# Frequency-Orthogonal LoRA: Improving Multitask Adaptation Efficiency

**Anonymous Authors**[1]

## Abstract

This document provides a basic paper template and submission guidelines. Abstracts must be a single paragraph, ideally between 4–6 sentences long. Gross violations will trigger corrections at the camera-ready phase.

## 1. Methodology

**Greedy Frequency Allocation and Per-task Scaling.** We propose a two-stage procedure to construct task-specific masks $M_t$ and scaling vectors $G_t$ such that task subspaces $\mathcal{S}_t$ remain approximately orthogonal and each task achieves performance comparable to single-task training.

**Stage 1: Greedy Frequency Allocation.** Given a total frequency budget $\Omega = \{1, \ldots, d\}$, we sequentially allocate disjoint frequency subsets to each task. Formally, for $T$ tasks, we initialize $\Omega_1 = \Omega$ and iterate:

$$M_t(i) = \begin{cases} 1, & \text{if } i \in \arg\max_{S \subseteq \Omega_t, |S|=k} \Phi_t(S), \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

$$\Omega_{t+1} = \Omega_t \setminus \{i : M_t(i) = 1\}, \tag{2}$$

where $\Phi_t(S)$ is a task-specific utility score (e.g., validation accuracy or gradient alignment) for selecting frequency set $S$ of size $k$.

**Stage 2: Per-task Least Squares Scaling.** After masks $M_t$ are fixed, we compute the scaling vector $G_t \in \mathbb{R}^d$ for each task by solving a least squares problem:

$$G_t^\star = \arg\min_{G \in \mathbb{R}^d} \left\| Y_t - U \cdot \mathcal{F}^{-1}\big(M_t \odot (F(V) \odot G)\big) X_t \right\|_2^2, \tag{3}$$

where $(X_t, Y_t)$ denote the task training data.

This construction ensures that task subspaces $\mathcal{S}_t$ have minimal overlap due to disjoint frequency allocation, while $G_t$ adapts to individual task statistics to recover near single-task performance.

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## 2. Experiments

### 2.1. Main Experiments

## 3. Electronic Submission

Submission to ICML 2025 will be entirely electronic, via a web site (not email). Information about the submission process and LaTeX templates are available on the conference web site at:

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|---|---|---|---|---|---|---|---|---|---|---|---|
| Qwen2.5-1.5B | | | | | | | | | | | |
| LoRA | 0.8m | 14.8 | | | | | | | | | |
| VeRA | 0.8m | 14.8 | | | | | | | | | |
| LoRI | 1.8m | 10.6 | | | | | | | | | |
| FMoRA (ours) | 1.0m | 11.1 | | | | | | | | | |
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| FMoRA (ours) | 2.2m | 19.2 | | | | | | | | | |
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| LoRA | 2.6m | 33.1 | | | | | | | | | |
| VeRA | 0.8m | 14.8 | | | | | | | | | |
| LoRI | 4.3m | 19.3 | | | | | | | | | |
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| VeRA | 0.8m | 14.8 | | | | | | | | | |
| LoRI | 4.3m | 19.3 | | | | | | | | | |
| FMoRA (ours) | 2.2m | 19.2 | | | | | | | | | |

*Table 1.*

- Keep your abstract brief and self-contained, one paragraph and roughly 4–6 sentences. Gross violations will require correction at the camera-ready phase. The title should have content words capitalized.

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```
dvips -Ppdf -tletter -G0 -o paper.ps paper.dvi
ps2pdf paper.ps
```

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The footnote, "Preliminary work. Under review by the International Conference on Machine Learning (ICML). Do not distribute." must be modified to "*Proceedings of the $42^{nd}$ International Conference on Machine Learning*, Vancouver, Canada, PMLR 267, 2025. Copyright 2025 by the author(s)."

For those using the LaTeX style file, this change (and others) is handled automatically by simply changing `\usepackage{icml2025}` to

```
\usepackage[accepted]{icml2025}
```

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# 4. Format of the Paper

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The text of the paper should be formatted in two columns, with an overall width of 6.75 inches, height of 9.0 inches, and 0.25 inches between the columns. The left margin should be 0.75 inches and the top margin 1.0 inch (2.54 cm). The right and bottom margins will depend on whether you print on US letter or A4 paper, but all final versions must be produced for US letter size. Do not write anything on the margins.

The paper body should be set in 10 point type with a vertical spacing of 11 points. Please use Times typeface throughout the text.

## 4.2. Title

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## 4.3. Author Information for Submission

ICML uses double-blind review, so author information must not appear. If you are using LaTeX and the `icml2025.sty` file, use `\icmlauthor{...}` to specify authors and `\icmlaffiliation{...}` to specify affiliations. (Read the TeX code used to produce this document for an example usage.) The author information will not be printed unless `accepted` is passed as an argument to the style file. Submissions that include the author information will not be reviewed.

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Each distinct affiliations should be listed once. If an author has multiple affiliations, multiple superscripts should be placed after the name, separated by thin spaces. If the authors would like to highlight equal contribution by multiple first authors, those authors should have an asterisk placed after their name in superscript, and the term "*Equal contribution" should be placed in the footnote block ahead of the list of affiliations. A list of corresponding authors and their emails (in the format Full Name <email@domain.com>) can follow the list of affiliations. Ideally only one or two names should be listed.

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You can use footnotes[1] to provide readers with additional information about a topic without interrupting the flow of the paper. Indicate footnotes with a number in the text where the point is most relevant. Place the footnote in 9 point type at the bottom of the column in which it appears. Precede the first footnote in a column with a horizontal rule of 0.8 inches.[2]

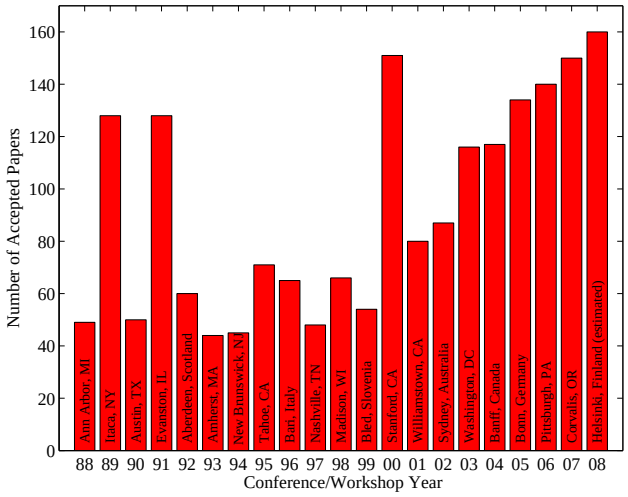


*Figure 1.* Historical locations and number of accepted papers for International Machine Learning Conferences (ICML 1993 – ICML 2008) and International Workshops on Machine Learning (ML 1988 – ML 1992). At the time this figure was produced, the number of accepted papers for ICML 2008 was unknown and instead estimated.

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You may want to include figures in the paper to illustrate your approach and results. Such artwork should be centered, legible, and separated from the text. Lines should be dark and at least 0.5 points thick for purposes of reproduction, and text should not appear on a gray background.

Label all distinct components of each figure. If the figure takes the form of a graph, then give a name for each axis and include a legend that briefly describes each curve. Do not include a title inside the figure; instead, the caption should serve this function.

Number figures sequentially, placing the figure number and caption *after* the graphics, with at least 0.1 inches of space before the caption and 0.1 inches after it, as in Figure 1. The figure caption should be set in 9 point type and centered unless it runs two or more lines, in which case it should be flush left. You may float figures to the top or bottom of a column, and you may set wide figures across both columns (use the environment `figure*` in LaTeX). Always place two-column figures at the top or bottom of the page.

[1] Footnotes should be complete sentences.

[2] Multiple footnotes can appear in each column, in the same order as they appear in the text, but spread them across columns and pages if possible.

**Algorithm 1** Bubble Sort

**Input:** data $x_i$, size $m$
**repeat**
  Initialize $noChange = true$.
  **for** $i = 1$ **to** $m - 1$ **do**
    **if** $x_i > x_{i+1}$ **then**
      Swap $x_i$ and $x_{i+1}$
      $noChange = false$
    **end if**
  **end for**
**until** $noChange$ is $true$

*Table 2.* Classification accuracies for naive Bayes and flexible Bayes on various data sets.

| DATA SET | NAIVE | FLEXIBLE | BETTER? |
|---|---|---|---|
| BREAST | 95.9± 0.2 | 96.7± 0.2 | √ |
| CLEVELAND | 83.3± 0.6 | 80.0± 0.6 | × |
| GLASS2 | 61.9± 1.4 | 83.8± 0.7 | √ |
| CREDIT | 74.8± 0.5 | 78.3± 0.6 | |
| HORSE | 73.3± 0.9 | 69.7± 1.0 | × |
| META | 67.1± 0.6 | 76.5± 0.5 | √ |
| PIMA | 75.1± 0.6 | 73.9± 0.5 | |
| VEHICLE | 44.9± 0.6 | 61.5± 0.4 | √ |

### 4.7. Algorithms

If you are using LaTeX, please use the "algorithm" and "algorithmic" environments to format pseudocode. These require the corresponding stylefiles, algorithm.sty and algorithmic.sty, which are supplied with this package. Algorithm 1 shows an example.

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You may also want to include tables that summarize material. Like figures, these should be centered, legible, and numbered consecutively. However, place the title *above* the table with at least 0.1 inches of space before the title and the same after it, as in Table 2. The table title should be set in 9 point type and centered unless it runs two or more lines, in which case it should be flush left.

Tables contain textual material, whereas figures contain graphical material. Specify the contents of each row and column in the table's topmost row. Again, you may float tables to a column's top or bottom, and set wide tables across both columns. Place two-column tables at the top or bottom of the page.

### 4.9. Theorems and such

The preferred way is to number definitions, propositions, lemmas, etc. consecutively, within sections, as shown below.

**Definition 4.1.** A function $f : X \to Y$ is injective if for any $x, y \in X$ different, $f(x) \neq f(y)$.

Using Theorem 4.1 we immediate get the following result:

**Proposition 4.2.** *If $f$ is injective mapping a set $X$ to another set $Y$, the cardinality of $Y$ is at least as large as that of $X$*

*Proof.* Left as an exercise to the reader. □

Theorem 4.3 stated next will prove to be useful.

**Lemma 4.3.** *For any $f : X \to Y$ and $g : Y \to Z$ injective functions, $f \circ g$ is injective.*

**Theorem 4.4.** *If $f : X \to Y$ is bijective, the cardinality of $X$ and $Y$ are the same.*

An easy corollary of Theorem 4.4 is the following:

**Corollary 4.5.** *If $f : X \to Y$ is bijective, the cardinality of $X$ is at least as large as that of $Y$.*

**Assumption 4.6.** The set $X$ is finite.

*Remark* 4.7. According to some, it is only the finite case (cf. Theorem 4.6) that is interesting.

### 4.10. Citations and References

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2000), edited volumes (Michalski et al., 1983), technical reports (Mitchell, 1980), and dissertations (Kearns, 1989).

Alphabetize references by the surnames of the first authors, with single author entries preceding multiple author entries. Order references for the same authors by year of publication, with the earliest first. Make sure that each reference includes all relevant information (e.g., page numbers).

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Authors are **required** to include a statement of the potential broader impact of their work, including its ethical aspects and future societal consequences. This statement should be in an unnumbered section at the end of the paper (co-located with Acknowledgements – the two may appear in either order, but both must be before References), and does not count toward the paper page limit. In many cases, where the ethical impacts and expected societal implications are those that are well established when advancing the field of Machine Learning, substantial discussion is not required, and a simple statement such as the following will suffice:

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## References

Author, N. N. Suppressed for anonymity, 2021.

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# A. Frequency Domain Analysis

# B. Experimental Hyperparameters

# C. MT Bench Case Study

# D. Proof of Rank Increasement

# E. Rank Increasement Analysis

**Theorem E.1** ( under nontrivial frequency masking)**.** *Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, denoting respectively the field of real or complex numbers. Let $\boldsymbol{W}_{UV} = \sum_{i=1}^{r} \boldsymbol{U}_i \boldsymbol{V}_i^\top$ with $\boldsymbol{U}_i \in \mathbb{K}^m$ and $\boldsymbol{V}_i \in \mathbb{K}^n$ (so $\operatorname{rank}(\boldsymbol{W}_{UV}) \le r$). Let $\boldsymbol{M}_t \in \mathbb{C}^{m\times n}$ be a frequency-domain mask and define its inverse DFT $h = \mathcal{F}^{-1}(\boldsymbol{M}_t) \in \mathbb{C}^{m\times n}$, interpreted as the corresponding spatial convolution kernel.*

*Assume that $h$ has finite support on $t$ distinct circular shifts, i.e.*

$$h = \sum_{k=1}^{t} \alpha_k \, S_{s_k,t_k}(\Delta), \tag{4}$$

*where $\alpha_k \in \mathbb{C} \setminus \{0\}$, $\Delta$ denotes the Kronecker delta at the origin, and $S_{s,t} : \mathbb{K}^{m\times n} \to \mathbb{K}^{m\times n}$ denotes the two-dimensional circular shift operator defined by*

$$[S_{s,t}(\boldsymbol{X})]_{i,j} = \boldsymbol{X}_{(i-s) \bmod m,\,(j-t) \bmod n}. \tag{5}$$

*For a vector $\boldsymbol{U} \in \mathbb{K}^m$, we define $S_s(\boldsymbol{U}) \in \mathbb{K}^m$ analogously as the circular shift by $s$ along its entries, so that $S_{s,t}(\boldsymbol{U}\boldsymbol{V}^\top) = S_s(\boldsymbol{U})\, S_t(\boldsymbol{V})^\top$.*

*Then the frequency-masked transform of $\boldsymbol{W}_{UV}$ satisfies*

$$\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV}) = \sum_{k=1}^{t} \alpha_k \, S_{s_k,t_k}(\boldsymbol{W}_{UV}) = \sum_{i=1}^{r}\sum_{k=1}^{t} \alpha_k \, S_{s_k}(\boldsymbol{U}_i)\, S_{t_k}(\boldsymbol{V}_i)^\top. \tag{6}$$

*Suppose further that the following* nondegeneracy (genericity) conditions *hold:*

1. *For each $i$, the set of shifted vectors $\{S_{s_k}(\boldsymbol{U}_i)\}_{k=1}^{t}$ is linearly independent over $\mathbb{K}$ (or at least spans a $t_i$-dimensional subspace with $t_i \ge 2$), and similarly $\{S_{t_k}(\boldsymbol{V}_i)\}_{k=1}^{t}$ are in general position.*
2. *The families $\{S_{s_k}(\boldsymbol{U}_i)\}_{i,k}$ and $\{S_{t_k}(\boldsymbol{V}_i)\}_{i,k}$ corresponding to different $i$ are in general position, i.e. their Kruskal ranks satisfy the usual generic full-rank condition.*[3]

*Then generically*

$$\operatorname{rank}\big(\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV})\big) = \min\big(rt,\ \min(m,n)\big). \tag{7}$$

*In particular, whenever $t > 1$ and the above nondegeneracy conditions are satisfied,*

$$\operatorname{rank}\big(\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV})\big) > \operatorname{rank}(\boldsymbol{W}_{UV}). \tag{8}$$

**Lemma E.2** (Khatri–Rao representation and generic full rank)**.** *For any $u \in \mathbb{K}^m$ and $v \in \mathbb{K}^n$,*

$$\operatorname{vec}\big(S_s(u)\, S_t(v)^\top\big) = S_t(v) \odot S_s(u), \tag{9}$$

*where $\odot$ denotes the columnwise Kronecker (Khatri–Rao) product. For a collection $\{S_{s_k}(\boldsymbol{U}_i) S_{t_k}(\boldsymbol{V}_i)^\top\}_{i=1,\dots,r;\,k=1,\dots,t}$, define the matrix*

$$A := \big[\, S_{t_1}(\boldsymbol{V}_1) \odot S_{s_1}(\boldsymbol{U}_1),\ S_{t_2}(\boldsymbol{V}_1) \odot S_{s_2}(\boldsymbol{U}_1),\ \dots,\ S_{t_t}(\boldsymbol{V}_r) \odot S_{s_t}(\boldsymbol{U}_r) \,\big] \in \mathbb{K}^{mn\times(rt)}. \tag{10}$$

[3] That is, for an open dense subset of $(\boldsymbol{U}_i, \boldsymbol{V}_i)$ in $\mathbb{K}^{m\times r} \times \mathbb{K}^{n\times r}$ the set of rank–1 outer products $\{S_{s_k}(\boldsymbol{U}_i) S_{t_k}(\boldsymbol{V}_i)^\top\}_{i,k}$ is linearly independent up to the ambient limit.

*If $A$ has full column rank, then the corresponding rank–1 atoms $\{S_{s_k}(\boldsymbol{U}_i)S_{t_k}(\boldsymbol{V}_i)^\top\}_{i,k}$ are linearly independent, and hence*

$$\operatorname{rank}\big(\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV})\big) = \min(rt,\, \min(m,n)). \tag{11}$$

*Moreover, full column rank of $A$ holds for an open dense subset of $(\boldsymbol{U}_i, \boldsymbol{V}_i) \in \mathbb{K}^{m\times r}\times\mathbb{K}^{n\times r}$, since $\det(A^\top A)$ is a nonzero polynomial in the entries of $\boldsymbol{U}_i, \boldsymbol{V}_i$.*

**Assumption E.3** (Nondegeneracy)**.** We assume the following conditions hold:

1. The singular vectors $\{u_i\}_{i=1}^r$ and $\{v_i\}_{i=1}^r$ of $\boldsymbol{W}_{UV} = \sum_{i=1}^r \boldsymbol{U}_i\boldsymbol{V}_i^\top$ are linearly independent, i.e.,
$$\operatorname{rank}\big([\boldsymbol{U}_1,\ldots,\boldsymbol{U}_r]\big) = r, \quad \operatorname{rank}\big([\boldsymbol{V}_1,\ldots,\boldsymbol{V}_r]\big) = r.$$
2. The frequency mask $\boldsymbol{M}_t \in \mathbb{C}^{m\times n}$ is *nontrivial*, meaning that it is not proportional to the all-one matrix and contains at least one element with a distinct complex phase or magnitude, formally,
$$\exists (p,q),(p',q') \text{ such that } \boldsymbol{M}_t(p,q) \neq \alpha\, \boldsymbol{M}_t(p',q') \text{ for any } \alpha \in \mathbb{C}.$$

*Proof.* **Step 1 (Convolution representation).** By the convolution theorem, multiplication by $\boldsymbol{M}_t$ in the frequency domain corresponds to circular convolution by its inverse DFT $h$ in the spatial domain:

$$\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV}) = h \star_{\text{circ}} \boldsymbol{W}_{UV}. \tag{12}$$

If $h$ is supported on $t$ distinct shifts, i.e. $h = \sum_{k=1}^t \alpha_k S_{s_k,t_k}(\Delta)$, then by linearity of convolution,

$$h \star_{\text{circ}} \boldsymbol{W}_{UV} = \sum_{k=1}^t \alpha_k\, S_{s_k,t_k}(\boldsymbol{W}_{UV}), \tag{13}$$

where $S_{s_k,t_k}(\boldsymbol{W}_{UV})$ denotes the circularly shifted copy of $\boldsymbol{W}_{UV}$ by $(s_k,t_k)$.

**Step 2 (Rank–1 component).** For a single rank–1 matrix $\boldsymbol{W}_{UV} = \boldsymbol{U}\boldsymbol{V}^\top$,

$$\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{U}\boldsymbol{V}^\top) = \sum_{k=1}^t \alpha_k\, S_{s_k}(\boldsymbol{U})\, S_{t_k}(\boldsymbol{V})^\top. \tag{14}$$

Each term is rank–1. If the family of shifted vectors $\{S_{s_k}(\boldsymbol{U})\}$ is linearly independent and the shifted $\{S_{t_k}(\boldsymbol{V})\}$ are in general position, then the $t$ outer products are generically linearly independent (up to $\min(m,n)$). Hence the rank of the resulting sum equals $\min(t, \min(m,n))$ for generic parameters.

**Step 3 (General rank–$r$ case).** For $\boldsymbol{W}_{UV} = \sum_{i=1}^r \boldsymbol{U}_i\boldsymbol{V}_i^\top$,

$$\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV}) = \sum_{i=1}^r \mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{U}_i\boldsymbol{V}_i^\top) = \sum_{i=1}^r\sum_{k=1}^t \alpha_k\, S_{s_k}(\boldsymbol{U}_i)\, S_{t_k}(\boldsymbol{V}_i)^\top. \tag{15}$$

The right-hand side is a linear combination of $r\cdot t$ rank–1 atoms. Under assumption E.3, the set of these rank–1 atoms is generically linearly independent up to the ambient limit, so the rank equals

$$\operatorname{rank}\left(\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV})\right) = \min(rt, \min(m,n)). \tag{16}$$

**Step 4 (Existence of such masks).** For any prescribed $t > 1$, one can explicitly construct $h = \sum_{k=1}^t \alpha_k S_{s_k,t_k}(\Delta)$ with distinct shifts $(s_k,t_k)$ and nonzero $\alpha_k$, and define $\boldsymbol{M}_t = \mathcal{F}(h)$. For generic $\boldsymbol{U}_i, \boldsymbol{V}_i$, the corresponding transform $\mathcal{T}_{\boldsymbol{M}_t}$ then strictly increases rank.

**Step 5 (Field consistency).** If all $\alpha_k$, $\boldsymbol{U}_i$, $\boldsymbol{V}_i$ are real, the argument holds over $\mathbb{R}$; if some $\alpha_k$ are complex, rank is interpreted over $\mathbb{C}$. Either way, the statement remains valid with the appropriate field. □

*Remark* E.4 (Discussion and genericity). 1. The assumption that $h$ is a finite sum of shifted deltas is a constructive special case. More generally, if $h$ has $t$ well-separated dominant coefficients (effective support), the same argument applies approximately, replacing exact rank by numerical rank determined by the singular values.

2. The conclusion is *generic*: for an open dense subset of $(\boldsymbol{U}_i, \boldsymbol{V}_i, \alpha_k)$ in $\mathbb{K}^{m\times r} \times \mathbb{K}^{n\times r} \times (\mathbb{C} \setminus \{0\})^t$, the linear independence of the rank–1 atoms $\{S_{s_k}(\boldsymbol{U}_i) S_{t_k}(\boldsymbol{V}_i)^\top\}_{i,k}$ holds, and the rank achieves the upper bound $\min(rt, \min(m, n))$. Degenerate counterexamples occur if $\boldsymbol{U}_i$ or $\boldsymbol{V}_i$ are shift-invariant (e.g. constant or periodic), in which case the rank may fail to increase.

3. In vectorized operator form,

$$\operatorname{vec}\big(\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV})\big) = C_{\boldsymbol{M}_t} \operatorname{vec}(\boldsymbol{W}_{UV}), \quad C_{\boldsymbol{M}_t} := \mathcal{F}^{-1} \operatorname{diag}(\operatorname{vec}(\boldsymbol{M}_t)) \mathcal{F}. \tag{17}$$

The operator $C_{\boldsymbol{M}_t}$ has rank equal to the number of nonzero entries in $\boldsymbol{M}_t$. If all frequencies are retained ($\boldsymbol{M}_t$ has no zeros), $C_{\boldsymbol{M}_t}$ is invertible (full rank). When $\boldsymbol{M}_t$ is a nontrivial mask (some zeros), $C_{\boldsymbol{M}_t}$ is not full rank, yet it can still map a low-rank $\boldsymbol{W}_{UV}$ to a higher-rank output because the linear mixing introduced by the circular shifts breaks the low-rank structure.

## F. ADD

**Genericity clarification.** Condition (b) may be stated equivalently as: the matrix $A$ defined in Lemma E.2 has full column rank. Equivalently, for all nonzero coefficient tensors $\{\beta_{i,k}\}$,

$$\sum_{i=1}^{r} \sum_{k=1}^{t} \beta_{i,k}\, S_{s_k}(\boldsymbol{U}_i)\, S_{t_k}(\boldsymbol{V}_i)^\top = 0 \quad \Longrightarrow \quad \beta_{i,k} = 0 \; \forall i, k. \tag{18}$$

This condition is satisfied for an open dense set of $(\boldsymbol{U}_i, \boldsymbol{V}_i)$, so the rank formula holds *generically*.