Frequency-Orthogonal LoRA: Improving Multitask Adaptation Efficiency

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Abstract

This document provides a basic paper template and submission guidelines. Abstracts must be a single paragraph, ideally between 4–6 sentences long. Gross violations will trigger corrections at the camera-ready phase.

1. Methodology

Greedy Frequency Allocation and Per-task Scaling. We propose a two-stage procedure to construct task-specific masks M_t and scaling vectors G_t such that task subspaces $\mathcal{S}_t$ remain approximately orthogonal and each task achieves performance comparable to single-task training.

Stage 1: Greedy Frequency Allocation. Given a total frequency budget $\Omega = \{1, \ldots, d\}$, we sequentially allocate disjoint frequency subsets to each task. Formally, for T tasks, we initialize $\Omega_1 = \Omega$ and iterate:

$$M_t(i) = \begin{cases} 1, & \text{if } i \in \arg\max_{S \subseteq \Omega_t,\, |S|=k} \Phi_t(S), \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

$$\Omega_{t+1} = \Omega_t \setminus \{i : M_t(i) = 1\}, \tag{2}$$

where $\Phi_t(S)$ is a task-specific utility score (e.g., validation accuracy or gradient alignment) for selecting frequency set S of size k.

Stage 2: Per-task Least Squares Scaling. After masks M_t are fixed, we compute the scaling vector $G_t \in \mathbb{R}^d$ for each task by solving a least squares problem:

$$G_t^\star = \arg\min_{G \in \mathbb{R}^d} \left\| Y_t - U \cdot \mathcal{F}^{-1}\big(M_t \odot (F(V) \odot G)\big) X_t \right\|_2^2, \tag{3}$$

where (X_t, Y_t) denote the task training data.

This construction ensures that task subspaces $\mathcal{S}_t$ have minimal overlap due to disjoint frequency allocation, while G_t adapts to individual task statistics to recover near single-task performance.

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2. Experiments

2.1. Main Experiments

3. Electronic Submission

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- Submissions must be in PDF.
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Qwen2.5-1.5B											
LoRA	0.8m	14.8									
VeRA	0.8m	14.8									
LoRI	1.8m	10.6									
FMoRA (ours)	1.0m	11.1									
Qwen2.5-7B											
LoRA	2.6m	33.1									
VeRA	0.8m	14.8									
LoRI	4.3m	19.3									
FMoRA (ours)	2.2m	19.2									
Llama2-7B											
LoRA	2.6m	33.1									
VeRA	0.8m	14.8									
LoRI	4.3m	19.3									
FMoRA (ours)	2.2m	19.2									
Mistral0.3-7B											
LoRA	2.6m	33.1									
VeRA	0.8m	14.8									
LoRI	4.3m	19.3									
FMoRA (ours)	2.2m	19.2									

Table 1.

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dvips -Ppdf -tletter -G0 -o paper.ps paper.dvi
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```

It is a zero following the "-G", which tells dvips to use the config.pdf file. Newer TeX distributions don't always need this option.

Using `pdflatex` rather than `latex`, often gives better results. This program avoids the Type-3 font problem, and supports more advanced features in the `microtype` package.

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The final versions of papers accepted for publication should follow the same format and naming convention as initial submissions, except that author information (names and affiliations) should be given. See Section 4.3.2 for formatting instructions.

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The text of the paper should be formatted in two columns, with an overall width of 6.75 inches, height of 9.0 inches, and 0.25 inches between the columns. The left margin should be 0.75 inches and the top margin 1.0 inch (2.54 cm). The right and bottom margins will depend on whether you print on US letter or A4 paper, but all final versions must be produced for US letter size. Do not write anything on the margins.

The paper body should be set in 10 point type with a vertical spacing of 11 points. Please use Times typeface throughout the text.

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The paper title should be set in 14 point bold type and centered between two horizontal rules that are 1 point thick, with 1.0 inch between the top rule and the top edge of the page. Capitalize the first letter of content words and put the rest of the title in lower case.

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If a paper is accepted, a final camera-ready copy must be prepared. For camera-ready papers, author information should start 0.3 inches below the bottom rule surrounding the title. The authors' names should appear in 10 point bold type, in a row, separated by white space, and centered. Author names should not be broken across lines. Unbolded superscripted numbers, starting 1, should be used to refer to affiliations.

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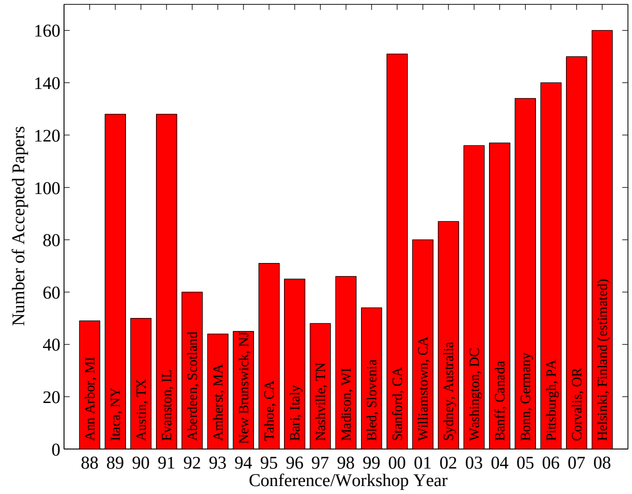

Figure 1. Historical locations and number of accepted papers for International Machine Learning Conferences (ICML 1993 – ICML 2008) and International Workshops on Machine Learning (ML 1988 – ML 1992). At the time this figure was produced, the number of accepted papers for ICML 2008 was unknown and instead estimated.

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You may want to include figures in the paper to illustrate your approach and results. Such artwork should be centered, legible, and separated from the text. Lines should be dark and at least 0.5 points thick for purposes of reproduction, and text should not appear on a gray background.

Label all distinct components of each figure. If the figure takes the form of a graph, then give a name for each axis and include a legend that briefly describes each curve. Do not include a title inside the figure; instead, the caption should serve this function.

Number figures sequentially, placing the figure number and caption *after* the graphics, with at least 0.1 inches of space before the caption and 0.1 inches after it, as in Figure 1. The figure caption should be set in 9 point type and centered unless it runs two or more lines, in which case it should be flush left. You may float figures to the top or bottom of a column, and you may set wide figures across both columns (use the environment `figure*` in LaTeX). Always place two-column figures at the top or bottom of the page.

[1]Footnotes should be complete sentences.

[2]Multiple footnotes can appear in each column, in the same order as they appear in the text, but spread them across columns and pages if possible.

Algorithm 1 Bubble Sort

```
Input: data x_i, size m
repeat
  Initialize noChange = true.
  for i = 1 to m − 1 do
    if x_i > x_{i+1} then
      Swap x_i and x_{i+1}
      noChange = false
    end if
  end for
until noChange is true
```

Table 2. Classification accuracies for naive Bayes and flexible Bayes on various data sets.

DATA SET	NAIVE	FLEXIBLE	BETTER?
BREAST	95.9± 0.2	96.7± 0.2	√
CLEVELAND	83.3± 0.6	80.0± 0.6	×
GLASS2	61.9± 1.4	83.8± 0.7	√
CREDIT	74.8± 0.5	78.3± 0.6	
HORSE	73.3± 0.9	69.7± 1.0	×
META	67.1± 0.6	76.5± 0.5	√
PIMA	75.1± 0.6	73.9± 0.5	
VEHICLE	44.9± 0.6	61.5± 0.4	√

4.7. Algorithms

If you are using LaTeX, please use the "algorithm" and "algorithmic" environments to format pseudocode. These require the corresponding stylefiles, algorithm.sty and algorithmic.sty, which are supplied with this package. Algorithm 1 shows an example.

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You may also want to include tables that summarize material. Like figures, these should be centered, legible, and numbered consecutively. However, place the title *above* the table with at least 0.1 inches of space before the title and the same after it, as in Table 2. The table title should be set in 9 point type and centered unless it runs two or more lines, in which case it should be flush left.

Tables contain textual material, whereas figures contain graphical material. Specify the contents of each row and column in the table's topmost row. Again, you may float tables to a column's top or bottom, and set wide tables across both columns. Place two-column tables at the top or bottom of the page.

4.9. Theorems and such

The preferred way is to number definitions, propositions, lemmas, etc. consecutively, within sections, as shown below.

Definition 4.1. A function $f : X \to Y$ is injective if for any $x, y \in X$ different, $f(x) \neq f(y)$.

Using Theorem 4.1 we immediate get the following result:

Proposition 4.2. *If f is injective mapping a set X to another set Y, the cardinality of Y is at least as large as that of X*

Proof. Left as an exercise to the reader. □

Theorem 4.3 stated next will prove to be useful.

Lemma 4.3. *For any $f : X \to Y$ and $g : Y \to Z$ injective functions, $f \circ g$ is injective.*

Theorem 4.4. *If $f : X \to Y$ is bijective, the cardinality of X and Y are the same.*

An easy corollary of Theorem 4.4 is the following:

Corollary 4.5. *If $f : X \to Y$ is bijective, the cardinality of X is at least as large as that of Y.*

Assumption 4.6. The set X is finite.

Remark 4.7. According to some, it is only the finite case (cf. Theorem 4.6) that is interesting.

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Authors should cite their own work in the third person in the initial version of their paper submitted for blind review. Please refer to Section 4.3 for detailed instructions on how to cite your own papers.

Use an unnumbered first-level section heading for the references, and use a hanging indent style, with the first line of the reference flush against the left margin and subsequent lines indented by 10 points. The references at the end of this document give examples for journal articles (Samuel, 1959), conference publications (Langley, 2000), book chapters (Newell & Rosenbloom, 1981), books (Duda et al., 2000), edited volumes (Michalski et al., 1983), technical reports (Mitchell, 1980), and dissertations (Kearns, 1989).

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Authors are **required** to include a statement of the potential broader impact of their work, including its ethical aspects and future societal consequences. This statement should be in an unnumbered section at the end of the paper (co-located with Acknowledgements – the two may appear in either order, but both must be before References), and does not count toward the paper page limit. In many cases, where the ethical impacts and expected societal implications are those that are well established when advancing the field of Machine Learning, substantial discussion is not required, and a simple statement such as the following will suffice:

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References

Author, N. N. Suppressed for anonymity, 2021.

Duda, R. O., Hart, P. E., and Stork, D. G. *Pattern Classification*. John Wiley and Sons, 2nd edition, 2000.

Kearns, M. J. *Computational Complexity of Machine Learning*. PhD thesis, Department of Computer Science, Harvard University, 1989.

Langley, P. Crafting papers on machine learning. In Langley, P. (ed.), *Proceedings of the 17th International Conference on Machine Learning (ICML 2000)*, pp. 1207–1216, Stanford, CA, 2000. Morgan Kaufmann.

Michalski, R. S., Carbonell, J. G., and Mitchell, T. M. (eds.). *Machine Learning: An Artificial Intelligence Approach, Vol. I*. Tioga, Palo Alto, CA, 1983.

Mitchell, T. M. The need for biases in learning generalizations. Technical report, Computer Science Department, Rutgers University, New Brunswick, MA, 1980.

Newell, A. and Rosenbloom, P. S. Mechanisms of skill acquisition and the law of practice. In Anderson, J. R. (ed.), *Cognitive Skills and Their Acquisition*, chapter 1, pp. 1–51. Lawrence Erlbaum Associates, Inc., Hillsdale, NJ, 1981.

Samuel, A. L. Some studies in machine learning using the game of checkers. *IBM Journal of Research and Development*, 3(3):211–229, 1959.

A. Frequency Domain Analysis

B. Experimental Hyperparameters

C. MT Bench Case Study

D. Proof of Rank Increasement

E. Rank Increasement Analysis

Theorem E.1 (Rank increase under nontrivial frequency masking)**.** *Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Consider a rank–r matrix*

$$\boldsymbol{W}_{UV} = \sum_{i=1}^{r} \boldsymbol{U}_i \boldsymbol{V}_i^{\top}, \quad \boldsymbol{U}_i \in \mathbb{K}^m,\ \boldsymbol{V}_i \in \mathbb{K}^n,$$

so that $\operatorname{rank}(\boldsymbol{W}_{UV}) \leq r$. Let $\boldsymbol{M}_t \in \mathbb{C}^{m\times n}$ be the frequency-domain mask for task t, and define its inverse DFT $h = \mathcal{F}^{-1}(\boldsymbol{M}_t) \in \mathbb{C}^{m\times n}$, interpreted as the spatial convolution kernel.

Assume that h has finite support on e distinct circular shifts, i.e.

$$h = \sum_{k=1}^{e} \alpha_k\, S_{s_k,e_k}(\Delta), \quad \alpha_k \in \mathbb{C}\setminus\{0\},$$

where $S_{s,e}$ denotes the two-dimensional circular shift operator:

$$[S_{s,e}(\boldsymbol{X})]_{i,j} = \boldsymbol{X}_{(i-s) \bmod m,\,(j-e) \bmod n}.$$

For $\boldsymbol{U} \in \mathbb{K}^m$, let $S_s(\boldsymbol{U})$ denote the corresponding 1D circular shift, so that $S_{s,e}(\boldsymbol{U}\boldsymbol{V}^{\top}) = S_s(\boldsymbol{U})S_e(\boldsymbol{V})^{\top}$.

Then the frequency-masked transform of $\boldsymbol{W}_{UV}$ satisfies

$$\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV}) = \sum_{k=1}^{e} \alpha_k\, S_{s_k,e_k}(\boldsymbol{W}_{UV}) = \sum_{i=1}^{r}\sum_{k=1}^{e} \alpha_k\, S_{s_k}(\boldsymbol{U}_i)\, S_{e_k}(\boldsymbol{V}_i)^{\top}.$$

Suppose the following nondegeneracy conditions *hold:*

(a) *For each i, the shifted vectors $\{S_{s_k}(\boldsymbol{U}_i)\}_{k=1}^{e}$ are linearly independent (or span a subspace of dimension $t_i \geq 2$), and similarly $\{S_{e_k}(\boldsymbol{V}_i)\}_{k=1}^{e}$ are in general position.*

(b) *The combined families $\{S_{s_k}(\boldsymbol{U}_i)\}_{i,k}$ and $\{S_{e_k}(\boldsymbol{V}_i)\}_{i,k}$ are in general position across different i, i.e. their Kruskal ranks satisfy the standard generic full-rank condition.*[3]

Then, generically,

$$\operatorname{rank}\big(\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV})\big) = \min\big(re,\ \min(m,n)\big).$$

In particular, whenever $e > 1$ and conditions (a)–(b) hold,

$$\operatorname{rank}\big(\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV})\big) > \operatorname{rank}(\boldsymbol{W}_{UV}).$$

Lemma E.2 (Khatri–Rao representation and generic full rank)**.** *For any $u \in \mathbb{K}^m$ and $v \in \mathbb{K}^n$,*

$$\operatorname{vec}\big(S_s(u)\, S_e(v)^{\top}\big) = S_e(v) \odot S_s(u),$$

where $\odot$ denotes the columnwise Kronecker (Khatri–Rao) product. Let

$$A := \big[\, S_{e_1}(\boldsymbol{V}_1)\odot S_{s_1}(\boldsymbol{U}_1),\ \ldots,\ S_{e_e}(\boldsymbol{V}_r)\odot S_{s_e}(\boldsymbol{U}_r) \,\big] \in \mathbb{K}^{mn\times(re)}.$$

[3] Formally, for an open dense subset of $(\boldsymbol{U}_i, \boldsymbol{V}_i) \in \mathbb{K}^{m\times r}\times\mathbb{K}^{n\times r}$, the outer products $\{S_{s_k}(\boldsymbol{U}_i)S_{e_k}(\boldsymbol{V}_i)^{\top}\}_{i,k}$ are linearly independent up to the ambient limit.

If A has full column rank, the corresponding rank–1 atoms $\{S_{s_k}(\boldsymbol{U}_i)S_{e_k}(\boldsymbol{V}_i)^\top\}_{i,k}$ are linearly independent. Consequently,

$$\operatorname{rank}\big(\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV})\big) = \min(re,\, \min(m,n)).$$

Moreover, A has full column rank for an open dense subset of $(\boldsymbol{U}_i, \boldsymbol{V}_i) \in \mathbb{K}^{m\times r} \times \mathbb{K}^{n\times r}$, since $\det(A^\top A)$ is a nonzero polynomial in the entries of $\boldsymbol{U}_i, \boldsymbol{V}_i$.

Assumption E.3 (Nondegeneracy)**.** We assume:

1. The singular random variable vectors $\{\boldsymbol{U}_i\}_{i=1}^r$ and $\{\boldsymbol{V}_i\}_{i=1}^r$ of $\boldsymbol{W}_{UV}$ are linearly independent:

$$\operatorname{rank}\big([\boldsymbol{U}_1, \dots, \boldsymbol{U}_r]\big) = r, \quad \operatorname{rank}\big([\boldsymbol{V}_1, \dots, \boldsymbol{V}_r]\big) = r.$$

2. The frequency mask $\boldsymbol{M}_t$ is *nontrivial*, meaning it is not proportional to the all-one matrix and contains at least one entry with distinct complex magnitude or phase:

$$\exists (p,q), (p',q') \text{ such that } \boldsymbol{M}_t(p,q) \neq \alpha\, \boldsymbol{M}_t(p',q') \text{ for any } \alpha \in \mathbb{C}.$$

Proof. **Step 1 (Convolution representation).** By the convolution theorem, elementwise multiplication by $\boldsymbol{M}_t$ in the frequency domain corresponds to circular convolution by its inverse DFT h in the spatial domain:

$$\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV}) = h \star_{\text{circ}} \boldsymbol{W}_{UV}.$$

If $h = \sum_{k=1}^e \alpha_k S_{s_k,e_k}(\Delta)$, linearity yields

$$h \star_{\text{circ}} \boldsymbol{W}_{UV} = \sum_{k=1}^{e} \alpha_k\, S_{s_k,e_k}(\boldsymbol{W}_{UV}).$$

Step 2 (Rank–1 case). For a single rank–1 factor $\boldsymbol{U}\boldsymbol{V}^\top$,

$$\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{U}\boldsymbol{V}^\top) = \sum_{k=1}^{e} \alpha_k\, S_{s_k}(\boldsymbol{U})\, S_{e_k}(\boldsymbol{V})^\top.$$

Each term is rank–1. Under condition (a), the family $\{S_{s_k}(\boldsymbol{U})\, S_{e_k}(\boldsymbol{V})^\top\}_{k=1}^e$ is generically linearly independent up to $\min(e, \min(m,n))$, implying

$$\operatorname{rank}\left(\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{U}\boldsymbol{V}^\top)\right) = \min(e, \min(m,n)).$$

Step 3 (General rank–r case). For $\boldsymbol{W}_{UV} = \sum_{i=1}^r \boldsymbol{U}_i \boldsymbol{V}_i^\top$,

$$\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV}) = \sum_{i=1}^{r} \mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{U}_i\boldsymbol{V}_i^\top) = \sum_{i=1}^{r}\sum_{k=1}^{e} \alpha_k\, S_{s_k}(\boldsymbol{U}_i)\, S_{e_k}(\boldsymbol{V}_i)^\top.$$

This is a linear combination of re rank–1 matrices. By Lemma E.2, their vectorizations form the columns of A. Under assumption E.3, A has full column rank generically, hence

$$\operatorname{rank}\left(\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV})\right) = \min(re,\, \min(m,n)).$$

Step 4 (Existence of suitable masks). For any $e > 1$, we can explicitly construct $h = \sum_{k=1}^e \alpha_k S_{s_k,e_k}(\Delta)$ with distinct shifts and nonzero α_k, and define $\boldsymbol{M}_t = \mathcal{F}(h)$. By Step 3, such a nontrivial mask generically increases rank.

Step 5 (Field consistency). The argument holds over $\mathbb{R}$ or $\mathbb{C}$ depending on α_k. In both cases, the rank formula remains valid. □

Remark E.4 (Discussion and genericity). 1. The assumption that h is a finite sum of shifted deltas is a constructive instance. More generally, if h has e dominant coefficients (effective support), the same reasoning holds approximately when rank is replaced by numerical rank.

2. The result is *generic*: for an open dense subset of $(\boldsymbol{U}_i, \boldsymbol{V}_i, \alpha_k) \in \mathbb{K}^{m\times r} \times \mathbb{K}^{n\times r} \times (\mathbb{C} \setminus \{0\})^e$, the rank–1 atoms $\{S_{s_k}(\boldsymbol{U}_i) S_{e_k}(\boldsymbol{V}_i)^\top\}_{i,k}$ are linearly independent, yielding the upper bound $\min(re, \min(m,n))$. Degeneracies occur only for highly structured $\boldsymbol{U}_i$ or $\boldsymbol{V}_i$ (e.g., constant or periodic).

3. In vectorized operator form,

$$\operatorname{vec}\big(\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV})\big) = C_{\boldsymbol{M}_t} \operatorname{vec}(\boldsymbol{W}_{UV}), \quad C_{\boldsymbol{M}_t} := \mathcal{F}^{-1} \operatorname{diag}(\operatorname{vec}(\boldsymbol{M}_t))\mathcal{F}.$$

The operator $C_{\boldsymbol{M}_t}$ has rank equal to the number of nonzero entries in $\boldsymbol{M}_t$. Even when $C_{\boldsymbol{M}_t}$ is not full-rank, the induced linear mixing via circular shifts can increase the rank of $\boldsymbol{W}_{UV}$, thereby breaking the original low-rank structure.