Laboratory Work #1
Equilibria, their classification, first integrals, the structure of the phase plane, basic bifurcations

General information about the first Laboratory Work

The purpose of this laboratory work is to get "hands-on" experience with hybrid mathematical-computational investigation techniques for basic models of nonlinear oscillators.

Every student should prepare a report of the laboratory work. The report should contain all programs, the system output (text plus figures), and answers to questions. All tasks (#1 to #8) must be solved by every student. Do not skip the tasks – solve then in the sequential order – because the results from the previous task will be required for solving the next tasks. Do not forget to give answers to questions and requests to discuss the results. Every student has to go through the tasks individually – working in groups is not promoted at this stage of the Course.

The list of tasks

#1. Given the model of the mathematical pendulum: $\frac{d^2x}{dt^2} + a\sin(x) = 0;\ a > 0$, construct a schematic diagram of the phase space (x-axis stands for x; y-axis stands for $\frac{dx}{dt}$). Write a computer code for plotting the diagram. Identify the separatrix, finite and infinite solutions. Do not integrate the differential equation – use the analytic expression of the first integral.

#2. Use a numerical integrator to solve the following initial value problem: $\frac{d^2x}{dt^2} + \sin(x) = 0;\ x(0) = 0;$ $\left.\frac{dx}{dt}\right|_{t=0} = 2$. Use the computer code from task #1 and plot the integrated trajectory on top of the schematic diagram. Comment the results. Investigate the impact of the integration step to the results. Fix the time step and try to modify both initial conditions in such a way that the computationally reconstructed trajectory would directly hit the saddle point. Comment the results.

#3. Measure (using computational tools) the period of finite and infinite solutions to the oscillator $\frac{d^2x}{dt^2} + \sin(x) = 0$. Compare the results to the period of solutions to the linear oscillator $\frac{d^2x}{dt^2} + x = 0$. Keep in mind that the system is Hamiltonian (without damping) – so if you start the integration process at $x(0) = 0;$ $\left.\frac{dx}{dt}\right|_{t=0} = A$, you will not return to the point $(0; A)$ in the phase plane due to the inevitable accumulation of integration errors. The recommendation is to set up a circle (the radius of the circle should be small) around the point $(0; A)$ and wait until the integrated trajectory crosses this point after the first period of oscillation. Comment the results. Discuss how to assess the period of infinite solutions. Plot the graph of the measured period where the x-axis stands for A and the y-axis stands for the period of oscillation.

#4. Construct the system of separatrix for the damped oscillator $\frac{d^2x}{dt^2} + h\frac{dx}{dt} + a\sin(x) = 0;\ h > 0;\ a > 0$. Use time forward and time backward integration from the surroundings of unstable saddle points for the construction of the map of separatrix (use the sample program available in Moodle). Try to vary parameters h and a. Determine when stable equilibria changes from a stable spiral to a stable node. Investigate the limit when h tends to zero. Comment the results.

#5. Construct the schematic diagram of the phase space for the oscillator $\frac{d^2x}{dt^2} + a\sin(x) - \omega^2\sin(x)\cos(x) = 0;\ a > 0$ before and after the pitchfork bifurcation. Do not integrate the differential equation – use the analytic expression of the first integral. Comment the results.

#6. Construct the system of separatrix for the damped oscillator $\frac{d^2x}{dt^2} + h\frac{dx}{dt} + a\sin(x) - \omega^2\sin(x)\cos(x) = 0;\ h > 0;\ a > 0$ before and after the pitchfork bifurcation. Use time forward and time backward integration from the surroundings of unstable saddle points for the construction of the map of separatrix. Comment the results.

#7. Construct the schematic diagram of the phase space for the oscillator $\frac{d^2x}{dt^2} + a\sin(x) = b; a > 0; b > 0,$ before and after the saddle-node bifurcation. Do not integrate the differential equation – use the analytic expression of the first integral. Comment the results.

#8. The final task. This is a difficult problem. The first part is simple - construct the system of separatrix for the damped oscillator $\frac{d^2x}{dt^2} + h\frac{dx}{dt} + a\sin(x) = b; h > 0; a > 0; b > 0$. Use time forward and time backward integration from the surroundings of unstable saddle points for the construction of the map of separatrix. Vary parameters h, a and b. Is there any chance you could observe some other type of bifurcation than the saddle-node bifurcation? How could you describe this bifurcation? (Those of you who will be able to give a sensible description of this new bifurcation will get extra points in the cumulative assessment of the course).