*1. Analyze the logical forms of the following statements.
(a) Anyone who has forgiven at least one person is a saint.
(b) Nobody in the calculus class is smarter than everybody in the discrete math class.
(c) Everyone likes Mary, except Mary herself.
(d) Jane saw a police officer, and Roger saw one too.
(e) Jane saw a police officer, and Roger saw him too.

a) $S(x) = x$ é um santo
$P(x,y) = x$ perdoou y
$\forall x(\exists y P(x,y) \rightarrow S(x))$

b) $C =$ sala de cálculo
$D =$ sala de matemática discreta
$I(x,y) = x$ é mais inteligente que y
$\neg\exists x(x \in C \wedge \forall y(y \in D \rightarrow I(x,y)))$

c) $L(x,y) = x$ gosta de y
$\forall x(x \neq Mary \rightarrow L(x, mary))$

d) $P(x,y) = x$ viu um policial y
$\exists y P(Jane, y) \wedge \exists z P(Roger, z)$

e) $P(x,y) = x$ viu um policial
$\exists y(P(Jane, y) \wedge P(Roger, y))$

2. Analyze the logical forms of the following statements.
(a) Anyone who has bought a Rolls Royce with cash must have a rich uncle.
(b) If anyone in the dorm has the measles, then everyone who has a friend in the dorm will have to be quarantined.
(c) If nobody failed the test, then everybody who got an A will tutor someone who got a D.
(d) If anyone can do it, Jones can.
(e) If Jones can do it, anyone can.

a) $C(x,y,z) = x$ comprou y utilizando z
$U(x,y) = x$ é primo de y
$R(x) = x$ é rico
$\forall x(C(x, Rolls\ Royce, dinheiro) \rightarrow \exists y(U(y,x) \wedge R(y)))$

b) $S(x) = x$ esta com sarampo
$A(x,y) = x$ é amigo de y
$Q(x) = x$ sera internado
$D(x) = x$ é do dormitório
$\exists x(D(x) \wedge S(x)) \rightarrow (\forall y(\exists z(D(z) \wedge A(y,z)) \rightarrow Q(y))$

c) $F(x) = x$ falhou no teste
$O(x,y) = x$ tirou y
$T(x,y) = x$ irá tutorar y
$\neg\exists x F(x) \rightarrow \forall y(O(y,A) \rightarrow \exists z(O(z,D) \wedge T(y,z)))$

d) $C(x) = x$ pode fazer isso
$\forall x(C(x) \rightarrow C(Jones))$

e) $C(Jones) \rightarrow \forall x C(x)$

3. Analyze the logical forms of the following statements. The universe of discourse is $\mathbb{R}$. What are the free variables in each statement?
(a) Every number that is larger than x is larger than y.
(b) For every number a, the equation $ax^2 + 4x - 2 = 0$ has at least one solution iff $a \geq -2$.
(c) All solutions of the inequality $x^3 - 3x < 3$ are smaller than 10.
(d) If there is a number x such that $x^2 + 5x = w$ and there is a number y such that $4 - y^2 = w$, then w is strictly between -10 and 10.

a) $\forall n(n > x \rightarrow n > y)$ y e x são variaveis livres

b) $\forall a \exists x[ax^2 + 4x - 2 = 0 \leftrightarrow a \geq -2]$ não ha variaveis livres

c) $\forall x(x^3 - 3x < 3 \rightarrow x < 10)$ não ha variaveis livres

d) $[\exists x(x^2 + 5x = \omega) \wedge \exists y(4 - y^2 = \omega)] \rightarrow (-10 \leq \omega \leq 10)$ ω é variavel livre

*4. Translate the following statements into idiomatic English.
(a) $\forall x[(H(x) \wedge \neg\exists y M(x,y)) \rightarrow U(x)]$, where $H(x)$ means "x is a man," $M(x,y)$ means "x is married to y," and $U(x)$ means "x is unhappy."
(b) $\exists z(P(z,x) \wedge S(z,y) \wedge W(y))$, where $P(z,x)$ means "z is a parent of x," $S(z,y)$ means "z and y are siblings," and $W(y)$ means "y is a woman."

a) Para todo x se x é homem e não existe y tal que x seja casado y, então x é infeliz.
b) Existe um z, tal que z é pai/mãe de x, z e y sejam irmãos e y seja uma mulher.

5. Translate the following statements into idiomatic mathematical English.
(a) $\forall x[(P(x) \wedge \neg(x = 2)) \rightarrow O(x)]$, where $P(x)$ means "x is a prime number" and $O(x)$ means "x is odd."
(b) $\exists x[P(x) \wedge \forall y(P(y) \rightarrow y \leq x)]$, where $P(x)$ means "x is a perfect number."

a) Para todo x, se x é primo e é diferente de dois, então x é impar.
b) Existe um número x perfeito, tal que para qualquer outro número y perfeito então $y \leq x$. (FINITUDE DOS NÚMEROS PERFEITOS)

6. Translate the following statements into idiomatic mathematical English. Are they true or false? The universe of discourse is $\mathbb{R}$.
(a) $\neg\exists x(x^2 + 2x + 3 = 0 \wedge x^2 + 2x - 3 = 0)$.
(b) $\neg[\exists x(x^2 + 2x + 3 = 0) \wedge \exists x(x^2 + 2x - 3 = 0)]$.
(c) $\neg\exists x(x^2 + 2x + 3 = 0) \wedge \neg\exists x(x^2 + 2x - 3 = 0)$.

a) Não existe x real tal que $x^2 + 2x + 3 = 0$ e $x^2 + 2x - 3 = 0$ simultaneamente

$$\frac{-2 \pm \sqrt{4 - 4 \cdot 3}}{2} = -1 \pm \sqrt{-2} = 1 \pm \sqrt{2}\,i$$

Verdadeiro ja não existe $x \in \mathbb{R}$ que satisfaça essa eq.

b) Não e verdade que; existe x real tal que $x^2 + 2x + 3 = 0$ e outro x tal que $x^2 + 2x - 3 = 0$

$$\frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot (-3)}}{2} = -1 \pm 2 = -3 \text{ ou } 1$$

Verdadeiro, pois realmente não existe x real que satisfaça a primeira eq. mesmo que para a segunda haja.

c) Não existe x real que satisfaça $x^2 + 2x + 3 = 0$ e não existe x real que satisfaça $x^2 + 2x - 3 = 0$
Falso, pois existem x reais que satisfazem a segunda igualdade

7. Are these statements true or false? The universe of discourse is the set of all people, and $P(x,y)$ means "x is a parent of y."
(a) $\exists x \forall y P(x,y)$.
(b) $\forall x \exists y P(x,y)$.
(c) $\neg\exists x \exists y P(x,y)$.
(d) $\exists x \neg\exists y P(x,y)$.
(e) $\exists x \exists y \neg P(x,y)$.

a) Existe x para todo y, tal que x é pai de y. (Falso, não existe um x que é pai de todo mundo).
b) Para todo x existe y, tal que x é pai de y. (Falso, nem todo mundo é pai).
c) não existe x tal que para algum y x é pai. (Falso, existem pessoas que são pai).
d) Existe x tal que não existe um y que x é pai. (Verdadeiro, existem pessoas que não são pai)
e) Existe x tal que existe y, tal que x não é pai de y. (Verdadeiro, existem pessoas não tem relação pai-filho).

*8. Are these statements true or false? The universe of discourse is $\mathbb{N}$.
(a) $\forall x \exists y(2x - y = 0)$.
(b) $\exists y \forall x(2x - y = 0)$.
(c) $\forall x \exists y(x - 2y = 0)$.
(d) $\forall x(x < 10 \rightarrow \forall y(y < x \rightarrow y < 9))$.
(e) $\exists y \exists z(y + z = 100)$.
(f) $\forall x \exists y(y > x \wedge \exists z(y + z = 100))$.

a) Verdadeiro
b) Falso
c) Se $x = 3$ o y precisaria ser 1,5; então falso.
d) Verdadeiro
e) Verdadeiro
f) Falso, se pegarmos $x = 100$ no minimo y teria que ser 101 e não existiria um z que satisfaça a condição

9. Same as exercise 8 but with $\mathbb{R}$ as the universe of discourse.

a) Verdadeiro
b) Falso
c) Verdadeiro
d) Falso, no caso dos $\mathbb{R}$ x pode ser 9,9 e y 9,8.
e) Verdadeiro
f) Verdadeiro

10. Same as exercise 8 but with $\mathbb{Z}$ as the universe of discourse.

a) Verdadeiro
b) Falso
c) Se $x = 3$ o y precisaria ser 1,5; então falso.
d) Verdadeiro
e) Verdadeiro
f) Verdadeiro

*1. Negate these statements and then reexpress the results as equivalent positive statements. (See Example 2.2.1.)
(a) Everyone who is majoring in math has a friend who needs help with his or her homework.
(b) Everyone has a roommate who dislikes everyone.
(c) $A \cup B \subseteq C \setminus D$.
(d) $\exists x \forall y[y > x \to \exists z(z^2 + 5z = y)]$.

a) $\forall x(M(x) \to \exists y(F(x,y) \wedge H(y))$

$\exists x \neg(\neg M(x) \vee \exists y(F(x,y) \wedge H(y))) = \exists x(M(x) \wedge \forall y \neg(F(x,y) \wedge H(y))) = \exists x(M(x) \wedge \forall y(F(x,y) \to \neg H(y)))$

b) $\forall x \exists y(R(x,y) \wedge \forall z \neg L(y,z))$

$\exists x \forall y \neg(R(x,y) \wedge \forall z \neg L(y,z)) = \exists x \forall y(R(x,y) \to \exists z L(y,z))$

c) $\forall x(x \in \{x \mid x \in A \vee x \in B\} \to x \in \{x \mid x \in C \wedge x \notin D\})$

$\exists x \neg(\neg x \in \{x \mid x \in A \vee x \in B\} \vee x \in \{x \mid x \in C \wedge x \notin D\}) = \exists x(x \in \{x \mid x \in A \vee x \in B\} \wedge \neg x \in \{x \mid x \in C \wedge x \notin D\}) =$

$\exists x(x \in \{x \mid x \in A \vee x \in B\} \wedge x \in \{x \mid x \notin C \vee x \in D\}) = \exists x[x \in (A \cup B) \cap (D \setminus C)]$

d) $\forall x \exists y \neg[\neg(y > x) \vee \exists z(z^2 + 5z = y)] = \forall x \exists y[(y > x) \wedge \forall z(z^2 + 5z \neq y)]$

2. Negate these statements and then reexpress the results as equivalent positive statements. (See Example 2.2.1.)
(a) There is someone in the freshman class who doesn't have a roommate.
(b) Everyone likes someone, but no one likes everyone.
(c) $\forall a \in A \exists b \in B(a \in C \leftrightarrow b \in C)$.
(d) $\forall y > 0 \exists x(ax^2 + bx + c = y)$.

a) $\exists x(F(x) \wedge \forall y \neg R(x,y))$

$\forall x \neg(F(x) \wedge \forall y \neg R(x,y)) = \forall x(\neg F(x) \vee \exists y R(x,y)) = \forall x(F(x) \to \exists y R(x,y))$

b) $\forall x \exists y L(x,y) \wedge \neg \exists x \forall y L(x,y)$

$\neg(\forall x \exists y L(x,y) \wedge \neg \exists x \forall y L(x,y)) = \exists x \forall y \neg L(x,y) \vee \exists x \forall y L(x,y)$

c) $\forall a(a \in A \to \exists b(b \in B \wedge (a \in C \leftrightarrow b \in C))) = \forall a(a \notin A \vee \exists b(b \in B \wedge (a \notin C \vee b \in C) \wedge (a \in C \vee b \notin C)))$

$\exists a(a \in A \wedge \neg \exists b(b \in B \wedge (a \notin C \vee b \in C) \wedge (a \in C \vee b \notin C))) = \exists a(a \in A \wedge \forall b \neg(b \in B \wedge (a \notin C \vee b \in C) \wedge (a \in C \vee b \notin C))) =$

$\exists a(a \in A \wedge \forall b(b \notin B \vee \neg(a \notin C \vee b \in C) \vee \neg(a \in C \vee b \notin C))) = \exists a(a \in A \wedge \forall b(b \notin B \vee (a \in C \wedge b \notin C) \vee (a \notin C \wedge b \in C)))$

d) $\forall y \exists x(y > 0 \to (ax^2 + bx + c = y))$

$\exists y \forall x \neg(\neg y > 0 \vee (ax^2 + bx + c = y)) = \exists y \forall x(y > 0 \wedge (ax^2 + bx + c \neq y))$

3. Are these statements true or false? The universe of discourse is $\mathbb{N}$.
(a) $\forall x(x < 7 \to \exists a \exists b \exists c(a^2 + b^2 + c^2 = x))$.
(b) $\exists! x((x - 4)^2 = 9)$.
(b) $\exists! x(x^2 + 3 = 4x)$.
(c) $\exists! x(x^2 = 4x + 5)$.
(d) $\exists x \exists y(x^2 = 4x + 5 \wedge y^2 = 4y + 5)$.

a) $1 = 1^2 + 0^2 + 0^2$ $\quad 3 = 1^2 + 1^2 + 1^2$ $\quad 5 = 2^2 + 1^2 + 0^2$ $\quad$ Verdadeiro

$2 = 1^2 + 1^2 + 0^2$ $\quad 4 = 2^2 + 0^2 + 0^2$ $\quad 6 = 2^2 + 1^2 + 1^2$

d) $x^2 - 4x - 5 = 0$ $\quad$ Verdadeiro.

$y^2 - 4y - 5 = 0$

b) $\Delta = 16 - 4.1.3 = 4$ $\quad \dfrac{-(-4) \pm \sqrt{4}}{2.1} = 2 \pm 1 = 3$ ou 1 $\quad$ Falso, existe dois x's que satisfazem a expressão.

$x^2 - 4x + 3 = 0$

c) $\Delta = 16 - 4.(-5).1 = 36$ $\quad \dfrac{-(-4) \pm \sqrt{36}}{2} = 2 \pm 3 = 5$ ou -1 $\quad$ Verdadeiro, apenas uma das soluções é natural

$x^2 - 4x - 5 = 0$

*4. Show that the second quantifier negation law, which says that $\neg \forall x P(x)$ is equivalent to $\exists x \neg P(x)$, can be derived from the first, which says that $\neg \exists x P(x)$ is equivalent to $\forall x \neg P(x)$. (Hint: Use the double negation law.)

$\neg \forall x P(x) = \neg \forall x(\neg\neg P(x)) = \neg\{\forall x[\neg(\neg P(x))]\} = \neg[\neg \exists(\neg P(x))] = \neg\neg \exists \neg P(x) = \exists \neg P(x)$

5. Show that $\neg \exists x \in A\, P(x)$ is equivalent to $\forall x \in A \neg P(x)$.

$\neg \exists x \in A\, P(x) = \neg \exists x(x \in A \wedge P(x)) = \forall x \neg(x \in A \wedge P(x)) = \forall x(\neg x \in A \vee \neg P(x)) = \forall x(x \in A \to \neg P(x)) = \forall x \in A(\neg P(x))$

*6. Show that the existential quantifier distributes over disjunction. In other words, show that $\exists x(P(x) \vee Q(x))$ is equivalent to $\exists x P(x) \vee \exists x Q(x)$. (Hint: Use the fact, discussed in this section, that the universal quantifier distributes over conjunction.)

$\exists x(P(x) \vee Q(x)) = \exists x \neg(\neg P(x) \wedge \neg Q(x)) = \neg \forall x[(\neg P(x)) \wedge (\neg Q(x))] = \neg[\forall x(\neg P(x)) \wedge \forall x(\neg Q(x))] = \neg \forall x(\neg P(x)) \vee \neg \forall x(\neg Q(x)) =$

$\exists x \neg(\neg P(x)) \vee \exists x \neg(\neg Q(x)) = \exists x(P(x)) \vee \exists x(Q(x)) = \exists x P(x) \vee \exists x Q(x)$

7. Show that $\exists x(P(x) \to Q(x))$ is equivalent to $\forall x P(x) \to \exists x Q(x)$.

$\exists x(P(x) \to Q(x)) = \exists x(\neg P(x) \vee Q(x)) = \exists x \neg P(x) \vee \exists x Q(x) = \neg \forall x P(x) \vee \exists x Q(x) = \forall x P(x) \to \exists x Q(x)$

*8. Show that $(\forall x \in A\, P(x)) \wedge (\forall x \in B\, P(x))$ is equivalent to $\forall x \in (A \cup B)\, P(x)$. (Hint: Start by writing out the meanings of the bounded quantifiers in terms of unbounded quantifiers.)

$\forall x \in A\, P(x) \wedge \forall x \in B\, P(x) = \forall x[(x \in A \to P(x))] \wedge \forall x[(x \in B \to P(x))] = \forall x[(\neg x \in A \vee P(x)) \wedge (\neg x \in B \vee P(x))] = \forall x[(\neg x \in A \wedge x \in B) \vee P(x)] =$

$\forall x[\neg(x \in A \vee x \in B) \vee P(x)] = \forall x[(x \in A \vee x \in B) \to P(x)] = \forall x[x \in A \cup B \to P(x)] = \forall x \in (A \cup B)\, P(x)$

9. Is $\forall x(P(x) \vee Q(x))$ equivalent to $\forall x P(x) \vee \forall x Q(x)$? Explain. (Hint: Try assigning meanings to $P(x)$ and $Q(x)$.)

Não são equivalentes, por exemplo, se tomarmos:

$P(x)$ = x é macho

$Q(x)$ = x é femea

E considerarmos o universo de discurso todos os cães domésticos do mundo o significado seria:

$\forall x(P(x) \vee Q(x))$ = Todo cão é macho ou femea. (Verdadeiro).

$\forall x P(x) \vee \forall x Q(x)$ = Todos os cães são machos ou todos os cães são femeas. (Falso).

10. (a) Show that $\exists x \in A\, P(x) \vee \exists x \in B\, P(x)$ is equivalent to $\exists x \in (A \cup B)\, P(x)$.
(b) Is $\exists x \in A\, P(x) \wedge \exists x \in B\, P(x)$ equivalent to $\exists x \in (A \cap B)\, P(x)$? Explain.

a) $\exists x \in A\, P(x) \vee \exists x \in B\, P(x) = \exists x(x \in A \wedge P(x)) \vee \exists x(x \in B \wedge P(x)) = \exists x[(x \in A \wedge P(x)) \vee (x \in B \wedge P(x))] = \exists x[(x \in A \vee x \in B) \wedge P(x)] = \exists x \in (A \cup B)\, P(x)$

b) Não é equivalente, pois existir um x de A que satisfaça $P(x)$ e um x de B que satisfaça $P(x)$ não garante que eles tenham um elemento x em comum e que satisfaça $P(x)$ simultaneamente.

*11. Show that the statements $A \subseteq B$ and $A \setminus B = \emptyset$ are equivalent by writing each in logical symbols and then showing that the resulting formulas are equivalent.

$A \subseteq B = \forall x(x \in A \to x \in B) = \forall x(\neg x \in A \vee x \in B) = \forall x \neg(x \in A \wedge \neg x \in B) = \neg \exists x(x \in A \wedge x \notin B) = \neg \exists x[x \in (A \setminus B)] = \neg \exists x \in (A \setminus B) = [A \setminus B = \emptyset]$

Ou seja se não existe x que pertença $A \setminus B$, logo $A \setminus B$ é vazio: $A \setminus B = \emptyset$

12. Show that the statements $C \subseteq A \cup B$ and $C \setminus A \subseteq B$ are equivalent by writing each in logical symbols and then showing that the resulting formulas are equivalent.

$C \subseteq A \cup B = \forall x(x \in C \to (x \in A \vee x \in B)) = \forall x(\neg x \in C \vee x \in A \vee x \in B) = \forall x[\neg(x \in C \wedge x \notin A) \vee x \in B] = \forall x((x \in C \wedge x \notin A) \to x \in B) = C \setminus A \subseteq B$

13. (a) Show that the statements $A \subseteq B$ and $A \cup B = B$ are equivalent by writing each in logical symbols and then showing that the resulting formulas are equivalent. (Hint: You may find exercise 11 from Section 1.5 useful.)
(b) Show that the statements $A \subseteq B$ and $A \cap B = A$ are equivalent.

a) $A \subseteq B = \forall x(x \in A \to x \in B) = \forall x((x \in A \vee x \in B) \leftrightarrow x \in B) = (A \cup B = B)$

b) $A \subseteq B = \forall x(x \in A \to x \in B) = \forall x((x \in A \wedge x \in B) \leftrightarrow x \in A) = (A \cap B = A)$

Relação de igualdade

$\forall x(x \in P \leftrightarrow x \in Q) \equiv P = Q$

*14. Show that the statements $A \cap B = \emptyset$ and $A \setminus B = A$ are equivalent.

$\neg \exists x(x \in A \wedge x \in B) = \forall x \neg(x \in A \wedge x \in B) = \forall x[\neg x \in A \vee x \notin B] = \forall x(x \in A \to x \notin B) = \forall x[(x \notin B \wedge x \in A) \leftrightarrow x \in A] = \forall x[x \in (A \setminus B) \leftrightarrow x \in A] = [(A \setminus B) = A]$

15. Let $T(x, y)$ mean "x is a teacher of y." What do the following statements mean? Under what circumstances would each one be true? Are any of them equivalent to each other?
(a) $\exists! y T(x, y)$.
(b) $\exists x \exists! y T(x, y)$.
(c) $\exists! x \exists y T(x, y)$.
(d) $\exists y \exists! x T(x, y)$.
(e) $\exists! x \exists! y T(x, y)$.
(f) $\exists x \exists y[T(x, y) \wedge \neg \exists u \exists v(T(u, v) \wedge (u \neq x \vee v \neq y))]$.

NECESSÁRIO REVISAR COM MAIS PACIÊNCIA, MAS APARENTEMENTE APENAS F E E EQUIVALEM

a) Existe um único aluno y para o professor x. Verdadeiro apenas para professores particulares.

b) Existe professores que possuem apenas um aluno. Verdadeiro como dito acima.

c) Existe um único professor que é professor de algum y. Falso, existem mais professores do que apenas um

d) Existem alunos que possuem um único professor. Verdadeiro, em caso de pessoas que tem apenas uma aula.

e) Existe apenas um professor que possui um único aluno. Falso, existe mais de um professor particular.

f) Existe x e y, tal que x é professor de y e também não existe u tal que u seja professor de algum v e u seja diferente de x e v diferente de y.

$\exists x \exists y \{T(x,y) \land [\forall u \forall v (\neg T(u,v) \lor \neg(u \neq x \lor v \neq y))]\} \equiv \exists x \exists y\, T(x,y) \land \forall u \forall v\, (T(u,v) \rightarrow (u = x \land v = y)) \equiv \exists! x \exists! y\, T(x,y)$

Example 2.3.6. Analyze the logical forms of the following statements.

4. $x \in \bigcup\{\mathscr{P}(A) \mid A \in \mathcal{F}\}$.

- $\bigcup\{P(A) \mid A \in \mathcal{F}\} = \{y \mid \exists Y \in \{P(A) \mid A \in \mathcal{F}\}(y \in Y)\}$
- $x \in \{y \mid \exists Y \in \{P(A) \mid A \in \mathcal{F}\}(y \in Y)\} \equiv \exists Y \in \{P(A) \mid A \in \mathcal{F}\}(x \in Y) \equiv \exists Y \in \{Z \mid \exists A \in \mathcal{F}(Z = P(A))\}(x \in Y)$

$\exists Y[Y \in \{Z \mid \exists A \in \mathcal{F}(Z = P(A))\} \land (x \in Y)] \equiv \exists Y[\exists A \in \mathcal{F}(Y = P(A)) \land (x \in Y)] \equiv \exists A \in \mathcal{F}(x \in P(A)) \equiv \exists A \in \mathcal{F}(x \subseteq A) \equiv \exists A \in \mathcal{F}[\forall w(w \in x \rightarrow w \in A)]$

Example 2.3.8. For this example our universe of discourse will be the set S of all students. Let $L(x,y)$ stand for "x likes y" and $A(x,y)$ for "x admires y." For each student s, let L_s be the set of all students that s likes. In other words $L_s = \{t \in S \mid L(s,t)\}$. Similarly, let $A_s = \{t \in S \mid A(s,t)\}$ = the set of all students that s admires. Describe the following sets.

1. $\bigcap_{s \in S} L_s$.
2. $\bigcup_{s \in S} L_s$.
3. $\bigcup_{s \in S} L_s \setminus \bigcup_{s \in S} A_s$.
4. $\bigcup_{s \in S}(L_s \setminus A_s)$.
5. $(\bigcap_{s \in S} L_s) \cap (\bigcap_{s \in S} A_s)$.
6. $\bigcap_{s \in S}(L_s \cap A_s)$.
7. $\bigcup_{b \in B} L_b$, where $B = \bigcap_{s \in S} A_s$.

1) Conjunto dos alunos que são gostados por todos os alunos
2) Conjunto dos alunos que são gostados por alguém
3) Conjunto dos alunos que são gostados por alguém, porém não admirados por ninguém.
4) Conjunto dos alunos que alguém gosta, porém não admira.
5) Conjunto dos alunos que é gostado por todo mundo e também admirado por todo mundo.
6) Conjunto dos alunos que são tanto gostados quanto admirados por todo mundo
7) Conjunto dos alunos que algum dos alunos admirado por todo mundo gosta.
B= Conjunto dos alunos admirados por todo mundo

*1. Analyze the logical forms of the following statements. You may use the symbols $\in, \notin, =, \neq, \land, \lor, \rightarrow, \leftrightarrow, \forall$ and $\exists$ in your answers, but not $\subseteq, \nsubseteq, \mathscr{P}, \cap, \cup, \setminus, \{\,\}$, or $\neg$. (Thus, you must write out the definitions of some set theory notation, and you must use equivalences to get rid of any occurrences of $\neg$.)
(a) $\mathcal{F} \subseteq \mathscr{P}(A)$.
(b) $A \subseteq \{2n+1 \mid n \in \mathbb{N}\}$.
(c) $\{n^2+n+1 \mid n \in \mathbb{N}\} \subseteq \{2n+1 \mid n \in \mathbb{N}\}$.
(d) $\mathscr{P}(\bigcup_{i \in I} A_i) \nsubseteq \bigcup_{i \in I} \mathscr{P}(A_i)$.

a) $\mathcal{F} \subseteq P(A) \equiv \forall X(X \in \mathcal{F} \rightarrow X \in P(A)) \equiv \forall X(X \in \mathcal{F} \rightarrow X \subseteq A) \equiv \forall X(X \in \mathcal{F} \rightarrow \forall x(x \in X \rightarrow x \in A))$

b) $A \subseteq \{2n+1 \mid n \in \mathbb{N}\} \equiv A \subseteq \{m \mid \exists n \in \mathbb{N}(m = 2n+1)\} \equiv \forall x(x \in A \rightarrow x \in \{m \mid \exists n \in \mathbb{N}(m = 2n+1)\}) \equiv$
$\forall x(x \in A \rightarrow \exists n \in \mathbb{N}(x = 2n+1))$

c) $\{n^2+n+1 \mid n \in \mathbb{N}\} \subseteq \{2n+1 \mid n \in \mathbb{N}\} \equiv \{p \mid \exists n \in \mathbb{N}(p = n^2+n+1)\} \subseteq \{q \mid \exists n \mathbb{N}(q = 2n+1)\} \equiv \forall x[x \in \{p \mid \exists n \in \mathbb{N}(p = n^2+n+1)\} \rightarrow x \in \{q \mid \exists n \mathbb{N}(q = 2n+1)\}] \equiv$
$\forall x[\exists n \in \mathbb{N}(x = n^2+n+1) \rightarrow \exists n \in \mathbb{N}(x = 2n+1)]$

d) $P(\{x \mid \exists X \in A_i(x \in X)\}) \nsubseteq \{y \mid \exists Y \in P(A_i)(y \in Y)\} \equiv \neg[\forall Z(Z \in P(\{x \mid \exists X \in A_i(x \in X)\}) \rightarrow Z \in \{y \mid \exists Y \in P(A_i)(y \in Y)\})] \equiv$

$\neg[\forall Z(Z \subseteq \{x \mid \exists X \in A_i(x \in X)\} \rightarrow \exists Y \in P(A_i)(Z \in Y))] \equiv \neg\forall Z[\forall z(z \in Z \rightarrow z \in \{x \mid \exists X \in A_i(x \in X)\}) \rightarrow \exists Y \in P(A_i)(Z \in Y)] \equiv$

$\neg\forall Z[\forall z(z \in Z \rightarrow \exists X \in A_i(z \in X)) \rightarrow \exists Y \in P(A_i)(Z \in Y)] \equiv \exists Z \neg[\forall z(z \in Z \rightarrow \exists X \in A_i(z \in X)) \rightarrow \exists Y \in P(A_i)(Z \in Y)] \equiv$

$\exists Z[\forall z(z \in Z \rightarrow \exists X \in A_i(z \in X)) \land \neg\exists Y \in P(A_i)(Z \in Y)] \equiv \exists Z[\forall z(z \in Z \rightarrow \exists X \in A_i(z \in X)) \land \neg\exists Y(Y \in P(A_i) \land Z \in Y)] \equiv$

$\exists Z[\forall z(z \in Z \rightarrow \exists X \in A_i(z \in X)) \land \neg\exists Y(Y \subseteq A_i \land Z \in Y)] \equiv \exists Z[\forall z(z \in Z \rightarrow \exists X \in A_i(z \in X)) \land \neg\exists Y(\forall y(y \in Y \rightarrow y \in A_i) \land Z \in Y)] \equiv$

$\exists Z[\forall z(z \in Z \rightarrow \exists X \in A_i(z \in X)) \land \forall Y(\exists y(y \in Y \land y \notin A_i) \lor Z \notin Y)]$

2. Analyze the logical forms of the following statements. You may use the symbols $\in, \notin, =, \neq, \land, \lor, \rightarrow, \leftrightarrow, \forall$ and $\exists$ in your answers, but not $\subseteq, \nsubseteq, \mathscr{P}, \cap, \cup, \setminus, \{\,\}$, or $\neg$. (Thus, you must write out the definitions of some set theory notation, and you must use equivalences to get rid of any occurrences of $\neg$.)
(a) $x \in \bigcup\mathcal{F} \setminus \bigcup\mathcal{G}$.
(b) $\{x \in B \mid x \notin C\} \in \mathscr{P}(A)$.
(c) $x \in \bigcap_{i \in I}(A_i \cup B_i)$.
(d) $x \in (\bigcap_{i \in I} A_i) \cup (\bigcap_{i \in I} B_i)$.

a) $x \in (\{p \mid \exists P \in \mathcal{F}(p \in P)\} \setminus \{q \mid \exists Q \in \mathcal{G}(q \in Q)\}) \equiv x \in \{p \mid \exists P \in \mathcal{F}(p \in P)\} \land x \notin \{p \mid \exists Q \in \mathcal{G}(p \in Q)\} \equiv$
$x \in \{p \mid \exists P \in \mathcal{F}(p \in P)\} \land \neg x \in \{p \mid \exists Q \in \mathcal{G}(p \in Q)\} \equiv \exists P \in \mathcal{F}(x \in P) \land \neg\exists Q \in \mathcal{G}(x \in Q) \equiv \exists P \in \mathcal{F}(x \in P) \land \forall Q \in \mathcal{G}(x \notin Q)$

b) $\{x \in B \mid x \notin C\} \in P(A) \equiv \{x \mid x \in B \land x \notin C\} \subseteq A \equiv \forall y[y \in \{x \mid x \in B \land x \notin C\} \rightarrow y \in A] \equiv \forall y[(y \in B \land y \notin C) \rightarrow y \in A]$

c) $x \in \bigcap_{i \in I}(A_i \cup B_i) \equiv x \in \{y \mid \forall_{i \in I}[y \in (A_i \cup B_i)]\} \equiv x \in \{y \mid \forall_{i \in I}[y \in A_i \lor y \in B_i]\} \equiv \forall_{i \in I}[x \in A_i \lor x \in B_i]$

d) $x \in (\bigcap_{i \in I} A_i) \cup (\bigcap_{i \in I} B_i) \equiv x \in \{y \mid \forall_{i \in I}(y \in A_i)\} \lor x \in \{z \mid \forall_{i \in I}(z \in B_i)\} \equiv \forall_{i \in I}(x \in A_i) \lor \forall_{i \in I}(x \in B_i)$

3. We've seen that $\mathscr{P}(\varnothing) = \{\varnothing\}$, and $\{\varnothing\} \neq \varnothing$. What is $\mathscr{P}(\{\varnothing\})$?

$P(\varnothing) = \{\varnothing, \{\varnothing\}\}$

*4. Suppose $\mathcal{F}$ = {{red, green, blue}, {orange, red, blue}, {purple, red, green, blue}}. Find $\bigcap\mathcal{F}$ and $\bigcup\mathcal{F}$.

$\bigcap\mathcal{F}$ = {red, blue} $\bigcup\mathcal{F}$ = {red, green, blue, orange, purple}

5. Suppose $\mathcal{F} = \{\{3,7,12\}, \{5,7,16\}, \{5,12,23\}\}$. Find $\bigcap\mathcal{F}$ and $\bigcup\mathcal{F}$.

$\bigcap\mathcal{F} = \varnothing$ $\bigcup\mathcal{F} = \{3,5,7,12,16,23\}$

6. Let $I = \{2,3,4,5\}$, and for each $i \in I$ let $A_i = \{i, i+1, i-1, 2i\}$.
(a) List the elements of all the sets A_i, for $i \in I$.
(b) Find $\bigcap_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$.

a) $A_2 = \{1,2,3,4\}$ $A_3 = \{2,3,4,6\}$ $A_4 = \{3,4,5,8\}$ $A_5 = \{4,5,6,10\}$

b) $\bigcap_{i \in I} A_i = \{4\}$ $\bigcup_{i \in I} A_i = \{1,2,3,4,5,6,8,10\}$

7. Let P = {Johann Sebastian Bach, Napoleon Bonaparte, Johann Wolfgang von Goethe, David Hume, Wolfgang Amadeus Mozart, Isaac Newton, George Washington} and let $Y = \{1750, 1751, 1752, \ldots, 1759\}$. For each $y \in Y$, let $A_y = \{p \in P \mid$ the person p was alive at some time during the year $y\}$. Find $\bigcup_{y \in Y} A_y$ and $\bigcap_{y \in Y} A_y$.

Bach [1750] Hume [1750–1759] Washington [1750–1759]
Bonaparte [ ] Mozart [1756–1759]
Goethe [1750–1759] Newton [ ]

$\bigcap_{y \in Y} A_y$ = {Goethe, Hume, Washington}

$\bigcup_{y \in Y} A_y$ = {Goethe, Hume, Washington, Mozart, Bach}

*8. Let $I = \{2,3\}$, and for each $i \in I$ let $A_i = \{i, 2i\}$ and $B_i = \{i, i+1\}$.
(a) List the elements of the sets A_i and B_i for $i \in I$.
(b) Find $\bigcap_{i \in I}(A_i \cup B_i)$ and $(\bigcap_{i \in I} A_i) \cup (\bigcap_{i \in I} B_i)$. Are they the same?
(c) In parts (c) and (d) of exercise 2 you analyzed the statements $x \in \bigcap_{i \in I}(A_i \cup B_i)$ and $x \in (\bigcap_{i \in I} A_i) \cup (\bigcap_{i \in I} B_i)$. What can you conclude from your answer to part (b) about whether or not these statements are equivalent?

a) $A_2 = \{2,4\}$ $B_2 = \{2,3\}$
$A_3 = \{3,6\}$ $B_3 = \{3,4\}$

b) $\bigcap_{i \in \{2,3\}}(A_i \cup B_i) = \{3,4\}$ $(\bigcap_{i \in \{2,3\}} A_i) \cup (\bigcap_{i \in \{2,3\}} B_i) = \varnothing \cup \{3\} = \{3\}$

c) Não são equivalentes como foi demonstrado
$\forall_{i \in I}(x \in A_i) \lor \forall_{i \in I}(x \in B_i) \not\equiv \forall_{i \in I}[x \in A_i \lor x \in B_i]$

9. (a) Analyze the logical forms of the statements $x \in \bigcup_{i\in I}(A_i \setminus B_i)$, $x \in (\bigcup_{i\in I} A_i)\setminus(\bigcup_{i\in I} B_i)$, and $x \in (\bigcup_{i\in I} A_i)\setminus(\bigcap_{i\in I} B_i)$. Do you think that any of these statements are equivalent to each other?
(b) Let I, A_i, and B_i be defined as in exercise 8. Find $\bigcup_{i\in I}(A_i\setminus B_i)$, $(\bigcup_{i\in I} A_i)\setminus(\bigcup_{i\in I} B_i)$, and $(\bigcup_{i\in I} A_i)\setminus(\bigcap_{i\in I} B_i)$. Now do you think any of the statements in part (a) are equivalent?

Não são equivalentes

a) • $x\in \bigcup_{i\in I}(A_i\setminus B_i) \equiv x\in\{y \mid \exists Y\in(A_i\setminus B_i)(y\in Y)\} \equiv \exists Y(Y\in A_i \wedge Y\notin B_i \wedge x\in Y)$

• $x\in[\bigcup_{i\in I}(A_i)\setminus\bigcup_{i\in I}(B_i)] \equiv x\in[\{y\mid\exists Y\in A_i(y\in Y)\}\setminus\{z\mid\exists Z\in B_i(z\in Z)\}] \equiv$
$x\in\{y\mid \exists Y\in A_i(y\in Y)\}\wedge\neg x\in\{z\mid\exists Z\in B_i(z\in Z)\} \equiv \exists Y\in A_i(x\in Y)\wedge\neg\exists Z\in B_i(x\in Z)\equiv$
$\exists Y(Y\in A_i\wedge x\in Y)\wedge\forall Z\in B_i(x\notin Z)\equiv\exists Y(Y\in A_i\wedge x\in Y)\wedge\forall Z(Z\in B_i\to x\notin Z)$

• $x\in[\bigcup_{i\in I}(A_i)\setminus\bigcap_{i\in I}(B_i)] \equiv x\in[\{y\mid\exists Y\in A_i(y\in Y)\}\setminus\{z\mid\forall Z\in B_i(z\in Z)\}] \equiv$
$x\in\{y\mid \exists Y\in A_i(y\in Y)\}\wedge\neg x\in\{z\mid\forall Z\in B_i(z\in Z)\} \equiv \exists Y\in A_i(x\in Y)\wedge\neg\forall Z\in B_i(x\in Z)\equiv$
$\exists Y\in A_i(x\in Y)\wedge\exists Z\in B_i(x\notin Z)\equiv\exists Y(Y\in A_i\wedge x\in Y)\wedge\exists Z(Z\in B_i\wedge x\notin Z)$

b) $A_2=\{2,4\}$ $B_2=\{2,3\}$
$A_3=\{3,6\}$ $B_3=\{3,4\}$
$A_2\setminus B_2=\{4\}$
$A_3\setminus B_3=\{6\}$

$\bigcup_{i\in\{2,3\}}(A_i\setminus B_i)=\{4,6\}$
$\bigcup_{i\in\{2,3\}}A_i\setminus\bigcup_{i\in\{2,3\}}B_i=\{6\}$
$\bigcup_{i\in\{2,3\}}A_i\setminus\bigcap_{i\in\{2,3\}}B_i=\{2,4,6\}$

Continuam não equivalentes

10. Give an example of an index set I and indexed families of sets $\{A_i \mid i\in I\}$ and $\{B_i\mid i\in I\}$ such that $\bigcup_{i\in I}(A_i\cap B_i)\neq(\bigcup_{i\in I}A_i)\cap(\bigcup_{i\in I}B_i)$.

$A_i=\{i,2i\}$ $A_0=$ $B_0=$
$B_i=\{i,i+1\}$

n	A	B	A∩B
0	{0}	{0,1}	{0}
1	{1,2}	{1,2}	{1,2}

I	U(A∩B)	UA∩UB
{0}	{1,2}	{1,2}
{0,1}	{0,1,2}	{0,1,2}

*No caso de $I=\{0,1\}$ funciona

11. Show that for any sets A and B, $\mathscr{P}(A\cap B)=\mathscr{P}(A)\cap\mathscr{P}(B)$, by showing that the statements $x\in\mathscr{P}(A\cap B)$ and $x\in \mathscr{P}(A)\cap\mathscr{P}(B)$ are equivalent. (See Example 2.3.3.)

$X\in P(A\cap B)\equiv X\subseteq A\cap B\equiv\forall x(x\in X\to(x\in A\wedge x\in B))\equiv\forall x[\neg x\in X\vee(x\in A\wedge x\in B)]\equiv$
$\forall x[(\neg x\in X\vee x\in A)\wedge(\neg x\in X\vee x\in B)]\equiv\forall x(\neg x\in X\vee x\in A)\wedge\forall x(\neg x\in X\vee x\in B)\equiv X\subseteq A\wedge X\subseteq B\equiv X\in P(A)\wedge X\in P(B)$

*12. Give examples of sets A and B for which $\mathscr{P}(A\cup B)\neq\mathscr{P}(A)\cup\mathscr{P}(B)$.

$A=\varnothing$ $P(A)=\{\varnothing\}$
$B=\varnothing$ $P(B)=\{\varnothing\}$
$A\cup B=\varnothing$ $P(A\cup B)=\{\varnothing\}$
$P(A)\cup P(B)=\{\varnothing\}$

$A=\{1\}$ $P(A)=\{\varnothing,\{1\}\}$
$B=\varnothing$ $P(B)=\{\varnothing\}$
$A\cup B=\{1\}$ $P(A\cup B)=\{\varnothing,\{1\}\}$
$P(A)\cup P(B)=\{\varnothing,\{1\}\}$

$A=\{1\}$ $P(A)=\{\varnothing,\{1\}\}$
$B=\{2\}$ $P(B)=\{\varnothing,\{2\}\}$
$A\cup B=\{1,2\}$ $P(A\cup B)=\{\varnothing,\{1\},\{2\},\{1,2\}\}$
$P(A)\cup P(B)=\{\varnothing,\{1\},\{2\}\}$

13. Verify the following identities by writing out (using logical symbols) what it means for an object x to be an element of each set and then using logical equivalences.
(a) $\bigcup_{i\in I}(A_i\cup B_i)=(\bigcup_{i\in I}A_i)\cup(\bigcup_{i\in I}B_i)$.
(b) $(\bigcap\mathcal{F})\cap(\bigcap\mathcal{G})=\bigcap(\mathcal{F}\cup\mathcal{G})$.
(c) $\bigcap_{i\in I}(A_i\setminus B_i)=(\bigcap_{i\in I}A_i)\setminus(\bigcup_{i\in I}B_i)$.

a) $x\in\bigcup_{i\in I}(A_i\cup B_i)\equiv x\in\bigcup_{i\in I}(A_i\cup B_i)\equiv x\in\{y\mid\exists i\in I(y\in A_i\cup B_i)\}\equiv$
$\exists i\in I(x\in A_i\cup B_i)\equiv\exists i\in I(x\in A_i\vee x\in B_i)\equiv\exists i\in I(x\in A_i)\vee\exists i\in I(x\in B_i)\equiv$
$x\in\{z\mid\exists i\in I(z\in A_i)\}\vee x\in\{w\mid\exists i\in I(w\in B_i)\}\equiv x\in(\bigcup_{i\in I}A_i\cup\bigcup_{i\in I}B_i)\equiv x\in[\bigcup_{i\in I}(A_i)\cup\bigcup_{i\in I}(B_i)]$

b) $x\in\bigcap(\mathcal F\cup\mathcal G)\equiv x\in\{w\mid\forall W\in(\mathcal F\cup\mathcal G)(w\in W)\}\equiv\forall W\in(\mathcal F\cup\mathcal G)(x\in W)\equiv\forall W[(W\in\mathcal F\vee W\in\mathcal G)\to x\in W]\equiv\forall W[\neg(W\in\mathcal F\vee W\in\mathcal G)\vee x\in W]\equiv\forall W[(W\notin\mathcal F\wedge W\notin\mathcal G)\vee x\in W]\equiv$
$\forall W[(W\notin\mathcal F\vee x\in W)\wedge(W\notin\mathcal G\vee x\in W)]\equiv\forall W(\neg W\in\mathcal F\vee x\in W)\wedge\forall W(\neg W\in\mathcal G\vee x\in W)\equiv\forall W\in\mathcal F(x\in W)\wedge\forall W\in\mathcal G(x\in W)\equiv x\in\{y\mid\forall W\in\mathcal F(y\in W)\}\wedge x\in\{z\mid\forall W\in\mathcal G(z\in W)\}\equiv$
$x\in[\{y\mid\forall W\in\mathcal F(y\in W)\}\cap\{z\mid\forall W\in\mathcal G(z\in W)\}]\equiv x\in[(\bigcap\mathcal F)\cap(\bigcap\mathcal G)]$

c) $x\in\bigcap_{i\in I}(A_i\setminus B_i)\equiv x\in\{y\mid\forall i\in I[y\in(A_i\setminus B_i)]\}\equiv\forall i\in I[x\in(A_i\setminus B_i)]\equiv\forall i\in I(x\in A_i\wedge x\notin B_i)\equiv\forall i\in I(x\in A_i)\wedge\forall i\in I(x\notin B_i)\equiv\forall i\in I(x\in A_i)\wedge\forall i\in I(\neg x\in B_i)\equiv$
$\forall i\in I(x\in A_i)\wedge\neg\exists i\in I(x\in B_i)\equiv(x\in\{y\mid\forall i\in I(y\in A_i)\})\wedge\neg(x\in\{z\mid\exists i\in I(z\in B_i)\})\equiv(x\in\{y\mid\forall_{i\in I}(y\in A_i)\})\wedge(x\notin\{z\mid\exists i\in I(z\in B_i)\})\equiv x\in[(x\in\{y\mid\forall_{i\in I}(y\in A_i)\})\setminus(x\in\{z\mid\exists i\in I(z\in B_i)\})]\equiv$
$x\in[\bigcap_{i\in I}A_i\setminus\bigcup_{i\in I}B_i]$

*14. Sometimes each set in an indexed family of sets has *two* indices. For this problem, use the following definitions: $I=\{1,2\}$, $J=\{3,4\}$. For each $i\in I$ and $j\in J$, let $A_{i,j}=\{i,j,i+j\}$. Thus, for example, $A_{2,3}=\{2,3,5\}$.
(a) For each $j\in J$ let $B_j=\bigcup_{i\in I}A_{i,j}=A_{1,j}\cup A_{2,j}$. Find B_3 and B_4.
(b) Find $\bigcap_{j\in J}B_j$. (Note that, replacing B_j with its definition, we could say that $\bigcap_{j\in J}B_j=\bigcap_{j\in J}(\bigcup_{i\in I}A_{i,j})$.)
(c) Find $\bigcup_{i\in I}(\bigcap_{j\in J}A_{i,j})$. (Hint: You may want to do this in two steps, corresponding to parts (a) and (b).) Are $\bigcap_{j\in J}(\bigcup_{i\in I}A_{i,j})$ and $\bigcup_{i\in I}(\bigcap_{j\in J}A_{i,j})$ equal?
(d) Analyze the logical forms of the statements $x\in\bigcap_{j\in J}(\bigcup_{i\in I}A_{i,j})$ and $x\in\bigcup_{i\in I}(\bigcap_{j\in J}A_{i,j})$. Are they equivalent?

a) $B_j=A_{1,j}\cup A_{2,j}$ $A_{1,3}=\{1,3,4\}$ $A_{2,3}=\{2,3,5\}$ $A_{1,4}=\{1,4,5\}$ $A_{2,4}=\{2,4,6\}$

$B_3=A_{1,3}\cup A_{2,3}=\{1,2,3,4,5\}$ $B_4=A_{1,4}\cup A_{2,4}=\{1,2,4,5,6\}$

b) $\bigcap_{j\in\{3,4\}}B_j=\{1,2,4,5\}$

c) $\bigcap_{j\in\{3,4\}}A_{i,j}=\{i,3,i+3\}\cap\{i,4,i+4\}=C_i=\{i\}$
$\bigcup_{i\in\{1,2\}}C_i=\{1,4\}\cup\{2\}=\{1,2,4\}$
$\bigcup_{i\in\{1,2\}}A_{i,j}=\{1,j,1+j\}\cup\{2,j,2+j\}=D_j=\{1,2,j,1+j,2+j\}$
$\bigcap_{j\in\{3,4\}}D_j=\{1,2,3,4,5\}\cap\{1,2,4,5,6\}=\{1,2,4,5\}$
* Não são iguais.

d) $x\in\bigcap_{j\in J}(\bigcup_{i\in I}A_{i,j})\equiv x\in\bigcap_{j\in J}\{y\mid\exists i\in I(y\in A_{i,j})\}\equiv$
$x\in\{z\mid\forall j\in J(z\in\{y\mid\exists i\in I(y\in A_{i,j})\})\}\equiv\forall j\in J(x\in\{y\mid\exists i\in I(y\in A_{i,j})\})\equiv$
$\forall j\in J(\exists i\in I(x\in A_{i,j}))\equiv\forall j\in J[\exists i(i\in I\wedge x\in A_{i,j})]\equiv$
$\forall j[j\in J\to\exists i(i\in I\wedge x\in A_{i,j})]$

$x\in\bigcup_{i\in I}(\bigcap_{j\in J}A_{i,j})\equiv x\in\bigcup_{i\in I}\{z\mid\forall j\in J(z\in A_{i,j})\}\equiv$
$x\in\{y\mid\exists i\in I(y\in\{z\mid\forall j\in J(z\in A_{i,j})\})\}\equiv\exists i\in I(x\in\{z\mid\forall j\in J(z\in A_{i,j})\})\equiv$
$\exists i\in I(\forall j\in J(x\in A_{i,j}))\equiv\exists i\in I[\forall j(j\in J\to x\in A_{i,j})]\equiv\exists i[i\in I\wedge\forall j(j\in J\to x\in A_{i,j})]$

* Não são equivalentes

15. (a) Show that if $\mathcal{F} = \varnothing$, then the statement $x \in \bigcup \mathcal{F}$ will be false no matter what x is. It follows that $\bigcup \varnothing = \varnothing$.

(b) Show that if $\mathcal{F} = \varnothing$, then the statement $x \in \bigcap \mathcal{F}$ will be true no matter what x is. In a context in which it is clear what the universe of discourse U is, we might therefore want to say that $\bigcap \varnothing = U$. However, this has the unfortunate consequence that the notation $\bigcap \varnothing$ will mean different things in different contexts. Furthermore, when working with sets whose elements are sets, mathematicians often do not use a universe of discourse at all. (For more on this, see the next exercise.) For these reasons, some mathematicians consider the notation $\bigcap \varnothing$ to be meaningless. We will avoid this problem in this book by using the notation $\bigcap \mathcal{F}$ only in contexts in which we can be sure that $\mathcal{F} \neq \varnothing$.

a) $x \in \{y \mid \exists Y \in \mathcal{F}(y \in Y)\} \equiv \exists Y \in \mathcal{F}(x \in Y) \equiv \exists Y(Y \in \mathcal{F} \wedge x \in Y)$, falso pois não existe $Y \in \mathcal{F}$ se $\mathcal{F} = \varnothing$.
Logo $\bigcup \mathcal{F} = \varnothing$.

b) $x \in \bigcap \mathcal{F} \equiv x \in \{y \mid \forall Y \in \mathcal{F}(y \in Y)\} \equiv \forall Y(Y \in \mathcal{F} \to y \in Y) \equiv \forall Y(Y \in \varnothing \to y \in Y)$, sendo sempre verdadeiro pois $Y \in \mathcal{F}$ é falso e de uma falsidade se segue que tudo é verdade, logo $\bigcap \mathcal{F}$ contera todos os elementos quando $\mathcal{F} = \varnothing$.

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16. In Section 2.3 we saw that a set can have other sets as elements. When discussing sets whose elements are sets, it might seem most natural to consider the universe of discourse to be the set of all sets. However, as we will see in this problem, assuming that there is such a set leads to contradictions.

Suppose U were the set of all sets. Note that in particular U is a set, so we would have $U \in U$. This is not yet a contradiction; although most sets are not elements of themselves, perhaps some sets are elements of themselves. But it suggests that the sets in the universe U could be split into two categories: the unusual sets that, like U itself, are elements of themselves, and the more typical sets that are not. Let R be the set of sets in the second category. In other words, $R = \{A \in U \mid A \notin A\}$. This means that for any set A in the universe U, A will be an element of R iff $A \notin A$. In other words, we have $\forall A \in U(A \in R \leftrightarrow A \notin A)$.

(a) Show that applying this last fact to the set R itself (in other words, plugging in R for A) leads to a contradiction. This contradiction was discovered by Bertrand Russell (1872–1970) in 1901, and is known as *Russell's paradox*.

(b) Think some more about the paradox in part (a). What do you think it tells us about sets?

a) $X \in R \equiv X \in \{Y \mid Y \in U \wedge Y \notin Y\} \equiv X \in U \wedge X \notin X$
$R \in R \equiv R \in U \wedge R \notin R$

Como $R \in U$ ja que é o conjunto de todos conjuntos é inevitavel concluir que $R \in R \equiv R \notin R$ e $R \notin R \equiv R \in R$, uma contradição

$X \notin R \equiv X \notin U \vee X \in X$
$R \notin R \equiv R \notin U \vee R \in R$

b) Significa que a teoria ingenua de conjuntos é insuficiente, precisamos seguir para a teoria ZFC de conjuntos para evitar contradições.