第二节 换元积分法

一、第一类换元法

二、第二类换元法

基本思路

设 $F'(u)=f(u)$， $u=\varphi(x)$ 可导，则有

$$\mathrm{d}F[\varphi(x)]=f[\varphi(x)]\varphi'(x)\mathrm{d}x$$

$$\therefore\quad \int f[\varphi(x)]\varphi'(x)\mathrm{d}x=F[\varphi(x)]+C=F(u)+C\Big|_{u=\varphi(x)}$$

$$=\int f(u)\mathrm{d}u\Big|_{u=\varphi(x)}$$

$$\int f[\varphi(x)]\varphi'(x)\,\mathrm{d}x \underset{\text{第二类换元法}}{\overset{\text{第一类换元法}}{\rightleftarrows}} \int f(u)\,\mathrm{d}u$$

一、第一类换元法

定理1. 设 $f(u)$ 有原函数, $u = j(x)$ 可导, 则有换元公式

$$\int f[\varphi(x)]\underline{\varphi'(x)\mathrm{d}x} = \int f(u)\mathrm{d}u \Big|_{u=j(x)}$$

即 $$\int f[j(x)]j'(x)\mathrm{d}x = \int f(j(x))\mathrm{d}j(x)$$

(也称**配元法, 凑微分法**)

例1. 求 $\int (ax+b)^m \mathrm{d}x \quad (m \neq -1).$

解： 令 $u = ax+b$，则 $\mathrm{d}u = a\mathrm{d}x$，故

$$\text{原式} = \int u^m \frac{1}{a} \mathrm{d}u = \frac{1}{a} \cdot \frac{1}{m+1} u^{m+1} + C$$

$$= \frac{1}{a(m+1)} (ax+b)^{m+1} + C$$

注意换回原变量

注： 当 $m = -1$ 时

$$\int \frac{\mathrm{d}x}{ax+b} = \frac{1}{a} \ln|ax+b| + C$$

例2. 求 $\int \frac{\mathrm{d}x}{a^2+x^2}$.

想到公式

$$\int \frac{\mathrm{d}u}{1+u^2} = \arctan u + C$$

解: $\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a^2}\int \frac{\mathrm{d}x}{1+(\frac{x}{a})^2}$

令 $u=\frac{x}{a}$, 则 $\mathrm{d}u=\frac{1}{a}\mathrm{d}x$

$$= \frac{1}{a}\int \frac{\mathrm{d}u}{1+u^2} = \frac{1}{a}\arctan u + C$$

$$= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

例3. 求 $\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} \ (a > 0).$

解: $$\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \int \frac{\mathrm{d}\,x}{a\sqrt{1-(\frac{x}{a})^2}} = \int \frac{\mathrm{d}(\frac{x}{a})}{\sqrt{1-(\frac{x}{a})^2}}$$

$$= \arcsin \frac{x}{a} + C$$

想到 $\int \frac{\mathrm{d}\,u}{\sqrt{1-u^2}} = \arcsin u + C$

$$\int f[j\,(x)]j\,'(x)\mathrm{d}x = \int f(j\,(x))\mathrm{d}\,j\,(x)$$ (直接配元)

例4. 求 $\int \tan x\mathrm{d}x$.

解: $$\int \tan x\mathrm{d}x = \int \frac{\sin x}{\cos x}\mathrm{d}x = -\int \frac{\mathrm{d}\cos x}{\cos x}$$

$$= -\ln\left|\cos x\right| + C$$

类似

$$\int \cot x\mathrm{d}x = \int \frac{\cos x\,\mathrm{d}x}{\sin x} = \int \frac{\mathrm{d}\sin x}{\sin x}$$

$$= \ln\left|\sin x\right| + C$$

例5. 求 $\int \frac{\mathrm{d}x}{x^2 - a^2}$.

解:

$$\because \quad \frac{1}{x^2 - a^2} = \frac{1}{2a}\,\frac{(x+a)-(x-a)}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$

$$\therefore \ \text{原式} = \frac{1}{2a}\left[\int \frac{\mathrm{d}x}{x-a} - \int \frac{\mathrm{d}x}{x+a}\right]$$

$$= \frac{1}{2a}\left[\int \frac{\mathrm{d}(x-a)}{x-a} - \int \frac{\mathrm{d}(x+a)}{x+a}\right]$$

$$= \frac{1}{2a}\left[\ln|x-a| - \ln|x+a|\right] + C = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

常用的几种配元形式:

1) $\int f(ax+b)\,\mathrm{d}x = \frac{1}{a}\int f(ax+b)\,\mathrm{d}(ax+b)$

2) $\int f(x^n)\,x^{n-1}\,\mathrm{d}x = \frac{1}{n}\int f(x^n)\,\mathrm{d}x^n$

3) $\int f(x^n)\frac{1}{x}\,\mathrm{d}x = \frac{1}{n}\int f(x^n)\,\frac{1}{x^n}\,\mathrm{d}x^n$

万能凑幂法

4) $\int f(\sin x)\cos x\,\mathrm{d}x = \int f(\sin x)\,\mathrm{d}\sin x$

5) $\int f(\cos x)\sin x\,\mathrm{d}x = -\int f(\cos x)\,\mathrm{d}\cos x$

6) $\int f(\tan x)\sec^2 x\mathrm{d}x = \int f(\tan x)\,\mathrm{d}\tan x$

7) $\int f(\mathrm{e}^x)\mathrm{e}^x\mathrm{d}x = \int f(\mathrm{e}^x)\,\mathrm{d}\mathrm{e}^x$

8) $\int f(\ln x)\frac{1}{x}\mathrm{d}x = \int f(\ln x)\,\mathrm{d}\ln x$

例6. 求 $\int \frac{\mathrm{d}x}{x(1+2\ln x)}$.

解: 原式 $= \int \frac{\mathrm{d}\ln x}{1+2\ln x} = \frac{1}{2}\int \frac{\mathrm{d}(1+2\ln x)}{1+2\ln x}$

$$= \frac{1}{2}\ln\left|1+2\ln x\right| + C$$

例7. 求 $\int \frac{e^{3\sqrt{x}}}{\sqrt{x}} dx.$

解: 原式 $= 2\int e^{3\sqrt{x}} d\sqrt{x} = \frac{2}{3}\int e^{3\sqrt{x}} d(3\sqrt{x})$

$$= \frac{2}{3} e^{3\sqrt{x}} + C$$

例8. 求 $\int \sec^6 x dx.$

解: 原式 $= \int (\tan^2 x + 1)^2 \, d\tan x$

$$= \int (\tan^4 x + 2\tan^2 x + 1) \, d\tan x$$

$$= \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C$$

例9. 求 $\int \frac{\mathrm{d}x}{1+\mathrm{e}^x}$.

解法1

$$\int \frac{\mathrm{d}x}{1+\mathrm{e}^x} = \int \frac{(1+\mathrm{e}^x)-\mathrm{e}^x}{1+\mathrm{e}^x}\,\mathrm{d}x = \int \mathrm{d}x - \int \frac{\mathrm{d}(1+\mathrm{e}^x)}{1+\mathrm{e}^x}$$

$$= x - \ln(1+\mathrm{e}^x) + C$$

解法2

$$\int \frac{\mathrm{d}x}{1+\mathrm{e}^x} = \int \frac{\mathrm{e}^{-x}}{1+\mathrm{e}^{-x}}\,\mathrm{d}x = -\int \frac{\mathrm{d}(1+\mathrm{e}^{-x})}{1+\mathrm{e}^{-x}}$$

$$= -\ln(1+\mathrm{e}^{-x}) + C$$

$$-\ln(1+\mathrm{e}^{-x}) = -\ln[\mathrm{e}^{-x}(\mathrm{e}^x+1)]$$ 两法结果一样

例10. 求 $\int \sec x \mathrm{d}x$.

解法1

$$\int \sec x \mathrm{d}x = \int \frac{\cos x}{\cos^2 x} \mathrm{d}x = \int \frac{\mathrm{d}\sin x}{1-\sin^2 x}$$

$$= \frac{1}{2}\int\left[\frac{1}{1+\sin x} + \frac{1}{1-\sin x}\right] \mathrm{d}\sin x$$

$$= \frac{1}{2}\left[\ln\left|1+\sin x\right| - \ln\left|1-\sin x\right|\right] + C$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

解法 2 $\int \sec x \mathrm{d}x = \int \frac{\sec x\,(\sec x + \tan x)}{\sec x + \tan x}\mathrm{d}x$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\mathrm{d}x$$

$$= \int \frac{\mathrm{d}(\sec x + \tan x)}{\sec x + \tan x}$$

$$= \ln\left| \sec x + \tan x \right| + C$$

同样可证

$$\int \csc x \mathrm{d}x = \ln\left| \csc x - \cot x \right| + C$$

或 $\int \csc x \mathrm{d}x = \ln\left| \tan \frac{x}{2} \right| + C$ (P199 例18)

208 -34

求 $\int \frac{\mathrm{d}x}{(x+1)(x-2)}$

解 $\int \frac{\mathrm{d}x}{(x+1)(x-2)} = \frac{1}{3}\int (\frac{1}{x-2} - \frac{1}{x+1})\mathrm{d}x$

$$= \frac{1}{3}\int \frac{\mathrm{d}(x-2)}{x-2} - \frac{1}{3}\int \frac{\mathrm{d}(x+1)}{x+1}$$

$$= \frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C$$

$$= \frac{1}{3}\ln\left|\frac{x-2}{x+1}\right| + C$$

$$\int(\tan x)^{10}\sec^2 x\mathrm{d}x=\int(\tan x)^{10}\mathrm{d}\tan x=\frac{1}{11}\tan^{11}x+C$$

$$\int\frac{1}{(\arcsin x)^2\sqrt{1-x^2}}\mathrm{d}x=\int\frac{1}{(\arcsin x)^2}\mathrm{d}\arcsin x$$

$$=-\frac{1}{\arcsin x}+C$$

$$\int\frac{10^{2\arccos x}}{\sqrt{1-x^2}}\mathrm{d}x=-\frac{1}{2}\int 10^{2\arccos x}\mathrm{d}2\arccos x=-\frac{10^{2\arccos x}}{2\ln 10}+C$$

$$\int \tan\sqrt{1-x^2}\,\frac{x\mathrm{d}x}{\sqrt{1-x^2}}\mathrm{d}x=\int \tan\sqrt{1-x^2}\,\frac{\mathrm{d}x^2}{\sqrt{1-x^2}}$$

$$=-\int \tan\sqrt{1-x^2}\,\frac{\mathrm{d}(1-x^2)}{\sqrt{1-x^2}}$$

$$=-\frac{1}{2}\int \tan\sqrt{1-x^2}\,\mathrm{d}\sqrt{1-x^2}$$

$$=-\frac{1}{2}\ln\left|\cos\sqrt{1-x^2}\right|+C$$

巧用三角函数

$$\int \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}\,\mathrm{d}x = \frac{1}{2}\int \frac{\arctan\sqrt{x}}{(1+x)}\,\mathrm{d}\sqrt{x}$$

$$=\frac{1}{2}\int \frac{\arctan\sqrt{x}}{[1+(\sqrt{x})^2]}\,\mathrm{d}\sqrt{x}$$

$$=\frac{1}{2}\int \arctan\sqrt{x}\,\mathrm{d}\arctan\sqrt{x}$$

$$=(\arctan\sqrt{x})^2+C$$

208页23

$$\int \frac{\ln \tan x}{\sin x \cos x} \mathrm{d}x = \int \frac{\cos x \ln \tan x}{\sin x \cos^2 x} \mathrm{d}x$$

$$= \int \cot x \cdot \ln \tan x \cdot \sec^2 x \mathrm{d}x = \int \cot x \cdot \ln \tan x \mathrm{d}\tan x$$

$$= \int \frac{\ln \tan x}{\tan x} \mathrm{d}\tan x = \int \ln \tan x \mathrm{d} \ln \tan x$$

$$= \frac{1}{2} (\ln \tan x)^2 + C$$

$$\int f[\phi(x)]\phi'(x)\mathrm{d}x \overset{u=\varphi(x)}{=} \int f(u)\mathrm{d}u$$

例11. 求 $\displaystyle\int \frac{x^3}{(x^2+a^2)^{3/2}}\mathrm{d}x$.

解: 原式 $\displaystyle= \frac{1}{2}\int \frac{x^2\,\mathrm{d}x^2}{(x^2+a^2)^{3/2}} = \frac{1}{2}\int \frac{(x^2+a^2)-a^2}{(x^2+a^2)^{3/2}}\mathrm{d}x^2$

$$= \frac{1}{2}\int (x^2+a^2)^{-1/2}\,\mathrm{d}(x^2+a^2)$$

$$-\frac{a^2}{2}\int (x^2+a^2)^{-3/2}\,\mathrm{d}(x^2+a^2)$$

$$= \sqrt{x^2+a^2} + \frac{a^2}{\sqrt{x^2+a^2}} + C$$

例12. 求 $\int \cos^4 x \, \mathrm{d}x$.

解： $\mathrm{Q} \cos^4 x = (\cos^2 x)^2 = (\frac{1+\cos 2x}{2})^2$

$$= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$$

$$= \frac{1}{4}(1 + 2\cos 2x + \frac{1+\cos 4x}{2})$$

$$= \frac{1}{4}(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x)$$

$$\therefore \quad \int \cos^4 x \, \mathrm{d}x = \frac{1}{4}\int (\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x)\, \mathrm{d}x$$

$$= \frac{1}{4}\left[\frac{3}{2}\int \mathrm{d}x + \int \cos 2x \, \mathrm{d}(2x) + \frac{1}{8}\int \cos 4x \, \mathrm{d}(4x)\right]$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

例

$$\int \sin^n x \cos x dx = \int \sin^n x d\sin x$$

$$= \frac{1}{n+1} \sin^{n+1} x + C$$

$$\int \cos^n x \sin x dx = -\int \cos^n x d\cos x$$

$$= -\frac{1}{n+1} \cos^{n+1} x + C$$

例 降幂后，运用二项展开式定理

$$\int \sin^{2n} x \cos^{2m} x\mathrm{d}x = \int (\sin^2 x)^n (\cos^2 x)^m \mathrm{d}x$$

$$\int \sin^{2n} x \cos^{2m} x\mathrm{d}x = \int (\sin^2 x)^n (\cos^2 x)^m \mathrm{d}x$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^n \left(\frac{1+\cos 2x}{2}\right)^m \mathrm{d}x$$

$$\int \cos 3x \cos 5x\mathrm{d}x = \frac{1}{2}\int (\cos x + \cos 4x)\mathrm{d}x$$

$$= \frac{1}{2}\left(\sin x + \frac{1}{4}\sin 4x\right) + C$$

例

$$\int \sin^n x\cos^{2m+1} x dx=\int \sin^n x\cos^{2m} x d\sin x$$

$$=\int \sin^n x(1-\sin^2 x)^m d\sin x \overset{u=\sin x}{=} \int u^n(1-u^2)^m du$$

$$\int \cos^n x\sin^{2m+1} x dx=-\int \cos^n x\sin^{2m} x d\cos x$$

$$=-\int \cos^n x(1-\cos^2 x)^m d\cos x \overset{u=\cos x}{=} -\int u^n(1-u^2)^m du$$

例13. 求 $\int \sin^2 x \cos^2 3x \, \mathrm{d}x$.

解: $\because \sin^2 x \cos^2 3x = [\frac{1}{2}(\sin 4x - \sin 2x)]^2$

$$= \frac{1}{4}\sin^2 4x - \frac{1}{4}\cdot 2\sin 4x \sin 2x + \frac{1}{4}\sin^2 2x$$

$$= \frac{1}{8}(1-\cos 8x) - \sin^2 2x \cos 2x + \frac{1}{8}(1-\cos 4x)$$

$$\therefore \text{原式} = \frac{1}{4}\int \mathrm{d}x - \frac{1}{64}\int \cos 8x \, \mathrm{d}(8x)$$

$$- \frac{1}{2}\int \sin^2 2x \, \mathrm{d}(\sin 2x) - \frac{1}{32}\int \cos 4x \, \mathrm{d}(4x)$$

$$= \frac{1}{4}x - \frac{1}{64}\sin 8x - \frac{1}{6}\sin^3 2x - \frac{1}{32}\sin 4x + C$$

例求(208-14) $\int \frac{\sin x+\cos x}{\sqrt[3]{\sin x-\cos x}}\mathrm{d}x$

$$\int f[\phi(x)]\phi'(x)\mathrm{d}x \overset{u=\varphi(x)}{=} \int f(u)\mathrm{d}u$$

解:

$$\int \frac{\sin x+\cos x}{\sqrt[3]{\sin x-\cos x}}\mathrm{d}x=\int \frac{\mathrm{d}(\sin x-\cos x)}{\sqrt[3]{\sin x-\cos x}}$$

$$=\frac{1}{1-\frac{1}{3}}(\sin x-\cos x)^{1-\frac{1}{3}}+C$$

$$=\frac{3}{2}(\sin x-\cos x)^{\frac{2}{3}}+C$$

例求(208-16) $\int \frac{\mathrm{d}x}{x\ln x\ln\ln x}$

解:

$$\int \frac{\mathrm{d}x}{x\ln x\ln\ln x}=\int \frac{\mathrm{d}\ln x}{\ln x\ln\ln x}$$

$$=\int \frac{\mathrm{d}\ln\ln x}{\ln\ln x}=\ln\ln\ln x+C$$

HIGHER EDUCATION PRESS

目录 上页 下页 返回 结束

例14. 求 $\int \frac{x+1}{x(1+x\mathrm{e}^x)}\mathrm{d}x.$

解: 原式 $=\int \frac{(x+1)\ \mathrm{e}^x}{x\mathrm{e}^x(1+x\mathrm{e}^x)}\mathrm{d}x=\int (\frac{1}{x\mathrm{e}^x}-\frac{1}{1+x\mathrm{e}^x})\mathrm{d}(x\mathrm{e}^x)$

$$=\ln\left|x\mathrm{e}^x\right|-\ln\left|1+x\mathrm{e}^x\right|+C$$

$$=x+\ln\left|x\right|-\ln\left|1+x\mathrm{e}^x\right|+C$$

分析: $\frac{1}{x\mathrm{e}^x(1+x\mathrm{e}^x)}=\frac{1+x\mathrm{e}^x-x\mathrm{e}^x}{x\mathrm{e}^x(1+x\mathrm{e}^x)}=\frac{1}{x\mathrm{e}^x}-\frac{1}{1+x\mathrm{e}^x}$

$$(x+1)\mathrm{e}^x\mathrm{d}x=x\mathrm{e}^x\mathrm{d}x+\mathrm{e}^x\mathrm{d}x=\mathrm{d}(x\mathrm{e}^x)$$

例15. 求 $\int\left[\frac{f(x)}{f'(x)}-\frac{f''(x)f^2(x)}{f'^3(x)}\right]\mathrm{d}\,x.$

解：原式 $=\int\frac{f(x)}{f'(x)}\left[1-\frac{f''(x)f(x)}{f'^2(x)}\right]\mathrm{d}x$

$$=\int\frac{f(x)}{f'(x)}\cdot\frac{f'^2(x)-f''(x)f(x)}{f'^2(x)}\mathrm{d}x$$

$$=\int\frac{f(x)}{f'(x)}\,\mathrm{d}\left(\frac{f(x)}{f'(x)}\right)$$

$$=\frac{1}{2}\left[\frac{f(x)}{f'(x)}\right]^2+C$$

小结 常用简化技巧:

(1) 分项积分: 利用积化和差; 分式分项;

$$1=\sin^2 x+\cos^2 x \text{等}$$

(2) 降低幂次: 利用倍角公式, 如

$$\cos^2 x=\tfrac{1}{2}(1+\cos 2x);\qquad \sin^2 x=\tfrac{1}{2}(1-\cos 2x);$$

万能凑幂法
$$\begin{cases}\int f(x^n)x^{n-1}\,\mathrm{d}x=\frac{1}{n}\int f(x^n)\,\mathrm{d}\,x^n\\ \int f(x^n)\frac{1}{x}\,\mathrm{d}x=\frac{1}{n}\int f(x^n)\frac{1}{x^n}\,\mathrm{d}\,x^n\end{cases}$$

(3) 统一函数: 利用三角公式; 配元方法

(4) 巧妙换元或配元

思考与练习　1. 下列各题求积方法有何不同?

$$(1)\ \int\frac{\mathrm{d}x}{4+x}=\int\frac{\mathrm{d}(4+x)}{4+x}\qquad(2)\ \int\frac{\mathrm{d}x}{4+x^2}=\frac{1}{2}\int\frac{\mathrm{d}(\frac{x}{2})}{1+(\frac{x}{2})^2}$$

$$(3)\ \int\frac{x}{4+x^2}\mathrm{d}x=\frac{1}{2}\int\frac{\mathrm{d}(4+x^2)}{4+x^2}$$

$$(4)\ \int\frac{x^2}{4+x^2}\mathrm{d}x=\int[1-\frac{4}{4+x^2}]\mathrm{d}\ x$$

$$(5)\ \int\frac{\mathrm{d}x}{4-x^2}=\frac{1}{4}\int[\frac{1}{2-x}+\frac{1}{2+x}]\mathrm{d}x$$

$$(6)\ \int\frac{\mathrm{d}x}{\sqrt{4x-x^2}}=\int\frac{\mathrm{d}(x-2)}{\sqrt{4-(x-2)^2}}$$

HIGHER EDUCATION PRESS

2. 求 $\int \frac{\mathrm{d}x}{x(x^{10}+1)}$.

提示:

法1 $\int \frac{\mathrm{d}x}{x(x^{10}+1)} = \int \frac{(x^{10}+1)-x^{10}}{x(x^{10}+1)}\mathrm{d}x$

法2 $\int \frac{\mathrm{d}x}{x(x^{10}+1)} = \frac{1}{10}\int \frac{\mathrm{d}\,x^{10}}{x^{10}(x^{10}+1)}$

法3 $\int \frac{\mathrm{d}x}{x(x^{10}+1)} = \int \frac{\mathrm{d}\,x}{x^{11}(1+x^{-10})} = \frac{-1}{10}\int \frac{\mathrm{d}\,x^{-10}}{1+x^{-10}}$

二、第二类换元法

第一类换元法解决的问题

$$\underset{\text{难求}}{\int f[\varphi(x)]\varphi'(x)\mathrm{d}x} = \underset{\text{易求}}{\int f(u)\mathrm{d}u}\Big|_{u=\varphi(x)}$$

若所求积分 $\int f(u)\mathrm{d}u$ 难求,

$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x$ 易求,

则得第二类换元积分法 .

定理2. 设 $x=\psi(t)$ 是单调可导函数，且 $\psi'(t)\neq 0$，$f[\psi(t)]\psi'(t)$ 具有原函数，则有换元公式

$$\int f(x)\,\mathrm{d}x=\int f[\psi(t)]\psi'(t)\,\mathrm{d}t\Big|_{t=\psi^{-1}(x)}$$

其中 $t=y^{-1}(x)$ 是 $\underline{x=y(t)}$ 的反函数.

证: 设 $f[y(t)]y'(t)$ 的原函数为 $F(t)$，令

$$F(x)=F[y^{-1}(x)]$$

则 $$F'(x)=\frac{\mathrm{d}F}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=f[y(t)]\cancel{y'}(t)\cdot\frac{1}{\cancel{y'}(t)}=f(x)$$

$$\therefore\quad \int f(x)\,\mathrm{d}x=F(x)+C=F[y^{-1}(x)]+C$$

$$=\int f[y(t)]y'(t)\,\mathrm{d}t\Big|_{t=y^{-1}(x)}$$

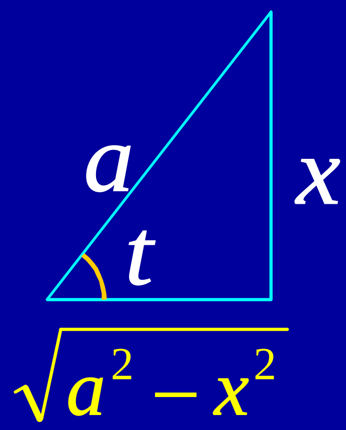
例 计算 $\int x\sqrt{a^2-x^2}\,\mathrm{d}x$

$$\int x\sqrt{a^2-x^2}\,\mathrm{d}x=\frac{1}{2}\int\sqrt{a^2-x^2}\,\mathrm{d}x^2$$

$$=\frac{1}{2}\int\sqrt{a^2-x^2}\,\mathrm{d}(a^2-x^2)$$

$$=\frac{1}{2}\frac{2}{3}\sqrt{(a^2-x^2)^3}+C$$

例16. 求 $\int\sqrt{a^2-x^2}\,\mathrm{d}x \quad (a>0)$.



解: 令 $x=a\sin t,\ t\in(-\frac{\pi}{2},\frac{\pi}{2})$, 则

$$\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2 t}=a\cos t$$

$$\mathrm{d}x=a\cos t\,\mathrm{d}t$$

$\therefore$ 原式 $=\int a\cos t\cdot a\cos t\,\mathrm{d}t=a^2\int\cos^2 t\,\mathrm{d}t$

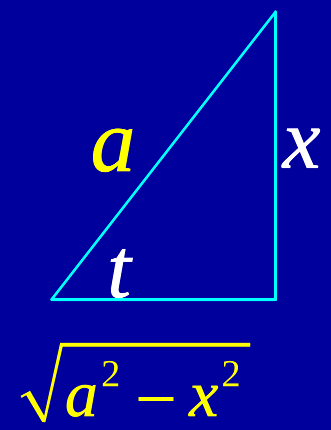
$$=a^2\int\frac{1+\cos 2t}{2}\mathrm{d}t \;=a^2\left(\frac{t}{2}+\frac{\sin 2t}{4}\right)+C$$

$$\sin 2t=2\sin t\cos t=2\cdot\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}$$

$$=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}+C$$

例计算208-36 $\int \frac{x^2 \mathrm{d}x}{\sqrt{a^2 - x^2}}, \quad a > 0$

解 令 $x = a\sin t, \quad \mathrm{d}x = a\cos t\, \mathrm{d}t$



$$\int \frac{x^2 \mathrm{d}x}{\sqrt{a^2 - x^2}} = \int \frac{a^2 \sin^2 t\; a\cos t\, \mathrm{d}t}{\sqrt{a^2 - a^2 \sin^2 t}}$$

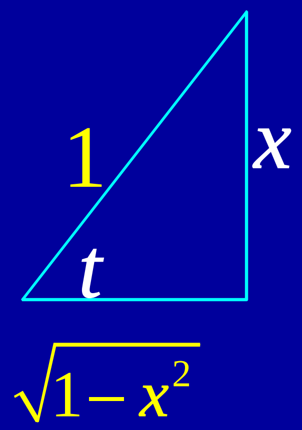
$$= \int \frac{a^2 \sin^2 t\; a\cos t}{a\cos t}\, \mathrm{d}t = \int a^2 \sin^2 t\, \mathrm{d}t$$

$$= a^2 \int \frac{1 - \cos 2t}{2}\, \mathrm{d}t = \frac{a^2}{2}\left(t - \frac{1}{2}\sin 2t\right) + C$$

$$= \frac{a^2}{2}(t - \sin t \cos t) + C = \frac{a^2}{2}\left(\arcsin \frac{x}{a} - \frac{x}{a}\frac{\sqrt{a^2 - x^2}}{a}\right) + C$$

例计算208-42 $\displaystyle\int\frac{dx}{x+\sqrt{1-x^2}}$,

解 令 $x=\sin t$, $dx=\cos t\,dt$



$$\int\frac{dx}{x+\sqrt{1-x^2}}=\int\frac{\cos t\,dt}{\sin t+\sqrt{1-\sin^2 t}}$$

$$=\int\frac{\cos t\,dt}{\sin t+\cos t}=\frac{1}{2}\int\frac{2\cos t\,dt}{\sin t+\cos t}$$

$$=\frac{1}{2}\int\frac{(\cos t+\sin t+\cos t-\sin t)dt}{\sin t+\cos t}$$

$$=\frac{1}{2}\int\left[1+\frac{\cos t-\sin t}{\sin t+\cos t}\right]dt=\frac{1}{2}(t+\ln\left|\sin t+\cos t\right|)+C$$

$$=\frac{1}{2}(\arcsin x+\ln\left|x+\sqrt{1-x^2}\right|)+C$$

例17. 求 $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} \quad (a>0).$

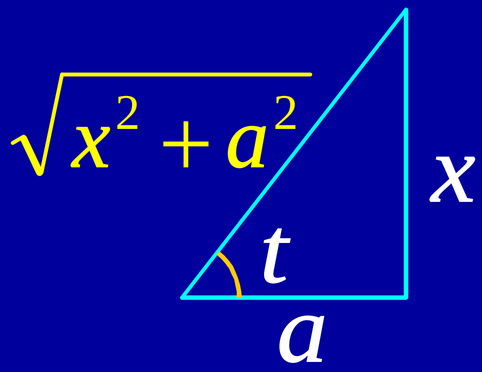
解: 令 $x = a\tan t,\ t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, 则

$$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2 t + a^2} = a\sec t$$

$$\mathrm{d}x = a\sec^2 t\,\mathrm{d}t$$

$$\therefore \text{原式} = \int \frac{a\sec^2 t}{a\sec t}\,\mathrm{d}t = \int \sec t\,\mathrm{d}t$$

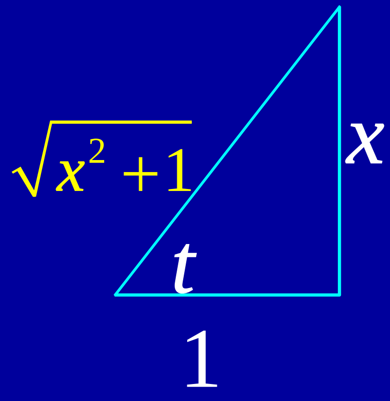
$$= \ln\left|\sec t + \tan t\right| + C_1$$



$$= \ln\left[\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right] + C_1$$

$$= \ln\left[x + \sqrt{x^2+a^2}\right] + C \qquad (C = C_1 - \ln a)$$

例计算208-38 $\int \frac{\mathrm{d}x}{\sqrt{(x^2+1)^3}}$



解 令 $x = \tan t, \quad \mathrm{d}x = \sec^2 t \, \mathrm{d}t$

$$\int \frac{\mathrm{d}x}{\sqrt{(x^2+1)^3}} = \int \frac{\sec^2 t}{\sqrt{(\tan^2 t+1)^3}} \, \mathrm{d}t$$

$$= \int \frac{\sec^2 t}{\sqrt{(\sec^2 t)^3}} \, \mathrm{d}t = \int \frac{\sec^2 t}{\sec^3 t} \, \mathrm{d}t$$

$$= \int \cos t \, \mathrm{d}t = \sin t + C$$

$$= \frac{x}{\sqrt{x^2+1}} + C$$

例18. 求 $\displaystyle\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}\ (a>0).$

解: 当 $x>a$ 时, 令 $x=a\sec t,\ t\in(0,\frac{\pi}{2})$, 则

$$\sqrt{x^2-a^2}=\sqrt{a^2\sec^2 t-a^2}=a\tan t$$

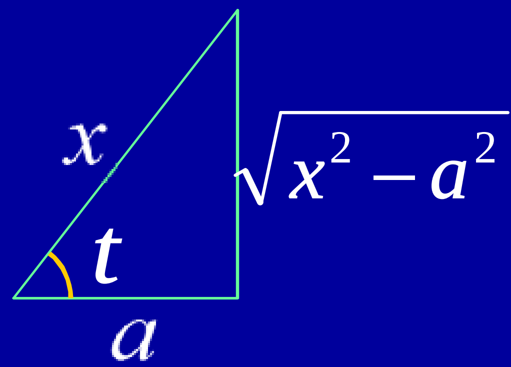
$$\mathrm{d}x=a\sec t\tan t\,\mathrm{d}t$$

$$\therefore\ \text{原式}=\int\frac{a\sec t\tan t}{a\tan t}\mathrm{d}t=\int\sec t\,\mathrm{d}t$$

$$=\ln\left|\sec t+\tan t\right|+C_1$$

$$=\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+C_1$$

$$=\ln\left|x+\sqrt{x^2-a^2}\right|+C\quad(C=C_1-\ln a)$$



当$x<-a$时，令 $x=-u$，则$u>a$，于是

$$\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}=-\int\frac{\mathrm{d}u}{\sqrt{u^2-a^2}}=-\ln\left|u+\sqrt{u^2-a^2}\right|+C_1$$

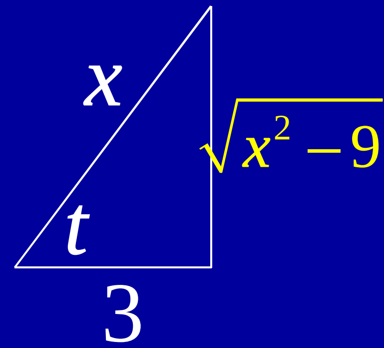
$$=-\ln\left|-x+\sqrt{x^2-a^2}\right|+C_1$$

$$=-\ln\left|\frac{a^2}{-x-\sqrt{x^2-a^2}}\right|+C_1$$

$$=\ln\left|x+\sqrt{x^2-a^2}\right|+C \quad (C=C_1-2\ln a)$$

$x>a$时，$\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}=\ln\left|x+\sqrt{x^2-a^2}\right|+C$

例计算208-39 $\int \frac{\sqrt{x^2-9}}{x}\mathrm{d}x$



解 令 $x=3\sec t$, $\mathrm{d}x=3\sec t\tan t\,\mathrm{d}t$,

$$\int\frac{\sqrt{x^2-9}}{x}\mathrm{d}x=\int\frac{\sqrt{9\sec^2 t-9}}{3\sec t}\cdot 3\sec t\tan t\mathrm{d}t$$

$$=\int\frac{3\tan t}{3\sec t}\cdot 3\sec t\tan t\mathrm{d}t=3\int\tan t^2\mathrm{d}t$$

$$=3\int(\sec^2 t-1)\mathrm{d}t=3(\tan t-t)+C$$

$$=\sqrt{x^2-9}-3\arccos\frac{3}{|x|}+C$$

说明:

1. 被积函数含有 $\sqrt{a^2-x^2}$ 时，可令

$x = a\sin t$ 去掉根号进行积分

2. 被积函数含有 $\sqrt{x^2+a^2}$ 时，可令

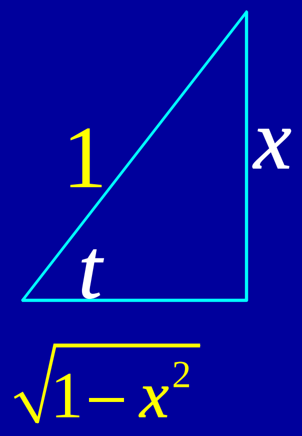
$x = a\tan t$ 去掉根号进行积分

3. 被积函数含有 $\sqrt{x^2-a^2}$

$x = a\sec t$ 去掉根号进行积分

例计算208-41 $\displaystyle\int\frac{\mathrm{d}x}{1+\sqrt{1-x^2}}$

解 令 $x=\sin t,\quad \mathrm{d}x=\cos t\ \mathrm{d}t$



$$\int\frac{\mathrm{d}x}{1+\sqrt{1-x^2}}=\int\frac{\cos t}{1+\sqrt{1-\sin^2 t}}\ \mathrm{d}t$$

$$=\int\frac{\cos t}{1+\cos t}\ \mathrm{d}t=\int\frac{\cos t(1-\cos t)}{(1+\cos t)(1-\cos t)}\ \mathrm{d}t$$

$$=\int\frac{\cos t(1-\cos t)}{1-\cos^2 t}\ \mathrm{d}t=\int\frac{\cos t(1-\cos t)}{\sin^2 t}\ \mathrm{d}t$$

$$=\int(\frac{\cos t}{\sin^2 t}-\frac{\cos^2 t}{\sin^2 t})\mathrm{d}t=\int\frac{\cos t}{\sin^2 t}\mathrm{d}t-\int\frac{\cos^2 t}{\sin^2 t}\ \mathrm{d}t$$

$$=\int\frac{\mathrm{dsin}t}{\sin^2 t}-\int\cot^2\ \mathrm{d}t=-\frac{1}{\sin t}-\int(\csc^2 t\ -1)\mathrm{d}t$$

$$=-\frac{1}{\sin t}+\cot t+t+C==-\frac{1}{x}+\frac{\sqrt{1-x^2}}{x}+\arcsin x+C$$

$$\int \frac{\cos t}{1+\cos t}\,\mathrm{d}t = \int \frac{\cos t + 1 - 1}{1+\cos t}\,\mathrm{d}t = \int \left(1 - \frac{1}{1+\cos t}\right)\mathrm{d}t$$

$$= t - \int \frac{1}{2\cos^2 \frac{t}{2}}\,\mathrm{d}t = t - \int \sec^2 \frac{t}{2}\,\mathrm{d}\frac{t}{2}$$

$$= t - \tan\frac{t}{2} + C = t - \frac{\sin\frac{t}{2}}{\cos\frac{t}{2}} + C$$

$$= t - \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} + C = t - \frac{\sin t}{1+\cos t} + C$$

$$= t - \frac{\sin t(1-\cos t)}{(1+\cos t)(1-\cos t)} + C$$

例19. 求 $\int \frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x$.

解: 令 $x=\frac{1}{t}$,则 $\mathrm{d}x=\frac{-1}{t^2}\mathrm{d}\,t$

$$\text{原式}=\int \frac{\sqrt{a^2-\frac{1}{t^2}}}{\frac{1}{t^4}}\cdot\frac{-1}{t^2}\mathrm{d}\,t=-\int(a^2t^2-1)^{\frac{1}{2}}\,|\,t\,|\,\mathrm{d}\,t$$

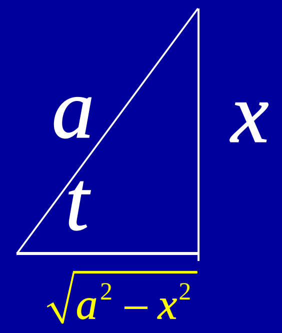
当$x>0$时,

$$\text{原式}=-\frac{1}{2a^2}\int(a^2t^2-1)^{\frac{1}{2}}\,\mathrm{d}(a^2t^2-1)$$

$$=-\frac{(a^2t^2-1)^{\frac{3}{2}}}{3a^2}+C=-\frac{(a^2-x^2)^{\frac{3}{2}}}{3a^2x^3}+C$$

当 $x<0$ 时, 类似可得同样结果 .

例19. 求 $\int \frac{\sqrt{a^2-x^2}}{x^4}\,\mathrm{d}x$.



解: 令 $x = a\sin t$, $\mathrm{d}x = a\cos t\mathrm{d}t$,

原式 $= \int \frac{\sqrt{a^2-a^2\sin^2 t}}{a^4\sin^4 t}\cdot a\cos t\,\mathrm{d}\,t \quad = \int \frac{a^2\cos^2 t}{a^4\sin^4 t}\mathrm{d}\,t$

$$= \frac{1}{a^2}\int \cot^2 t\csc^2 t\mathrm{d}t = -\frac{1}{a^2}\int \cot^2 t\mathrm{d}\cot t$$

$$= -\frac{1}{3a^2}\cot^3 t + C$$

$$= -\frac{1}{3a^2}\left(\frac{\sqrt{a^2-x^2}}{x}\right)^3 + C$$

例求(208-40) $\int \frac{\mathrm{d}x}{1+\sqrt{2x}}$

解：令 $t=\sqrt{2x},\ x=\frac{1}{2}t^2,\ \mathrm{d}x=t\mathrm{d}t,$

$$\int \frac{\mathrm{d}x}{1+\sqrt{2x}}=\int \frac{t\mathrm{d}t}{1+t}=\int \frac{t+1-1}{1+t}\mathrm{d}t$$

$$=\int (1-\frac{1}{1+t})\mathrm{d}t=t-\ln|1+t|+C$$

$$=\sqrt{2x}-\ln(1+\sqrt{2x})+C$$

小结:

1. 第二类换元法常见类型:

1) $\int f(x, \sqrt[n]{ax+b})\,\mathrm{d}x$, 令 $t = \sqrt[n]{ax+b}$

2) $\int f(x, \sqrt[n]{\frac{ax+b}{cx+d}})\,\mathrm{d}x$, 令 $t = \sqrt[n]{\frac{ax+b}{cx+d}}$

$\left.\begin{matrix}1)\\2)\end{matrix}\right\}$ 第四节讲

3) $\int f(x, \sqrt{a^2 - x^2})\,\mathrm{d}x$, 令 $x = a\sin t$ 或 $x = a\cos t$

4) $\int f(x, \sqrt{a^2 + x^2})\,\mathrm{d}x$, 令 $x = a\tan t$ 或 $x = a\,\mathrm{sh}\,t$

5) $\int f(x, \sqrt{x^2 - a^2})\,\mathrm{d}x$, 令 $x = a\sec t$ 或 $x = a\,\mathrm{ch}\,t$

6) $\int f(a^x)\,\mathrm{d}x$，令 $t=a^x$

7) 分母中因子次数较高时，可试用**倒代换**

2. 常用基本积分公式的补充 (P205 ~ P206)

(16) $\int \tan x\,\mathrm{d}x = -\ln|\cos x| + C$

(17) $\int \cot x\,\mathrm{d}x = \ln|\sin x| + C$

(18) $\int \sec x\,\mathrm{d}x = \ln|\sec x + \tan x| + C$

(19) $\int \csc x\,\mathrm{d}x = \ln|\csc x - \cot x| + C$

(20) $\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a} + C$

(21) $\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

(22) $\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\frac{x}{a} + C$

(23) $\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \ln(x+\sqrt{x^2+a^2}) + C$

(24) $\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \ln\left|x+\sqrt{x^2-a^2}\right| + C$

例20. 求 $\int \frac{\mathrm{d}x}{x^2+2x+3}$.

解: 原式 $= \int \frac{1}{(x+1)^2+(\sqrt{2})^2}\mathrm{d}(x+1)$

$$= \frac{1}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}}+C$$ (P206 公式 (20))

例21. 求 $I = \int \frac{\mathrm{d}x}{\sqrt{4x^2+9}}$.

解: $I = \frac{1}{2}\int \frac{\mathrm{d}(2x)}{\sqrt{(2x)^2+3^2}} = \frac{1}{2}\ln\left|2x+\sqrt{4x^2+9}\right|+C$

(P206 公式 (23))

例22. 求 $\int \frac{\mathrm{d}x}{\sqrt{1+x-x^2}}$.

解: 原式 $= \int \frac{\mathrm{d}\left(x-\frac{1}{2}\right)}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}} = \arcsin\frac{2x-1}{\sqrt{5}}+C$

(P206 公式 (22))

例23. 求 $\int \frac{\mathrm{d}x}{\sqrt{\mathrm{e}^{2x}-1}}$.

解: 原式 $= -\int \frac{\mathrm{d}\,\mathrm{e}^{-x}}{\sqrt{1-\mathrm{e}^{-2x}}} = -\arcsin \mathrm{e}^{-x}+C$

(P206 公式 (22))

HIGHER EDUCATION PRESS

目录 上页 下页 返回 结束

例24. 求 $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+a^2}}$.

解: 令 $x=\frac{1}{t}$, 得

$$\text{原式} = -\int \frac{t}{\sqrt{a^2t^2+1}}\,\mathrm{d}t$$

$$= -\frac{1}{2a^2}\int \frac{\mathrm{d}(a^2t^2+1)}{\sqrt{a^2t^2+1}} = -\frac{1}{a^2}\sqrt{a^2t^2+1}+C$$

$$= -\frac{\sqrt{x^2+a^2}}{a^2x}+C$$

例25. 求 $\int \frac{\mathrm{d}x}{(x+1)^3\sqrt{x^2+2x}}$.

解: 原式 $= \int \frac{\mathrm{d}x}{(x+1)^3\sqrt{(x+1)^2-1}}$

令 $x+1=\frac{1}{t}$

$$= \int \frac{t^3}{\sqrt{\frac{1}{t^2}-1}}\left(-\frac{1}{t^2}\right)\mathrm{d}t = -\int \frac{t^2}{\sqrt{1-t^2}}\,\mathrm{d}t$$

$$= \int \frac{(1-t^2)-1}{\sqrt{1-t^2}}\,\mathrm{d}t = \boxed{\int \sqrt{1-t^2}\,\mathrm{d}t} - \int \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t$$

$$= \frac{1}{2}t\sqrt{1-t^2} + \frac{1}{2}\arcsin t - \arcsin t + C$$

例16

$$= \frac{1}{2}\frac{\sqrt{x^2+2x}}{(x+1)^2} - \frac{1}{2}\arcsin\frac{1}{x+1} + C$$

HIGHER EDUCATION PRESS

思考与练习

1. 下列积分应如何换元才使积分简便？

(1) $\int \frac{x^5}{\sqrt{1+x^2}}\mathrm{d}x$

令 $t=\sqrt{1+x^2}$

(2) $\int \frac{\mathrm{d}x}{\sqrt{1+\mathrm{e}^x}}$

令 $t=\sqrt{1+\mathrm{e}^x}$

(3) $\int \frac{\mathrm{d}x}{x(x^7+2)}$

令 $t=\frac{1}{x}$

2. 已知 $\int x^5 f(x)\,\mathrm{d}x = \sqrt{x^2-1}+C$, 求 $\int f(x)\,\mathrm{d}x$.

解: 两边求导, 得 $x^5 f(x) = \dfrac{x}{\sqrt{x^2-1}}$, 则

$$\int f(x)\,\mathrm{d}x = \int \frac{\mathrm{d}x}{x^4\sqrt{x^2-1}} \quad (令\ t=\frac{1}{x})$$

$$= \int \frac{-t^3\,\mathrm{d}t}{\sqrt{1-t^2}} = \frac{1}{2}\int \frac{(1-t^2)-1}{\sqrt{1-t^2}}\,\mathrm{d}t^2$$

$$= \frac{-1}{2}\int (1-t^2)^{\frac{1}{2}}\,\mathrm{d}(1-t^2) + \frac{1}{2}\int (1-t^2)^{-\frac{1}{2}}\,\mathrm{d}(1-t^2)$$

$$= \frac{-1}{3}(1-t^2)^{\frac{3}{2}} + (1-t^2)^{\frac{1}{2}} + C = \ \blacksquare.$$

(代回原变量)

作业

P207

2 (4) , (5) , (9) , (11) , (12) , (16) , (20) ,

(21) , (23) , (28) , (29) , (30) , (32) , (33) ,

(35) , (36) , (38), (40) , (42) , (44)

备用题 1. 求下列积分:

1) $$\int x^2 \frac{1}{\sqrt{x^3+1}}\,\mathrm{d}x = \frac{1}{3}\int \frac{1}{\sqrt{x^3+1}}\,\mathrm{d}(x^3+1)$$

$$= \frac{2}{3}\sqrt{x^3+1}+C$$

2) $$\int \frac{2x+3}{\sqrt{1+2x-x^2}}\,\mathrm{d}x = \int \frac{-(2-2x)+5}{\sqrt{1+2x-x^2}}\,\mathrm{d}x$$

$$= -\int \frac{\mathrm{d}(1+2x-x^2)}{\sqrt{1+2x-x^2}} + 5\int \frac{\mathrm{d}(x-1)}{\sqrt{2-(x-1)^2}}$$

$$= -2\sqrt{1+2x-x^2} + 5\arcsin\frac{x-1}{\sqrt{2}} + C$$

2. 求不定积分 $\int \frac{2\sin x \cos x \sqrt{1+\sin^2 x}}{2+\sin^2 x} \mathrm{d}x.$

解: 利用凑微分法，得

$$\text{原式} = \int \frac{\sqrt{1+\sin^2 x}}{2+\sin^2 x} \mathrm{d}(1+\sin^2 x)$$

令 $t = \sqrt{1+\sin^2 x}$

$$= \int \frac{2t^2}{1+t^2} \mathrm{d}t = 2\int (1 - \frac{1}{1+t^2}) \mathrm{d}t$$

$$= 2t - 2\arctan t + C$$

$$= 2[\sqrt{1+\sin^2 x} - \arctan\sqrt{1+\sin^2 x}] + C$$

HIGHER EDUCATION PRESS

3. 求不定积分 $\int \frac{1}{(1+x^2)\sqrt{1-x^2}}\mathrm{d}x.$

解： 令 $x=\sin t$，$1+x^2=1+\sin^2 t$，$\mathrm{d}x=\cos t\,\mathrm{d}t$

$$\text{原式}=\int \frac{\cos t}{(1+\sin^2 t)\cos t}\mathrm{d}t=\int \frac{1}{1+\sin^2 t}\mathrm{d}t$$

分子分母同除以 $\cos^2 t$

$$=\int \frac{\sec^2 t}{\sec^2 t+\tan^2 t}\mathrm{d}t=\int \frac{1}{1+2\tan^2 t}\mathrm{d}\tan t$$

$$=\frac{1}{\sqrt{2}}\int \frac{1}{1+(\sqrt{2}\tan t)^2}\mathrm{d}\sqrt{2}\tan t$$

$$=\frac{1}{\sqrt{2}}\arctan(\sqrt{2}\tan t)+C=\frac{1}{\sqrt{2}}\arctan\frac{\sqrt{2}x}{\sqrt{1-x^2}}+C$$