Assignment 7 (for Lecture 7) Solutions

April 16, 2017

A1. False. Counterexample. An ostrich is bird that cannot fly.

A2. False. Counterexample. Choose $x = y = 0$. It is not the case that $(0 - 0) = 0 > 0$.

A3. True. Proof by direct construction. Suppose there are two rationals $a/b, p/q$ such that $a/b < p/q$ $(b \neq 0 \wedge q \neq 0)$. Then there exists another rational between the two rationals given by: $\frac{aq+bp}{2bq}$. We show that this construction obeys the following relation: $a/b < \frac{aq+bp}{2bq} < p/q$. We have $a/b = \frac{aq}{bq}$, and $\frac{aq}{bq} < \frac{aq+bp}{2bq} \iff \frac{1}{2}\frac{aq}{bq} < \frac{1}{2}\frac{bp}{bq} \iff a/b < p/q$, which is true by assumption. Similarly, we have $p/q = \frac{bp}{bq}$, and $\frac{aq+bp}{2bq} < \frac{bp}{bq} \iff \frac{1}{2}\frac{aq}{bq} < \frac{1}{2}\frac{bp}{bq} \iff a/b < p/q$,which is true by assumption. Thus we have found a rational between the two rationals.

A4. If we construct a truth table for the following expression: $[(\phi \implies \psi) \wedge (\psi \implies \phi)] \implies (\phi \iff \psi)$, we will obtain a tautology.

A5. We know that $[(\neg\phi) \implies (\neg\psi)] \iff (\psi \implies \phi)$ because it is the contrapositive. Then from A4, we have that $[(\phi \implies \psi) \wedge (\neg\phi \implies \neg\psi)] \implies (\phi \iff \psi)$.

A6. The only way to ensure that each investor receives the least possible amount of money is to split the payout evenly. But if we split the payout evenly, each investor will receive \$400,000. Since this is the only way to ensure each investor receives the least amount of money, we cannot do better. And so, at least one investor receives at least \$400,000.

A7. Suppose that $\sqrt{3}$ is rational. Then it can be represented by a fraction written in lowest terms, a/b. Then we have $\sqrt{3} = a/b \iff 3 = a^2/b^2 \iff 3b^2 = a^2$. This means that $3|a^2$, but if a prime divides a^2, then$3|a$. Thus, $a = 3k$ for some $k \in \mathbb{N}$. And so we have $3b^2 = 9k^2 \iff b^2 = 3k^2$. This means that $3|b^2$, and so $3|b$. Thus, $b = 3n$ for some $n \in \mathbb{N}$. But if $(a = 3k) \wedge (b = 3n)$, then a and b have a common factor, and thus a/b is not in lowest terms. Our original assumption must've been false, and so $\sqrt{3}$ is irrational.

A8.

(a) If the Yuan rises, the Dollar will fall.

(b) $(\forall x, y \in \mathbb{R})[(-y < -x) \implies (x < y)]$

(c) If two triangles have the same area they are congruent.

(d) If $b^2 \geq 4ac$, then $ax^2 + bx + c = 0$. ($\forall x, b, c \in \mathbb{R}$ and $a \in R \setminus \{0\}$)

(e) The opposite angles in a quadrilateral ABCD are pairwise equal iff the opposite sides of ABCD are pairwise equal is true.

(f) Not sure what the converse is here...

(g) $(\forall n \in \mathbb{N})[3|(n^2 + 5) \implies \neg(3|n)]$

A9.

(b) $(\forall x, y \in \mathbb{R})[x < y \implies -y < -x]$ is true. If $x < y \iff 0 < y - x \iff -y < -x$. $(\forall x, y \in \mathbb{R})[-y < -x \implies x < y]$ is also true. If $-y < -x \iff 0 < y - x \iff x < y$. Thus, they are equivalent.

(c) The statement is true, but the converse is not true. Implication: If two triangles are congruent, then there side lengths must be equal, and if their side lengths are equal, then they must have the same area. Converse: If two triangles have the same area a^2, one can construct one triangle which is an equilateral triangle with side length $\frac{2a}{\sqrt[4]{3}}$ and another isosceles triangle with side lengths a and $\frac{\sqrt{17}a}{2}$. These triangles will have the same area, but they are not congruent.

(d) $(\forall b, c \in \mathbb{R} \wedge \forall a \in \mathbb{R} \setminus \{0\})[(b^2 \geq 4ac) \iff (\exists x \in \mathbb{R})(ax^2+bx+c = 0)]$ is true. Proof by direct construction of the solution $ax^2+bx+c = 0 \iff x^2+\frac{bx}{a}+\frac{c}{a} = 0 \iff (x + \frac{b}{2a})^2 + \frac{c}{a} = \frac{b^2}{4a^2} \iff (x + \frac{b}{2a})^2 = \frac{b^2-4ac}{4a^2} \iff x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a} \iff x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$. Since we require that $x \in \mathbb{R}$, $b^2 - 4ac \geq 0 \iff b^2 \geq 4ac$. Thus, if there exists a solution, $b^2 \geq 4ac$, and the converse implies a solution as well.

(e) If a shape has congruent opposite sides, we have a parallelogram. Since a parallelogram has necessarily congruent opposite angles, we have the implication. If the opposite angles are pairwise equal, then we have a parallelogram. Since a parallelogram has necessarily congruent opposite sides, we have the converse implication.

(f) ...

(g) If $3 \nmid n$, then $n = (3k+1) \vee (3k+2)$. We then have $n^2 + 5 = [3(3k^2 + 2k + 2)] \vee [3(3k^2 + 4k + 3)]$, but this means that $3|(n^2+5)$. Hence, we have proved that implication. The converse, however, is false. We prove this by constructing a class of counterexamples. If we have $3|(n^2+5)$, then we can write $n^2+5 = (3k+1) \vee (3k+2)$. Suppose we consider only the case where $n^2+5 = 3k+2$, then $n^2 = 3k-3 = 3(k-1)$, thus $3|n^2$, but by Euclid's Lemma we have $3|n$. This is a class of counterexamples, and hence the converse implication is false.

A10. The statement is false because the converse is false. The implication is true by the following argument: if $n|12$, then we have $n = 12k$, $\forall k \in \mathbb{Z}$. Since $n^3 = 12^3k^3$, we have $12|n^3$. The converse is false by counterexample. Let $n = 6$, then $12|n^3 = 216$, but $\neg 12|n = 6$.

A11.

(1) Irrational. Proof by contradiction. Suppose $r+3$ is rational. Then we have $r+3 = {}^{p}\!/_{q} \iff r = {}^{p}\!/_{q} - 3 = \frac{p-3q}{q} \in \mathbb{Q}$. But it was assumed that r was irrational. Hence we have a contradiction, and $r+3$ must be irrational.

(2) Irrational. Proof by contradiction. Suppose $5r$ is rational. then we have $5r = {}^{p}\!/_{q} \iff r = \frac{p}{5q} \in \mathbb{Q}$. But it was assumed that r was irrational. Hence we have a contradiction, and $5r$ must be irrational.

(3) Could be rational. Counterexample. Let $r = -s = \sqrt{2}$, then we have $r + s = 0 \in \mathbb{Q}$.

(4) Could be rational. Counterexample. Let $r = s = \sqrt{2}$, then we have $rs = 2 \in \mathbb{Q}$.

(5) Irrational. Proof by contradiction. Suppose $\sqrt{r}$ is rational. Then we have $\sqrt{r} = {}^{p}\!/_{q} \iff r = {}^{p^2}\!/_{q^2} \in \mathbb{Q}$. But it was assumed that r was irrational. Hence we have a contradiction, and $\sqrt{r}$ must be irrational.

(6) Could be rational. Counterexample. If $r = \sqrt{2}^{\sqrt{2}}$ is rational, then let $r = s = \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$. If $r = \sqrt{2}^{\sqrt{2}}$ is irrational, let $r = \sqrt{2}^{\sqrt{2}}$, $s = \sqrt{2}$, then we have $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{2} = 2 \in \mathbb{Q}$.

A12.

(a) If m and n are even integers, then they can be written as $m = 2k,\ n = 2l,\ \forall k, l \in \mathbb{Z}$. Then we have $m + n = 2k + 2l = 2(k + 1)$, or $2|(m + n)$. Thus, $m + n$ is even.

(b) If m and n are even integers, then they can be written as $m = 2k,\ n = 2l,\ \forall k, l \in \mathbb{Z}$. Then we have $mn = 4kl$, or $2|(mn)$. Thus, mn is even.

(c) If m and n are odd integers, then they can be written as $m = 2k + 1,\ n = 2l + 1,\ \forall k, l \in \mathbb{Z}$. Then we have $m + n = 2k + 1 + 2l + 1 = 2k + 2l + 2 = 2(k + l + 1)$, or $2|(m + n)$. Thus, $m + n$ is even.

(d) If one of m, n is even and the other is odd (let's assume m is odd), then they can be written as $m = 2k + 1,\ n = 2l,\ \forall k, l \in \mathbb{Z}$. Then we have $m + n = 2k + 1 + 2l = 2(k + l) + 1$, or $\neg 2|(m + n)$. The same argument can be made when m is even and n is odd. Thus, $m + n$ is even.

(e) If one of m, n is even and the other is odd (let's assume m is odd), then they can be written as $m = 2k + 1,\ n = 2l,\ \forall k, l \in \mathbb{Z}$. Then we have $mn = (2k+1)(2l) = 2(2kl + l)$, or $2|(mn)$. The same argument can be made when m is even and n is odd. Thus, mn is even.

OPTIONAL PROBLEM

A1.

(a) $(\exists x, y \in \mathbb{R})[x + y = y]$ is true. Choose $x = 0$ and y to be any element of the Reals.

(b) $(\forall x)(\exists y)[x + y = 0]$ is true. Choose $y = -x$.

(c) $(\forall a, b, c \in \mathbb{N})[(a|bc) \implies (a|b) \vee (a|c)]$ is false. Let $a = 4,\ b = 2,\ c = 2$. Then we $4 = a|bc = 4$, but it is not the case that $4 = a|b = 2$ or $4 = a|c = 2$.

(d) $(\forall x, y \in \mathbb{R})(Rational(x) \wedge Irrational(y) \implies Irrational(x + y)]$ is true.

(e) $(\forall x, y \in \mathbb{R})[Irrational(x+y) \implies Irrational(x) \vee Irrational(y)]$ is true.

(f) $(\forall x, y \in \mathbb{R})[Rational(x+y) \implies Rational(x) \vee Rational(y)]$ is false. Let $x = \sqrt{2},\ y = -\sqrt{2}$. Then $x + y$ is rational, but neither x or y are rational.