# Question 2

(a) Given the probability that the question $j$ is correctly answered by student $i$ is:

$$p_{ij} = \frac{\exp(\theta_i - \beta_j)}{1 + \exp(\theta_i - \beta_j)}$$

The log-likelihood for all students is derived as follows:

$$\begin{aligned}
\log p(\mathbf{C}|\boldsymbol{\theta}, \boldsymbol{\beta}) &= \sum_{i,j} (c_{ij} \log p_{ij} + (1 - c_{ij}) \log(1 - p_{ij})) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \left( c_{ij} \log \left( \frac{\exp(\theta_i - \beta_j)}{1 + \exp(\theta_i - \beta_j)} \right) + (1 - c_{ij}) \log \left( 1 - \frac{\exp(\theta_i - \beta_j)}{1 + \exp(\theta_i - \beta_j)} \right) \right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} (c_{ij}(\theta_i - \beta_j) - \log(1 + \exp(\theta_i - \beta_j))),
\end{aligned}$$

where $c_{ij}$ is the binary response of student $i$ to question $j$.

The log-likelihood with respect to $\theta_i$ is:

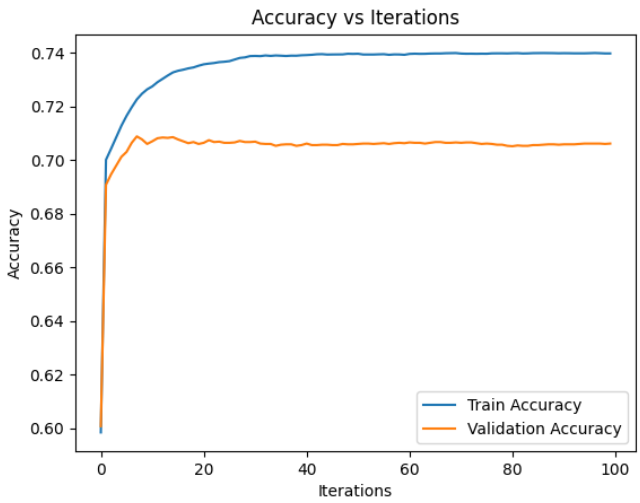
$$\begin{aligned}
\frac{\partial \log p(\mathbf{C}|\boldsymbol{\theta}, \boldsymbol{\beta})}{\partial \theta_i} &= \sum_{j=1}^{m} \left( c_{ij} - \frac{\exp(\theta_i - \beta_j)}{1 + \exp(\theta_i - \beta_j)} \right) \\
&= \sum_{j=1}^{m} (c_{ij} - p_{ij}).
\end{aligned}$$

The log-likelihood with respect to $\beta_j$ is:

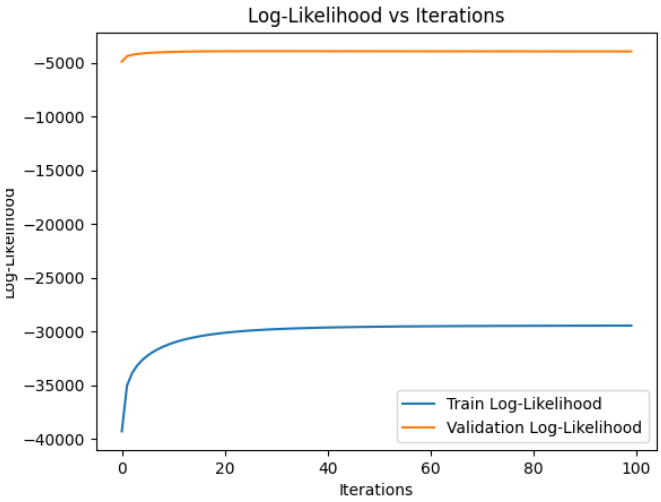
$$\begin{aligned}
\frac{\partial \log p(\mathbf{C}|\boldsymbol{\theta}, \boldsymbol{\beta})}{\partial \beta_j} &= \sum_{i=1}^{n} \left( c_{ij} - \frac{\exp(\theta_i - \beta_j)}{1 + \exp(\theta_i - \beta_j)} \right) \\
&= \sum_{i=1}^{n} (c_{ij} - p_{ij}).
\end{aligned}$$

(b) The hyperparameters I selected are: learning rate = 0.01 and iterations = 100.

The training and validation accuracies vs iterations are in the graph below:



The log-likelihoods vs iterations are in the graph below:

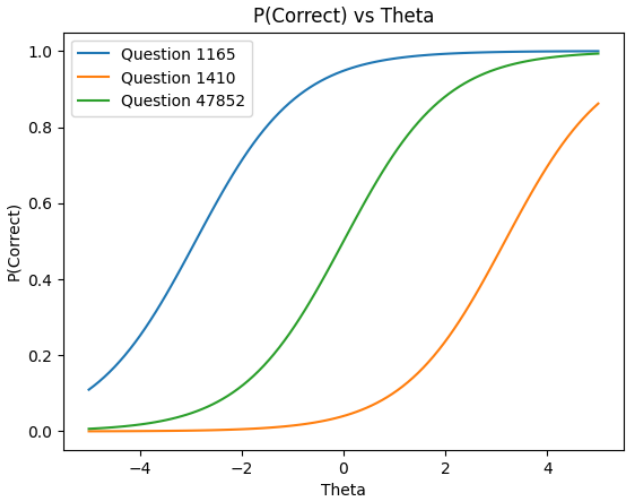



(c) The Final Validation Accuracy: 0.7063223257126728

The Final Test Accuracy: 0.707310189105278

(d) I select the lowest difficulty question $j_1$ (Question 1165), the highest difficulty question $j_2$ (Question 47852) and the average difficulty question $j_3$ (Question 1410).

The probability of the correct response is in the graph below:




The shape of the curves are like the sigmoid function as expected.

Fix a question $j$. As $\theta_i$ increases, the probability of the correct response $p_{ij}$ increases. This means if a student has a higher ability, the probability of the correct response increases.

Fix a student $i$. As $\beta_j$ increases, the probability of the correct response $p_{ij}$ decreases. This means if a question has a higher difficulty, the probability of the correct response decreases.