Introduction to Formal Methods
Chapter 09: SAT-Based Bounded Model Checking

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Outline

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SAT-based Bounded Model Checking

- Key problems with BDD's:
  - they can explode in space
  - an expert user can make the difference (e.g. reordering, algorithms)
- A possible alternative:
  - Propositional Satisfiability Checking (SAT)
  - SAT technology is very advanced
- Advantages:
  - reduced memory requirements
  - limited sensitivity: one good setting, does not require expert users
  - much higher capacity (more variables) than BDD based techniques

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SAT-based Bounded Model Checking [cont.]

Key ideas:

- look for counter-example paths of increasing length k
  $\Longrightarrow$ oriented to finding bugs
- for each k, builds a Boolean formula that is satisfiable iff there is a counter-example of length k
  - can be expressed using $k \cdot |\mathbf{s}|$ variables
  - formula construction is not subject to state explosion
- satisfiability of the Boolean formulas is checked using a SAT procedure
  - can manage complex formulae on several 100K variables
  - returns satisfying assignment (i.e., a counter-example)

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Outline

DPLL

- Davis-Putnam-Longeman-Loveland procedure (DPLL)
- Tries to build recursively an assignment μ satisfying φ;
- At each recursive step assigns a truth value to (all instances of) one atom.
- Performs deterministic choices first.

DPLL Algorithm

```
function DPLL(φ, μ)
    if φ = ⊤                                       /* base      */
        then return True;
    if φ = ⊥                                       /* backtrack */
        then return False;
    if {a unit clause (l) occurs in φ}             /* unit      */
        then return DPLL(assign(l, φ), μ ∧ l);
    (...)
    l := choose-literal(φ);                        /* split     */
    return  DPLL(assign(l, φ), μ ∧ l)  or
            DPLL(assign(¬l, φ), μ ∧ ¬l);
```

"Classic" chronological backtracking

- variable assignments (literals) stored in a stack
- each variable assignments labeled as "unit", "open", "closed"
- when a conflict is encountered, the stack is popped up to the most recent open assignment l
- l is toggled, is labeled as "closed", and the search proceeds.

Classic chronological backtracking – example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
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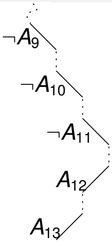
...



$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
(initial assignment)

Classic chronological backtracking – example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
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$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ $\surd$
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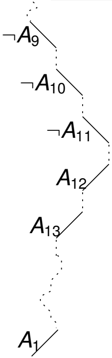
...



$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., A_1\}$
... (branch on A_1)

Classic chronological backtracking – example

$c_1 : \neg A_1 \vee A_2$ $\surd$
$c_2 : \neg A_1 \vee A_3 \vee A_9$ $\surd$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
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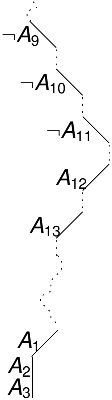

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., A_1, A_2, A_3\}$
(unit A_2, A_3)

Classic chronological backtracking – example

$c_1 : \neg A_1 \vee A_2$ $\surd$

$c_2 : \neg A_1 \vee A_3 \vee A_9$ $\surd$

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ $\surd$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

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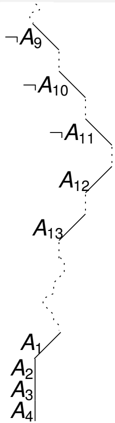
...



$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., A_1, A_2, A_3, A_4\}$
(unit A_4)

Classic chronological backtracking – example

$c_1 : \neg A_1 \lor A_2$ $\surd$

$c_2 : \neg A_1 \lor A_3 \lor A_9$ $\surd$

$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$ $\surd$

$c_4 : \neg A_4 \lor A_5 \lor A_{10}$ $\surd$

$c_5 : \neg A_4 \lor A_6 \lor A_{11}$ $\surd$

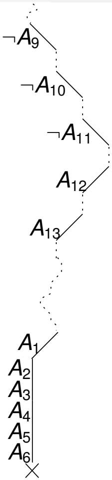
$c_6 : \neg A_5 \lor \neg A_6$ $\times$

$c_7 : A_1 \lor A_7 \lor \neg A_{12}$ $\surd$

$c_8 : A_1 \lor A_8$ $\surd$

$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$

...



$\{..., \neg A_9, \neg A_{10}, \neg A_{1 \neg A_4 1}, A_{12}, A_{13}, ..., A_1, A_2, A_3, A_4, A_5, A_6\}$

(unit A_5, A_6)$\Longrightarrow$ conflict

Classic chronological backtracking – example

$c_1 : \neg A_1 \vee A_2$
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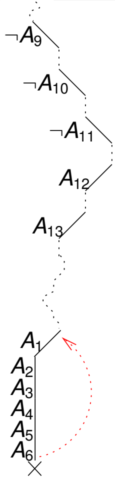

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
$\Longrightarrow$ backtrack up to A_1

Classic chronological backtracking – example

$c_1 : \neg A_1 \vee A_2$ $\surd$

$c_2 : \neg A_1 \vee A_3 \vee A_9$ $\surd$

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

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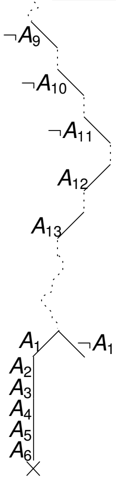

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., \neg A_1\}$
(unit $\neg A_1$)

Classic chronological backtracking – example

$c_1 : \neg A_1 \lor A_2$ $\surd$

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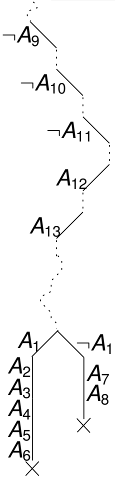

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., \neg A_1, A_7, A_8\}$

(unit A_7, A_8) $\Longrightarrow$ conflict

Classic chronological backtracking – example

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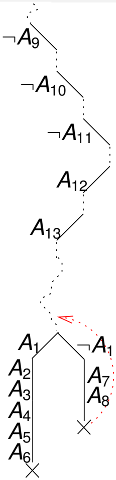

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
$\Longrightarrow$ backtrack to the most recent open branching point

Classic chronological backtracking – example

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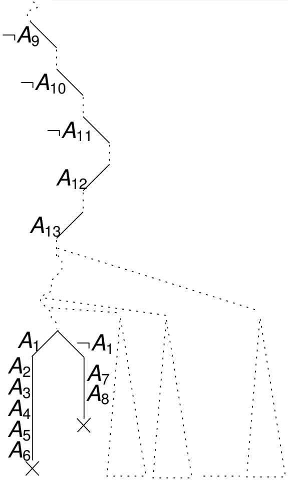

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$

$\Longrightarrow$ lots of useless search before backtracking up to A_{13}!

Classic chronological backtracking: drawbacks

- often the branch heuristic delays the “right” choice
- chronological backtracking always backtracks to the most recent branching point, even though a higher backtrack could be possible
  $\Longrightarrow$ lots of useless search!

Modern DPLL implementations [Silva & Sakallah '96, Moskewicz et al. '01]

Conflict-Driven Clause-Learning (CDCL) DPLL solvers:

- Non-recursive: stack-based representation of data structures
- Efficient data structures for doing and undoing assignments
- Perform conflict-driven backtracking (backjumping) and learning
- May perform search restarts
- Reason on total assignments

Dramatically efficient: solve industrial-derived problems with $\approx 10^7$ Boolean variables and $\approx 10^7 - 10^8$ clauses

Conflict-directed backtracking (backjumping) and learning

- Idea: when a branch μ fails,
  (i) conflict analysis: reveal the sub-assignment $\eta \subseteq \mu$ causing the failure (conflict set η):
    - find $\eta \subseteq \mu$ by generating the conflict clause $C \stackrel{\text{def}}{=} \neg\eta$ via resolution from the falsified clause (e.g. , via the 1^{st}UIP strategy)

  (ii) learning: add the conflict clause C to the clause set
  (iii) backjumping: backtrack to the highest branching point s.t. the stack contains all-but-one literals in η, and then unit-propagate the unassigned literal on C
- may jump back up much more than one decision level in the stack $\Longrightarrow$ may avoid lots of redundant search!!.

State-of-the-art backjumping and learning: intuitions

- Backjumping: climb up to many decision levels in the stack
  - intuition: " go back to the oldest decision where you'd have done something different if only you had known C"

  $\Longrightarrow$ may avoid lots of redundant search
- Learning: in future branches, when all-but-one literals in η are assigned, the remaining literal is assigned to false by unit-propagation:
  - intuition: "when you're about to repeat the mistake, do the opposite of the last step"

  $\Longrightarrow$ avoid finding the same conflict again

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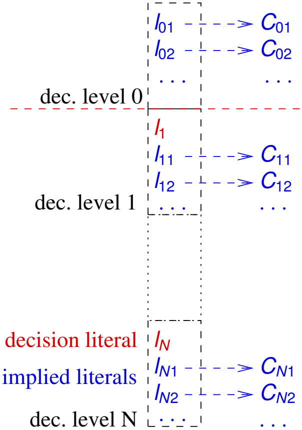
Stack-based representation of a truth assignment μ

- stack partitioned into decision levels:
  - one decision literal
  - its implied literals
  - each implied literal tagged with the clause causing its unit-propagation (antecedent clause)
- equivalent to an implication graph:
  - a node without incoming edges represent a decision literal
  - the graph contains $l_1 \xrightarrow{c} l, ..., l_n \xrightarrow{c} l$ iff $c \stackrel{\text{def}}{=} \bigvee_{j=1}^{n} \neg l_i \vee l$ is the antecedent clause of l

representation of the dependencies between literals in μ



Implication graph - example

c_1 :	$\neg A_1 \vee A_2$	$\surd$
c_2 :	$\neg A_1 \vee A_3 \vee A_9$	$\surd$
c_3 :	$\neg A_2 \vee \neg A_3 \vee A_4$	$\surd$
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c_5 :	$\neg A_4 \vee A_6 \vee A_{11}$	$\surd$
c_6 :	$\neg A_5 \vee \neg A_6$	×
c_7 :	$A_1 \vee A_7 \vee \neg A_{12}$	$\surd$
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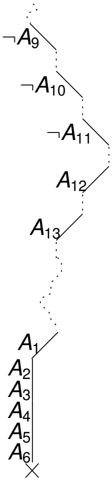
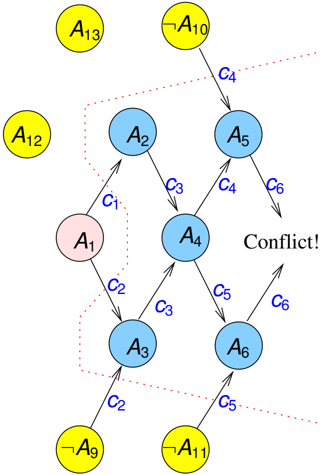
...





Building a conflict set/clause by resolution

1. $C :=$ conflicting clause
2. repeat
   (i) resolve current clause C with the antecedent clause of the last unit-propagated literal l in C

   until C verifies some given termination criteria
   (e.g., until C contains only decision literals)

$$\dfrac{\neg A_1 \vee A_2 \qquad \dfrac{\neg A_1 \vee A_3 \vee A_9 \qquad \dfrac{\neg A_2 \vee \neg A_3 \vee A_4 \qquad \dfrac{\neg A_4 \vee A_5 \vee A_{10} \qquad \dfrac{\neg A_4 \vee A_6 \vee A_{11} \qquad \overbrace{\neg A_5 \vee \neg A_6}^{\textit{Conflicting cl.}}}{\neg A_4 \vee \neg A_5 \vee A_{11}}\ (A_6)}{\neg A_4 \vee A_{10} \vee A_{11}}\ (A_5)}{\neg A_2 \vee \neg A_3 \vee A_{10} \vee A_{11}}\ (A_4)}{\neg A_2 \vee \neg A_1 \vee A_9 \vee A_{10} \vee A_{11}}\ (A_3)}{\neg A_1 \vee A_9 \vee A_{10} \vee A_{11}}\ (A_2)$$

Idea: “Undo” unit-propagations.

Building a conflict set/clause by resolution

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Idea: “Undo” unit-propagations.

State-of-the-art in backjumping & learning

First Unique Implication Point (1st UIP) strategy:

- corresponds to consider the first clause encountered containing one literal of the current level (1st UIP).

$$\frac{\neg A_4 \vee A_5 \vee A_{10} \qquad \dfrac{\neg A_4 \vee A_6 \vee A_{11} \qquad \overbrace{\neg A_5 \vee \neg A_6}^{Conflicting\ cl.}}{\neg A_4 \vee \neg A_5 \vee A_{11}}\ (A_6)}{\underbrace{\neg A_4}_{1st\ UIP} \vee A_{10} \vee A_{11}}\ (A_5)$$

1st UIP strategy – example

$c_1 : \neg A_1 \vee A_2$ ✓

$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓

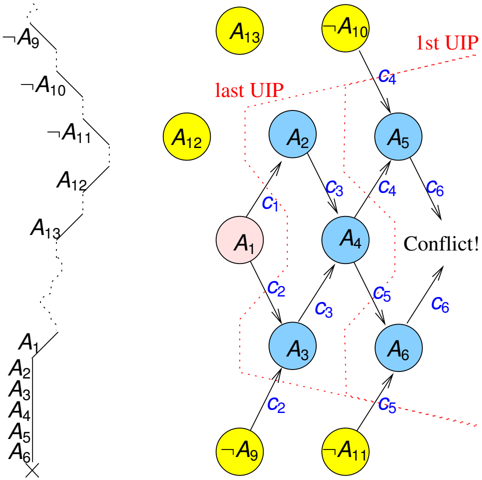
$c_6 : \neg A_5 \vee \neg A_6$ ×

$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

$c_8 : A_1 \vee A_8$ ✓

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

...



$\Longrightarrow$ Conflict set: $\{\neg A_{10}, \neg A_{11}, A_4\}$, learn $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$

1st UIP strategy and backjumping

- The added conflict clause states the reason for the conflict
- The procedure backtracks to the most recent decision level of the variables in the conflict clause which are not the UIP.
- then the conflict clause forces the negation of the UIP by unit propagation.

E.g.: $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$
$\Longrightarrow$ backtrack to A_{11}, then assign $\neg A_4$

1st UIP strategy – example (7)

$c_1 : \neg A_1 \lor A_2$ ✓

$c_2 : \neg A_1 \lor A_3 \lor A_9$ ✓

$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$ ✓

$c_4 : \neg A_4 \lor A_5 \lor A_{10}$ ✓

$c_5 : \neg A_4 \lor A_6 \lor A_{11}$ ✓

$c_6 : \neg A_5 \lor \neg A_6$ ×

$c_7 : A_1 \lor A_7 \lor \neg A_{12}$ ✓

$c_8 : A_1 \lor A_8$ ✓

$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$

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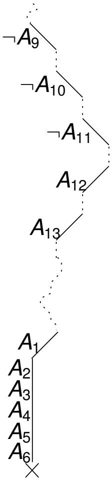

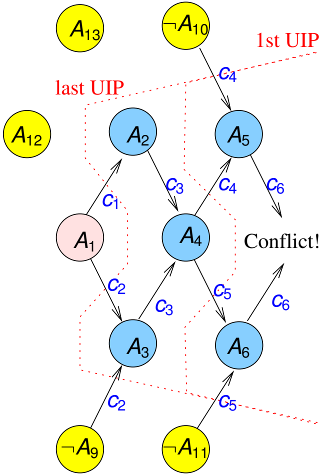

$\Longrightarrow$ Conflict set: $\{\neg A_{10}, \neg A_{11}, A_4\}$, learn $c_{10} := A_{10} \lor A_{11} \lor \neg A_4$

1st UIP strategy – example (8)

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
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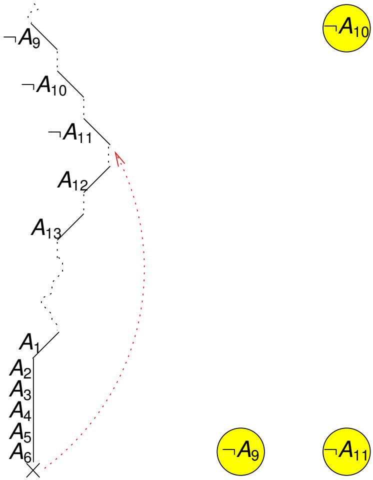

$\Longrightarrow$ backtrack up to $A_{11} \Longrightarrow \{..., \neg A_9, \neg A_{10}, \neg A_{11}\}$

1st UIP strategy – example (9)

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ $\surd$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ $\surd$
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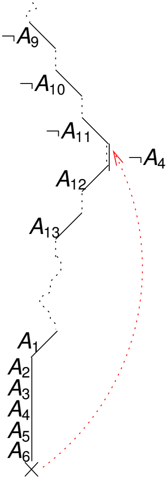

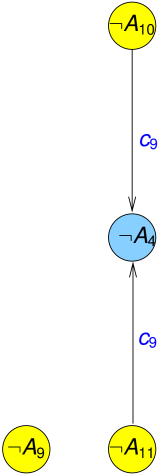

$\Longrightarrow$ unit propagate $\neg A_4 \Longrightarrow \{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_4\}$...

Learning – example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
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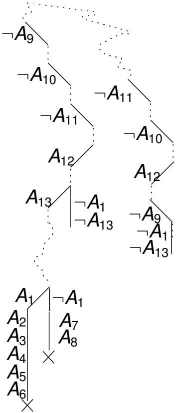
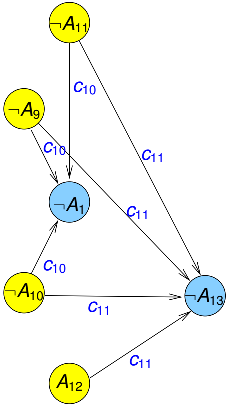
$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ $\surd$
$c_{11} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$ $\surd$
...
$\Longrightarrow$ Unit: $\{\neg A_1, \neg A_{13}\}$





Remark: the “quality” of conflict sets

- Different ideas of “good” conflict set
  - Backjumping: if causes the highest backjump (“local” role)
  - Learning: if causes the maximum pruning (“global” role)
- Many different strategies implemented

Drawbacks of Learning

- Prunes drastically the search.
- Problem: may cause a blowup in space

$\Longrightarrow$ techniques to drop learned clauses when necessary
  - according to their size
  - according to their activity.

Definition

A clause is currently active if it occurs in the current implication graph (i.e., it is the antecedent clause of a literal in the current assignment).

Property

In order to guarantee correctness, completeness & termination of a CDCL solver, it suffices to keep each clause until it is active.
$\Longrightarrow$ CDCL solvers require polynomial space

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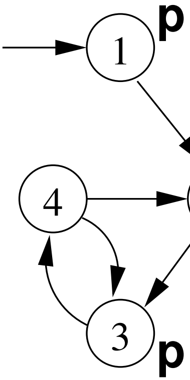
Many applications of SAT Solvers

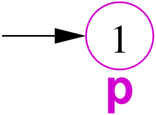
- Many successful applications of SAT:
  - Boolean circuits
  - (Bounded) Planning
  - (Bounded) Model Checking
  - Cryptography
  - Scheduling
  - ...
- All NP-complete problem can be (polynomially) converted to SAT.
- Key issue: find an efficient encoding.

Outline

Bounded Model Checking: Example

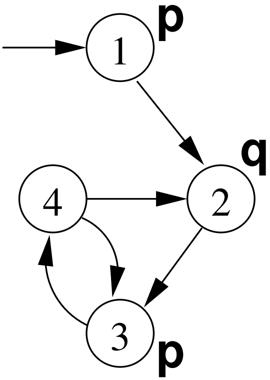


- LTL Formula: $\mathbf{G}(p \rightarrow \mathbf{F}q)$
- Negated Formula (violation): $\mathbf{F}(p \wedge \mathbf{G}\neg q)$
- $k = 0$:

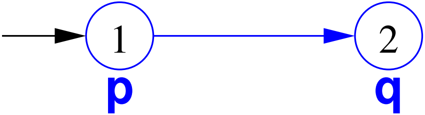


- No counter-example found.

Bounded Model Checking: Example



- LTL Formula: $\mathbf{G}(p \rightarrow \mathbf{F}q)$
- Negated Formula (violation): $\mathbf{F}(p \wedge \mathbf{G}\neg q)$
- $k = 1$:



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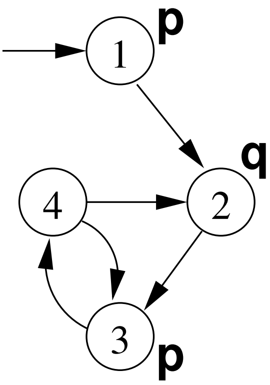
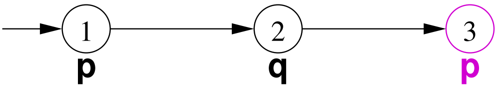


- LTL Formula: $\mathbf{G}(p \rightarrow \mathbf{F}q)$
- Negated Formula (violation): $\mathbf{F}(p \wedge \mathbf{G}\neg q)$
- $k = 2$:



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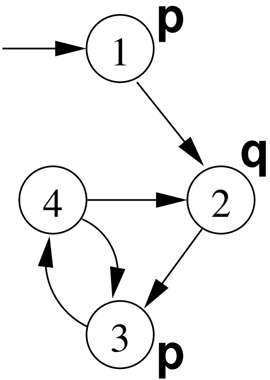
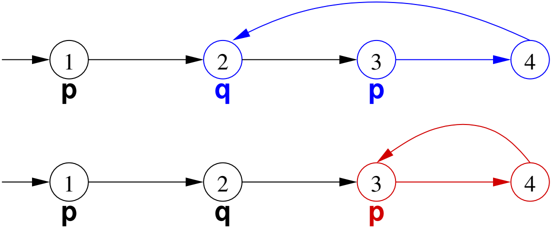
Bounded Model Checking: Example



- LTL Formula: $\mathbf{G}(p \rightarrow \mathbf{F}q)$
- Negated Formula (violation): $\mathbf{F}(p \wedge \mathbf{G}\neg q)$
- $k = 3$:



- The 2nd trace is a counter-example!

Outline

The problem [Biere et al, 1999]

Ingredients:

- A system written as a Kripke structure $M := \langle S, I, T, \mathcal{L} \rangle$
- A property f written as a LTL formula:
- an integer $k \geq 0$ (bound)

Problem

Is there a (possibly-partial) execution path π of M of length k satisfying the temporal property f?

- the check is repeated for increasing values of $k = 1, 2, 3, ...$

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The encoding

Equivalent to the satisfiability problem of a Boolean formula $[[M, f]]_k$ defined as follows:

$$[[M, f]]_k := [[M]]_k \wedge [[f]]_k \quad (1)$$

$$[[M]]_k := I(s^0) \wedge \bigwedge_{i=0}^{k-1} R(s^i, s^{i+1}), \quad (2)$$

$$[[f]]_k := (\neg \bigvee_{l=0}^{k} R(s^k, s^l) \wedge [[f]]_k^0) \vee \bigvee_{l=0}^{k} (R(s^k, s^l) \wedge {}_l[[f]]_k^0), \quad (3)$$

- the vector s of propositional variables is replicated k+1 times $s^0, s^1, ..., s^k$
- $[\![M]\!]_k$ encodes the fact that the k-path is an execution of M
- $[\![f]\!]_k$ encodes the fact that the k-path satisfies f

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The Encoding [cont.]

In general, the encoding for a formula f with k steps, $[[f]]_k$ is the disjunction of

- the constraints needed to express a model without loopback: $(\neg(\bigvee_{l=0}^{k} R(s^k, s^l)) \wedge [[f]]_k^0)$
  - $[[f]]_k^i$, $i \in [0, k]$: encodes the fact that f holds in s^i under the assumption that $s^0, ..., s^k$ is a no-loopback path
- the constraints needed to express a given loopback, for all possible points of loopback: $\bigvee_{l=0}^{k}(R(s^k, s^l) \wedge {}_l[[f]]_k^0)$
  - ${}_l[[f]]_k^0$, $i \in [0, k]$: encodes the fact that f holds in s^i under the assumption that $s^0, ..., s^k$ is a path with a loopback from s^k to s^l

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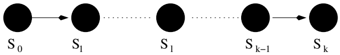

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$$\bigvee_{l=0}^{k}(R(s^k, s^l) \wedge \ _l[[f]]_k^0)$$

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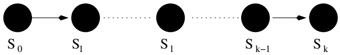

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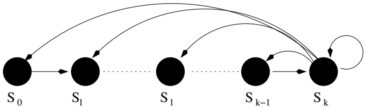

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The encoding of $[[f]]_k^i$ and ${}_l[[f]]_k^i$

f	$[[f]]_k^i$	${}_l[[f]]_k^i$
p	p_i	p_i
$\neg p$	$\neg p_i$	$\neg p_i$
$h \wedge g$	$[[h]]_k^i \wedge [[g]]_k^i$	${}_l[[h]]_k^i \wedge {}_l[[g]]_k^i$
$h \vee g$	$[[h]]_k^i \vee [[g]]_k^i$	${}_l[[h]]_k^i \vee {}_l[[g]]_k^i$
$\mathbf{X}g$	$\begin{array}{ll} [[g]]_k^{i+1} & \textit{if } i < k \\ \bot & \textit{otherwise.} \end{array}$	$\begin{array}{ll} {}_l[[g]]_k^{i+1} & \textit{if } i < k \\ {}_l[[g]]_k^{l} & \textit{otherwise.} \end{array}$
$\mathbf{G}g$	$\bot$	$\bigwedge_{j=min(i,l)}^{k} {}_l[[g]]_k^j$
$\mathbf{F}g$	$\bigvee_{j=i}^{k} [[g]]_k^j$	$\bigvee_{j=min(i,l)}^{k} {}_l[[g]]_k^j$
$h\mathbf{U}g$	$\bigvee_{j=i}^{k} \left([[g]]_k^j \wedge \bigwedge_{n=i}^{j-1} [[h]]_k^n \right)$	$\bigvee_{j=i}^{k} \left({}_l[[g]]_k^j \wedge \bigwedge_{n=i}^{j-1} {}_l[[h]]_k^n \right) \vee$ $\bigvee_{j=l}^{i-1} \left({}_l[[g]]_k^j \wedge \bigwedge_{n=i}^{k} {}_l[[h]]_k^n \wedge \bigwedge_{n=l}^{j-1} {}_l[[h]]_k^n \right)$
$h\mathbf{R}g$	$\bigvee_{j=i}^{k} \left([[h]]_k^j \wedge \bigwedge_{n=i}^{j} [[g]]_k^n \right)$	$\bigwedge_{j=min(i,l)}^{k} {}_l[[g]]_k^j \vee$ $\bigvee_{j=i}^{k} \left({}_l[[h]]_k^j \wedge \bigwedge_{n=i}^{j} {}_l[[g]]_k^n \right) \vee$ $\bigvee_{j=l}^{i-1} \left({}_l[[h]]_k^j \wedge \bigwedge_{n=i}^{k} {}_l[[g]]_k^n \wedge \bigwedge_{n=l}^{j} {}_l[[g]]_k^n \right)$

Example: **F**p (reachability)

- $f := \mathbf{F}p$, s.t. p Boolean:
  is there a reachable state in which p holds?
- a finite path can show that the property holds
- $[[M, f]]_k$ is:

$$I(s^0) \wedge \bigwedge_{i=0}^{k-1} R(s^i, s^{i+1}) \wedge \bigvee_{j=0}^{k} p^j$$

Note

if done for increasing value of k, then it suffices that $[[M, f]]_k$ is:

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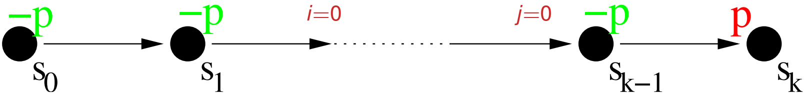

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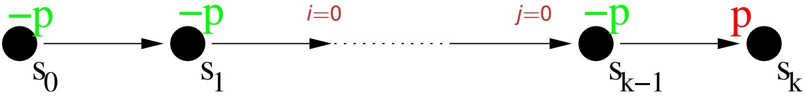
$$I(s^0) \wedge \bigwedge_{i=0}^{k-1} R(s^i, s^{i+1}) \wedge \bigvee_{j=0}^{k} p^j$$



Note

if done for increasing value of k, then it suffices that $[[M, f]]_k$ is:

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Example: $\mathbf{G}p$

- $f := \mathbf{G}p$, s.t. p Boolean: is there a path where p holds forever?
- We need to produce an infinite behaviour, with a finite number of transitions
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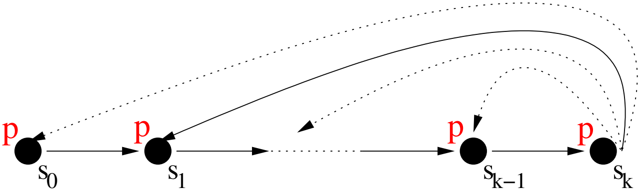

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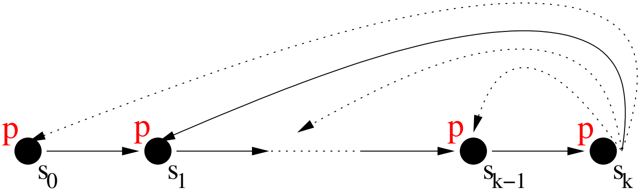

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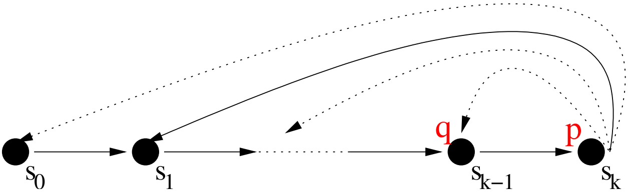

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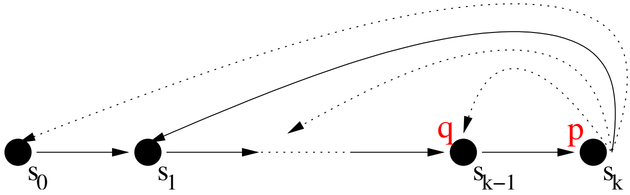

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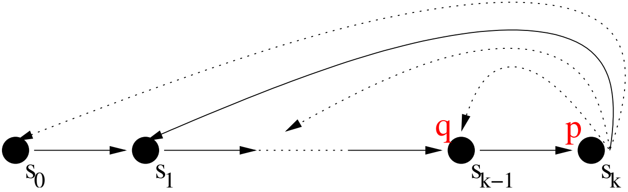

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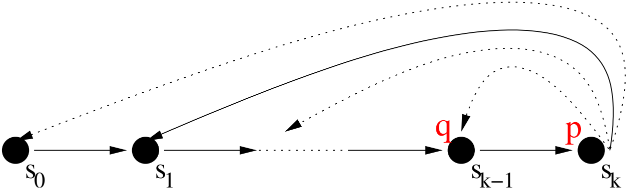

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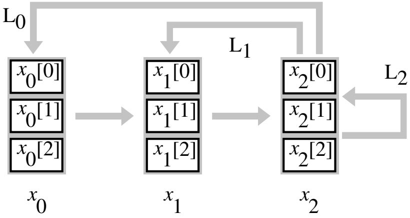
Example: a bugged 3-bit shift register [cont.]

$k = 2$:



$$[[M]]_2 : \quad \left(\begin{array}{l} (x_1[0] \leftrightarrow x_0[1]) \;\wedge\; (x_1[1] \leftrightarrow x_0[2]) \;\wedge\; (x_1[2] \leftrightarrow 1) \;\wedge \\ (x_2[0] \leftrightarrow x_1[1]) \;\wedge\; (x_2[1] \leftrightarrow x_1[2]) \;\wedge\; (x_2[2] \leftrightarrow 1) \end{array} \right) \wedge$$

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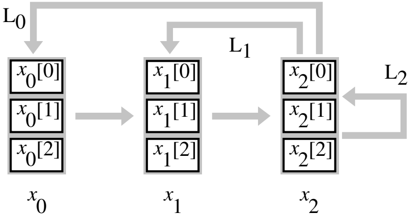

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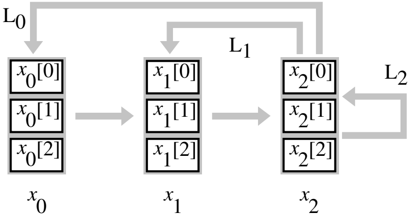

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- incomplete technique:
  - if you find all formulas unsatisfiable, it tells you nothing
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Outline

Basic bounds for k

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Let f be a LTL formula. $M \models \mathbf{E}f \iff M \models_k \mathbf{E}f$ for some $k \leq |M| \cdot 2^{|f|}$.

- $|M| \cdot 2^{|f|}$ is always a bound of k.
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Other bounds for k

ACTL & ECTL

- ACTL is a subset of CTL in which "**A**..." (resp. "**E**...") sub-formulas occur only positively (resp. negatively) in each formula.
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- Many frequently-used LTL properties $\neg f$ have equivalent ACTL representations $\neg f'$ (e.g. $\mathbf{G}(p \rightarrow \mathbf{GF}q)$ wrt. $\mathbf{AG}(p \rightarrow \mathbf{AGAF}q)$):

$$M \not\models_{LTL} \neg f \Leftrightarrow M \not\models_{CTL^*} \mathbf{A}\neg f \Leftrightarrow M \not\models_{CTL^*} \mathbf{A}\neg f' \Leftrightarrow M \models_{CTL^*} \mathbf{E}f'$$

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Other bounds for k (cont)

Theorem [Biere et al. TACAS 1999]

Let p be a Boolean formula and d be the diameter of M. Then $M \models \mathbf{EF}p \iff M \models_k \mathbf{EF}p$ for some $k \leq d$.

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The diameter

Definition: diameter

Given M, the diameter of M is the smallest integer d s.t. for every path $s_0, ..., s_{d+1}$ there exist a path $t_0, ..., t_l$ s.t. $l \leq d$, $t_0 = s_0$ and $t_l = s_{d+1}$.

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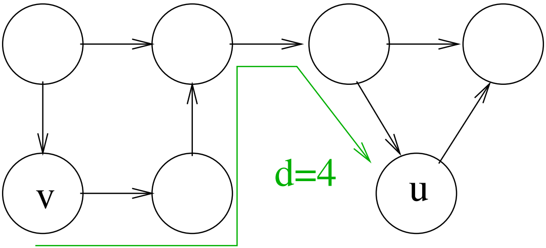

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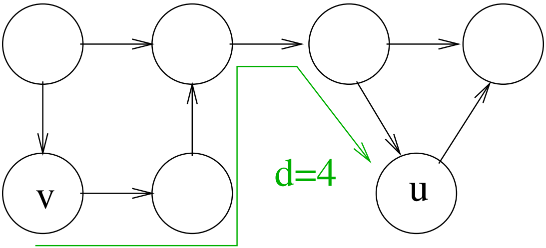

The diameter: computation

- d is the smallest integer d which makes the following formula true:

$$\forall s_0, \ldots, s_{d+1}.\exists t_0, \ldots, t_d.$$

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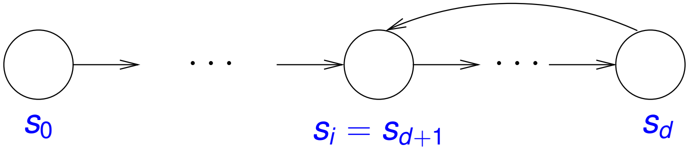

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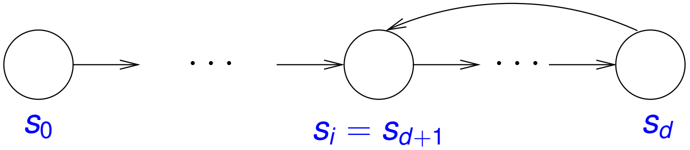

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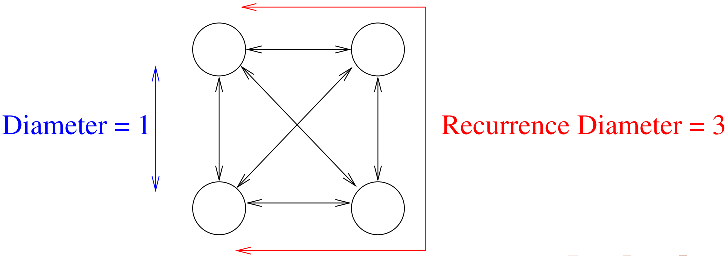

Outline

Inductive Reasoning on Invariants

Invariant: “**AG***Good*”, *Good* being a Boolean formula

(i) If all the initial states are good,

(ii) and if from good states we only go to good states

$\Rightarrow$ then we can conclude that the system is correct for all reachable states.

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SAT-based Inductive Reasoning on Invariants

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- Problem: Induction may fail because of unreachable states:
  - if $(Good(s^k) \wedge R(s^k, s^{k+1})) \rightarrow Good(s^{k+1})$ is not valid, this does not mean that the property does not hold
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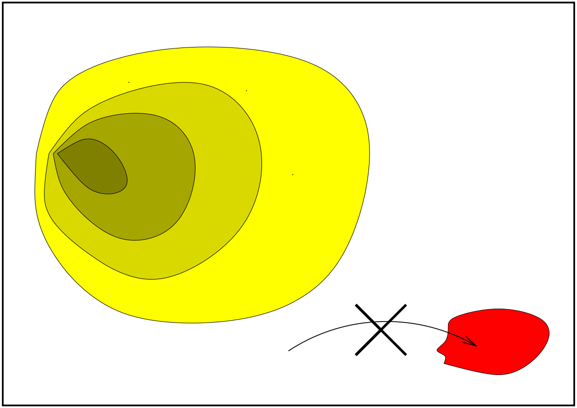

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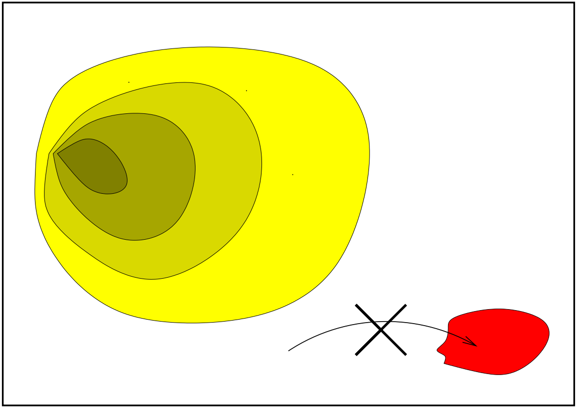

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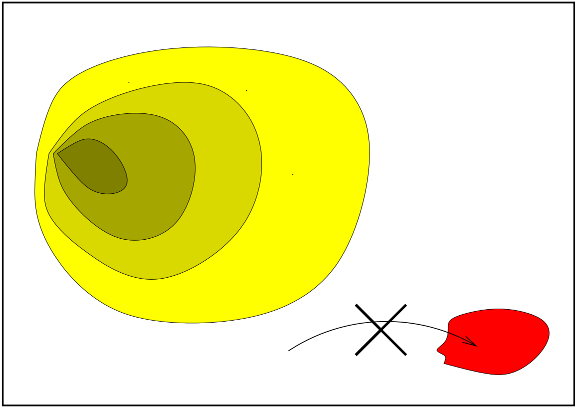

Strengthening of Invariants [cont.]

Solution:

- increase the depth of induction
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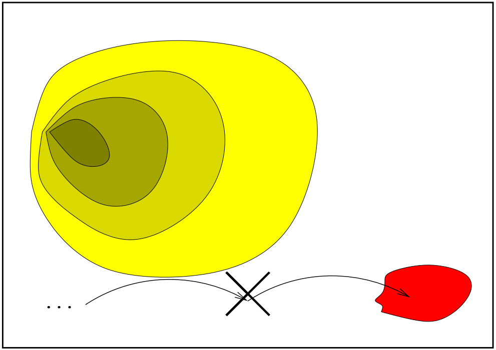

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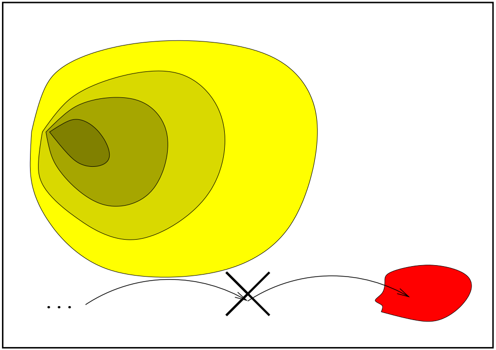

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Example: a correct 3-bit shift register

- System M:
  - $I(x) := (\neg x[0] \wedge \neg x[1] \wedge \neg x[2])$
  - $R(x, x') := ((x'[0] \leftrightarrow x[1]) \wedge (x'[1] \leftrightarrow x[2]) \wedge (x'[2] \leftrightarrow 0))$
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Example: a correct 3-bit shift register [cont.]

- Init: $((\neg x^0[0] \wedge \neg x^0[1] \wedge \neg x^0[2]) \wedge x^0[0]) \Longrightarrow$ unsat
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$\Longrightarrow$ (partly by unit-propagation)

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$\Longrightarrow$ not proved

Remark

Both $\{\neg x^k[0], \ x^k[1], \ x^k[2])\}$ and $\{ \ x^{k+1}[0], \ x^{k+1}[1], \neg x^{k+1}[2]\}$ are non-reachable.

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Example: a correct 3-bit shift register [cont.]

- Step 2:

$$\left(\begin{array}{l} (\neg x^k[0] \wedge ((x^{k+1}[0] \leftrightarrow x^k[1]) \wedge (x^{k+1}[1] \leftrightarrow x^k[2]) \wedge (x^{k+1}[2] \leftrightarrow 0))\wedge \\ \quad \neg x^{k+1}[0] \wedge ((x^{k+2}[0] \leftrightarrow x^{k+1}[1]) \wedge (x^{k+2}[1] \leftrightarrow x^{k+1}[2]) \wedge (x^{k+2}[2] \leftrightarrow 0)) \\ \ ) \wedge x^{k+2}[0] \end{array}\right)$$
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$\Longrightarrow$ not proved

Remark

$\{\neg x^k[0], \neg x^k[1], x^k[2]\}$, $\{\neg x^{k+1}[0], x^{k+1}[1], \neg x^{k+1}[2]\}$, and $\{x^{k+2}[0], \neg x^{k+2}[1], \neg x^{k+2}[2]\}$ are non-reachable.

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Remark

$\{\neg x^k[0], \neg x^k[1], \ x^k[2]\}$, $\{\neg x^{k+1}[0], \ x^{k+1}[1], \neg x^{k+1}[2]\}$, and $\{\ x^{k+2}[0], \neg x^{k+2}[1], \neg x^{k+2}[2]\}$ are non-reachable.

Example: a correct 3-bit shift register [cont.]

- Step 2:

$$
\left(\begin{array}{l}
(\neg x^k[0] \wedge ((x^{k+1}[0] \leftrightarrow x^k[1]) \wedge (x^{k+1}[1] \leftrightarrow x^k[2]) \wedge (x^{k+1}[2] \leftrightarrow 0))\wedge \\
\quad \neg x^{k+1}[0] \wedge ((x^{k+2}[0] \leftrightarrow x^{k+1}[1]) \wedge (x^{k+2}[1] \leftrightarrow x^{k+1}[2]) \wedge (x^{k+2}[2] \leftrightarrow 0)) \\
\ ) \wedge x^{k+2}[0]
\end{array}\right)
$$

$$
\begin{array}{l}
\wedge\neg((x^{k+1}[0] \leftrightarrow x^k[0]) \wedge (x^{k+1}[1] \leftrightarrow x^k[1]) \wedge (x^{k+1}[2] \leftrightarrow x^k[2])) \\
\wedge\neg((x^{k+2}[0] \leftrightarrow x^k[0]) \wedge (x^{k+2}[1] \leftrightarrow x^k[1]) \wedge (x^{k+2}[2] \leftrightarrow x^k[2])) \\
\wedge\neg((x^{k+2}[0] \leftrightarrow x^{k+1}[0]) \wedge (x^{k+2}[1] \leftrightarrow x^{k+1}[1]) \wedge (x^{k+2}[2] \leftrightarrow x^{k+1}[2]))
\end{array}
$$

$$
\Longrightarrow \text{ sat: } \left\{\begin{array}{ccc}
\neg x^k[0], & \neg x^k[1], & x^k[2] \\
\neg x^{k+1}[0], & x^{k+1}[1], & \neg x^{k+1}[2] \\
x^{k+2}[0], & \neg x^{k+2}[1], & \neg x^{k+2}[2]
\end{array}\right\}
$$

$\Longrightarrow$ not proved

Remark

$\{\neg x^k[0], \neg x^k[1], \ x^k[2]\}$, $\{\neg x^{k+1}[0], \ x^{k+1}[1], \neg x^{k+1}[2]\}$, and $\{\ x^{k+2}[0], \neg x^{k+2}[1], \neg x^{k+2}[2]\}$ are non-reachable.

Example: a correct 3-bit shift register [cont.]

- Step 2:

$$\left(\begin{array}{l} (\neg x^k[0] \wedge ((x^{k+1}[0] \leftrightarrow x^k[1]) \wedge (x^{k+1}[1] \leftrightarrow x^k[2]) \wedge (x^{k+1}[2] \leftrightarrow 0))\wedge \\ \quad \neg x^{k+1}[0] \wedge ((x^{k+2}[0] \leftrightarrow x^{k+1}[1]) \wedge (x^{k+2}[1] \leftrightarrow x^{k+1}[2]) \wedge (x^{k+2}[2] \leftrightarrow 0)) \\ \;) \wedge x^{k+2}[0] \end{array}\right)$$
$$\wedge\neg((x^{k+1}[0] \leftrightarrow x^k[0]) \wedge (x^{k+1}[1] \leftrightarrow x^k[1]) \wedge (x^{k+1}[2] \leftrightarrow x^k[2]))$$
$$\wedge\neg((x^{k+2}[0] \leftrightarrow x^k[0]) \wedge (x^{k+2}[1] \leftrightarrow x^k[1]) \wedge (x^{k+2}[2] \leftrightarrow x^k[2]))$$
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$$\Longrightarrow \text{ sat: } \left\{\begin{array}{ccc} \neg x^k[0], & \neg x^k[1], & x^k[2] \\ \neg x^{k+1}[0], & x^{k+1}[1], & \neg x^{k+1}[2] \\ x^{k+2}[0], & \neg x^{k+2}[1], & \neg x^{k+2}[2] \end{array}\right\}$$

$\Longrightarrow$ not proved

Remark

$\{\neg x^k[0], \neg x^k[1], \; x^k[2]\}$, $\{\neg x^{k+1}[0], \; x^{k+1}[1], \neg x^{k+1}[2]\}$, and $\{\; x^{k+2}[0], \neg x^{k+2}[1], \neg x^{k+2}[2]\}$ are non-reachable.

Example: a correct 3-bit shift register [cont.]

- Step 3:

$$\left(\begin{array}{l} (\neg x^k[0] \wedge ((x^{k+1}[0] \leftrightarrow x^k[1]) \wedge (x^{k+1}[1] \leftrightarrow x^k[2]) \wedge (x^{k+1}[2] \leftrightarrow 0))\wedge \\ \quad \neg x^{k+1}[0] \wedge ((x^{k+2}[0] \leftrightarrow x^{k+1}[1]) \wedge (x^{k+2}[1] \leftrightarrow x^{k+1}[2]) \wedge (x^{k+2}[2] \leftrightarrow 0))\wedge \\ \quad \neg x^{k+2}[0] \wedge ((x^{k+3}[0] \leftrightarrow x^{k+2}[1]) \wedge (x^{k+3}[1] \leftrightarrow x^{k+2}[2]) \wedge (x^{k+3}[2] \leftrightarrow 0)) \\ ) \wedge x^{k+3}[0] \end{array}\right)$$

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$$\wedge\neg((x^{k+3}[0] \leftrightarrow x^{k+2}[0]) \wedge (x^{k+3}[1] \leftrightarrow x^{k+2}[1]) \wedge (x^{k+3}[2] \leftrightarrow x^{k+2}[2]))$$

$\Longrightarrow$ (unit-propagation) $\{x^{k+3}[0], x^{k+2}[1], x^{k+1}[2]\}$

$\Longrightarrow$ unsat

$\Longrightarrow$ proved!

Example: a correct 3-bit shift register [cont.]

- Step 3:

$$\left(\begin{array}{l} (\neg x^k[0] \wedge ((x^{k+1}[0] \leftrightarrow x^k[1]) \wedge (x^{k+1}[1] \leftrightarrow x^k[2]) \wedge (x^{k+1}[2] \leftrightarrow 0))\wedge \\ \quad \neg x^{k+1}[0] \wedge ((x^{k+2}[0] \leftrightarrow x^{k+1}[1]) \wedge (x^{k+2}[1] \leftrightarrow x^{k+1}[2]) \wedge (x^{k+2}[2] \leftrightarrow 0))\wedge \\ \quad \neg x^{k+2}[0] \wedge ((x^{k+3}[0] \leftrightarrow x^{k+2}[1]) \wedge (x^{k+3}[1] \leftrightarrow x^{k+2}[2]) \wedge (x^{k+3}[2] \leftrightarrow 0)) \\ \ ) \wedge x^{k+3}[0] \end{array}\right)$$

$$\begin{array}{l} \wedge\neg((x^{k+1}[0] \leftrightarrow x^k[0]) \wedge (x^{k+1}[1] \leftrightarrow x^k[1]) \wedge (x^{k+1}[2] \leftrightarrow x^k[2])) \\ \wedge\neg((x^{k+2}[0] \leftrightarrow x^k[0]) \wedge (x^{k+2}[1] \leftrightarrow x^k[1]) \wedge (x^{k+2}[2] \leftrightarrow x^k[2])) \\ \wedge\neg((x^{k+3}[0] \leftrightarrow x^k[0]) \wedge (x^{k+3}[1] \leftrightarrow x^k[1]) \wedge (x^{k+3}[2] \leftrightarrow x^k[2])) \\ \wedge\neg((x^{k+2}[0] \leftrightarrow x^{k+1}[0]) \wedge (x^{k+2}[1] \leftrightarrow x^{k+1}[1]) \wedge (x^{k+2}[2] \leftrightarrow x^{k+1}[2])) \\ \wedge\neg((x^{k+3}[0] \leftrightarrow x^{k+1}[0]) \wedge (x^{k+3}[1] \leftrightarrow x^{k+1}[1]) \wedge (x^{k+3}[2] \leftrightarrow x^{k+1}[2])) \\ \wedge\neg((x^{k+3}[0] \leftrightarrow x^{k+2}[0]) \wedge (x^{k+3}[1] \leftrightarrow x^{k+2}[1]) \wedge (x^{k+3}[2] \leftrightarrow x^{k+2}[2])) \end{array}$$

$\Longrightarrow$ (unit-propagation) $\{x^{k+3}[0], x^{k+2}[1], x^{k+1}[2]\}$

$\Longrightarrow$ unsat

$\Longrightarrow$ proved!

Example: a correct 3-bit shift register [cont.]

- Step 3:

$$\left(\begin{array}{l}(\neg x^k[0] \wedge ((x^{k+1}[0] \leftrightarrow x^k[1]) \wedge (x^{k+1}[1] \leftrightarrow x^k[2]) \wedge (x^{k+1}[2] \leftrightarrow 0))\wedge \\ \quad \neg x^{k+1}[0] \wedge ((x^{k+2}[0] \leftrightarrow x^{k+1}[1]) \wedge (x^{k+2}[1] \leftrightarrow x^{k+1}[2]) \wedge (x^{k+2}[2] \leftrightarrow 0))\wedge \\ \quad \neg x^{k+2}[0] \wedge ((x^{k+3}[0] \leftrightarrow x^{k+2}[1]) \wedge (x^{k+3}[1] \leftrightarrow x^{k+2}[2]) \wedge (x^{k+3}[2] \leftrightarrow 0)) \\ \ ) \wedge x^{k+3}[0]\end{array}\right)$$

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Example: a correct 3-bit shift register [cont.]

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$\Longrightarrow$ unsat

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Mixed BMC & Inductive reasoning [Sheeran et al. 2000]

$$
\begin{array}{lll}
Base_n & := & I(\mathbf{s}_0) \wedge \bigwedge_{i=0}^{n-1} (R(\mathbf{s}_i, \mathbf{s}_{i+1}) \wedge \varphi(\mathbf{s}_i)) \wedge \neg\varphi(\mathbf{s}_n) \\
Step_n & := & \bigwedge_{i=0}^{n} (R(\mathbf{s}_i, \mathbf{s}_{i+1}) \wedge \varphi(\mathbf{s}_i)) \wedge \neg\varphi(\mathbf{s}_{n+1}) \\
Unique_n & := & \bigwedge_{0 \le i \le j \le n} \neg(\mathbf{s}_i = \mathbf{s}_{j+1})
\end{array}
$$

Algorithm

1. **function** CHECK_PROPERTY (I, R, φ)
2. **for** $n := 0, 1, 2, 3, ...$ **do**
3. **if** (DPLL($Base_n$) == SAT)
4. **then return** PROPERTY_VIOLATED;
5. **else if** (DPLL($Step_n \wedge Unique_n$) == UNSAT)
6. **then return** PROPERTY_VERIFIED;
7. **end for**;

$\Longrightarrow$ reuses previous search if DPLL is incremental!!

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$$\begin{array}{lcl} Base_n & := & I(\mathbf{s}_0) \wedge \bigwedge_{i=0}^{n-1} (R(\mathbf{s}_i, \mathbf{s}_{i+1}) \wedge \varphi(\mathbf{s}_i)) \wedge \neg\varphi(\mathbf{s}_n) \\ Step_n & := & \bigwedge_{i=0}^{n} (R(\mathbf{s}_i, \mathbf{s}_{i+1}) \wedge \varphi(\mathbf{s}_i)) \wedge \neg\varphi(\mathbf{s}_{n+1}) \\ Unique_n & := & \bigwedge_{0 \le i \le j \le n} \neg(\mathbf{s}_i = \mathbf{s}_{j+1}) \end{array}$$

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Other Successful SAT-based (UNbounded) MC Techniques

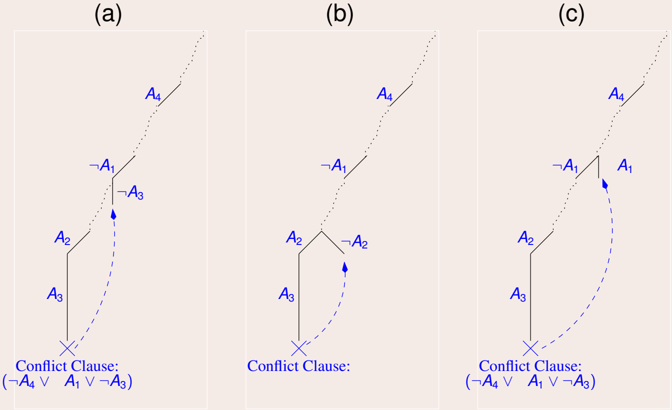
- Counter-example guided abstraction refinement (CEGAR)
  [Clarke et al. CAV 2002]
- Interpolant-based MC
  [Mc Millan, TACAS 2005]
- IC3/PDR
  [Bradley, VMCAI 2011]
- ...

For a survey see e.g.
[Amla et al., CHARME 2005, Prasad et al. STTT 2005].

Outline

Ex: CDCL SAT Solving

Which of the following figures may correspond to a modern DPLL 1st-UIP backjumping step?



[Solution: The correct answer is (a). (b) represents standard chronological backtracking, whilst (c) is nonsense.]

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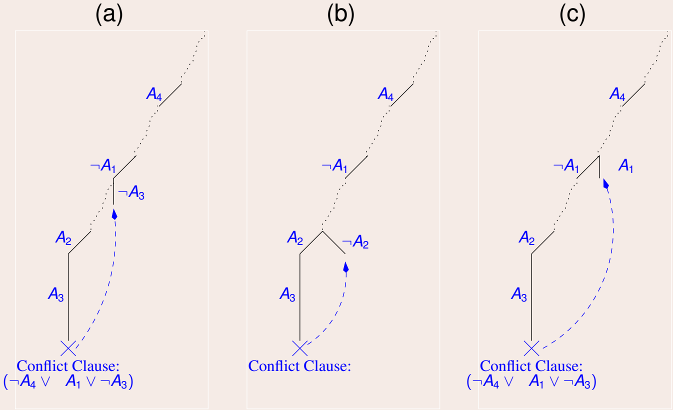
Which of the following figures may correspond to a modern DPLL 1st-UIP backjumping step?



[Solution: The correct answer is (a). (b) represents standard chronological backtracking, whilst (c) is nonsense.]

Ex: Bounded Model Checking

Given the symbolic representation of a FSM M, expressed in terms of the two Boolean formulas: $I(x,y) \stackrel{\text{def}}{=} \neg(x \vee \neg y)$, $T(x,y,x',y') \stackrel{\text{def}}{=} (x' \leftrightarrow (x \leftrightarrow \neg y)) \wedge (y' \leftrightarrow \neg y)$, and the LTL property: $\varphi \stackrel{\text{def}}{=} \neg\mathbf{F}(x \wedge y)$,

1. Write a Boolean formula whose solutions (if any) represent executions of M of length 2 which violate φ.

   [Solution: The question corresponds to the Bounded Model Checking problem $M \models_2 \mathbf{E}\,\mathbf{F}f$, s.t. $f(x,y) \stackrel{\text{def}}{=} (x \wedge y)$. Thus we have:

$$
\begin{array}{lll}
\neg(x_0 \vee \neg y_0) & \wedge & // \; I(x_0,y_0) \wedge \\
(x_1 \leftrightarrow (x_0 \leftrightarrow \neg y_0)) \wedge (y_1 \leftrightarrow \neg y_0) & \wedge & // \; T(x_0,y_0,x_1,y_1) \wedge \\
(x_2 \leftrightarrow (x_1 \leftrightarrow \neg y_1)) \wedge (y_2 \leftrightarrow \neg y_1) & \wedge & // \; T(x_1,y_1,x_2,y_2) \wedge \\
((x_0 \wedge y_0) & \vee & // \; (f(x_0,y_0) \vee \\
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(x_2 \wedge y_2)) & & // \; f(x_2,y_2))
\end{array}
$$

   ]

2. Is there a solution? If yes, find the corresponding execution; if no, show why.

   [Solution: Yes: $\{\neg x_0, y_0, x_1, \neg y_1, x_2, y_2\}$, corresponding to the execution: $(0,1) \rightarrow (1,0) \rightarrow (1,1)$]

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$$
\begin{array}{lcl}
\neg(x_0 \vee \neg y_0) & \wedge & // \; I(x_0,y_0) \wedge \\
(x_1 \leftrightarrow (x_0 \leftrightarrow \neg y_0)) \wedge (y_1 \leftrightarrow \neg y_0) & \wedge & // \; T(x_0,y_0,x_1,y_1) \wedge \\
(x_2 \leftrightarrow (x_1 \leftrightarrow \neg y_1)) \wedge (y_2 \leftrightarrow \neg y_1) & \wedge & // \; T(x_1,y_1,x_2,y_2) \wedge \\
((x_0 \wedge y_0) & \vee & // \; (f(x_0,y_0) \vee \\
(x_1 \wedge y_1) & \vee & // \; f(x_1,y_1) \vee \\
(x_2 \wedge y_2)) & & // \; f(x_2,y_2))
\end{array}
$$

   ]

2. Is there a solution? If yes, find the corresponding execution; if no, show why.

   [Solution: Yes: $\{\neg x_0, y_0, x_1, \neg y_1, x_2, y_2\}$, corresponding to the execution: $(0,1) \rightarrow (1,0) \rightarrow (1,1)$]

Ex: Bounded Model Checking

3. From the solutions to question #1 and #2 we can conclude that:
   (a) $M \models \varphi$
   (b) $M \not\models \varphi$
   (c) we can conclude nothing.

   [Solution: b)]

4. What are the diameter and the recurrence diameter of this system?

   [Solution:

   ]

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   [Solution:

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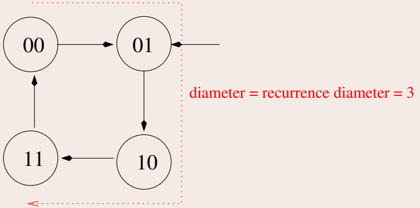
4. What are the diameter and the recurrence diameter of this system?

   [Solution:



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