1、已知$\vec{a}=(2,1,-2)$，$\vec{b}=(2,3,1)$，则$\vec{a}\cdot\vec{b}=5$　　王陈偲、210401　20212304

2、已知$\vec{a}=(1,-1,3)$，$\vec{b}=(2,2,-1)$，则$\vec{a}\times\vec{b}=(-5,7,4)$

3、求过点$(2,1,-2)$且与平面$2x+3y-2z-5=0$平行的平面方程

解：因为所求平面与已知平面平行　所以其法向量可取$\vec{n}=(2,3,-2)$

则所求平面方程为$2(x-2)+3(y-1)-2(z+2)=0$　即$2x+3y-2z-11=0$

4、过点$(1,-2,1)$且平行于直线$\frac{x-2}{2}=\frac{y-1}{4}=\frac{z+3}{5}$的直线方程

解：因为所求直线与已知直线平行　所以其方向向量可取$\vec{s}=(2,4,5)$

则所求直线方程为$\frac{x-1}{2}=\frac{y+2}{4}=\frac{z-1}{5}$

5、一平面过点$(4,1,2)$且平行于向量$\vec{a}=(3,1,2)$和$\vec{b}=(0,1,-1)$，则这平面方程

解：因为所求平面平行于$\vec{a}$和$\vec{b}$　所以其法向量可取$\vec{n}=\vec{a}\times\vec{b}=(-3,3,3)$

则所求平面方程为$-3(x-4)+3(y-1)+3(z-2)=0$　即$-x+y+z+1=0$

6、求过点$(3,0,0)$和$(0,0,1)$且与xOy面成$\frac{\pi}{3}$角的平面方程

解：设平面方程为$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$．因为过点$(3,0,0)$与$(0,0,1)$　所以$a=3$，$c=1$

则平面方程为$\frac{x}{3}+\frac{y}{b}+\frac{z}{1}=1$，法向量$\vec{n}=(\frac{1}{3},\frac{1}{b},1)$．因为平面与$xOy$面成$\frac{\pi}{3}$角

所以$\frac{|\vec{n}\cdot\vec{k}|}{|\vec{n}||\vec{k}|}=\frac{1}{\sqrt{(\frac{1}{3})^2+(\frac{1}{b})^2+1}}=\cos\frac{\pi}{3}=\frac{1}{2}$　解得$\frac{1}{b}=\pm\frac{\sqrt{26}}{3}$　故平面方程为$x+\sqrt{26}y+3z-3=0$或$x-\sqrt{26}y+3z-3=0$

7、设直线L的方程为$\begin{cases}x-y+z=1\\2x+y+z=4\end{cases}$，则L的参数方程为

解：根据题意已知直线的方向向量为$\vec{s}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\1&-1&1\\2&1&1\end{vmatrix}=(-2,1,3)$

取$x=1$，代入直线方程得$\begin{cases}-y+z=0\\y+z=2\end{cases}$　解得$y=1$，$z=1$　所以直线经过一点$(1,1,1)$

故直线的对称式方程为$\frac{x-1}{-2}=\frac{y-1}{1}=\frac{z-1}{3}$，则参数方程为$\begin{cases}x=1-2t\\y=1+t\\z=1+3t\end{cases}$

8、将xOz坐标面上的抛物线$z^2=2x$绕x轴旋转一周所生成的旋转曲面的方程为

解：$(\pm\sqrt{y^2+z^2})^2=2x$　，即$y^2+z^2=2x$

二、计算题

1、分别求母线平行于x轴及y轴而且通过曲线$\begin{cases}3x^2+y^2+z^2=9\\x^2+z^2-y^2=0\end{cases}$的柱面方程

解：将$x^2+z^2-y^2=0$两边乘以3得$3x^2+3z^2-3y^2=0$，母线平行于x轴的柱面方程为$3x^2+3z^2-3y^2-3x^2-y^2-z^2=-9$　，即$4y^2-2z^2=9$

将$3x^2+y^2+z^2=9$和$x^2+z^2-y^2=0$相加得$4x^2+2z^2=9$，所以母线平行于y轴的柱面方程为$4x^2+2z^2=9$

2、设$|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$，$\vec{a}=(3,-5,8)$，$\vec{b}=(-1,1,z)$，求$z$

解：因为$|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$　所以$(\vec{a}+\vec{b})^2=(\vec{a}-\vec{b})^2$　即$\vec{a}^2+2\vec{a}\cdot\vec{b}+\vec{b}^2=\vec{a}^2-2\vec{a}\cdot\vec{b}+\vec{b}^2$　所以$\vec{a}\cdot\vec{b}=0$

故$-3+(-5)\times1+8z=0$　所以$z=1$

(法二)$\vec{a}+\vec{b}=(2,-4,8+z)$，$\vec{a}-\vec{b}=(4,-6,8-z)$，因为$|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$　所以$\sqrt{2^2+(-4)^2+(8+z)^2}=\sqrt{4^2+(-6)^2+(8-z)^2}$

解得$z=1$

3、已知点 $A(1,0,0)$，$B(0,2,1)$，试在 z 轴上求一点 C，使 ΔABC 的面积最小

解：设 $C(0,0,z)$，$S_{\Delta ABC}=\frac{1}{2}|\vec{AB}\times\vec{AC}|$，$\vec{AB}\times\vec{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 2 & 1\\ -1 & 0 & z\end{vmatrix}=2z\vec{i}+(z-1)\vec{j}+2\vec{k}$

故 $S_{\Delta ABC}=\frac{1}{2}\sqrt{(2z)^2+(z-1)^2+2^2}=\frac{1}{2}\sqrt{5z^2-2z+5}$

令 $f(z)=5z^2-2z+5$，$f'(z)=10z-2$　令 $f'(z)=0$　得 $10z-2=0$　即 $z=\frac{1}{5}$

又因 $f''(z)=10$，$f''(\frac{1}{5})=10>0$　所以当 $z=\frac{1}{5}$ 时，$f(z)$ 取得最小值，即当 $C(0,0,\frac{1}{5})$ 时，$S_{\Delta ABC}$ 最小

4、求直线 $\begin{cases}2x-4y+z=0\\3x-y-2z=9\end{cases}$ 在平面 $4x-y+z=1$ 上的投影直线方程

解：过直线 $\begin{cases}2x-4y+z=0\\3x-y-2z=9\end{cases}$ 的平面束方程为 $2x-4y+z+\lambda(3x-y-2z-9)=0$

即 $(2+3\lambda)x+(-4-\lambda)y+(1-2\lambda)z-9\lambda=0$　该平面与平面 $4x-y+z=1$ 垂直的条件是 $(2+3\lambda)\cdot4+(-4-\lambda)\cdot(-1)+(1-2\lambda)\cdot1=0$　解得 $\lambda=-\frac{3}{11}$，代入平面束方程得 $17x+31y-37z-117=0$

因此所求投影直线方程为 $\begin{cases}17x+31y-37z-117=0\\4x-y+z=1\end{cases}$

5、求直线 $\begin{cases}x+2y+z=0\\x-y-2z=0\end{cases}$ 与平面 $x+y-z+1=0$ 的夹角

解：直线的方向向量 $\vec{s}=\vec{n}_1\times\vec{n}_2=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 2 & 1\\ 1 & -1 & -2\end{vmatrix}=(-3,3,-3)$，平面的法向量 $\vec{n}_3=(1,1,-1)$

故直线与平面的夹角 $\sin\varphi=|\cos(\widehat{\vec{s},\vec{n}})|=\frac{|-3+3+3|}{\sqrt{9+9+9}\sqrt{1+1+1}}=\frac{1}{3}$　所以夹角为 $\arcsin\frac{1}{3}$

一、填空题

1、极限 $\lim\limits_{(x,y)\to(0,0)}\frac{3-\sqrt{xy+9}}{xy}=(\quad)$ $\lim\limits_{(x,y)\to(0,0)}\frac{3xy}{\sqrt{5-e^{xy}}-2}=(\quad)$

$\lim\limits_{(x,y)\to(0,0)}\frac{3-\sqrt{xy+9}}{xy}=\lim\limits_{(x,y)\to(0,0)}\frac{3-\sqrt{xy+9}}{xy}\cdot\frac{3+\sqrt{xy+9}}{3+\sqrt{xy+9}}=\lim\limits_{(x,y)\to(0,0)}\frac{3^2-(xy+9)}{xy}\cdot\lim\limits_{(x,y)\to(0,0)}\frac{1}{3+\sqrt{xy+9}}=-1\cdot\frac{1}{6}=-\frac{1}{6}$

$\lim\limits_{(x,y)\to(0,0)}\frac{3xy}{\sqrt{5-e^{xy}}-2}=\lim\limits_{(x,y)\to(0,0)}\frac{3xy(\sqrt{5-e^{xy}}+2)}{(\sqrt{5-e^{xy}}-2)(\sqrt{5-e^{xy}}+2)}=\lim\limits_{(x,y)\to(0,0)}\frac{3xy\cdot4}{5-e^{xy}-4}=\lim\limits_{(x,y)\to(0,0)}\frac{12xy}{1-e^{xy}}=\lim\limits_{(x,y)\to(0,0)}\frac{12xy}{e^{xy}-1}=-12$

2、设 $z=x^4y+\ln^2(xy)$，则 $\frac{\partial z}{\partial x}=(\quad)$，$\frac{\partial z}{\partial y}=(\quad)$，$z=x^3+y^4-4xy$，则 $\frac{\partial z}{\partial x}=(\quad)$，$\frac{\partial z}{\partial y}=(\quad)$

(1) $\frac{\partial z}{\partial x}=4x^3y+2\ln(xy)\cdot\frac{1}{xy}\cdot y=4x^3y+2\ln(xy)\cdot\frac{1}{x}$，$\frac{\partial z}{\partial y}=x^4+2\ln(xy)\cdot\frac{1}{xy}\cdot x=x^4+2\ln(xy)\cdot\frac{1}{y}$

(2) $\frac{\partial z}{\partial x}=3x^2-4y$，$\frac{\partial z}{\partial y}=4y^3-4x$

3、函数 $u=f(x^2+y^2,xy)$，其中 f 具有二阶连续偏导数，则 $\frac{\partial u}{\partial x}=(\quad)$，$\frac{\partial^2 u}{\partial x\partial y}=(\quad)$，

函数 $u=f(x+y,xy)$，其中 f 具有二阶连续偏导数，则 $\frac{\partial u}{\partial x}=(\quad)$，$\frac{\partial^2 u}{\partial x\partial y}=(\quad)$

(1) $\frac{\partial u}{\partial x}=f_1'\cdot2x+f_2'y$，$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=(f_{11}''\cdot2x+f_{12}''\cdot y)\cdot2x+(f_{21}''\cdot2y+f_{22}''\cdot x)\cdot y+f_2'$

$=4x^2f_{11}''+2xyf_{12}''+2y^2f_{21}''+xyf_{22}''+f_2'=4x^2f_{11}''+2y(x+y)f_{12}''+xyf_{22}''+f_2'$

(2) $\frac{\partial u}{\partial x}=f_1'\cdot1+f_2'y$，$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=f_{11}''+f_{12}''x+(f_{21}''+f_{22}''x)y+f_2'=f_{11}''+f_{12}''x+f_{21}''y+f_{22}''xy+f_2'$

$=f_{11}''+f_{12}''(x+y)+f_{22}''xy+f_2'$

4、设 $x^2+3y+5z^2-xy^2z=0$，则 $\frac{\partial z}{\partial x}=(\quad)$，$\frac{\partial z}{\partial y}=(\quad)$；设 $\ln(x^2+y^2)+z=\sin(xyz)$，则 $\frac{\partial z}{\partial x}=(\quad)$，$\frac{\partial z}{\partial y}=(\quad)$

(1) 设 $F(x,y,z)=x^2+3y+5z^2-xy^2z$，$F_x=2x-y^2z$，$F_y=3-2xyz$，$F_z=10z-xy^2$

所以 $\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{2x-y^2z}{10z-xy^2}$，$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{3-2xyz}{10z-xy^2}$

(法二) 在所给方程两端对 x 求偏导数，得 $2x+10z\frac{\partial z}{\partial x}-(y^2z+xy^2\frac{\partial z}{\partial x})=0$

即 $-2x+y^2z=(10z-xy^2)\frac{\partial z}{\partial x}$ 所以 $\frac{\partial z}{\partial x}=\frac{-2x+y^2z}{10z-xy^2}$

在方程两端对 y 求偏导数，得 $3+10z\frac{\partial z}{\partial y}-(2xyz+xy^2\frac{\partial z}{\partial y})=0$，即 $\frac{\partial z}{\partial y}=\frac{2xyz-3}{10z-xy^2}$

(2) 令 $F(x,y,z)=\ln(x^2+y^2)+z-\sin(xyz)$，$F_x=\frac{2x}{x^2+y^2}-\cos(xyz)\cdot yz$，$F_y=\frac{2y}{x^2+y^2}-\cos(xyz)\cdot xz$

$F_z=1-\cos(xyz)\cdot xy$，$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{2x-(x^2+y^2)yz\cos(xyz)}{(x^2+y^2)[1-xy\cos(xyz)]}$，$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=\frac{2y-(x^2+y^2)xz\cos(xyz)}{(x^2+y^2)[1-\cos(xyz)xy]}$

5、设 $\begin{cases}x+2y+3z=0\\x^2+4y^2+z^2=1\end{cases}$，则 $\frac{dz}{dx}=(\quad)$，$\frac{dy}{dx}=(\quad)$；设 $\begin{cases}3z-2x^2-2y^2=0\\2x^2+3y^2+5z^2=30\end{cases}$，则 $\frac{dz}{dx}=(\quad)$，$\frac{dy}{dx}=(\quad)$

(1) 将所给方程组两端分别对 x 求导得 $\begin{cases}1+2\frac{dy}{dx}+3\frac{dz}{dx}=0\\2x+8y\frac{dy}{dx}+2z\frac{dz}{dx}=0\end{cases}$，即 $\begin{cases}2\frac{dy}{dx}+3\frac{dz}{dx}=-1\\4y\frac{dy}{dx}+z\frac{dz}{dx}=-x\end{cases}$

当 $D=\begin{vmatrix}2&3\\4y&z\end{vmatrix}=2z-12y\neq0$ 时，解方程组得 $\frac{dy}{dx}=\frac{\begin{vmatrix}-1&3\\-x&z\end{vmatrix}}{D}=\frac{-z-3x}{2z-12y}$，$\frac{dz}{dx}=\frac{\begin{vmatrix}2&-1\\4y&-x\end{vmatrix}}{D}=\frac{-2x+4y}{2z-12y}$

(2) 将所给方程组两端分别对 x 求导，得 $\begin{cases}3\frac{dz}{dx}-4x-4y\frac{dy}{dx}=0\\4x+6y\frac{dy}{dx}+10z\frac{dz}{dx}=0\end{cases}$，即 $\begin{cases}-4y\frac{dy}{dx}+3\frac{dz}{dx}=4x\\3y\frac{dy}{dx}+5z\frac{dz}{dx}=2x\end{cases}$

当 $D=\begin{vmatrix}-4y&3\\3y&5z\end{vmatrix}=-20zy-9y\neq0$ 时，解方程组得 $\frac{dy}{dx}=\frac{\begin{vmatrix}4x&3\\2x&5z\end{vmatrix}}{D}=-\frac{20xz-6x}{20yz+9y}$，$\frac{dz}{dx}=\frac{\begin{vmatrix}-4y&4x\\3y&2x\end{vmatrix}}{D}=-\frac{-8xy-12xy}{20zy+9y}=\frac{20x}{20z+9}$

6、设 $z=x^3y+xy^2-\frac{x}{y}$，则 $dz=(\quad)$；设 $z=e^{x+y}+\ln(1+x^2+y^2)$，则 $dz=(\quad)$

(1) $\frac{\partial z}{\partial x}=3x^2y+y^2-\frac{1}{y}$，$\frac{\partial z}{\partial y}=x^3+2xy+\frac{x}{y^2}$，$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=(3x^2y+y^2-\frac{1}{y})dx+(x^3+2xy+\frac{x}{y^2})dy$

(2) $\frac{\partial z}{\partial x}=e^{x+y}+\ln(1+x^2+y^2)\cdot2x$，$\frac{\partial z}{\partial y}=e^{x+y}+\ln(1+x^2+y^2)\cdot2y$，$dz=[e^{xy}+2x\ln(1+x^2+y^2)]dx+2y\ln(1+x^2+y^2)dy$

二、计算题

1、函数 $z=\frac{y^2+3x}{y^2-5x}$ 在何处是间断的？函数 $z=\frac{xy+5xy^2-7}{x^2+y^2-1}$ 在何处是间断的？

$y^2-5x=0$，即在 $x=\frac{y^2}{5}$ 处间断；$x^2+y^2-1=0$，即在 $x^2+y^2=1$ 处是间断的.

2、求曲线 $x=t, y=t^2, z=t^3$ 在对应于 $t_0=1$ 的点处的切线及法平面方程.

解：$x'=1, y'=2t, z'=3t^2$，所以切向量为 $(1,2,3)$，故切线方程为 $\frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{3}$

法平面方程为 $(x-1)+2(y-1)+3(z-1)=0$，即 $x+2y+3z=6$

3、求曲面 $e^z-z+xy=3$ 在点 $(2,1,0)$ 处的切平面与法线方程.

解：$F(x,y,z)=e^z-z+xy-3$，$F_x'=y$，$F_y'=x$，$F_z'=e^z-1$，把点 $(2,1,0)$ 代入得 $F_x'=1$，$F_y'=2$，$F_z'=0$，故切平面方程为 $(x-2)+2(y-1)+0=0$，法线为 $\frac{x-2}{1}=\frac{y-1}{2}=\frac{z-0}{0}$，即切平面为 $x+2y-4=0$，法线方程为 $\begin{cases}\frac{x-2}{1}=\frac{y-1}{2}\\ z=0\end{cases}$

4、求曲面 $x^2+2y^2+z^2=1$ 平行于平面 $x-y+2z=3$ 的切平面方程

解：令 $F(x,y,z)=x^2+2y^2+z^2-1$，$F_x'=2x$，$F_y'=4y$，$F_z'=2z$，设切点为 (x_0,y_0,z_0)

则切平面的法向量 $\vec{n}=2(x_0,2y_0,z_0)$，平面 $x-y+2z-3=0$ 的法向量 $\vec{m}=(1,-1,2)$

由题意得 $\vec{n}\parallel\vec{m}$，所以 $\frac{x_0}{1}=\frac{2y_0}{-1}=\frac{z_0}{2}=\lambda$，即 $x_0=\lambda$，$y_0=-\frac{\lambda}{2}$，$z_0=2\lambda$，切点为 $(\lambda,-\frac{\lambda}{2},2\lambda)$

又切点在曲面上 所以 $\lambda^2+2(-\frac{\lambda}{2})^2+(2\lambda)^2=1$，$\lambda=\pm\sqrt{\frac{2}{11}}$，故切平面方程为：

$(x-\lambda)\cdot1+(y+\frac{\lambda}{2})\cdot(-1)+(z-2\lambda)\cdot2=0$ 即 $x-y+2z=\pm\frac{\sqrt{22}}{2}$

5、将周长为 l 的矩形绕它的边旋转一周而构成一个圆柱体. 问矩形的长宽分别为多少时，可使圆柱体的体积最大.

解：设矩形的长宽分别为 x, y，则旋转体体积为 $V=\pi x^2y$

构造函数 $L(x,y,\lambda)=\pi x^2y+\lambda(x+y-\frac{l}{2})$，$\begin{cases}\frac{\partial L}{\partial x}=2\pi xy+\lambda=0\\ \frac{\partial L}{\partial y}=\pi x^2+\lambda=0\\ \frac{\partial L}{\partial\lambda}=x+y-\frac{l}{2}=0\end{cases}$ 解得 $x=\frac{l}{3}$，$y=\frac{l}{6}$

6、设 $\phi(u,v)$ 具有连续偏导数，证明方程 $\phi(cx-az,cy-bz)=0$ 所确定的函数 $z=f(x,y)$ 满足 $a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=c$

证明：令 $u=cx-az$，$v=cy-bz$，则 $\phi_x=\phi_u\cdot\frac{\partial u}{\partial x}=c\phi_u$，$\phi_y=\phi_v\cdot\frac{\partial v}{\partial y}=c\phi_v$，$\phi_z=\phi_u\cdot\frac{\partial u}{\partial z}+\phi_v\cdot\frac{\partial v}{\partial z}=-a\phi_u-b\phi_v$

所以 $\frac{\partial z}{\partial x}=-\frac{\phi_x}{\phi_z}=\frac{c\phi_u}{a\phi_u+b\phi_v}$，$\frac{\partial z}{\partial y}=-\frac{\phi_y}{\phi_z}=\frac{c\phi_v}{a\phi_u+b\phi_v}$，故 $a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=a\cdot\frac{c\phi_u}{a\phi_u+b\phi_v}+b\cdot\frac{c\phi_v}{a\phi_u+b\phi_v}=c$

(法二)在方程 $\phi(cx-az,cy-bz)=0$ 两边对 x 求偏导得：$(c-a\frac{\partial z}{\partial x})\phi_1+(0-b\frac{\partial z}{\partial x})\phi_2=0$

解得 $\frac{\partial z}{\partial x}=\frac{c\phi_1}{a\phi_1+b\phi_2}$，在方程 $\phi(cx-az,cy-bz)=0$ 两边对 y 求偏导得：$(0-a\frac{\partial z}{\partial y})\phi_1+(c-b\frac{\partial z}{\partial y})\phi_2=0$

解得 $\frac{\partial z}{\partial y}=\frac{c\phi_2}{a\phi_1+b\phi_2}$，$a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=a\frac{c\phi_1}{a\phi_1+b\phi_2}+b\frac{c\phi_2}{a\phi_1+b\phi_2}=c$

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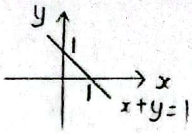
一、填空题

1、已知D是由两坐标轴及直线x+y=1所围成的闭区域，则$\iint_D (x+y)dxdy=$

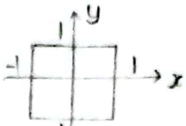


$\iint_D (x+y)dxdy=\int_0^1 dx\int_0^{1-x} x+y\,dy=\frac{1}{2}\int_0^1 1-x^2dx=\frac{1}{2}(1-\frac{1}{3})=\frac{1}{3}$

2、改变积分次序$\int_0^2 dy\int_{y^2}^{2y} f(x,y)dx=$

由题可知：积分区域$D=\{(x,y)|0\le y\le 2, y^2\le x\le 2y\}$ 故$D=\{(x,y)|0\le x\le 4, \frac{x}{2}\le y\le \sqrt{x}\}$

所以$\int_0^2 dy\int_{y^2}^{2y} f(x,y)dx=\int_0^4 dx\int_{\frac{x}{2}}^{\sqrt{x}} f(x,y)dy$



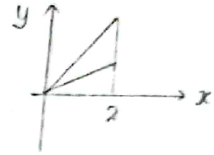
3、设$D=\{(x,y)|-1\le x\le 1, -1\le y\le 1\}$，则$\iint_D xy(x+y)dxdy=$

(法一)$\iint_D xy(x+y)dxdy=\iint_D x^2y\,dxdy+\iint_D xy^2dxdy$. 由图可知，在D上，$f(x,-y)=-f(x,y)$，$f(-x,y)=-f(x,y)$

所以$\iint_D x^2y\,dxdy=0$，$\iint_D xy^2dxdy=0$ 即原式$=0$

(法二)$\iint_D xy(x+y)dxdy=\int_{-1}^1 dx\int_{-1}^1 xy(x+y)dy=\int_{-1}^1 dx\int_{-1}^1 (x^2y+xy^2)dy=\int_{-1}^1 dx(\int_{-1}^1 xy^2dy+\int_{-1}^1 x^2y\,dy)=\int_{-1}^1 dx(\frac{1}{2}x^2\cdot y^2|_{-1}^1+\frac{1}{3}x\cdot y^3|_{-1}^1)=\int_{-1}^1\frac{2}{3}x\,dx=\frac{2}{3}\cdot\frac{1}{2}x^2|_{-1}^1=0$

4、将下列二重积分化为极坐标形式：$\int_0^2 dx\int_x^{\sqrt{3}x} f(\sqrt{x^2+y^2})dy$



$x=2\Rightarrow r\cos\theta=2\Rightarrow r=2\sec\theta$ 所以$\int_0^2 dx\int_x^{\sqrt{3}x} f(\sqrt{x^2+y^2})dy=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}d\theta\int_0^{2\sec\theta} f(r)r\,dr$

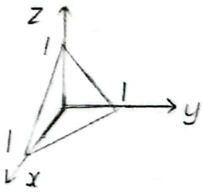
5、已知Ω是由三个坐标面及平面x+y+z=1所围成的闭区域，则$\iiint_\Omega x\,dxdydz=$

(法一)$\iiint_\Omega x\,dxdydz=\int_0^1 x\,dx\iint_{D_{yz}} dydz=\int_0^1\frac{1}{2}x(1-x)^2dx=\frac{1}{2}\int_0^1 x-2x^2+x^3dx=\frac{1}{2}(\frac{1}{2}-\frac{2}{3}+\frac{1}{4})=\frac{1}{24}$

(法二)$D=\{(x,y,z)|0\le x\le 1, 0\le y\le 1-x, 0\le z\le 1-x-y\}$

$\iiint_\Omega x\,dxdydz=\iint_D dxdy\int_0^{1-x-y} x\,dz=\int_0^1 x\,dx\int_0^{1-x}dy\int_0^{1-x-y}dz=\int_0^1 x\,dx\int_0^{1-x}(1-x-y)dy=\frac{1}{2}\int_0^1 x(1-x^2)dx=\frac{1}{2}\int_0^1(x-2x^2+x^3)dx=\frac{1}{24}$

6、球面$x^2+y^2+z^2=1$含在圆柱面$x^2+y^2=x$内部的那部分面积是



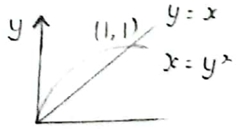
上半球面$z=\sqrt{1-x^2-y^2}$，$\frac{\partial z}{\partial x}=\frac{-x}{\sqrt{1-x^2-y^2}}$，$\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{1-x^2-y^2}}$，$\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}=\frac{1}{\sqrt{1-x^2-y^2}}$

由曲面的对称性所求面积$S=4\iint_{x^2+y^2\le x}\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}dxdy=4\iint_{x^2+y^2\le x}\frac{1}{\sqrt{1-x^2-y^2}}dxdy$

$=4\iint_{\rho\le\cos\theta}\frac{\rho}{\sqrt{1-\rho^2}}d\rho d\theta=4\int_0^{\frac{\pi}{2}}d\theta\int_0^{\cos\theta}\frac{\rho}{\sqrt{1-\rho^2}}d\rho=-2\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta}\frac{1}{\sqrt{1-\rho^2}}d(1-\rho^2)=-2\int_0^{\frac{\pi}{2}}2(1-\rho^2)^{\frac{1}{2}}|_0^{\cos\theta}d\theta$

$=-4\int_0^{\frac{\pi}{2}}[(1-\cos^2\theta)^{\frac{1}{2}}-1]d\theta=-4\int_0^{\frac{\pi}{2}}(\sin\theta-1)d\theta=2(\pi-2)=2\pi-4$

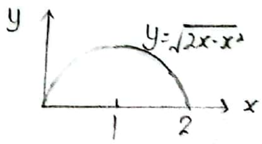
二、计算题



$D\begin{cases}y^2\le x\le y\\0\le y\le 1\end{cases}$

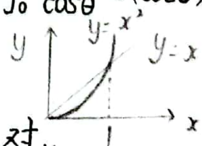
1、$\int_0^1 dx\int_x^{\sqrt{x}}\frac{\sin y}{y}dy=\int_0^1 dy\int_{y^2}^{y}\frac{\sin y}{y}dx=\int_0^1\frac{\sin y}{y}(y-y^2)dy=\int_0^1\sin y\,dy-\int_0^1 y\sin y\,dy=[-\cos y]_0^1-[-y\cos y+\sin y]_0^1=1-\sin 1$

2、$\int_0^2 dx\int_0^{\sqrt{2x-x^2}}\sqrt{x^2+y^2}dy=\int_0^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}r^2dr=\frac{1}{3}\int_0^{\frac{\pi}{2}}8\cos^3\theta d\theta=\frac{8}{3}\cdot\frac{2}{3}=\frac{16}{9}$



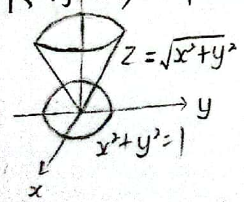
3、$\int_0^1 dx\int_{x^2}^{x}(x^2+y^2)^{-\frac{1}{2}}dy=\int_0^{\frac{\pi}{4}}d\theta\int_0^{\frac{\sin\theta}{\cos^2\theta}}\frac{1}{r}\cdot r\,dr=\int_0^{\frac{\pi}{4}}\frac{\sin\theta}{\cos^2\theta}d\theta=-\int_0^{\frac{\pi}{4}}\frac{1}{\cos^2\theta}d(\cos\theta)$

$=\frac{1}{\cos\theta}|_0^{\frac{\pi}{4}}=\sqrt{2}-1$ $\quad r\sin\theta=r^2\cos^2\theta\Rightarrow r=\frac{\sin\theta}{\cos^2\theta}$

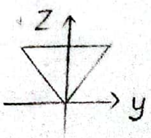


$\begin{cases}0\le x\le 1\\x^2\le y\le x\end{cases}$

4、将积分$\int_{-1}^1 dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dy\int_{\sqrt{x^2+y^2}}^1 f(x,y,z)dz$，换成先对x，再对y，最后对z变量的积分



原式$=\iint_D dydz\int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}}dx=\int_0^1 dz\int_{-z}^{z}dy\int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}}f(x,y,z)dx$



$\begin{cases}\sqrt{x^2+y^2}\le z\le 1\\-\sqrt{1-x^2}\le y\le\sqrt{1-x^2}\\-1\le x\le 1\end{cases}$

5、$\iiint_\Omega z\,dxdydz$，$\Omega=\{(x,y,z)\mid x^2+y^2+z^2\le 1,z\ge 0\}$

(法一) $\iiint_\Omega z\,dxdydz = \iint_D dxdy\int_0^1 z\,dz = \int_0^{\sqrt{1-z^2}} r\,dr\int_0^{2\pi}d\theta\int_0^1 z\,dz = 2\pi\int_0^1 z(\frac{1}{2}r^2)\Big|_0^{\sqrt{1-z^2}}dz = 2\pi\int_0^1\frac{1}{2}z(1-z^2)dz = \pi(\frac{z^2}{2}-\frac{z^4}{4})\Big|_0^1$
$=\frac{\pi}{4}$

(法二) $\iiint_\Omega z\,dxdydz = \int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}z\,d\varphi\int_0^1 r^2\sin\varphi\,dr = 2\pi\cdot\int_0^{\frac{\pi}{2}}\sin\varphi\cos\varphi\,d\varphi\int_0^1 r^3dr = 2\pi\cdot\frac{1}{2}\cdot\frac{1}{4}=\frac{\pi}{4}$

6. $\iiint_\Omega\sqrt{x^2+y^2}\,dxdydz$，$\Omega$由曲面$z=\sqrt{x^2+y^2}$和$z=1$围成

$\iiint_\Omega\sqrt{x^2+y^2}\,dxdydz = \iint_{D_{xy}}\sqrt{x^2+y^2}\,dxdy\int_{\sqrt{x^2+y^2}}^1dz = \iint_{D_{xy}}\sqrt{x^2+y^2}(1-\sqrt{x^2+y^2})dxdy = \int_0^{2\pi}d\theta\int_0^1 r(1-r)r\,dr$

$=2\pi\cdot\int_0^1 r^2-r^3\,dr = 2\pi\cdot(\frac{1}{3}-\frac{1}{4})=\frac{\pi}{6}$

三、证明$\iiint_{x^2+y^2+z^2\le1}f(u)dv=\pi\int_{-1}^1(1-u^2)f(u)du$

$\iiint_{x^2+y^2+z^2\le1}f(u)dv=\int_{-1}^1 f(z)dz\iint_{x^2+y^2\le1-z^2}dxdy=\pi(1-z^2)\int_{-1}^1f(z)dz=\pi(1-z^2)\int_{-1}^1(u)du=\pi\int_{-1}^1(1-u^2)f(u)du$

四、(1)计算以xOy面上的圆周$x^2+y^2=x$围成的闭区域为底，而以曲面$z=x^2+y^2$为顶的曲顶柱体的体积

$D_1=\{(x,y)\mid 0\le y\le\sqrt{x-x^2},0\le x\le1\}=\{(r,\theta)\mid 0\le r\le\cos\theta,0\le\theta\le\frac{\pi}{2}\}$

由于曲顶柱体关于xoz面对称，故$V=2\iint_{D_1}(x^2+y^2)dxdy=2\iint_{D_1}r^2\cdot r\,drd\theta=2\int_0^{\frac{\pi}{2}}d\theta\int_0^{\cos\theta}r^3dr$

$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos^4\theta\,d\theta=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{3}{32}\pi$　　$V=\iint_D(x^2+y^2)dxdy=\iint_D r^2\cdot r\,drd\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{\cos\theta}r^3dr$

(2)求曲面$z=\sqrt{2-x^2-y^2}$和$z=x^2+y^2$所围成的几何体的体积

$\begin{cases}x^2+y^2=z\\\sqrt{2-x^2-y^2}=z\end{cases}$ 得$(x^2+y^2)=2-(x^2+y^2)$，所以在xOy面上的投影区域为$D_{xy}=\{(x,y)\mid x^2+y^2\le1\}$

(法一) $V=\iiint_\Omega dv=\iint_{x^2+y^2\le1}dxdy\int_{x^2+y^2}^{\sqrt{2-x^2-y^2}}dz=\iint_{x^2+y^2}[\sqrt{2-x^2-y^2}-(x^2+y^2)]dxdy=\int_0^{2\pi}d\theta\int_0^1(\sqrt{2-\rho^2}-\rho^2)\rho\,d\rho$

$=\int_0^{2\pi}d\theta\int_0^1(\rho\sqrt{2-\rho^2}-\rho^3)d\rho=2\pi\cdot(\int_0^1\rho\sqrt{2-\rho^2}\,d\rho-\int_0^1\rho^3d\rho)=2\pi\cdot[-\frac{1}{2}\int_0^1\sqrt{2-\rho^2}\,d(2-\rho^2)-\int_0^1\frac{1}{4}d\rho^4]$

$=2\pi\cdot[-\frac{1}{3}(2-\rho^2)^{\frac{3}{2}}\Big|_0^1-\frac{1}{4}]=2\pi\cdot[-\frac{1}{3}(1-2\sqrt{2})-\frac{1}{4}]=2\pi\cdot(-\frac{7}{12}+\frac{2\sqrt{2}}{3})=\frac{4\sqrt{2}}{3}\pi-\frac{7}{6}\pi$

(法二) $V=\iint_{x^2+y^2\le1}\sqrt{2-x^2-y^2}\,dxdy-\iint_{x^2+y^2\le1}x^2+y^2\,dxdy=\iint_{x^2+y^2\le1}[\sqrt{2-x^2-y^2}-(x^2+y^2)]dxdy$

$\frac{1}{2}\int_0^{\frac{\pi}{2}}(\cos^2\theta)^2d\theta=\frac{1}{2}\int_0^{\frac{\pi}{2}}(\frac{\cos2\theta+1}{2})^2=\frac{1}{8}\int_0^{\frac{\pi}{2}}(1+2\cos2\theta+\cos^22\theta)d\theta=\frac{1}{8}\int_0^{\frac{\pi}{2}}(1+2\cos2\theta+\frac{1+\cos4\theta}{2})d\theta$

$=\frac{1}{8}(\int_0^{\frac{\pi}{2}}d\theta+2\int_0^{\frac{\pi}{2}}\cos2\theta\,d\theta+\frac{1}{2}\int_0^{\frac{\pi}{2}}d\theta+\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos4\theta\,d\theta)=\frac{1}{8}(\frac{\pi}{2}+0+\frac{\pi}{4}+0)=\frac{1}{8}\times\frac{3\pi}{4}=\frac{3}{32}\pi$