Test Flight Problem Set Solutions

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A1. The statement is false. If $n \geq 2$, then for any m, we have that $3m + 5n \geq 13$ (since $3m + 5n \geq 3m + 10 \geq 13$). Thus, the only way to find such a solution for n in the natural numbers would be when $n = 1$. Substituting, we have $3m + 5 \cdot 1 = 3m + 5 = 12$, or $3m = 7$. But since there is no natural number m satisfying this equation, we have proved the result. QED

A2. The statement is true. Without loss of generality, we may assume the consecutive five integers may be written in the form: $n-2,\ n-1,\ n,\ n+1,\ n+2$. If we sum these integers, we have $5n$, which is divisible by 5. Hence, we have proved the result. QED

A3. The statement is true. We may rewrite $n^2 + n + 1$, as $n \cdot (n+1) + 1$. If n is even, then $n + 1$ is odd. If n is odd, then $n + 1$ is even. In either case, $n \cdot (n + 1)$ is even because the product of an even and odd number is even. Hence, we may write $n \cdot (n + 1) + 1$ as $2k + 1$, which is odd. Hence, we have proved the result. QED

A4. Recall from the remainder theorem: if a, b are integers with $b > 0$, then there exist unique integers q, r such that$a = bq + r$ and $0 \leq r < b$. If we let $b = 4$ (and $n = q$), then we have the statement that $a = 4n + r$ with $0 \leq r < 4$. If $r = 0$ or 2, then we have $a = 4n$ or $a = 4n + 2$, which are even natural numbers. If $r = 1$ or 3, we have that $a = 4n+1$ or $a = 4n+3$, which are odd natural numbers. Since a is any odd natural number, satisfying the antecedent, we have that it must be of one of the following forms $a = 4n + 1$ or $a = 4n + 3$. Hence, we have proved the result. QED

A5. Recall from the remainder theorem: if a, b are integers with $b > 0$, then there exist unique integers q, r such that$a = bq + r$ and $0 \leq r < b$. If we take $b = 3$, then we have the statement that $a = 3q + r$ with $0 \leq r < 3$. Expanding out (and letting $n = a$), we have that $n = 3q$, or $n = 3q + 1$, or $n = 3q + 2$. Let's now write n, $n + 2$, and $n + 4$ in these forms: n is either $3q$, $3q + 1$, or $3q + 2$. $n + 2$ is either $3q + 2$, $3q + 3$, or $3q + 4$. $n + 4$ is either $3q + 4$, $3q + 5$, or $3q + 6$. But we see that in each of the forms, there exists an element which is divisible by 3 i.e. if n, $3|3q$ and if $n+2$, $3|(3q+3)$, and if $n+4$, $3|(3q+6)$. Hence, we have proved the result. QED

A6. We prove this by contradiction i.e. assume there exists $n > 3$, such that n, $n+2$, and $n+4$ are prime. But from the proof of #5, we have just shown that one of n, $n+2$, $n+4$ must be divisible by 3. And since $n > 3$, 3 is not one of the primes. Thus, one of n, $n+2$, $n+4$ is not prime. Hence, we have proved the result. QED

A7. Let the sum, $2+2^2+2^3+...+2^n$, be denoted by S. Multiplying by 2, we have that $2S = 2^2+2^3+...+2^{n+1}$. Substracting S from $2S$, we have that $S = 2^{n+1}-2$, which was to be proved. QED

A8. By the assumption, we have that for any given $\epsilon > 0$, there exists an n where for all $m \geq n$, $|a_m - L| < \epsilon$. The statement that Ma_n tends to ML as n tends to infinity is equivalent to saying that for any given $\epsilon_1 > 0$, there exists an n where all $m \geq n$, $|Ma_m - ML| < \epsilon_1$. This simplifies to $|M||a_m - L| \iff M|a_m - L| < \epsilon_1 \iff |a_m - L| < \frac{\epsilon_1}{M}$. This will be true if we take ${}^{\epsilon_1}/_{M} = \epsilon$, and find such an n. Since we can always do this, we have proved the result. QED

A9. Let $A_n = (0, {}^1/_n)$. We have that A_n is a subset of A_1 since $(0, {}^1/_n)$ is a subset of $(0,1)$. Suppose that x is an element of $(0,1)$. We can always find a natural number m such that ${}^1/_m < x$. But that means that x is not an element of A_m. Hence, x is not an element of the intersection of A_n where n is a natural number. Since we can always find this number m, we must necessarily have that intersection of A_n is empty. Hence, we have proved the result. QED

A10. Let $A_n = [0, {}^1/_n)$. We may write this set as $0 \cup B_n$, where $B_n = (0, {}^1/_n)$. The intersection of A_n for n in the natural numbers may thus be written as $0 \cup (\cap B_n)$. But since we have proved from #9 that intersection of all B_n is the empty set, we have that $0 \cup \emptyset = \{0\}$. Hence, we have proved the result. QED