利用神经网络求解2D-Helmholtz方程

问题描述

As an example, let us consider the Helmholtz equation in two space dimensions:

$$\Delta u(x,y) + k^2 u(x,y) = q(x,y), (x,y) \in \Omega := (-1,1)$$
$$u(x,y) = h(x,y), (x,y) \in \partial\Omega$$

Assuming that the source term is in the form like:

$$q(x,y) = -(a_1\pi)^2 sin(a_1\pi x) sin(a_2\pi y) - (a_2\pi)^2 sin(a_1\pi x) sin(a_2\pi y) + k^2 sin(a_1\pi x) sin(a_2\pi$$

We can easily fabricate an exact solution to this problem:

$$u(x,y) = sin(a_1\pi x) sin(a_2\pi y)$$

环境配置

```
python ==3.9
pytorch == 2.0.0+cu117
gpu == NVIDIA GeForce RTX 3050 Ti Laptop GPU(RAM=4GB)
```

优化目标

For a typical initial and boundary value problem, loss functions would take the form

$$
\begin{gathered}
L=\lambda_1 L_{pde}+\lambda_2 L_{bc}+\lambda_3 L_{init} \\
L_{pde}=\frac{1}{N_{pde}} \sum_{i=1}^{N_{pde}}\left[u\left(x_{pde}^i\right)\right]^2 \\
L_{bc}=\frac{1}{N_{bc}} \sum_{i=1}^{N_{bc}}\left[u\left(x_{bc}^i\right)-g_{bc}^i\right] \\
L_i=\frac{1}{N_i} \sum_{i=1}^{N_i}\left[u\left(x_{bc}^i, t=0\right)-h_{init}^i\right]
\end{gathered}
$$

For the problem in frequency domain, L_{init} is ignored , and the loss function can be simplified as:

$$
\begin{gathered}
L=\lambda_1 L_{pde}+\lambda_2 L_{bc} \\
L_{pde}=\frac{1}{N_{pde}} \sum_{i=1}^{N_{pde}}\left[u\left(x_{pde}^i, y_{pde}^i\right)\right]^2 \\
L_{bc}=\frac{1}{N_{bc}} \sum_{i=1}^{N_{bc}}\left[u\left(x_{bc}^i, y_{bc}^i\right)-g_{bc}^i\right]
\end{gathered}
$$

数据设计

According to the problem described before , we need to generate the data for PINN. We assume the area is $x \in(-1,1), y \in(-1,1)$. It is obviously that the more data points we use, the better the training effect. But huge amount of data will significantly increase the computational cost, thus an appropriate amount of data is necessary.

First , we seperate the area to the shape of 256*256 as test data.

```
x_1 = np.linspace(-1,1,256)  # 256 points between -1 and 1 [256x1]
x_2 = np.linspace(1,-1,256)  # 256 points between 1 and -1 [256x1]
```

Second , we conduct Latin sampling in the interval as PDE point , 'N_f' is the number of sampled data .

```
X_f = lb + (ub-lb)*lhs(2,N_f)
```

Thirdly , we sample boundary point randomly with the number 'N_u'

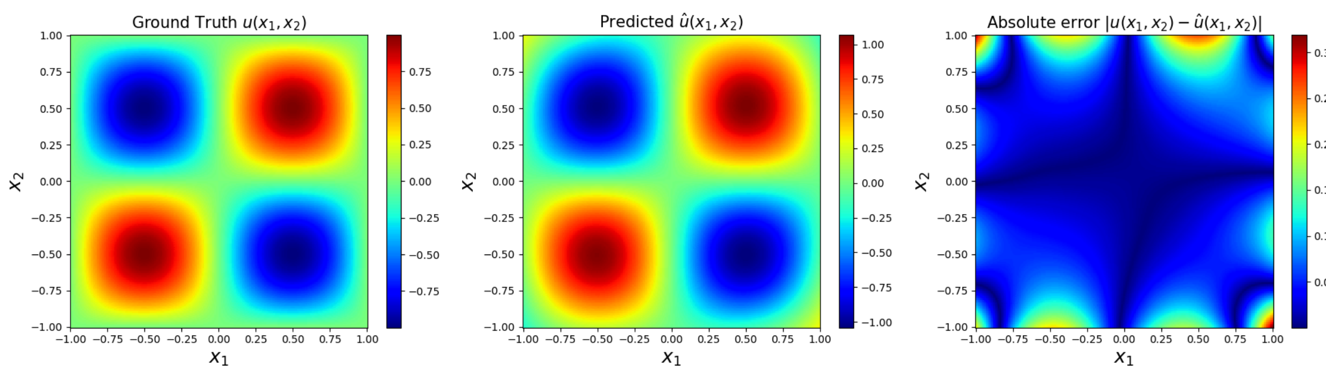
```
idx = np.random.choice(all_X_u_train.shape[0], N_u, replace=False)
X_u_train = all_X_u_train[idx[0:N_u], :] #choose indices from  set 'idx' (x,t)
u_train = all_u_train[idx[0:N_u],:]      #choose corresponding u
```

研究目标

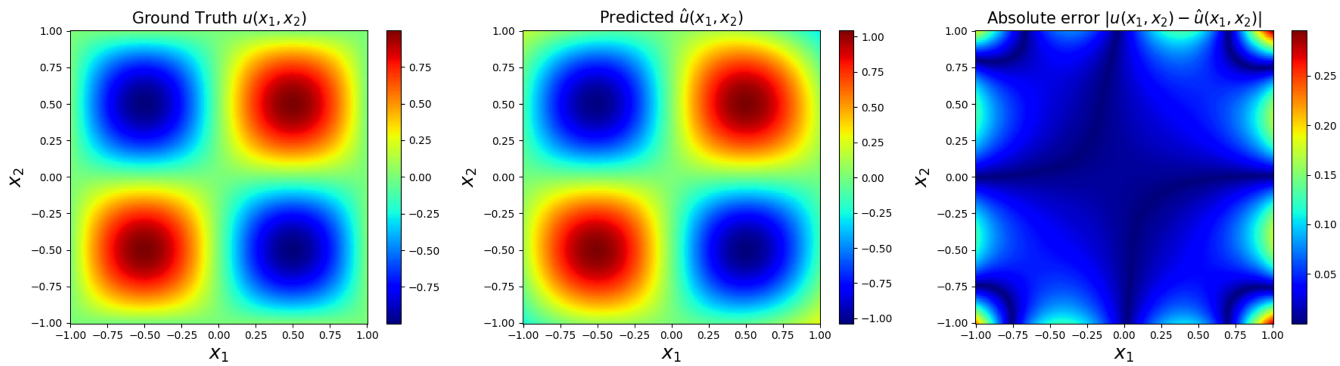
- Solve the 2D Helmholtz equation with source correctly ?
- Time used ?
- The influence of bc point number and pde point number ?

改变边界控制点数

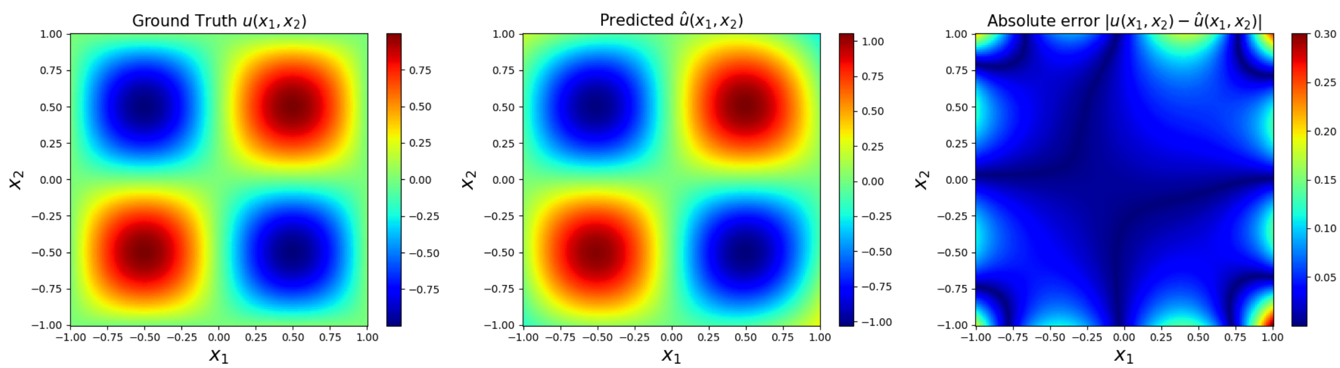
```
N_u = 100
N_f = 10000
Training time: 21.98
Test Error: 0.13519
```



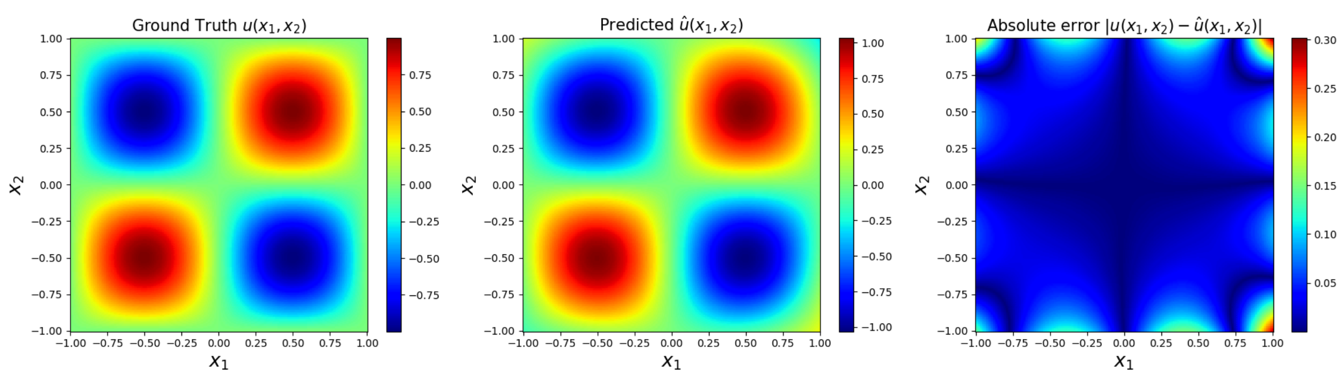

```
N_u = 200
N_f = 10000
Training time: 21.98
Test Error: 0.10051
```




```
N_u = 300
N_f = 10000
Training time: 22.74
Test Error: 0.10007
```

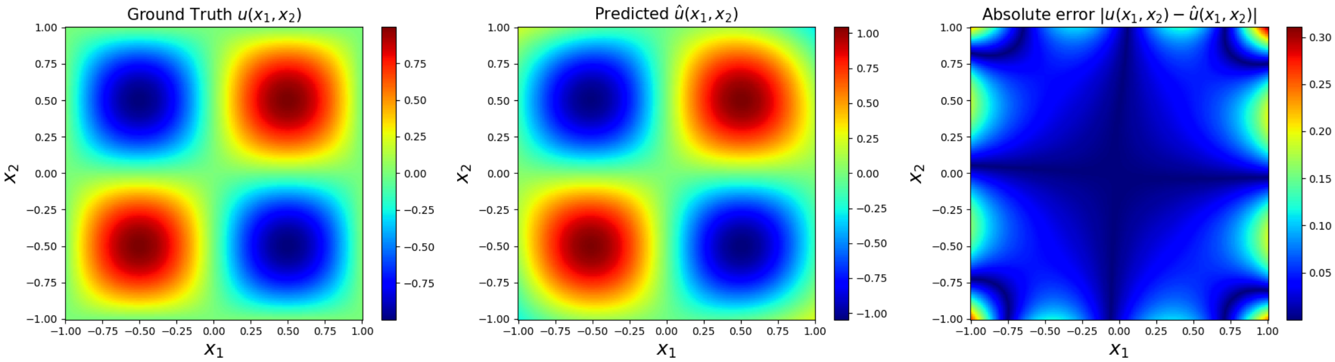



```
N_u = 400
N_f = 10000
Training time: 22.23
Test Error: 0.08802
```

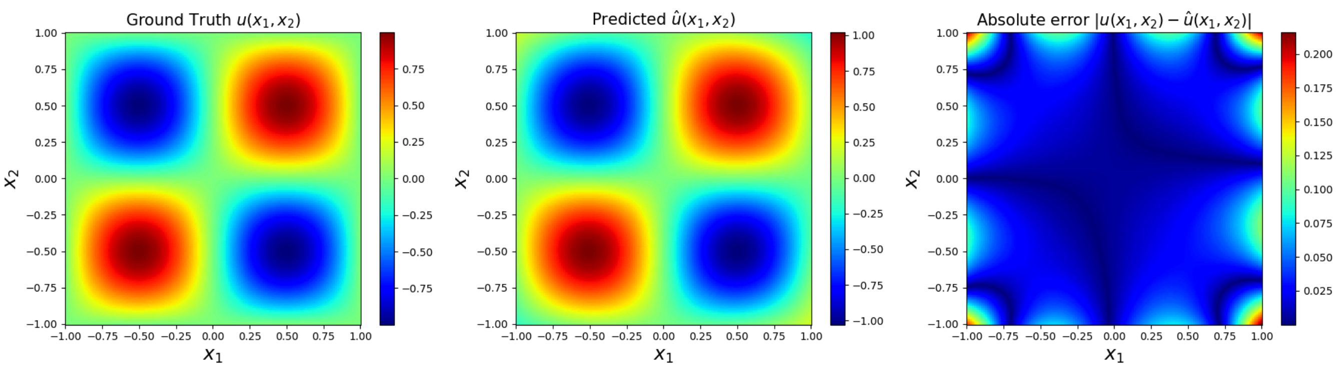


改变PDE控制点数

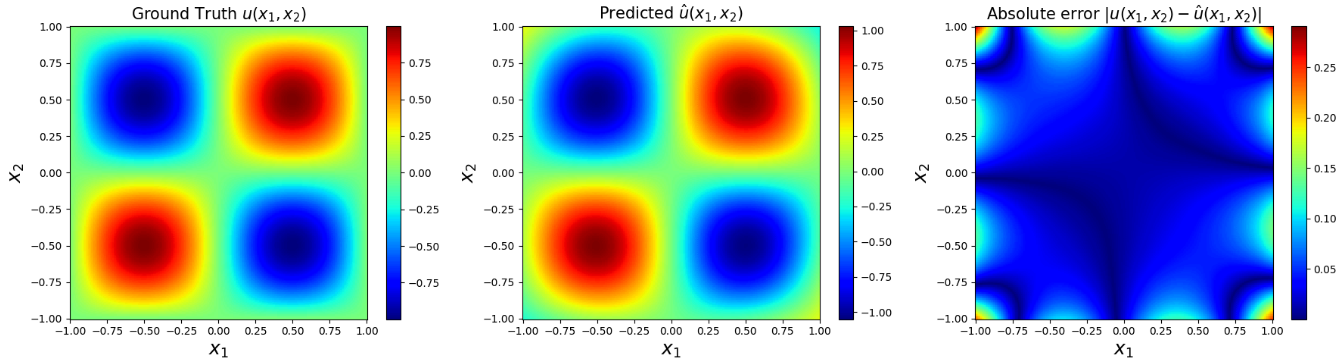
```
N_u = 400
N_f = 20000
Training time: 39.04
Test Error: 0.10225
```




```
N_u = 400
N_f = 30000
Training time: 50.46
Test Error: 0.07301
```




```
N_u = 400
N_f = 40000
Training time: 66.36
Test Error: 0.10518
```




```
N_u = 400
N_f = 50000
Training time: 81.16
Test Error: 0.08737
```