CS 3430: S19: SciComp with Py
Lecture 24

Linear Programming in 2 Variables
Part 3

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Review

Identity Matrix

An **identity** matrix is a square matrix (i.e., the number of rows is equal to the number of columns) where all diagonal elements are equal to 1 and the other elements are equal to 0. It is typically denoted as $\mathbf{I}_m$.

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Matrix

The **inverse** of an n x n matrix $\mathbf{A}$, denoted as $\mathbf{A}^{-1}$, is a matrix such that $\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$.

If we can find the inverse of some matrix $\mathbf{A}$, then $\mathbf{A}$ is **invertible**; if $\mathbf{A}$ cannot be inverted, it is called **singular**.

Matrix Transpose

The **transpose** of a matrix $\mathbf{A}$, denoted as $\mathbf{A}^T$ is a reordering of $\mathbf{A}$ where the rows are interchanged with columns, in order. Row 1 of $\mathbf{A}$ becomes column 1 of $\mathbf{A}^T$, row 2 of $\mathbf{A}$ becomes column 2 of $\mathbf{A}^T$, etc. In other words,

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}$$

$$\mathbf{A}^T = \begin{bmatrix} a_{1,1} & a_{2,1} & \dots & a_{m,1} \\ a_{1,2} & a_{2,2} & \dots & a_{m,2} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ a_{1,n} & a_{2,n} & \dots & a_{m,n} \end{bmatrix}$$

Matrix Transpose Properties

- $(\mathbf{A}^T)^T = A$ (the transpose of the transpose).
- $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$ (transpose of a sum).
- $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ (transpose of a product).

Augmented Matrix

An augmented matrix is the matrix in which rows or columns of another matrix of the appropriate order are appended to the original matrix. If $\mathbf{A}$ is augmented on the right with $\mathbf{B}$, the resultant matrix is denoted as $(\mathbf{A}|\mathbf{B})$. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 5 & 6 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}; \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $(\mathbf{A}|\mathbf{B}) = \left[\begin{array}{cc|c} 1 & 4 & 3 \\ 5 & 6 & 1 \end{array}\right]$ and $(\mathbf{A}|\mathbf{I}) = \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{array}\right]$.

Generic Linear System

A generic linear systems with m equations in n unknowns is written as follows:

$$\begin{gathered}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
. \\
. \\
. \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m
\end{gathered}$$

Since the system is determined by its $m \times n$ coefficient matrix $\mathbf{A} = [a_{ij}]$ and its column vector $\mathbf{b}$, it can be written as $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{x}$ is a column vector $(x_1, x_2, \ldots, x_n)$.

Generic Linear System as Augmented Matrix

A generic linear systems with m equations in n can be expressed with the following augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ . & . & \ldots & . & . \\ . & . & \ldots & . & . \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right]$$

The above matrix is typically shorthanded as $[\mathbf{A}|\mathbf{b}]$.

Row Equivalence

- If a matrix $\mathbf{B}$ can be obtained from a matrix $\mathbf{A}$ by a sequence of elementary row operations, then $\mathbf{B}$ is **row equivalent** to $\mathbf{A}$.
- Since each elementary row operation can be undone (reversed), if $\mathbf{B}$ is row equivalent to $\mathbf{A}$, denoted as $\mathbf{B} \sim \mathbf{A}$, then $\mathbf{A}$ is row equivalent to $\mathbf{B}$, i.e., $\mathbf{A} \sim \mathbf{B}$.
- The elementary row operations do not change the solution set of an augmented matrix.

A Fundamental Theorem of Linear Algebra

If $[\mathbf{A}|\mathbf{b}] \sim [\mathbf{H}|\mathbf{c}]$, then the corresponding linear systems $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{H}\mathbf{x} = \mathbf{c}$ have the same solution set.

Row Echelon Form and Pivot

A matrix is in **row echelon form** if:

- All rows containing only zeros appear below the rows containing nonzero entries.
- The first nonzero entry in any row appears in a column to the right of the first nonzero entry in any preceding row.

The first nonzero entry in a row of a row echelon form matrix is called the **pivot**.

A Generic Algorithm Fundamental for Solving a Linear System

Given a linear system $\mathbf{Ax} = \mathbf{b}$, obtain $[\mathbf{A}|\mathbf{b}]$, row-reduce it to $[\mathbf{H}|\mathbf{c}]$, where $\mathbf{H}$ is in row echelon form and use back substitution to find a solution (or not in case of inconsistency).

Computation of the Inverse Matrix $\mathbf{A}^{-1}$

The inverse of a matrix $\mathbf{A}$ is denoted as $\mathbf{A}^{-1}$. To find $\mathbf{A}^{-1}$, if it exists, proceed as follows:

1. From the augmented matrix $[\mathbf{A}|\mathbf{I}]$.
2. Apply the Gauss method to attempt to reduce $[\mathbf{A}|\mathbf{I}]$ to $[\mathbf{I}|\mathbf{C}]$. If the reduction can be carried out, then $\mathbf{A}^{-1} = \mathbf{C}$. Otherwise, $\mathbf{A}^{-1}$ does not exist.

Solving Linear Programming Problems in 2 Variables

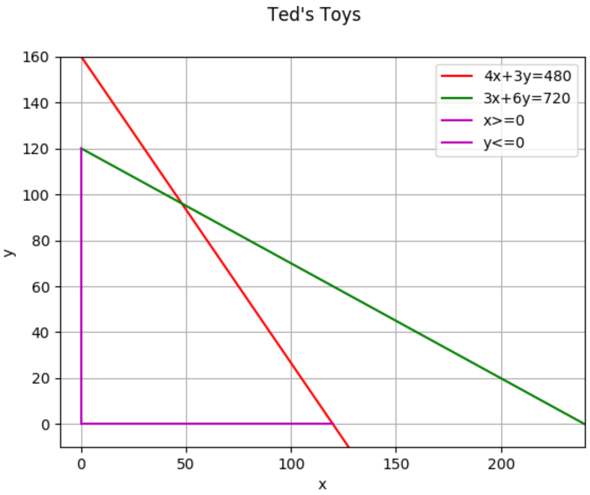
General Method of Solving LP Problems

1. Identify the feasible set;
2. Find the corner points;
3. Evaluate the objective function at the corner points;
4. Check if there is a solution.

Problem

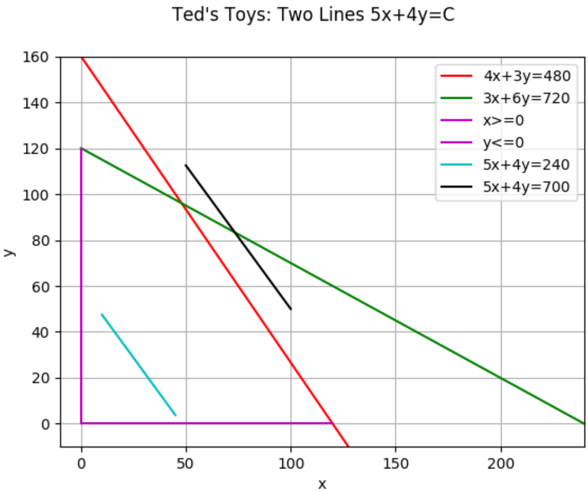
Ted's Toys makes toy cars and toy trucks using plastic and steel. Each car requires 4 ounces of plastic and 3 ounces of steel, while each truck requires 3 ounces of plastic and 6 ounces of steel. Each day Ted has 30 pounds of plastic and 45 pounds of steel to use in making toy cars and trucks, and he can sell all the cars and trucks he makes with these materials. His profit is \$5 per car and \$4 per truck. Ted would like to know how many cars and trucks he should make in order to maximize the total profit from the sale of these toys.

Solution: Identification of Feasible Set



Corner points of the feasible set: (0, 0), (0, 120), (120, 0), (48, 96).

Solution: Two Possible Lines $5x + 3y = C$



Corner points of the feasible set: (0, 0), (0, 120), (120, 0), (48, 96).

Solution: Evaluate Objective Function at Corner Points

Let's evaluate the objective function $5x + 4y$ at (0, 0), (0, 120), (120, 0), (48, 96):

1. $5 \cdot 0 + 4 \cdot 0 = 0$;
2. $5 \cdot 0 + 4 \cdot 120 = 480$;
3. $5 \cdot 120 + 4 \cdot 0 = 600$;
4. $5 \cdot 48 + 4 \cdot 96 = 624$.

So the maximum value is achieved at (48, 96).

Solution: Solving the Linear System

Let's solve the linear system:

1. $4x + 3y = 480;$
2. $3x + 6y = 720.$

Solution

We can do it with `numpy`:

```
A = np.array(
        [[4, 3],
         [3, 6]])
b = np.array([480, 720])
x = np.linalg.solve(A, b)
print(x)
assert np.allclose(np.dot(A, x), b)
```

Output:

```
[48.  96.]
```

Solution

Ted is supposed to make $x = 48$ cars and $y = 96$ trucks.

How about the use of materials?

1. $4(48) + 3(96) = 480$ (plastic);
2. $3(48) + 6(96) = 720$ (steel).

All plastic and steel is used.

Problem

Let's reformulate the Ted's Toys problem and assume that the profit per car changed from \$5 to \$6.

Ted's Toys makes toy cars and toy trucks using plastic and steel. Each car requires 4 ounces of plastic and 3 ounces of steel, while each truck requires 3 ounces of plastic and 6 ounces of steel. Each day Ted has 30 pounds of plastic and 45 pounds of steel to use in making toy cars and trucks, and he can sell all the cars and trucks he makes with these materials. His profit is \$6 per car and \$4 per truck. Ted would like to know how many cars and trucks he should make in order to maximize the total profit from the sale of these toys.

Solution

Since the constraints are the same, the corner points are the same. Let's evaluate $6x + 4y$ at the corner points (0, 0), (0, 120), (120, 0), (48, 96):

1. $6 \cdot 0 + 4 \cdot 0 = 0$;
2. $6 \cdot 0 + 4 \cdot 120 = 480$;
3. $6 \cdot 120 + 4 \cdot 0 = 720$;
4. $6 \cdot 48 + 4 \cdot 96 = 672$.

So the maximum value is achieved at (120, 0).

What about the material usage? $4 \cdot 120 + 0 = 480$, so all plastic is used; $3 \cdot 120 + 6 \cdot 0 = 360 < 720$, so not all steel is used. This amount of unused material is called **slack**.

Bounded and Unbounded Sets, Corner Points

A set of points in the plane is **bounded** if it is contained in some circle centered at $(0, 0)$. Otherwise, the set is **unbounded**.

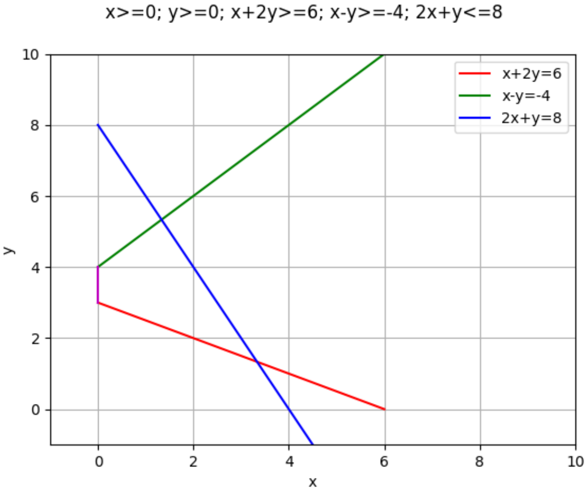
A point T is a **corner point** (aka **extreme point**) of a feasible set for an LP problem if every line segment that is contained in the set and that contains T has T as one of its end points.

Problem

Maximize $x + y$ that satisfies the following constraints:

1. $x \geq 0$;
2. $y \geq 0$;
3. $x + 2y \geq 6$;
4. $x - y \geq -4$;
5. $2x + y \leq 8$.

Solution



Corner points of the feasible set: (0, 4), (4/3, 16/3), (10/3, 4/3), (0, 3).

Solution

Let's evaluate $x + y$ at the corner points (0, 4), (4/3, 16/3), (10/3, 4/3), (0, 3):

1. $0 + 4 = 4$;
2. $\frac{4}{3} + \frac{16}{3} = \frac{20}{3}$;
3. $\frac{10}{3} + \frac{4}{3} = \frac{14}{3}$;
4. $0 + 3 = 3$.

So the maximum value is achieved at $(\frac{4}{3}, \frac{16}{3})$.

Problem

A hiker is planning her trail food, which is to include a snack mix of raisins and peanuts. Each day she wants 600 calories and 90 grams of carbohydrates from the mix. Each gram of raisins contains 0.8 grams of carbohydrates and 3 calories and costs 3 cents. Each gram of peanuts contains 0.2 grams of carbohydrates and 6 calories and costs 5 cents. Let's help the hiker to minimize the cost.

Solution

Let's define the decision variables:

x – number of grams of raisins;
y – number of grams of peanuts.

We know from the statement of the problem that $x \geq 0$ and $y \geq 0$.

Since each gram of raisins contains 0.8 grams of carbohydrates and each gram of peanuts contains 0.2 grams of carbohydrates, the first constraint is $0.8x + 0.2y \geq 90$.

Since each gram of raisins contains 3 calories and each gram of peanuts contains 6 calories, the second constraint is $3x + 6y \geq 600$.

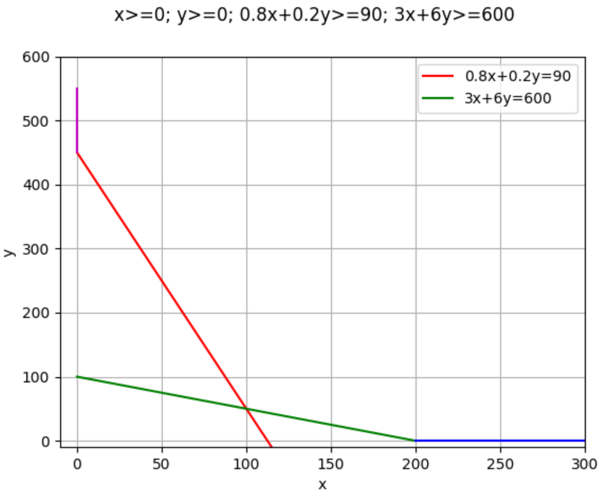
The cost function to minimize is $3x + 5y$.

Solution

Minimize $3x + 5y$ that satisfies the following constraints:

1. $x \geq 0$;
2. $y \geq 0$;
3. $0.8x + 0.2y \geq 90$;
4. $3x + 6y \geq 600$.

Solution



Corner points of the feasible set: (0, 450), (100, 50), (200, 0).

Solution

Let's evaluate $3x + 5y$ at (0, 450), (100, 50), (200, 0):

1. $3 \cdot 0 + 5 \cdot 450 = 2250$;
2. $3 \cdot 100 + 5 \cdot 50 = 550$;
3. $3 \cdot 200 + 5 \cdot 0 = 600$.

So, the minimum value is attained at $(100, 50)$. The hiker needs to take 100 grams of raisins and 50 grams of peanuts.

References

1. D.P. Maki, M. Thompson. *Finite Mathematics*, Ch. 7, McGraw-Hill.
2. `www.python.org`.
3. `docs.scipy.org`.