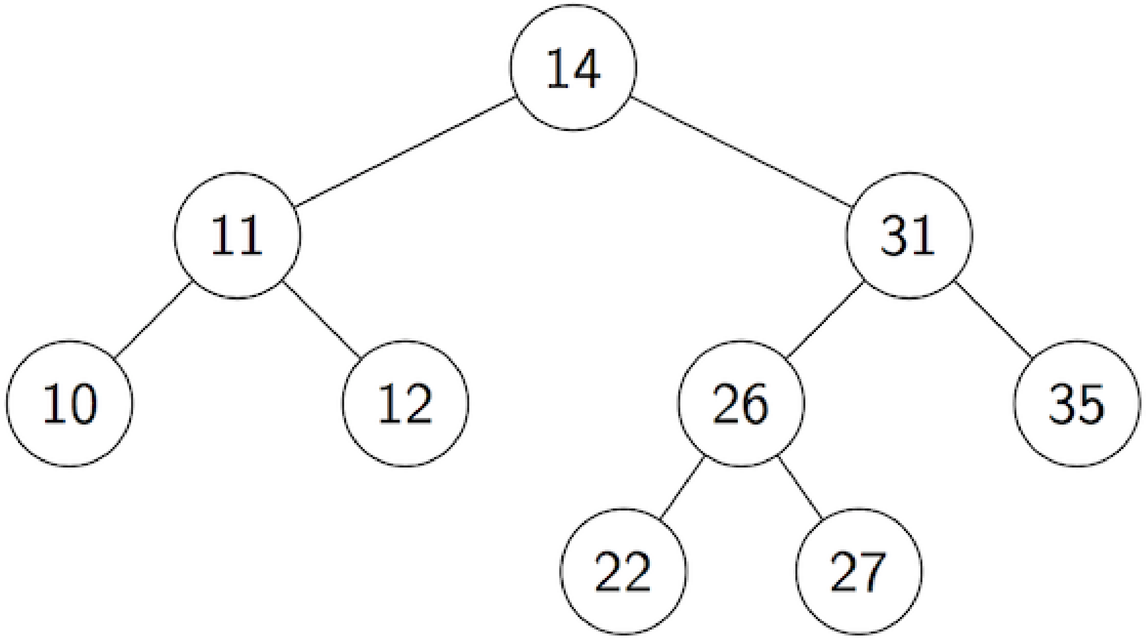
```
insertAVL(tree,item):
   Input  tree, item
   Output tree with item AVL-inserted

   if tree is empty then
      return new node containing item
   else if item=data(tree) then
      return tree
   else
      if item<data(tree) then
         left(tree)=insertAVL(left(tree),item)
      else if item>data(tree) then
         right(tree)=insertAVL(right(tree),item)
      end if
      if height(left(tree))-height(right(tree)) > 1 then
         tree=rotateRight(tree)
      else if height(right(tree))-height(left(tree)) > 1 then
         tree=rotateLeft(tree)
      end if
      return tree
   end if
```

Exercise #4: AVL Trees

Insert **27** into the AVL tree

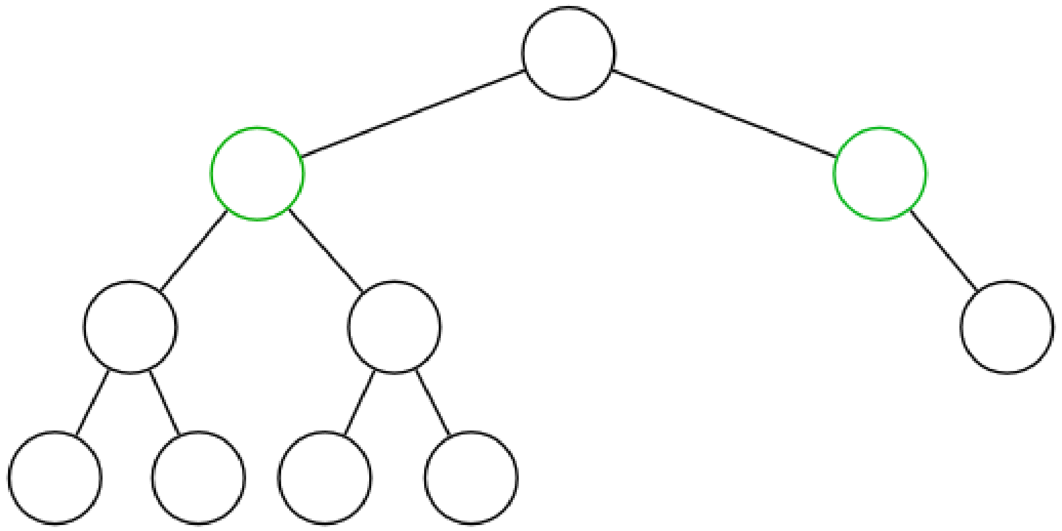



14
11
31
10
12
26
35
22
27

AVL Trees (cont)

Analysis of AVL trees:

- trees are height-balanced; subtree depths differ by +/-1
- average/worst-case search performance of *O(log n)*
- *require* extra data to be stored in each node (efficiency)
- may not be weight-balanced; subtree sizes may differ

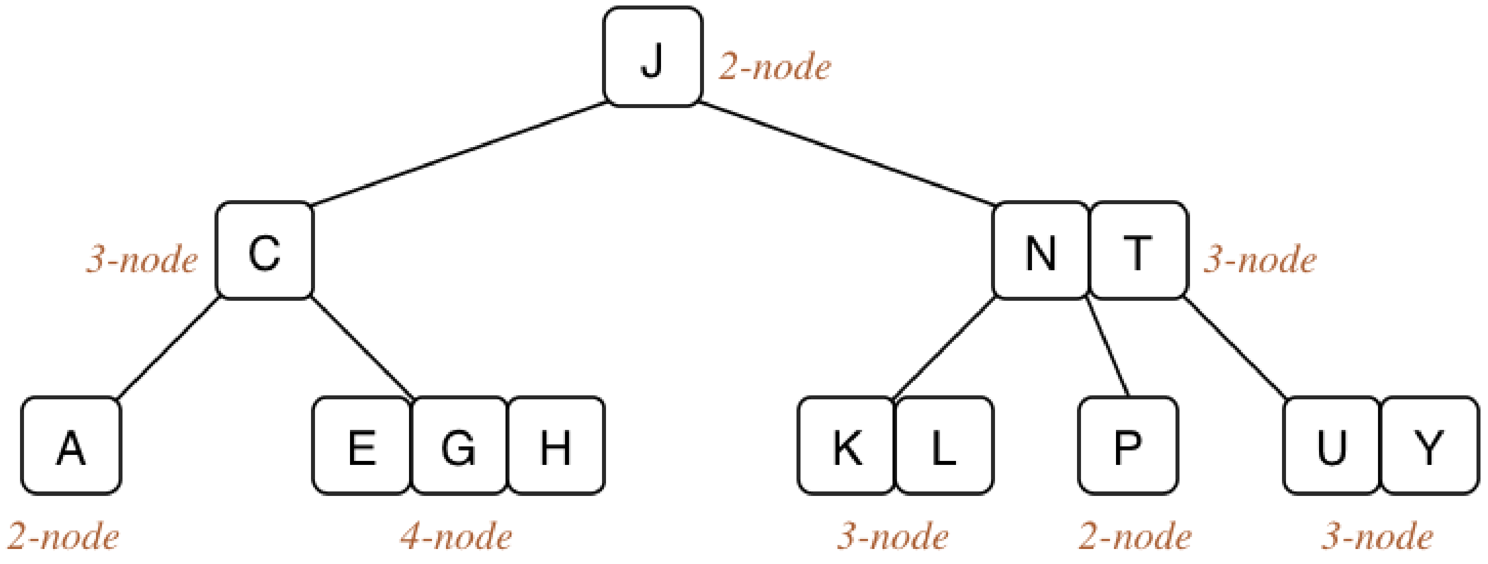


2-3-4 Trees

2-3-4 Trees

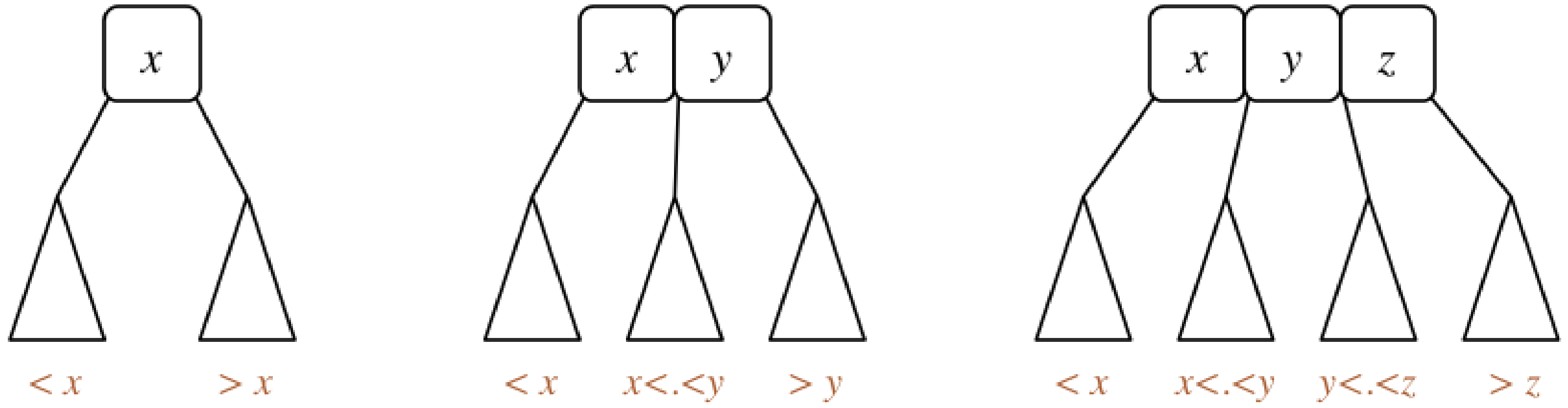
2-3-4 trees have three kinds of nodes

- 2-nodes, with two children (same as normal BSTs)
- 3-nodes, two values and three children
- 4-nodes, three values and four children



2-3-4 Trees (cont)

2-3-4 trees are ordered similarly to BSTs



In a balanced 2-3-4 tree:

- all leaves are at same distance from the root

2-3-4 trees grow "upwards" from the leaves.

2-3-4 Trees (cont)

Possible 2-3-4 tree data structure:

```
typedef struct node {
   int          order;     // 2, 3 or 4
   int          data[3];   // items in node
   struct node *child[4];  // links to subtrees
} node;
```

2-3-4 Trees (cont)

Searching in 2-3-4 trees:

```
Search(tree,item):
|  Input  tree, item
|  Output address of item if found in 2-3-4 tree
|         NULL otherwise
|
|  if tree is empty then
|     return NULL
|  else
|  |  i=0
|  |  while i<tree.order-1 ∧ item>tree.data[i] do
|  |     i=i+1    // find relevant slot in data[]
|  |  end while
|  |  if item=tree.data[i] then    // item found
|  |     return address of tree.data[i]
|  |  else          // keep looking in relevant subtree
|  |     return Search(tree.child[i],item)
|  |  end if
|  end if
```

2-3-4 Trees (cont)

2-3-4 tree searching cost analysis:

- as for other trees, worst case determined by height *h*
- 2-3-4 trees are always balanced ⇒ height is *O(log n)*
- worst case for height: all nodes are 2-nodes
  same case as for balanced BSTs, i.e. $h \cong \log_2 n$
- best case for height: all nodes are 4-nodes
  balanced tree with branching factor 4, i.e. $h \cong \log_4 n$

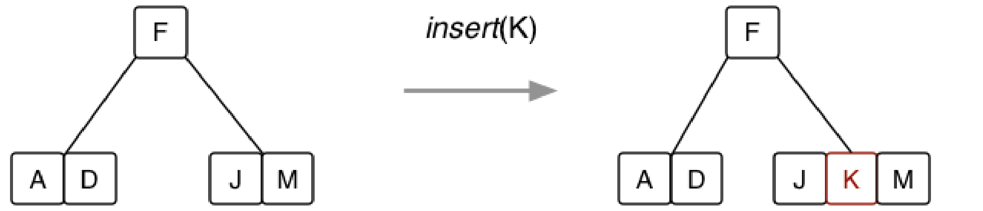
Insertion into 2-3-4 Trees

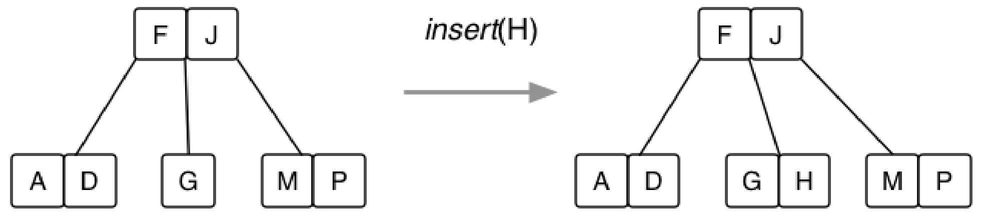
Insertion algorithm:

- find leaf node where Item belongs (via search)
- if not full (i.e. order < 4)
  - insert Item in this node, order++
- if node is full (i.e. contains 3 items)
  - split into two 2-nodes as leaves
  - promote middle element to parent
  - insert item into appropriate leaf 2-node
  - if parent is a 4-node
    - continue split/promote upwards
  - if promote to root, and root is a 4-node
    - split root node and add new root

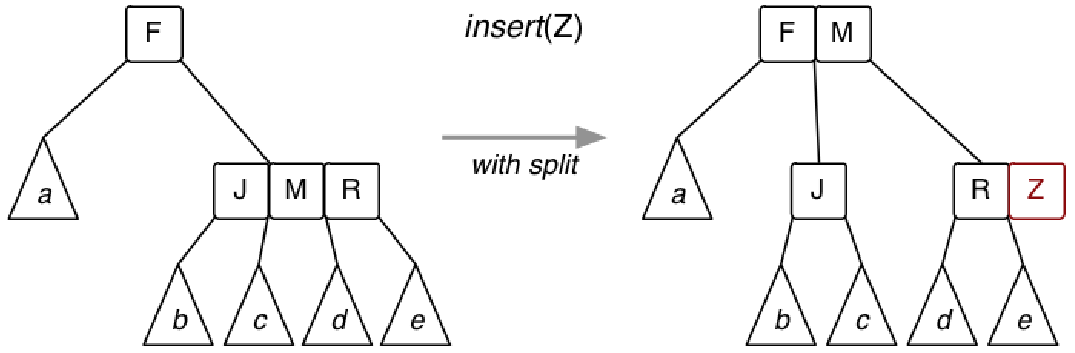
Insertion into 2-3-4 Trees (cont)

Insertion into a 2-node or 3-node:



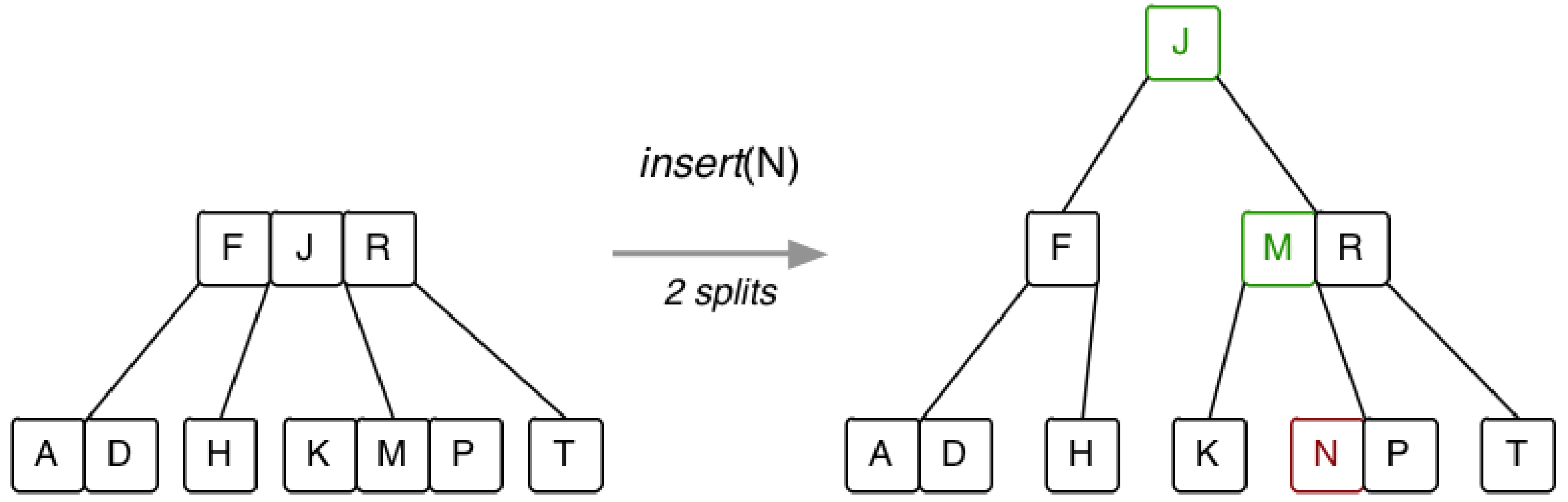


Insertion into a 4-node (requires a split):



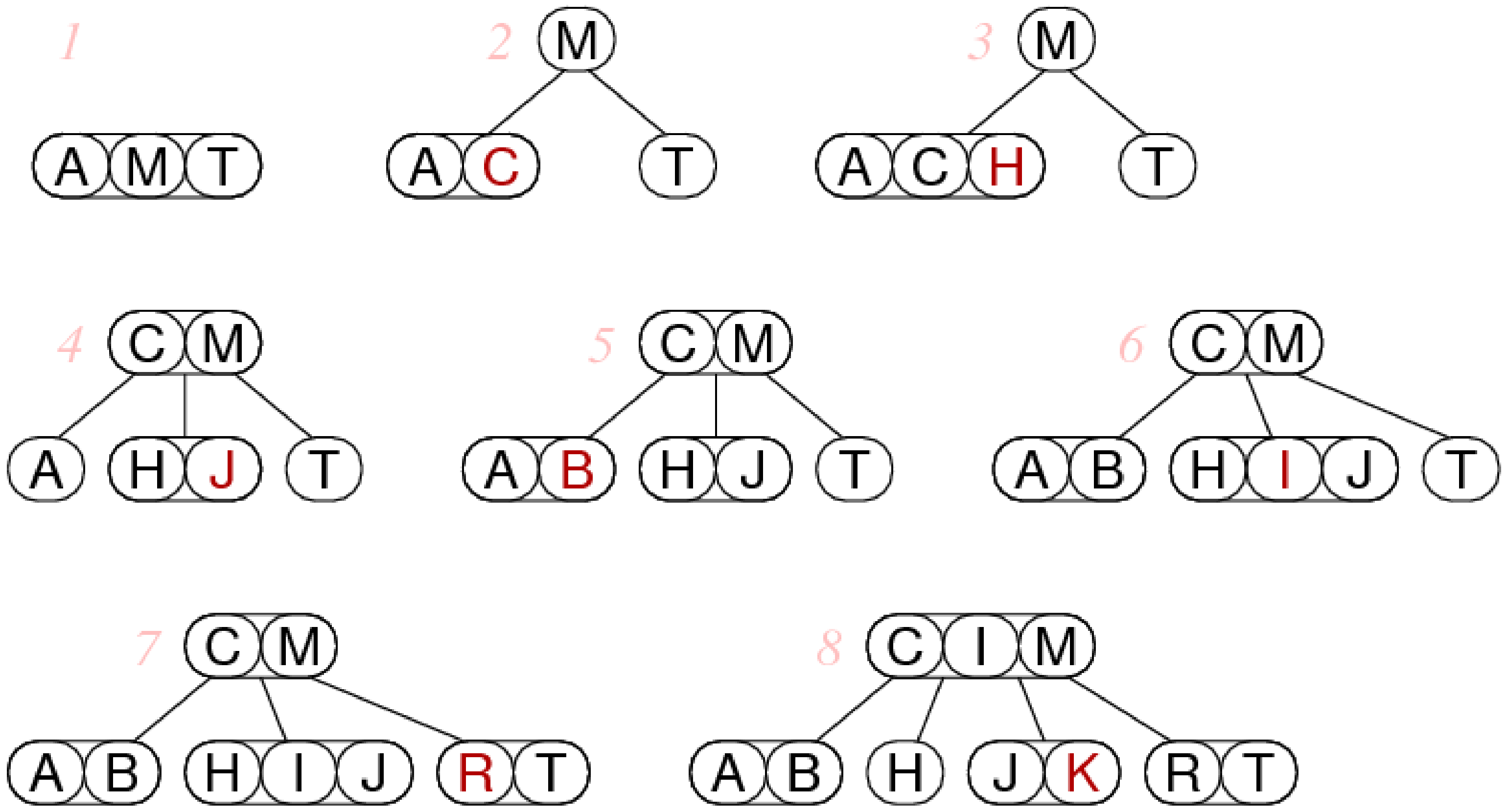
Insertion into 2-3-4 Trees (cont)

Splitting the root:



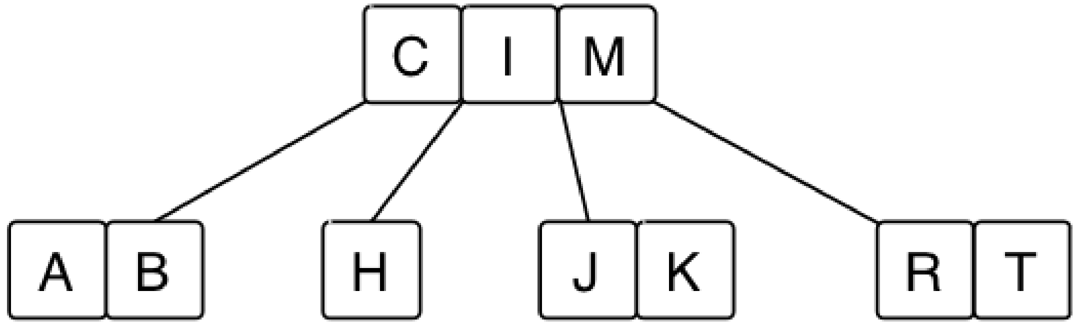
Insertion into 2-3-4 Trees (cont)

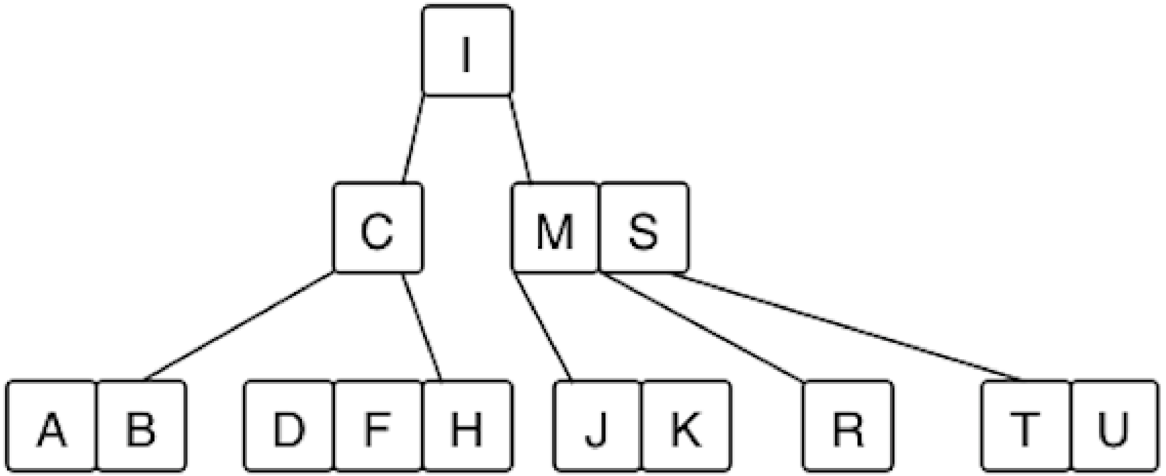
Building a 2-3-4 tree … 7 insertions:



Exercise #5: Insertion into 2-3-4 Tree

Show what happens when D, S, F, U are inserted into this tree:




I
C
M
S
A
B
D
F
H
J
K
R
T
U

Insertion into 2-3-4 Trees (cont)

Insertion algorithm:

```
insert(tree,item):
   Input  2-3-4 tree, item
   Output tree with item inserted

   if tree is empty then
      return new node containing item
   end if
   node=Search(tree,item)
   parent=parent of node
   if node.order<4 then
      insert item into node
      increment node.order
   else
      promote = node.data[1]      // middle value
      nodeL   = new node containing data[0]
      nodeR   = new node containing data[2]
      if item<node.data[1] then
         insert(nodeL,item)
      else
         insert(nodeR,item)
      end if
      insert(parent,promote)
      while parent.order=4 do
         continue promote/split upwards
      end while
      if parent is root ∧ parent.order=4 then
         split root, making new root
      end if
   end if
```

Insertion into 2-3-4 Trees (cont)

Variations on 2-3-4 trees …

Variation #1: why stop at 4? why not 2-3-4-5 trees? or *M*-way trees?

- allow nodes to hold up to *M-1* items, and at least *M/2*
- if each node is a disk-page, then we have a B-tree (databases)
- for B-trees, depending on `Item` size, *M > 100/200/400*

Variation #2: don't have "variable-sized" nodes

- use standard BST nodes, augmented with one extra piece of data
- implement similar strategy as 2-3-4 trees → red-black trees.

Red-Black Trees

Red-Black Trees

Red-black trees are a representation of 2-3-4 trees using BST nodes.

- each node needs one extra value to encode link type
- but we no longer have to deal with different kinds of nodes

Link types:

- red links … combine nodes to represent 3- and 4-nodes
- black links … analogous to "ordinary" BST links (child links)

Advantages:

- standard BST search procedure works unmodified
- get benefits of 2-3-4 tree self-balancing (although deeper)

Red-Black Trees

Definition of a red-black tree

- a BST in which each node is marked red or black
- no two red nodes appear consecutively on any path
- a red node corresponds to a 2-3-4 sibling of its parent
- a black node corresponds to a 2-3-4 child of its parent

Balanced red-black tree

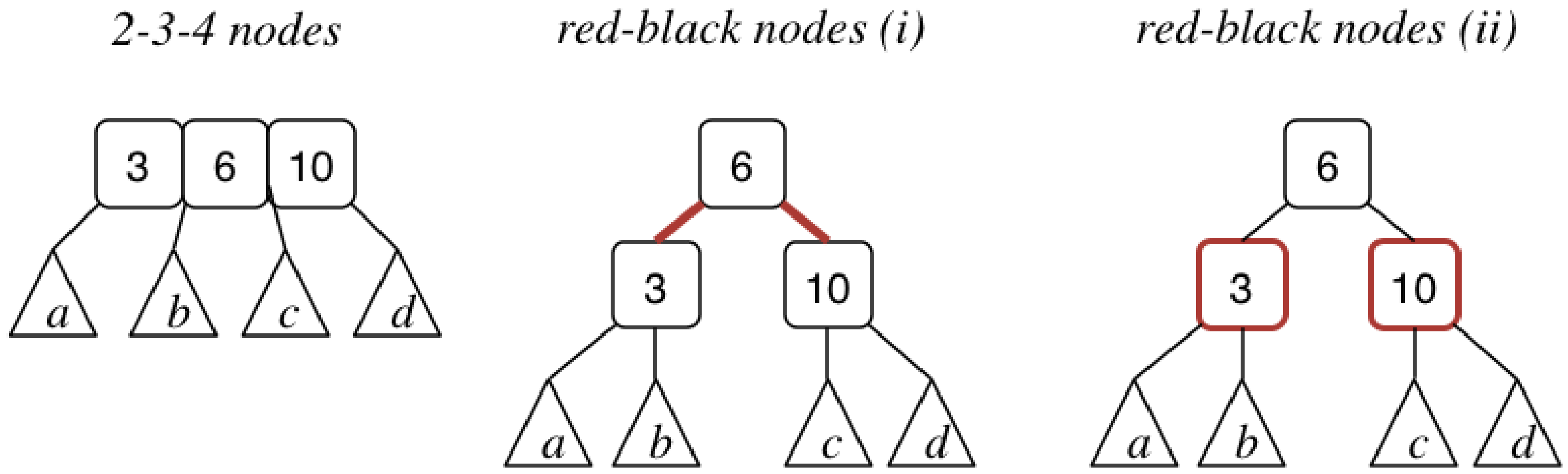
- all paths from root to leaf have same number of black nodes

Insertion algorithm: avoids worst case *O(n)* behaviour

Search algorithm: standard BST search

Red-Black Trees (cont)

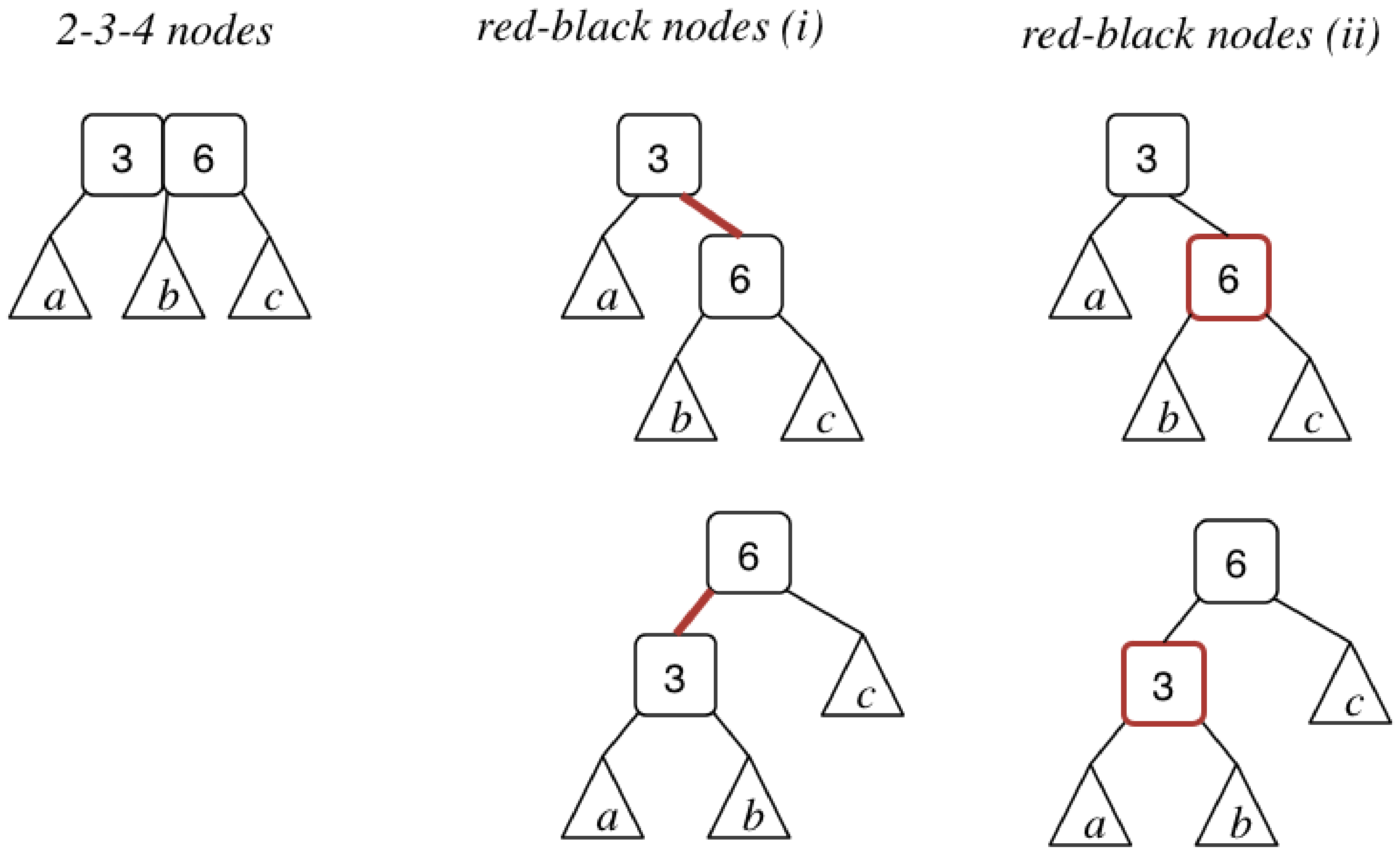
Representing 4-nodes in red-black trees:



Some texts colour the links rather than the nodes.

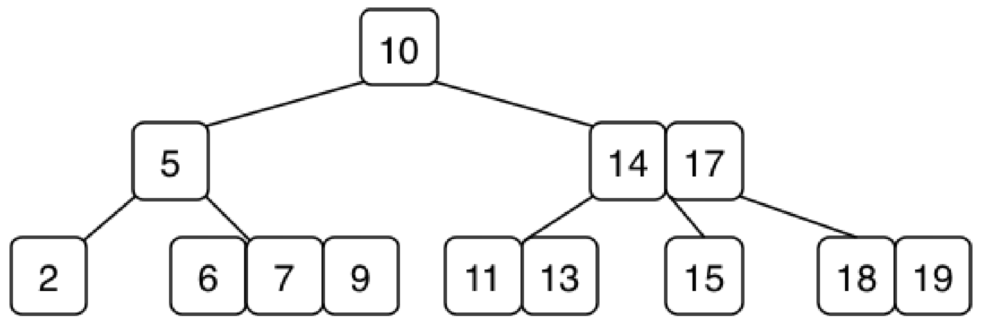
Red-Black Trees (cont)

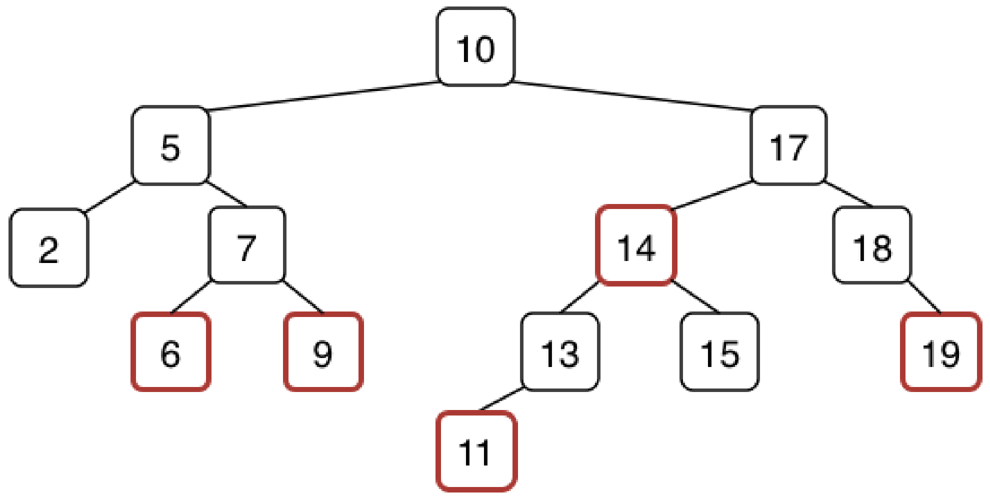
Representing 3-nodes in red-black trees (two possibilities):



Red-Black Trees (cont)

Equivalent trees (one 2-3-4, one red-black):





Red-Black Trees (cont)

Red-black tree implementation:

```
typedef enum {RED,BLACK} Colour;
typedef struct node *RBTree;
typedef struct node {
   int    data;     // actual data
   Colour colour;   // relationship to parent
   RBTree left;     // left subtree
   RBTree right;    // right subtree
} node;

#define colour(tree) ((tree)->colour)
#define isRed(tree)  ((tree) != NULL && (tree)->colour == RED)
```

RED = node is part of the same 2-3-4 node as its parent (sibling)

BLACK = node is a child of the 2-3-4 node containing the parent

Red-Black Trees (cont)

New nodes are always red:

```
RBTree newNode(Item it) {
   RBTree new = malloc(sizeof(Node));
   assert(new != NULL);
   data(new) = it;
   colour(new) = RED;
   left(new) = right(new) = NULL;
   return new;
}
```

Red-Black Trees (cont)

`Node.red` allows us to distinguish links

- black = parent node is a "real"parent
- red = parent node is a 2-3-4 neighbour

$$\begin{tikzcd}
& 2 \arrow[rr] \arrow[dl] & & 4 \arrow[dl] \arrow[dr] & \\
1 & & 3 & & 5
\end{tikzcd}
\qquad
\begin{tikzcd}
& 2 \arrow[dl] \arrow[dr] & & 4 \arrow[ll] \arrow[dr] & \\
1 & & 3 & & 5
\end{tikzcd}$$

$$\begin{tikzcd}
& 2 \arrow[dl] \arrow[dr] & & 4 \arrow[ll] \arrow[rr] & & 8 \arrow[dl] \arrow[dr] & \\
1 & & 3 & & 5 & & 12
\end{tikzcd}$$

Red-Black Trees (cont)

Search method is standard BST search:

```
SearchRedBlack(tree,item):
|   Input  tree, item
|   Output true if item found in red-black tree
|          false otherwise
|
|   if tree is empty then
|      return false
|   else if item<data(tree) then
|      return SearchRedBlack(left(tree),item)
|   else if item>data(tree) then
|      return SearchRedBlack(right(tree),item)
|   else             // found
|     return true
|   end if
```

Red-Black Tree Insertion

Insertion is more complex than for standard BSTs

- need to recall direction of last branch (L or R)
- need to recall whether parent link is red or black
- splitting/promoting implemented by rotateLeft/rotateRight
- several cases to consider depending on colour/direction combinations

Red-Black Tree Insertion (cont)

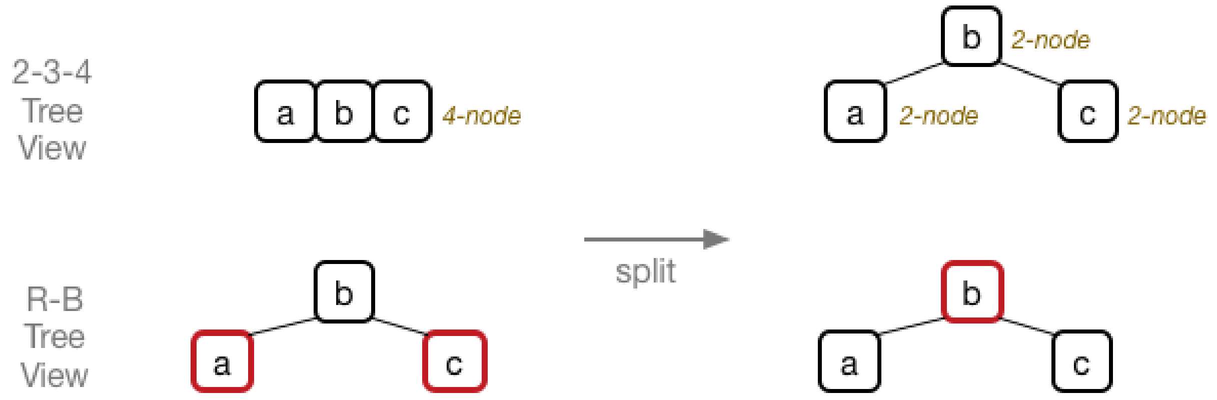
High-level description of insertion algorithm:

```
insertRB(tree,item,inRight):
|  Input  tree, item, inRight indicating direction of last branch
|  Output tree with it inserted
|
|  if tree is empty then
|     return newNode(item)
|  end if
|  if left(tree) and right(tree) both are RED then
|     split 4-node
|  end if
|  recursive insert cases (cf. regular BST)
|  re-arrange links/colours after insert
|  return modified tree

insertRedBlack(tree,item):
|  Input  red-black tree, item
|  Output tree with item inserted
|
|  tree=insertRB(tree,item,false)
|  colour(tree)=BLACK
|  return tree
```

Red-Black Tree Insertion (cont)

Splitting a 4-node, in a red-black tree:

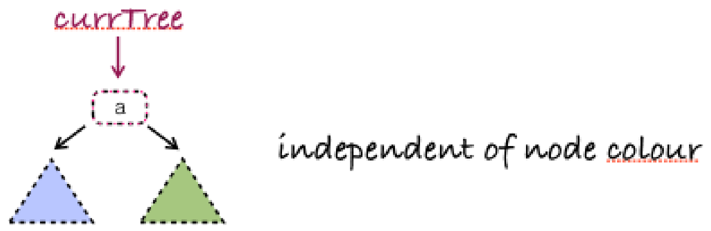


Algorithm:

```
if isRed(left(currentTree)) ∧ isRed(right(currentTree)) then
   colour(currentTree)=RED
   colour(left(currentTree))=BLACK
   colour(right(currentTree))=BLACK
end if
```

Red-Black Tree Insertion (cont)

Simple recursive insert (a la BST):



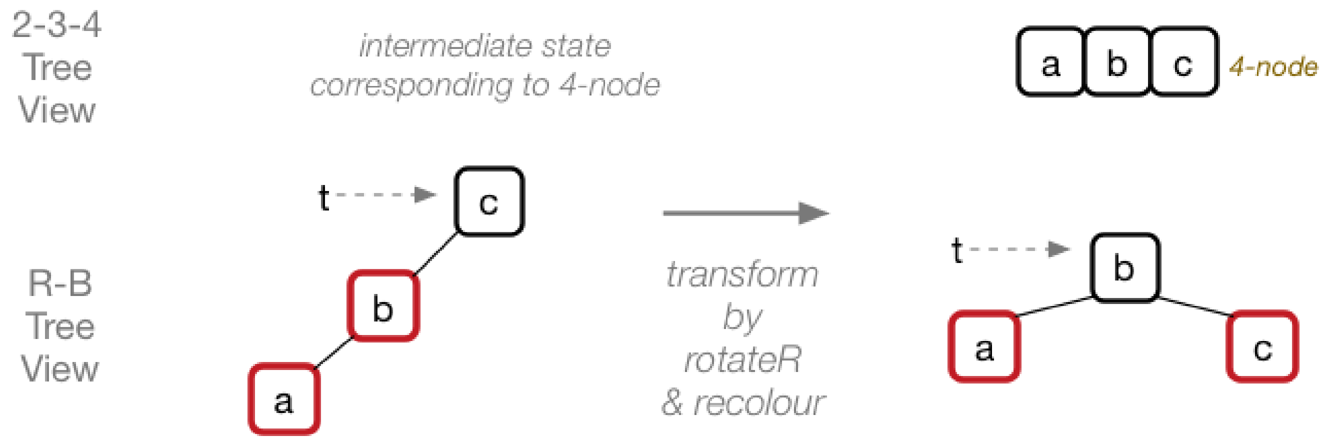
Algorithm:

```
if item<data(tree)) then
   left(tree)=insertRB(left(tree),item,false)
   re-arrange links/colours after insert
else             item larger than data in root
   right(tree)=insertRB(right(tree),item,true)
   re-arrange links/colours after insert
end if
```

Not affected by colour of **`tree`** node.

Red-Black Tree Insertion (cont)

Re-arrange after insert (1): two successive red links = newly-created 4-node

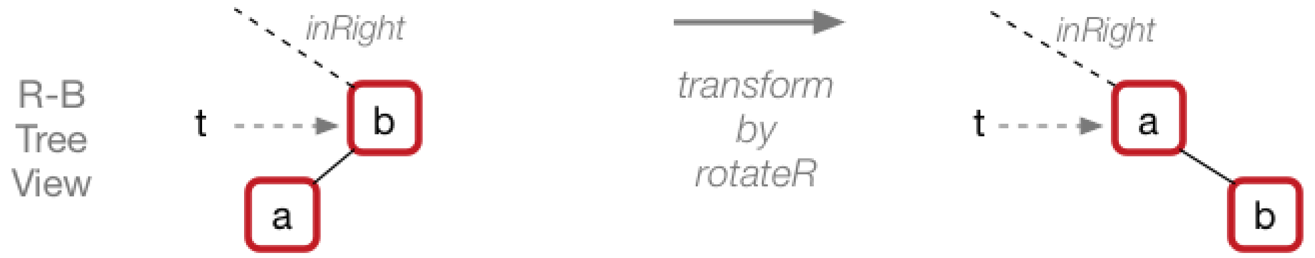

2-3-4 Tree View
intermediate state corresponding to 4-node
a
b
c
4-node
R-B Tree View
t
c
b
a
transform by rotateR & recolour
t
b
a
c

Algorithm:

```
if isRed(left(currentTree)) ∧ isRed(left(left(currentTree))) then
   currentTree=rotateRight(currentTree)
   colour(currentTree)=BLACK
   colour(right(currentTree))=RED
end if
```

Red-Black Tree Insertion (cont)

Re-arrange after insert (2): "normalise" direction of successive red links



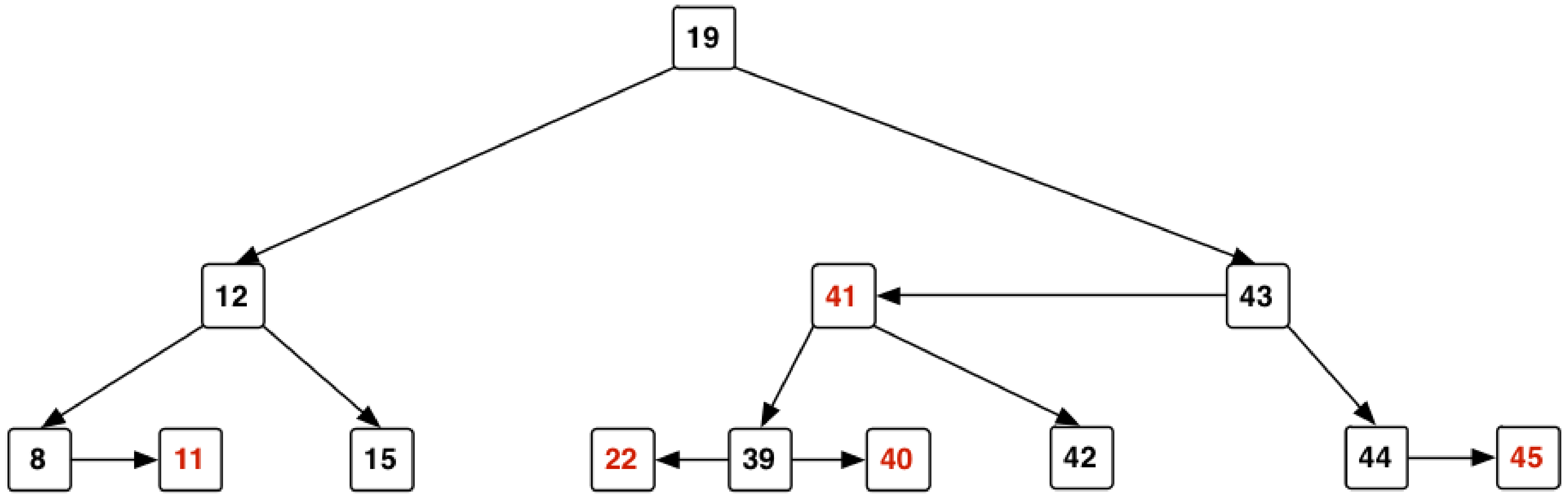
Algorithm:

```
if inRight ∧ isRed(currentTree) ∧ isRed(left(currentTree)) then
   currentTree=rotateRight(currentTree)
end if
```

Red-Black Tree Insertion (cont)

Example of insertion, starting from empty tree:

```
22, 12, 8, 15, 11, 19, 43, 44, 45, 42, 41, 40, 39
```



Red-black Tree Performance

Cost analysis for red-black trees:

- tree is well-balanced; worst case search is $O(\log_2 n)$
- insertion affects nodes down one path; max #rotations is $2 \cdot h$
  (where h is the height of the tree)

Only disadvantage is complexity of insertion/deletion code.

Note: red-black trees were popularised by Sedgewick.

Summary

- Random numbers
- Real balanced trees
  - Splay trees
  - AVL trees
  - 2-3-4 trees
  - Red-black trees

- Suggested reading:
  - Sedgewick, Ch.13.1-13.4