

# Frequency-Orthogonal LoRA: Improving Multitask Adaptation Efficiency

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## Abstract

This document provides a basic paper template and submission guidelines. Abstracts must be a single paragraph, ideally between 4–6 sentences long. Gross violations will trigger corrections at the camera-ready phase.

## 1. Methodology

**Greedy Frequency Allocation and Per-task Scaling.** We propose a two-stage procedure to construct task-specific masks  $M_t$  and scaling vectors  $G_t$  such that task subspaces  $\mathcal{S}_t$  remain approximately orthogonal and each task achieves performance comparable to single-task training.

**Stage 1: Greedy Frequency Allocation.** Given a total frequency budget  $\Omega = \{1, \dots, d\}$ , we sequentially allocate disjoint frequency subsets to each task. Formally, for  $T$  tasks, we initialize  $\Omega_1 = \Omega$  and iterate:

$$M_t(i) = \begin{cases} 1, & \text{if } i \in \arg \max_{S \subseteq \Omega_t, |S|=k} \Phi_t(S), \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

$$\Omega_{t+1} = \Omega_t \setminus \{i : M_t(i) = 1\}, \quad (2)$$

where  $\Phi_t(S)$  is a task-specific utility score (e.g., validation accuracy or gradient alignment) for selecting frequency set  $S$  of size  $k$ .

**Stage 2: Per-task Least Squares Scaling.** After masks  $M_t$  are fixed, we compute the scaling vector  $G_t \in \mathbb{R}^d$  for each task by solving a least squares problem:

$$G_t^* = \arg \min_{G \in \mathbb{R}^d} \|Y_t - U \cdot \mathcal{F}^{-1}(M_t \odot (F(V) \odot G))X_t\|_2^2, \quad (3)$$

where  $(X_t, Y_t)$  denote the task training data.

This construction ensures that task subspaces  $\mathcal{S}_t$  have minimal overlap due to disjoint frequency allocation, while  $G_t$  adapts to individual task statistics to recover near single-task performance.

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Preliminary work. Under review by the International Conference on Machine Learning (ICML). Do not distribute.

## 2. Experiments

### 2.1. Main Experiments

### 3. Electronic Submission

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- References must include page numbers whenever possible and be as complete as possible. Place multiple citations in chronological order.
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/	# Params	memory	BoolQ	PIQA	SIQA	OBQA	ARC-c	ARC-e	HellaS.	WinoG.	Avg.
Qwen2.5-1.5B											
LoRA	0.8m	14.8									
VeRA	0.8m	14.8									
LoRI	1.8m	10.6									
FMoRA (ours)	1.0m	11.1									
Qwen2.5-7B											
LoRA	2.6m	33.1									
VeRA	0.8m	14.8									
LoRI	4.3m	19.3									
FMoRA (ours)	2.2m	19.2									
Llama2-7B											
LoRA	2.6m	33.1									
VeRA	0.8m	14.8									
LoRI	4.3m	19.3									
FMoRA (ours)	2.2m	19.2									
Mistral0.3-7B											
LoRA	2.6m	33.1									
VeRA	0.8m	14.8									
LoRI	4.3m	19.3									
FMoRA (ours)	2.2m	19.2									

Table 1.

- Keep your abstract brief and self-contained, one paragraph and roughly 4–6 sentences. Gross violations will require correction at the camera-ready phase. The title should have content words capitalized.

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**Anonymous Submission:** ICML uses double-blind review: no identifying author information may appear on the title page or in the paper itself. Section 4.3 gives further details.

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```
dvips -Ppdf -tletter -G0 -o paper.ps paper.dvi
ps2pdf paper.ps
```

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The final versions of papers accepted for publication should follow the same format and naming convention as initial submissions, except that author information (names and affiliations) should be given. See Section 4.3.2 for formatting instructions.

The footnote, “Preliminary work. Under review by the International Conference on Machine Learning (ICML). Do not distribute.” must be modified to “*Proceedings of the 42<sup>nd</sup> International Conference on Machine Learning*, Vancouver, Canada, PMLR 267, 2025. Copyright 2025 by the author(s).”

For those using the  $\text{\LaTeX}$  style file, this change (and others) is handled automatically by simply changing `\usepackage{icml2025}` to

```
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```

Authors using **Word** must edit the footnote on the first page of the document themselves.

Camera-ready copies should have the title of the paper as running head on each page except the first one. The running title consists of a single line centered above a horizontal rule which is 1 point thick. The running head should be centered, bold and in 9 point type. The rule should be 10 points above the main text. For those using the  $\text{\LaTeX}$  style file, the original title is automatically set as running head using the `fancyhdr` package which is included in the ICML 2025 style file package. In case that the original title exceeds the size restrictions, a shorter form can be supplied by using

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## 4. Format of the Paper

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The text of the paper should be formatted in two columns, with an overall width of 6.75 inches, height of 9.0 inches, and 0.25 inches between the columns. The left margin should be 0.75 inches and the top margin 1.0 inch (2.54 cm). The right and bottom margins will depend on whether you print on US letter or A4 paper, but all final versions must be produced for US letter size. Do not write anything on the margins.

The paper body should be set in 10 point type with a vertical spacing of 11 points. Please use Times typeface throughout the text.

### 4.2. Title

The paper title should be set in 14 point bold type and centered between two horizontal rules that are 1 point thick, with 1.0 inch between the top rule and the top edge of the page. Capitalize the first letter of content words and put the rest of the title in lower case.

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ICML uses double-blind review, so author information must not appear. If you are using  $\text{\LaTeX}$  and the `icml2025.sty` file, use `\icmlauthor{...}` to specify authors and `\icmlaffiliation{...}` to specify

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If you are citing published papers for which you are an author, refer to yourself in the third person. In particular, do not use phrases that reveal your identity (e.g., “in previous work (Langley, 2000), we have shown ...”).

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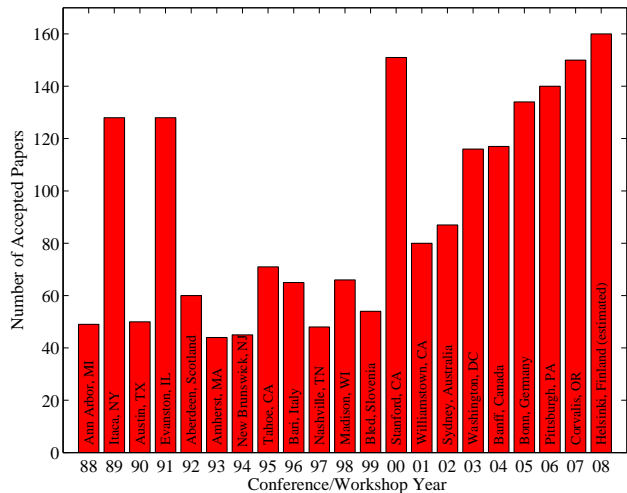


Figure 1. Historical locations and number of accepted papers for International Machine Learning Conferences (ICML 1993 – ICML 2008) and International Workshops on Machine Learning (ML 1988 – ML 1992). At the time this figure was produced, the number of accepted papers for ICML 2008 was unknown and instead estimated.

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You may want to include figures in the paper to illustrate your approach and results. Such artwork should be centered, legible, and separated from the text. Lines should be dark and at least 0.5 points thick for purposes of reproduction, and text should not appear on a gray background.

Label all distinct components of each figure. If the figure takes the form of a graph, then give a name for each axis and include a legend that briefly describes each curve. Do not include a title inside the figure; instead, the caption should serve this function.

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<sup>1</sup>Footnotes should be complete sentences.

<sup>2</sup>Multiple footnotes can appear in each column, in the same order as they appear in the text, but spread them across columns and pages if possible.

**Algorithm 1** Bubble Sort

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**Input:** data  $x_i$ , size  $m$

**repeat**

  Initialize  $noChange = true$ .

**for**  $i = 1$  **to**  $m - 1$  **do**

**if**  $x_i > x_{i+1}$  **then**

      Swap  $x_i$  and  $x_{i+1}$

$noChange = false$

**end if**

**end for**

**until**  $noChange$  is  $true$

---

Table 2. Classification accuracies for naive Bayes and flexible Bayes on various data sets.

DATA SET	NAIVE	FLEXIBLE	BETTER?
BREAST	95.9±0.2	96.7±0.2	✓
CLEVELAND	83.3±0.6	80.0±0.6	×
GLASS2	61.9±1.4	83.8±0.7	✓
CREDIT	74.8±0.5	78.3±0.6	
HORSE	73.3±0.9	69.7±1.0	×
META	67.1±0.6	76.5±0.5	✓
PIMA	75.1±0.6	73.9±0.5	
VEHICLE	44.9±0.6	61.5±0.4	✓

**4.7. Algorithms**

If you are using  $\text{\LaTeX}$ , please use the “algorithm” and “algorithmic” environments to format pseudocode. These require the corresponding stylefiles, algorithm.sty and algorithmic.sty, which are supplied with this package. Algorithm 1 shows an example.

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You may also want to include tables that summarize material. Like figures, these should be centered, legible, and numbered consecutively. However, place the title *above* the table with at least 0.1 inches of space before the title and the same after it, as in Table 2. The table title should be set in 9 point type and centered unless it runs two or more lines, in which case it should be flush left.

Tables contain textual material, whereas figures contain graphical material. Specify the contents of each row and column in the table’s topmost row. Again, you may float tables to a column’s top or bottom, and set wide tables across both columns. Place two-column tables at the top or bottom of the page.

**4.9. Theorems and such**

The preferred way is to number definitions, propositions, lemmas, etc. consecutively, within sections, as shown below.

**Definition 4.1.** A function  $f : X \rightarrow Y$  is injective if for any  $x, y \in X$  different,  $f(x) \neq f(y)$ .

Using Theorem 4.1 we immediately get the following result:

**Proposition 4.2.** *If  $f$  is injective mapping a set  $X$  to another set  $Y$ , the cardinality of  $Y$  is at least as large as that of  $X$*

*Proof.* Left as an exercise to the reader.  $\square$

Theorem 4.3 stated next will prove to be useful.

**Lemma 4.3.** *For any  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  injective functions,  $f \circ g$  is injective.*

**Theorem 4.4.** *If  $f : X \rightarrow Y$  is bijective, the cardinality of  $X$  and  $Y$  are the same.*

An easy corollary of Theorem 4.4 is the following:

**Corollary 4.5.** *If  $f : X \rightarrow Y$  is bijective, the cardinality of  $X$  is at least as large as that of  $Y$ .*

**Assumption 4.6.** The set  $X$  is finite.

*Remark 4.7.* According to some, it is only the finite case (cf. Theorem 4.6) that is interesting.

**4.10. Citations and References**

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Use an unnumbered first-level section heading for the references, and use a hanging indent style, with the first line of the reference flush against the left margin and subsequent lines indented by 10 points. The references at the end of this document give examples for journal articles (Samuel, 1959), conference publications (Langley, 2000), book chapters (Newell & Rosenbloom, 1981), books (Duda et al.,



2000), edited volumes (Michalski et al., 1983), technical reports (Mitchell, 1980), and dissertations (Kearns, 1989).

Alphabetize references by the surnames of the first authors, with single author entries preceding multiple author entries. Order references for the same authors by year of publication, with the earliest first. Make sure that each reference includes all relevant information (e.g., page numbers).

Please put some effort into making references complete, presentable, and consistent, e.g. use the actual current name of authors. If using bibtex, please protect capital letters of names and abbreviations in titles, for example, use {B}ayesian or {L}ipschitz in your .bib file.

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## Impact Statement

Authors are **required** to include a statement of the potential broader impact of their work, including its ethical aspects and future societal consequences. This statement should be in an unnumbered section at the end of the paper (co-located

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“This paper presents work whose goal is to advance the field of Machine Learning. There are many potential societal consequences of our work, none which we feel must be specifically highlighted here.”

The above statement can be used verbatim in such cases, but we encourage authors to think about whether there is content which does warrant further discussion, as this statement will be apparent if the paper is later flagged for ethics review.

## References

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## A. Frequency Domain Analysis

## B. Experimental Hyperparameters

## C. MT Bench Case Study

## D. Proof of Rank Increase

## E. Rank Increase Analysis

**Theorem E.1** (Rank increase under nontrivial frequency masking). *Let  $\mathbb{K}$  denote either  $\mathbb{R}$  or  $\mathbb{C}$ . Consider a rank- $r$  matrix*

$$\mathbf{W}_{UV} = \sum_{i=1}^r \mathbf{U}_i \mathbf{V}_i^\top, \quad \mathbf{U}_i \in \mathbb{K}^m, \mathbf{V}_i \in \mathbb{K}^n,$$

*so that  $\text{rank}(\mathbf{W}_{UV}) \leq r$ . Let  $\mathbf{M}_t \in \mathbb{C}^{m \times n}$  be the frequency-domain mask for task  $t$ , and define its inverse DFT  $h = \mathcal{F}^{-1}(\mathbf{M}_t) \in \mathbb{C}^{m \times n}$ , interpreted as the spatial convolution kernel.*

*Assume that  $h$  has finite support on  $e$  distinct circular shifts, i.e.*

$$h = \sum_{k=1}^e \alpha_k S_{s_k, e_k}(\Delta), \quad \alpha_k \in \mathbb{C} \setminus \{0\},$$

*where  $S_{s,e}$  denotes the two-dimensional circular shift operator:*

$$[S_{s,e}(\mathbf{X})]_{i,j} = \mathbf{X}_{(i-s) \bmod m, (j-e) \bmod n}.$$

*For  $\mathbf{U} \in \mathbb{K}^m$ , let  $S_s(\mathbf{U})$  denote the corresponding 1D circular shift, so that  $S_{s,e}(U\mathbf{V}^\top) = S_s(\mathbf{U})S_e(\mathbf{V})^\top$ .*

*Then the frequency-masked transform of  $\mathbf{W}_{UV}$  satisfies*

$$\mathcal{T}_{\mathbf{M}_t}(\mathbf{W}_{UV}) = \sum_{k=1}^e \alpha_k S_{s_k, e_k}(\mathbf{W}_{UV}) = \sum_{i=1}^r \sum_{k=1}^e \alpha_k S_{s_k}(\mathbf{U}_i) S_{e_k}(\mathbf{V}_i)^\top.$$

*Suppose the following nondegeneracy conditions hold:*

- (a) For each  $i$ , the shifted vectors  $\{S_{s_k}(\mathbf{U}_i)\}_{k=1}^e$  are linearly independent (or span a subspace of dimension  $t_i \geq 2$ ), and similarly  $\{S_{e_k}(\mathbf{V}_i)\}_{k=1}^e$  are in general position.*
- (b) The combined families  $\{S_{s_k}(\mathbf{U}_i)\}_{i,k}$  and  $\{S_{e_k}(\mathbf{V}_i)\}_{i,k}$  are in general position across different  $i$ , i.e. their Kruskal ranks satisfy the standard generic full-rank condition.<sup>3</sup>*

*Then, generically,*

$$\text{rank}(\mathcal{T}_{\mathbf{M}_t}(\mathbf{W}_{UV})) = \min(re, \min(m, n)).$$

*In particular, whenever  $e > 1$  and conditions (a)–(b) hold,*

$$\text{rank}(\mathcal{T}_{\mathbf{M}_t}(\mathbf{W}_{UV})) > \text{rank}(\mathbf{W}_{UV}).$$

**Lemma E.2** (Khatrı–Rao representation and generic full rank). *For any  $u \in \mathbb{K}^m$  and  $v \in \mathbb{K}^n$ ,*

$$\text{vec}(S_s(u) S_e(v)^\top) = S_e(v) \odot S_s(u),$$

*where  $\odot$  denotes the columnwise Kronecker (Khatrı–Rao) product. Let*

$$\mathbf{A} := [S_{e_1}(\mathbf{V}_1) \odot S_{s_1}(\mathbf{U}_1), \dots, S_{e_e}(\mathbf{V}_r) \odot S_{s_e}(\mathbf{U}_r)] \in \mathbb{K}^{mn \times (re)}.$$

<sup>3</sup>Formally, for an open dense subset of  $(\mathbf{U}_i, \mathbf{V}_i) \in \mathbb{K}^{m \times r} \times \mathbb{K}^{n \times r}$ , the outer products  $\{S_{s_k}(\mathbf{U}_i) S_{e_k}(\mathbf{V}_i)^\top\}_{i,k}$  are linearly independent up to the ambient limit.

If  $A$  has full column rank, the corresponding rank-1 atoms  $\{S_{s_k}(\mathbf{U}_i)S_{e_k}(\mathbf{V}_i)^\top\}_{i,k}$  are linearly independent. Consequently,

$$\text{rank}(\mathcal{T}_{M_t}(\mathbf{W}_{UV})) = \min(re, \min(m, n)).$$

Moreover,  $A$  has full column rank for an open dense subset of  $(\mathbf{U}_i, \mathbf{V}_i) \in \mathbb{K}^{m \times r} \times \mathbb{K}^{n \times r}$ , since  $\det(A^\top A)$  is a nonzero polynomial in the entries of  $\mathbf{U}_i, \mathbf{V}_i$ .

**Assumption E.3** (Nondegeneracy). We guarantee the following:

1. The singular random variable vectors  $\{\mathbf{U}_i\}_{i=1}^r$  and  $\{\mathbf{V}_i\}_{i=1}^r$  of  $\mathbf{W}_{UV}$  are linearly independent:

$$\text{rank}([\mathbf{U}_1, \dots, \mathbf{U}_r]) = r, \quad \text{rank}([\mathbf{V}_1, \dots, \mathbf{V}_r]) = r.$$

2. The frequency mask  $M_t$  is *nontrivial*, meaning it is not proportional to the all-one matrix and contains at least one entry with distinct complex magnitude or phase:

$$\exists(p, q), (p', q') \text{ such that } M_t(p, q) \neq \alpha M_t(p', q') \text{ for any } \alpha \in \mathbb{C}.$$

*Proof. Step 1 (Convolution representation).* By the convolution theorem, elementwise multiplication by  $M_t$  in the frequency domain corresponds to circular convolution by its inverse DFT  $h$  in the spatial domain:

$$\mathcal{T}_{M_t}(\mathbf{W}_{UV}) = h \star_{\text{circ}} \mathbf{W}_{UV}.$$

If  $h = \sum_{k=1}^e \alpha_k S_{s_k, e_k}(\Delta)$ , linearity yields

$$h \star_{\text{circ}} \mathbf{W}_{UV} = \sum_{k=1}^e \alpha_k S_{s_k, e_k}(\mathbf{W}_{UV}).$$

**Step 2 (Rank-1 case).** For a single rank-1 factor  $\mathbf{UV}^\top$ ,

$$\mathcal{T}_{M_t}(\mathbf{UV}^\top) = \sum_{k=1}^e \alpha_k S_{s_k}(\mathbf{U}) S_{e_k}(\mathbf{V})^\top.$$

Each term is rank-1. Under condition (a), the family  $\{S_{s_k}(\mathbf{U}) S_{e_k}(\mathbf{V})^\top\}_{k=1}^e$  is generically linearly independent up to  $\min(e, \min(m, n))$ , implying

$$\text{rank}(\mathcal{T}_{M_t}(\mathbf{UV}^\top)) = \min(e, \min(m, n)).$$

**Step 3 (General rank- $r$  case).** For  $\mathbf{W}_{UV} = \sum_{i=1}^r \mathbf{U}_i \mathbf{V}_i^\top$ ,

$$\mathcal{T}_{M_t}(\mathbf{W}_{UV}) = \sum_{i=1}^r \mathcal{T}_{M_t}(\mathbf{U}_i \mathbf{V}_i^\top) = \sum_{i=1}^r \sum_{k=1}^e \alpha_k S_{s_k}(\mathbf{U}_i) S_{e_k}(\mathbf{V}_i)^\top.$$

This is a linear combination of  $re$  rank-1 matrices. By Lemma E.2, their vectorizations form the columns of  $A$ . Under assumption E.3,  $A$  has full column rank generically, hence

$$\text{rank}(\mathcal{T}_{M_t}(\mathbf{W}_{UV})) = \min(re, \min(m, n)).$$

**Step 4 (Existence of suitable masks).** For any  $e > 1$ , we can explicitly construct  $h = \sum_{k=1}^e \alpha_k S_{s_k, e_k}(\Delta)$  with distinct shifts and nonzero  $\alpha_k$ , and define  $M_t = \mathcal{F}(h)$ . By Step 3, such a nontrivial mask generically increases rank.

**Step 5 (Field consistency).** The argument holds over  $\mathbb{R}$  or  $\mathbb{C}$  depending on  $\alpha_k$ . In both cases, the rank formula remains valid.  $\square$

*Remark E.4* (Discussion and genericity). 1. The assumption that  $h$  is a finite sum of shifted deltas is a constructive instance. More generally, if  $h$  has  $e$  dominant coefficients (effective support), the same reasoning holds approximately when rank is replaced by numerical rank.



2. The result is *generic*: for an open dense subset of  $(\mathbf{U}_i, \mathbf{V}_i, \alpha_k) \in \mathbb{K}^{m \times r} \times \mathbb{K}^{n \times r} \times (\mathbb{C} \setminus \{0\})^e$ , the rank-1 atoms  $\{S_{s_k}(\mathbf{U}_i)S_{e_k}(\mathbf{V}_i)^\top\}_{i,k}$  are linearly independent, yielding the upper bound  $\min(re, \min(m, n))$ . Degeneracies occur only for highly structured  $\mathbf{U}_i$  or  $\mathbf{V}_i$  (e.g., constant or periodic).

3. In vectorized operator form,

$$\text{vec}(\mathcal{T}_{\mathbf{M}_t}(\mathbf{W}_{UV})) = C_{\mathbf{M}_t} \text{vec}(\mathbf{W}_{UV}), \quad C_{\mathbf{M}_t} := \mathcal{F}^{-1} \text{diag}(\text{vec}(\mathbf{M}_t))\mathcal{F}.$$

The operator  $C_{\mathbf{M}_t}$  has rank equal to the number of nonzero entries in  $\mathbf{M}_t$ . Even when  $C_{\mathbf{M}_t}$  is not full-rank, the induced linear mixing via circular shifts can increase the rank of  $\mathbf{W}_{UV}$ , thereby breaking the original low-rank structure.