CS 3430: S19: SciComp with Py
Lecture 21

Linear Programming in 2 Variables
Part 1

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Linear Optimization Models

Linear Optimization Models

We had exposure to optimization problems that optimize (minimize/maximize) a function of 1 variable (recall local minima/maxima problems, inflection point problems).

Let's move on to optimization problems that require us to use functions of two variables (e.g., lines such as $ax + by = c$). Hence, the term **linear optimization**.

The generic term for optimization problems that use lines is **linear programming**.

Linear programming is one of the most widely applied and effective tools in social, life, and management sciences (e.g., marketing, manufacturing, finance, operations research, distribution, transportation, etc.).

LP Problem Formulation

The most important step in LP is problem formulation. In other words, we need to come up with a function to optimize and a set of constraints that the function must satisfy.

In formulating LP problems, we must be senstive to the implicit assumptions we make.

Example 1: when we formulate an LP problem on diet optimization, we often implicitly assume that we know the nutrious contents of ingredients regradless of where and how they are grown/produced.

Example 2: when we formulate an LP problem that optimizes item manufacturing, we often implicitly assume that the costs are linear, which is a great first approximation but which may or may not be true in a given situation (e.g., bulk discounts, sales, etc.).

LP Problem Formulation: Example 1

Ted's Toys makes toy cars and toy trucks using plastic and steel. Each car requires 4 ounces of plastic and 3 ounces of steel, while each truck requires 3 ounces of plastic and 6 ounces of steel. Each day Ted has 30 pounds of plastic and 45 pounds of steel to use in making toy cars and trucks, and he can sell all the cars and trucks he makes with these materials. His profit is \$5 per car and \$4 per truck. Ted would like to know how many cars and trucks he should make in order to maximize the total profit from the sale of these toys.

LP Problem Formulation: Example 1

Ted's goal is to decide how many cars and trucks to make, these quantities are **decision variables**. So, we introduce two decision variables:

x – number of cars to make each day;
y – number of trucks to make each day.

Each car requires 4 ounces of plastic (i.e., $4x$) and each truck requires 3 ounces of plastic (i.e., $3y$).

Since Ted has 30 pounds of plastic available, that is, $30 \cdot 16 = 480$ ounces. So the first constraint is $4x + 3y \leq 480$.

LP Problem Formulation: Example 2

x cars require $3x$ ounces of steel and y trucks require $6y$ ounces of steel, and there are 45 pounds (i.e., $45 \cdot 16 = 720$) of steel available. So the second constraint is $3x + 6y \leq 720$.

We know that x and y cannot be negative. We have the following constraints:

1. $x \geq 0$;
2. $y \geq 0$;
3. $4x + 3y \leq 480$;
4. $3x + 6y \leq 720$.

Ted's profit is \$5 per car and \$4 per truck, so Ted's profit function to maximize is $5x + 4y$.

LP Problem Formulation: Example 2

A hiker is planning her trail food, which is to include a snack mix of peanuts and raisins. Each day she wants 600 calories and 90 grams of carbohydrates from the mix. Each gram of raisins contains 0.8 grams of carbohydrates and 3 calories and costs 4 cents. Each gram of peanuts contains 0.2 grams of carbohydrates and 6 calories and costs 5 cents.

LP Problem Formulation: Example 2

Let's define the decision variables:

x – number of grams of raisins;
y – number of grams of peanuts.

We know from the statement of the problem that $x \geq 0$ and $y \geq 0$.

Since each gram of raisins contains 0.8 grams of carbohydrates and each gram of peanuts contains 0.2 grams of carbohydrates, the first constraint is $0.8x + 0.2y \geq 90$.

Since each gram of raisins contains 3 calories and each gram of peanuts contains 6 calories, the second constraint is $3x + 6y \geq 600$.

The cost function to minimize is $4x + 5y$.

LP Problem Formulation: Example 2

The formulation of Example 2 is as follows.

Minimize $4x + 5y$ subject to the following constraints:

1. $x \geq 0$;
2. $y \geq 0$;
3. $0.8x + 0.2y \geq 90$;
4. $3x + 6y \geq 600$.

LP Problem Formulation: Example 3

An executive divides his time between sales and support activities (paperwork and reading up on new products). Keeping up to date on new products requires at least 5 hours each week reading trade newspapers and magazines. Each hour he devotes to sales generates 0.1 hours of paperwork. He plans to devote at least half of his time to sales and at most 50 hours per week to his job. He estimates that the time he devotes to sales is worth \$15 per hour and that the time he devotes to support activities is worth \$10 per hour.

LP Problem Formulation: Example 3

Let x and y denote the number of hours devoted to sales and support activities, respectively.

The total number of hours worked is $x + y$, so the first constraint is $x + y \leq 50$.

Since he plans to devote at least half of his time to sales, the second constraint $x \geq 0.5(x + y)$ or $0.5x - 0.5y \geq 0$.

Since each hour devoted to sales generates 0.1 hour of paperwork and an additional 5 hours of support are required each week. Hence, the third constraint is $y \geq 5 + 0.1x$ or $-0.1x + y \geq 5$.

LP Problem Formulation: Example 3

The formulation of this problem is as follows.

Maximize the following function $15x + 10y$ subject to the following constraints:

1. $x \geq 0$;
2. $y \geq 0$;
3. $x + y \leq 50$;
4. $0.5x - 0.5y \geq 0$;
5. $-0.1 + y \geq 5$.

Systems of Linear Equalities in 2 Variables

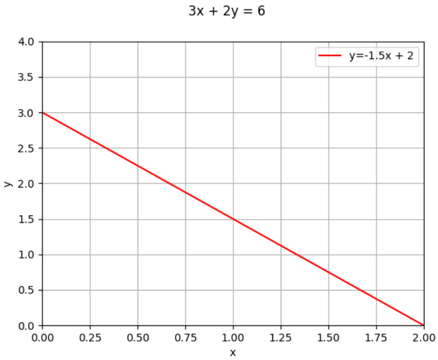
Geometric Solutions

Each line of the form $Ax + By \leq C$ or $Ax + By \geq C$ divides the plane into 2 *half planes*.

The line $Ax + By = C$ is the *boundary*.

We can use any point off the specified line $Ax + By = C$ to determine which half plane satisfies the inequality $Ax + By \leq C$ or $Ax + By \geq C$.

Example



The set of points (x, y) that satisfy $3x + 2y \leq 6$ lie below this line. We can use any point not on the line to check if that point satisfies $3x + 2y \leq 6$. For example, since $(0, 0)$ satisfies $3 \cdot 0 + 2 \cdot 0 \leq 6$.

Problem

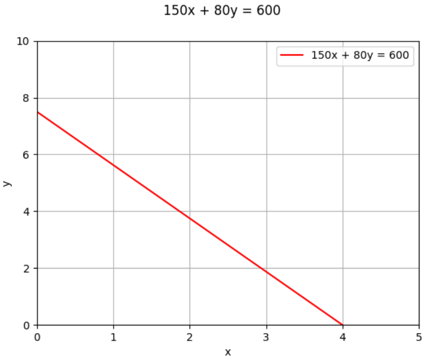
A hiker thinks that on a long hike she needs snacks with at least 600 calories. She plans to take chocolate and raisins. The chocolate has 150 calories per ounce. The raisins have 80 calories per ounce. Find and graph the system of inequalities for this problem and the set that satisfies them.

Solution

Let x be the number of ounces of chocolate and y the number of ounces of raisins. We have the following inequalities to constraint this problem.

1. $x \geq 0$;
2. $y \geq 0$;
3. $150x + 80y \geq 600$.

Solution



The set that satisfies this problem is $x \geq 0 \cap y \geq 0 \cap 150x + 80y \geq 600$.

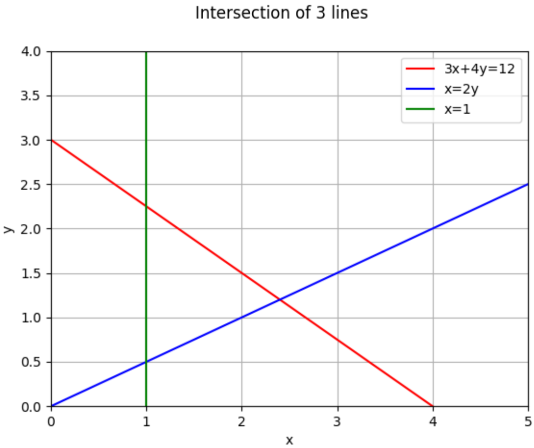
Problem

Graph the set of points which satisfy the inequalities:

1. $x \geq 1$;
2. $x \leq 2y$;
3. $3x + 4y \leq 12$.

Solution

The solution is the internal region of the intersection.



Constraints, Feasible Sets, and Objective Functions

The constraints we will work with will always be inclusive, i.e., the ineqalities expressed with $\geq$ and $\leq$.

The set of points satisfying the constraints of the problem is known as the **feasible set** of the problem.

The function to be maximized or minimized is known as the **objective function** for the problem.

LP Problem Formulation: 3 Steps

1) Define the decision variables.

2) Formulate the constraints using the variables.

3) Define the objective function using the decision variables.

References

1. D.P. Maki, M. Thompson. *Finite Mathematics*, Chapter 7. McGraw-Hill.
2. `www.python.org`.