Frequency-Orthogonal LoRA: Improving Multitask Adaptation Efficiency

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Abstract

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1. Methodology

Greedy Frequency Allocation and Per-task Scaling. We propose a two-stage procedure to construct task-specific masks M_t and scaling vectors G_t such that task subspaces $\mathcal{S}_t$ remain approximately orthogonal and each task achieves performance comparable to single-task training.

Stage 1: Greedy Frequency Allocation. Given a total frequency budget $\Omega = \{1, \ldots, d\}$, we sequentially allocate disjoint frequency subsets to each task. Formally, for T tasks, we initialize $\Omega_1 = \Omega$ and iterate:

$$M_t(i) = \begin{cases} 1, & \text{if } i \in \arg\max_{S \subseteq \Omega_t,\, |S|=k} \Phi_t(S), \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

$$\Omega_{t+1} = \Omega_t \setminus \{i : M_t(i) = 1\}, \tag{2}$$

where $\Phi_t(S)$ is a task-specific utility score (e.g., validation accuracy or gradient alignment) for selecting frequency set S of size k.

Stage 2: Per-task Least Squares Scaling. After masks M_t are fixed, we compute the scaling vector $G_t \in \mathbb{R}^d$ for each task by solving a least squares problem:

$$G_t^\star = \arg\min_{G \in \mathbb{R}^d} \left\| Y_t - U \cdot \mathcal{F}^{-1}\big(M_t \odot (F(V) \odot G)\big) X_t \right\|_2^2, \tag{3}$$

where (X_t, Y_t) denote the task training data.

This construction ensures that task subspaces $\mathcal{S}_t$ have minimal overlap due to disjoint frequency allocation, while G_t adapts to individual task statistics to recover near single-task performance.

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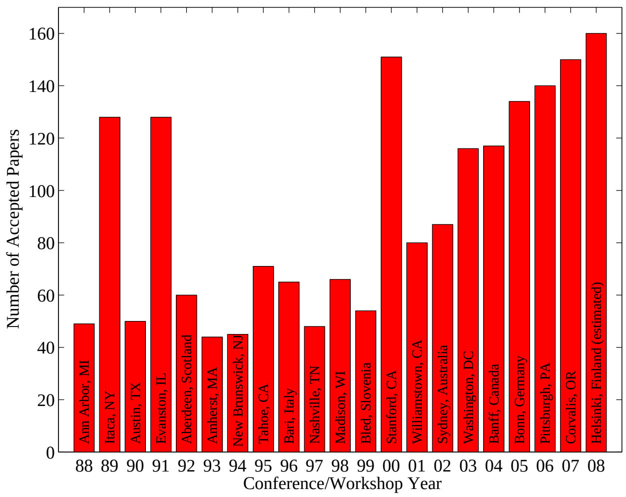

Figure 1. Historical locations and number of accepted papers for International Machine Learning Conferences (ICML 1993 – ICML 2008) and International Workshops on Machine Learning (ML 1988 – ML 1992). At the time this figure was produced, the number of accepted papers for ICML 2008 was unknown and instead estimated.

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Algorithm 1 Bubble Sort

Input: data x_i, size m
repeat
  Initialize $noChange = true$.
  for $i = 1$ **to** $m - 1$ **do**
    if $x_i > x_{i+1}$ **then**
      Swap x_i and x_{i+1}
      $noChange = false$
    end if
  end for
until $noChange$ is $true$

Table 1. Classification accuracies for naive Bayes and flexible Bayes on various data sets.

DATA SET	NAIVE	FLEXIBLE	BETTER?
BREAST	95.9$\pm$ 0.2	96.7$\pm$ 0.2	$\surd$
CLEVELAND	83.3$\pm$ 0.6	80.0$\pm$ 0.6	$\times$
GLASS2	61.9$\pm$ 1.4	83.8$\pm$ 0.7	$\surd$
CREDIT	74.8$\pm$ 0.5	78.3$\pm$ 0.6	
HORSE	73.3$\pm$ 0.9	69.7$\pm$ 1.0	$\times$
META	67.1$\pm$ 0.6	76.5$\pm$ 0.5	$\surd$
PIMA	75.1$\pm$ 0.6	73.9$\pm$ 0.5	
VEHICLE	44.9$\pm$ 0.6	61.5$\pm$ 0.4	$\surd$

3.7. Algorithms

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3.9. Theorems and such

The preferred way is to number definitions, propositions, lemmas, etc. consecutively, within sections, as shown below.

Definition 3.1. A function $f : X \to Y$ is injective if for any $x, y \in X$ different, $f(x) \neq f(y)$.

Using Theorem 3.1 we immediate get the following result:

Proposition 3.2. *If f is injective mapping a set X to another set Y, the cardinality of Y is at least as large as that of X*

Proof. Left as an exercise to the reader. $\square$

Theorem 3.3 stated next will prove to be useful.

Lemma 3.3. *For any $f : X \to Y$ and $g : Y \to Z$ injective functions, $f \circ g$ is injective.*

Theorem 3.4. *If $f : X \to Y$ is bijective, the cardinality of X and Y are the same.*

An easy corollary of Theorem 3.4 is the following:

Corollary 3.5. *If $f : X \to Y$ is bijective, the cardinality of X is at least as large as that of Y.*

Assumption 3.6. The set X is finite.

Remark 3.7. According to some, it is only the finite case (cf. Theorem 3.6) that is interesting.

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References

Author, N. N. Suppressed for anonymity, 2021.

Duda, R. O., Hart, P. E., and Stork, D. G. *Pattern Classification*. John Wiley and Sons, 2nd edition, 2000.

Kearns, M. J. *Computational Complexity of Machine Learning*. PhD thesis, Department of Computer Science, Harvard University, 1989.

Langley, P. Crafting papers on machine learning. In Langley, P. (ed.), *Proceedings of the 17th International Conference on Machine Learning (ICML 2000)*, pp. 1207–1216, Stanford, CA, 2000. Morgan Kaufmann.

Michalski, R. S., Carbonell, J. G., and Mitchell, T. M. (eds.). *Machine Learning: An Artificial Intelligence Approach, Vol. I*. Tioga, Palo Alto, CA, 1983.

Mitchell, T. M. The need for biases in learning generalizations. Technical report, Computer Science Department, Rutgers University, New Brunswick, MA, 1980.

Newell, A. and Rosenbloom, P. S. Mechanisms of skill acquisition and the law of practice. In Anderson, J. R. (ed.), *Cognitive Skills and Their Acquisition*, chapter 1, pp. 1–51. Lawrence Erlbaum Associates, Inc., Hillsdale, NJ, 1981.

Samuel, A. L. Some studies in machine learning using the game of checkers. *IBM Journal of Research and Development*, 3(3):211–229, 1959.

A. Frequency Domain Analysis

B. Experimental Hyperparameters

C. MT Bench Case Study

D. Proof of Rank Increasement

Let $\boldsymbol{W}_{UV} \in \mathbb{R}^{m\times n}$. Denote by $\mathcal{F} : \mathbb{C}^{m\times n} \to \mathbb{C}^{m\times n}$ the 2D DFT acting on the matrix entries (we use the convention that $\mathcal{F}$ is invertible and $\mathcal{F}^{-1}$ is the inverse DFT). For a frequency-domain mask $\boldsymbol{M}_t \in \mathbb{C}^{m\times n}$ define the frequency-domain multiplication operator for task t

$$\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV}) \coloneqq \mathcal{F}^{-1}\big(\boldsymbol{M}_t \odot \mathcal{F}(\boldsymbol{W}_{UV})\big), \tag{4}$$

where $\odot$ denotes elementwise (Hadamard) product. We will study $\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV})$ when $\boldsymbol{W}_{UV} = \boldsymbol{U}\boldsymbol{V}$ with $\boldsymbol{U} \in \mathbb{R}^{m\times r}, \boldsymbol{V} \in \mathbb{R}^{r\times n}$ (so $\operatorname{rank}(\boldsymbol{W}_{UV}) \le r$).

Observe that elementwise multiplication in frequency is equivalent to circular convolution in spatial domain: there exists a spatial kernel $h \in \mathbb{C}^{m\times n}$ (the inverse DFT of M) such that

$$\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV}) = h \star_{\text{circ}} \boldsymbol{W}_{UV}, \tag{5}$$

where $\star_{\text{circ}}$ denotes 2D circular convolution (convolution performed with wrap-around on both dimensions). The following theorem makes precise how the rank can increase.

Theorem D.1 (Rank increase under nontrivial frequency masking)**.** *Let $\boldsymbol{W}_{UV} = \sum_{i=1}^{r} u_i v_i^\top$ with $u_i \in \mathbb{R}^m$, $v_i \in \mathbb{R}^n$ (so $\operatorname{rank}(\boldsymbol{W}_{UV}) \le r$). Let $\boldsymbol{M}_t$ be a frequency mask and write $h = \mathcal{F}^{-1}(\boldsymbol{M}_t) \in \mathbb{C}^{m\times n}$ for its inverse DFT (the spatial kernel). Suppose h can be written as a finite sum of t distinct shifted deltas (i.e. it is supported on t distinct circular shifts):*

$$h = \sum_{k=1}^{t} \alpha_k S_{s_k,t_k}(\Delta), \tag{6}$$

where $\alpha_k \in \mathbb{C} \setminus \{0\}$, S_{s_k,t_k} denotes the circular shift operator by (s_k, t_k) in the two dimensions, and Δ is the Kronecker delta at the origin. (Equivalently, h has exactly t nonzero entries located at distinct coordinates.) Then

$$\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV}) \;=\; \sum_{k=1}^{t} \alpha_k\, S_{s_k,t_k}(\boldsymbol{W}_{UV}). \tag{7}$$

In particular, if for each fixed i the set of shifted vectors $\{\, S_{s_k} u_i : k = 1, \dots, t \,\}$ (row-shifts of u_i) contains at least t_i linearly independent vectors, and similarly the column-shifts of v_i are in general position, then generically

$$\operatorname{rank}\big(\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV})\big) \ge \min\big(r \cdot t,\ \min(m, n)\big). \tag{8}$$

Hence, for any X of rank r there exist masks M (equivalently kernels h with $t > 1$ nonzero shifts) such that

$$\operatorname{rank}\big(\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV})\big) > \operatorname{rank}(\boldsymbol{W}_{UV}). \tag{9}$$

Proof. Step 1 (convolution expansion). Since $h = \mathcal{F}^{-1}(M)$, by linearity of DFT/IDFT and the convolution theorem we have for any matrix $\boldsymbol{W}_{UV}$,

$$\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV}) = h \star_{\text{circ}} \boldsymbol{W}_{UV}. \tag{10}$$

If h is supported on t distinct circular shifts (i.e. has nonzero entries at exactly t coordinates), then the circular convolution with $\boldsymbol{W}_{UV}$ reduces to a finite sum of shifted copies of X weighted by those kernel entries. Writing the shifts explicitly,

$$h = \sum_{k=1}^{t} \alpha_k S_{s_k,t_k}(\Delta) \quad\Longrightarrow\quad h \star_{\text{circ}} \boldsymbol{W}_{UV} = \sum_{k=1}^{t} \alpha_k\, S_{s_k,t_k}(\boldsymbol{W}_{UV}), \tag{11}$$

where $S_{s_k,t_k}(\boldsymbol{W}_{UV})$ denotes $\boldsymbol{W}_{UV}$ circularly shifted by (s_k, t_k) (shift of rows by s_k and columns by t_k).

Step 2 (apply to rank-1 atoms). For a rank–1 matrix $\boldsymbol{W}_{UV} = uv^\top$ we then get

$$\mathcal{T}_{\boldsymbol{M}_t}(uv^\top) = \sum_{k=1}^{t} \alpha_k \, S_{s_k}(u) \, S_{t_k}(v)^\top \tag{12}$$

which is an explicit sum of t rank–1 matrices. If the set of row-shifted vectors $\{S_{s_k}(u)\}_{k=1}^t$ is linearly independent (or, more weakly, if the family of outer products produce a span of dimension t), then the rank of the right-hand side equals t (up to the ambient dimension limit). Thus a single rank–1 atom can be mapped to a matrix of rank up to $\min(t, \min(m, n))$.

Step 3 (general rank r). For general $\boldsymbol{W}_{UV} = \sum_{i=1}^r u_i v_i^\top$, by linearity

$$\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV}) = \sum_{i=1}^{r}\sum_{k=1}^{t} \alpha_k \, S_{s_k}(u_i) \, S_{t_k}(v_i)^\top, \tag{13}$$

a sum of $r \cdot t$ rank–1 terms. The rank of this sum is bounded above by $\min(r \cdot t, \min(m, n))$ and (by standard genericity arguments) is equal to that upper bound for a generic choice of the vectors $\{u_i\}$, $\{v_i\}$ and nondegenerate coefficients $\{\alpha_k\}$. Consequently, whenever $t > 1$ and the shifts produce new independent components (i.e. the shifted vectors are not all collinear), the rank strictly increases compared to the original rank $\leq r$.

Step 4 (existence). The final statement follows because one can always choose a mask M whose inverse DFT h is (for example) the sum of t distinct Kronecker deltas with nonzero weights; such an M exists (take $M = \mathcal{F}(h)$). For such masks, provided the data vectors are not in a degenerate alignment, the rank strictly increases.

This completes the proof. $\square$

Remark D.2.
1. The hypothesis that h is a sum of a finite number of shifted deltas is only one convenient constructive case; more generally if h has effective support on t well-separated locations (or can be approximated by such), the same reasoning applies approximately.
2. The result is *generic* in the sense that for an open dense set of masks M (or of input factors U, V) the inequality $\mathrm{rank}(\mathcal{T}_{\boldsymbol{M}_t}(UV)) > \mathrm{rank}(UV)$ holds whenever the spatial kernel has $t > 1$ effective shifts. Degenerate counterexamples exist (e.g. if U or V have shift-invariant structure that cancels the shifts), so one must state the nondegeneracy assumptions as above.
3. In operator notation: $\mathrm{vec}(\mathcal{T}_{\boldsymbol{M}_t}(\boldsymbol{W}_{UV})) = C_M \,\mathrm{vec}(X)$ where $C_M = \mathcal{F}^{-1}\,\mathrm{diag}(\mathrm{vec}(M))\mathcal{F}$ is a linear operator; for generic M the operator C_M is full (or high) rank, so it does not preserve low rank of X.

E. Gradient Analysis

Let the network output be

$$y = (W + \Delta)x + b, \qquad \Delta \in \mathbb{R}^{m\times n},$$

and loss $\ell(y)$. Denote $g \coloneqq \frac{\partial \ell}{\partial y} \in \mathbb{R}^m$ (column vector) and the input to the layer $x \in \mathbb{R}^n$. Then the matrix

$$G \coloneqq \frac{\partial \ell}{\partial \Delta} = g\, x^\top \in \mathbb{R}^{m\times n}.$$

We compare two parameterizations of Δ:

- **Direct LoRA (spatial):** $\Delta = UV$, with $U \in \mathbb{R}^{m\times r},\ V \in \mathbb{R}^{r\times n}$.
- **Freq-masked LoRA:** $\Delta = \mathcal{A}(UV)$, where $\mathcal{A} : \mathbb{C}^{m\times n} \to \mathbb{C}^{m\times n}$ is the linear operator

  $$\mathcal{A}(X) \coloneqq \mathcal{F}^{-1}\big(M \odot \mathcal{F}(X)\big) = F^{-1}\,\mathrm{diag}(\mathrm{vec}(M))\, F\, \mathrm{vec}^{-1}(\cdot)$$

  (we treat $\mathcal{F}$ as the linear DFT operator on $\mathrm{vec}(X)$).

Gradient formulae

For both parameterizations, by the chain rule we obtain closed-form gradients.

Lemma E.1 (Closed-form gradients)**.** *Let* $G = gx^\top$.

1. *If* $\Delta = UV$*, then*
$$\frac{\partial \ell}{\partial U} = GV^\top, \qquad \frac{\partial \ell}{\partial V} = U^\top G.$$
2. *If* $\Delta = \mathcal{A}(UV)$*, denote by* $\mathcal{A}^*$ *the adjoint of* $\mathcal{A}$ *under the Frobenius inner product. Then*
$$\frac{\partial \ell}{\partial U} = \mathcal{A}^*(G)\, V^\top, \qquad \frac{\partial \ell}{\partial V} = U^\top \mathcal{A}^*(G).$$
Moreover, when $\mathcal{A}(X) = \mathcal{F}^{-1}\big(M \odot \mathcal{F}(X)\big)$*,* $\mathcal{A}^*(G) = \mathcal{F}^{-1}\big(\overline{M} \odot \mathcal{F}(G)\big)$ *(where* $\overline{M}$ *denotes complex conjugate of* M*).*

Proof. (1) Standard: $\partial \mathrm{tr}(G^\top UV)/\partial U = GV^\top$ and symmetrically for V.

(2) By linearity and chain rule:
$$\frac{\partial \ell}{\partial U} = \left(\frac{\partial \ell}{\partial \Delta}\right) \frac{\partial \Delta}{\partial (UV)} \frac{\partial (UV)}{\partial U} = \mathcal{A}^*(G)\, V^\top,$$
and similarly for V. The explicit form of $\mathcal{A}^*$ follows because $\mathcal{F}$ is unitary (up to scaling) and the adjoint of elementwise multiplication by M is elementwise multiplication by $\overline{M}$. □