Assignment 8 (for Lecture 8) Solutions

April 18, 2017

A1. True. Construction of an example. If we let $m = 0$, we have the claim that n^2 is a perfect square, which is true as $(n)^2 = n^2$.

A2. True. Construction of an example. The claim is $(\forall m \in \mathbb{N})(\exists n \in \mathbb{N})[PerfectSquare(mn+1)]$. If we let $n = m+2$, $mn+1$ becomes $m^2+2m+1 = (m+1)^2$. Since $m \in \mathbb{N}$, we have that $n = m+2 \in \mathbb{N}$ also.

A3. True. Construction of an example The claim is $(\forall n \in \mathbb{N})(\exists b, c \in \mathbb{N})[Composite(f(n) = n^2 + bn + c)]$. If we let $b = 2,\ c = 1$, the quadratic becomes $f(n) = n^2 + 2n + 1 = (n+1)^2$. We know that $f(n)$ is composite because it is a perfect square i.e. the factors of $f(n)$ are $n+1$ and $n+1$.

A4. The statement is $(\forall n = 2k,\ k \in \{2, 3, 4, \ldots\})[(\exists p_1, p_2 \in \mathbb{P})(n = p_1 + p_2)] \implies (\forall m = 2k+1,\ k \in \{3, 4, 5, \ldots\})(\exists p_1, p_2, p_3 \in \mathbb{P})[m = p_1 + p_2 + p_3]$. We prove this by construction. If m is an odd number greater than 5, then it can be written as $m = 2k{+}1,\ k \in \{3, 4, 5, \ldots\}$. If we subtract 3 from m, we obtain $m - 3 = 2k + 1 - 3 = 2k - 2 = 2(k-1),\ k \in \{3, 4, 5, \ldots\}$, but we know that since $2|(m-3)$ and $m - 3 \in \{4, 6, 8, \ldots\}$, we can write $m - 3 = p_1 + p_2$, where $p_1, p_2 \in \mathbb{P}$. Thus we have $m - 3 = p_1 + p_2 \iff m = p_1 + p_2 + 3$, but since $3 \in \mathbb{P}$, we have $m = p_1 + p_2 + p_3$. Hence, we have proved that m can be written as the sum of three prime numbers.

A5. The claim is that $P(n) = \sum\limits_{i=1}^{n} 2i - 1 = n^2$. We prove this by induction. The base case is $P(1) = \sum\limits_{i=1}^{1} 2i - 1 = 2 - 1 = 1$, which is true. We now assume that for some k, $P(k)$ is true, and try to prove $P(k+1)$. For $P(k) = 1 + 3 + \ldots + (2k-1)$, we may obtain the next term by adding $2k+1$. By the induction hypothesis, $P(k) = k^2$, and adding the next term, we have $k^2 + 2k + 1 = (k+1)^2$, which is precisely $P(k+1)$. Hence, with the base case and induction hypothesis, we have proved $P(n)\ \forall n \in \mathbb{N}$.

A6. The claim is that $P(n) = 1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$. We prove this by induction. The base case is $1^2 = P(1) = \frac{1}{6}1(2)(3)$, which is true. We now assume for some k that $P(k)$ is true, and try to prove $P(k+1)$. For $P(k) = 1^2 + 2^2 + \ldots + k^2$, we may obtain the next term by adding $(k+1)^2 = k^2 + 2k + 1$. By the induction hypothesis, $P(k) = \frac{k^3}{3} + \frac{k^2}{2} + \frac{k}{6}$, and adding the next term, we have $\frac{k^3}{3} + \frac{k^2}{2} + \frac{k}{6} + k^2 + 2k + 1 = \frac{k^3}{3} + \frac{3k^2}{2} + \frac{13k}{6} + 1 = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$, which is precisely $P(k+1)$. Hence, with the base case and induction hypothesis, we have proved $P(n)\ \forall n \in \mathbb{N}$.

OPTIONAL PROBLEMS

A1. $(1+n) + (2+(n-1)) + (3+(n-2)) + \ldots + (n+1) = n(n+1) = 2S \iff S = \frac{1}{2}n(n+1)$

A2. We can prove this using induction. The statement is $(\forall n \in \mathbb{N} \setminus \{1,2\})P(n)$, where $P(n)$ means *for a collection of n points in a plane, not all collinear, there is a triangle having three points as its vertices, which contains none of the other points in its interior.* The base case is $P(n=3)$, which is true because 3 non-collinear points in a plane trivially form a triangle with no other points in its interior. Now for the induction hypothesis, we assume $P(k)$ is true for some k. We may obtain the next term by adding a point on the plane. There are two cases we must deal with. In the first case, the added point is *not* within the triangle, and thus $P(k+1)$ is trivially true. In the second case, the added point *is* within the triangle. But if this is so, we may form another triangle by connecting any two vertices of the previous triangle to the new point (Figure 1). This new triangle contains no other points in its interior, and thus $P(k+1)$ is also true. Hence, with the base case and induction hypothesis, we have proved $P(n)\ \forall n \in \mathbb{N} \setminus \{1,2\}$.

A3.

(a) The base case is $P(n=1)$, or $3|(4^1-1)$, which is true. We assume for some k, $P(k)$ is true i.e. $3|(4^k-1)$. The next term is given by $4^{k+1}-1 = 4 \cdot 4^k - 1$. We may rewrite this term as $4 \cdot 4^k - 4 + 3 = 4(4^k-1) + 3$. The second term is divisible by 3, but first term is also divisible by 3 since $3|(4^k-1)$ by the induction hypothesis. Hence, with the base case and induction hypothesis, we have proved $P(n)\ \forall n \in \mathbb{N}$.

(b) The base case is $P(n=5)$, or $6! = 720 > 2^8 = 256$, which is true. We assume for some k, $P(k)$ is true i.e. $(k+1)! > 2^{k+3}$. If we multiply both sides by $k+2$, we obtain $(k+2)(k+1)! = (k+2)! > (k+2) \cdot 2^{k+3} =$

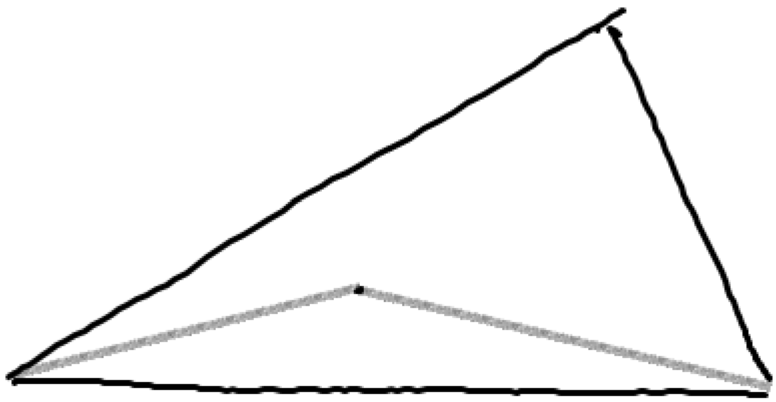

Figure 1:

$k \cdot 2^{k+3} + 2 \cdot 2^{k+3} = k \cdot 2^{k+3} + 2^{k+4} > 2^{(k+1)+3}$, which is precisely the induction hypothesis. Hence, with the base case the induction hypothesis, we have proved $P(n)\ \forall n \in \mathbb{N} \setminus \{1, 2, 3, 4\}$.

(c) The base case is $1 \cdot 1! = P(n = 1) = 2! - 1$, which is true. We assume for some k, $P(k)$ is true. We obtain the next term in the sum by adding $(k+1) \cdot (k+1)!$ to $P(k)$, so we get $(k+1)! - 1 + (k+1) \cdot (k+1)! = (k+1)![1 + (k+1)] - 1 = (k+2)! - 1$, which is precisely $P(k+1)$. Hence, with the base case and induction hypothesis, we have proved $P(n)\ \forall n \in \mathbb{N}$.