CTL Syntax

Assume a set *Atom* of atom propositions.

$$
\begin{aligned}
\phi \ ::= \ & p \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \\
& \mid \mathbf{AX}\ \phi \mid \mathbf{EX}\ \phi \mid \mathbf{AF}\ \phi \mid \mathbf{EF}\ \phi \mid \mathbf{AG}\ \phi \mid \mathbf{EG}\ \phi \\
& \mid \mathbf{A}[\phi\ \mathbf{U}\ \phi] \mid \mathbf{E}[\phi\ \mathbf{U}\ \phi]
\end{aligned}
$$

where $p \in Atom$.

Each temporal connective is a pair of a *path quantifier*:

A – for all paths

E – there exists a path

and an LTL-like temporal operator **X**, **F**, **G**, **U**.

Precedence (high-to-low): $(\mathbf{AX}, \mathbf{EX}, \mathbf{AF}, \mathbf{EF}, \mathbf{AG}, \mathbf{EG}, \neg), (\wedge, \vee), \to$

CTL Semantics 1: Transition Systems and Paths

(*This is the same as for LTL*)

Definition (Transition System)

A *transition system* $\mathcal{M} = \langle S, \rightarrow, L \rangle$ consists of:

$$\begin{array}{cl} S & \text{a finite set of states} \\ \rightarrow \subseteq S \times S & \text{transition relation} \\ L : S \rightarrow \mathcal{P}(Atom) & \text{a labelling function} \end{array}$$

such that $\forall s_1 \in S.\ \exists s_2 \in S.\ s_1 \rightarrow s_2$

Definition (Path)

A *path* π in a transition system $\mathcal{M} = \langle S, \rightarrow, L \rangle$ is an infinite sequence of states $s_0, s_1, \ldots$ such that $\forall i \geq 0.\ s_i \rightarrow s_{i+1}$.

Paths are written as: $\pi = s_0 \rightarrow s_1 \rightarrow s_2 \rightarrow \ldots$

CTL Semantics 2: Satisfaction Relation

Satisfaction relation $\mathcal{M}, s \models \phi$ read as

state s in model $\mathcal{M}$ satisfies CTL formula ϕ

We often leave $\mathcal{M}$ implicit.
The propositional connectives:

$$
\begin{array}{lllll}
s & \models & \top & & \\
s & \not\models & \bot & & \\
s & \models & p & \text{iff} & p \in L(s) \\
s & \models & \neg\phi & \text{iff} & s \not\models \phi \\
s & \models & \phi \land \psi & \text{iff} & s \models \phi \text{ and } s \models \psi \\
s & \models & \phi \lor \psi & \text{iff} & s \models \phi \text{ or } s \models \psi \\
s & \models & \phi \to \psi & \text{iff} & s \models \phi \text{ implies } s \models \psi
\end{array}
$$

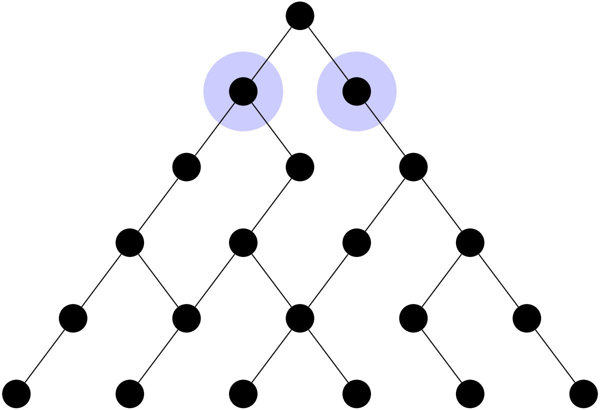
CTL Semantics 2: Satisfaction Relation

The temporal connectives, assuming path $\pi = s_0 \to s_1 \to s_2 \to \ldots$,

$$
\begin{array}{lllll}
s & \models & \mathbf{AX}\,\phi & \text{iff} & \forall\pi \text{ s.t. } s_0 = s.\ s_1 \models \phi \\
s & \models & \mathbf{EX}\,\phi & \text{iff} & \exists\pi \text{ s.t. } s_0 = s.\ s_1 \models \phi \\
s & \models & \mathbf{AG}\,\phi & \text{iff} & \forall\pi \text{ s.t. } s_0 = s.\ \forall i.\ s_i \models \phi \\
s & \models & \mathbf{EG}\,\phi & \text{iff} & \exists\pi \text{ s.t. } s_0 = s.\ \forall i.\ s_i \models \phi \\
s & \models & \mathbf{AF}\,\phi & \text{iff} & \forall\pi \text{ s.t. } s_0 = s.\ \exists i.\ s_i \models \phi \\
s & \models & \mathbf{EF}\,\phi & \text{iff} & \exists\pi \text{ s.t. } s_0 = s.\ \exists i.\ s_i \models \phi \\
s & \models & \mathbf{A}[\phi\ \mathbf{U}\ \psi] & \text{iff} & \forall\pi \text{ s.t. } s_0 = s. \\
 & & & & \qquad \exists i.\ s_i \models \psi \text{ and } \forall j < i.\ s_j \models \phi \\
s & \models & \mathbf{E}[\phi\ \mathbf{U}\ \psi] & \text{iff} & \exists\pi \text{ s.t. } s_0 = s. \\
 & & & & \qquad \exists i.\ s_i \models \psi \text{ and } \forall j < i.\ s_j \models \phi
\end{array}
$$

Note: The semantics for **AX** and **EX** is given differenttly in H&R.

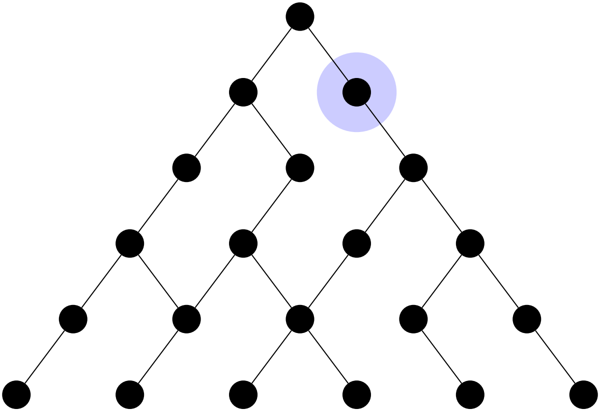
CTL in Pictures



AX ϕ

For *every* next state, ϕ holds.

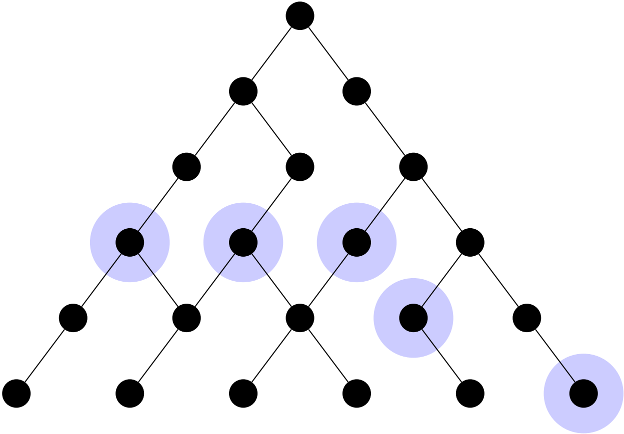
CTL in Pictures



EX ϕ

There *exists* a next state where ϕ holds.

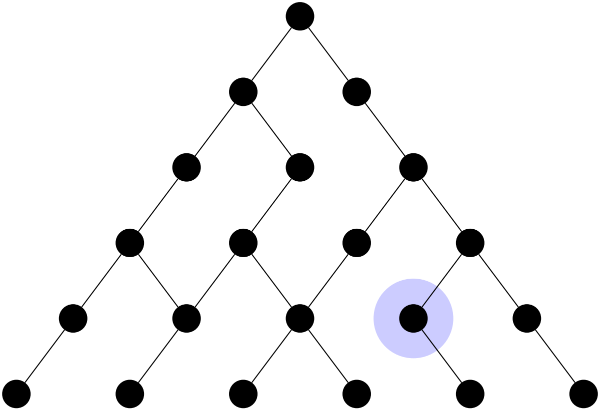
CTL in Pictures



AF ϕ

For all paths, there exists a future state where ϕ holds.

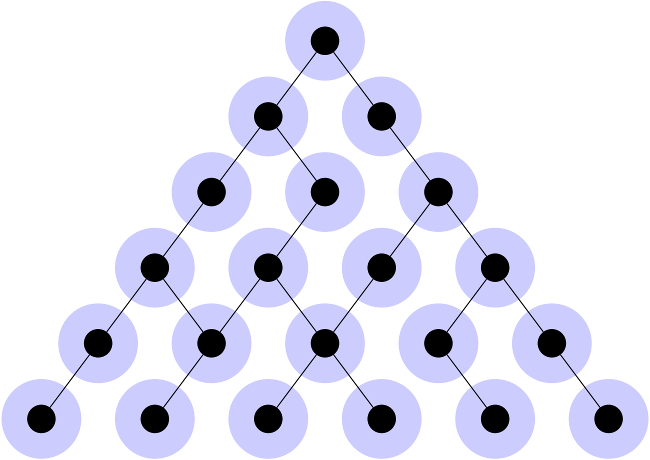
CTL in Pictures



EF ϕ

There exists a path with a future state where ϕ holds.

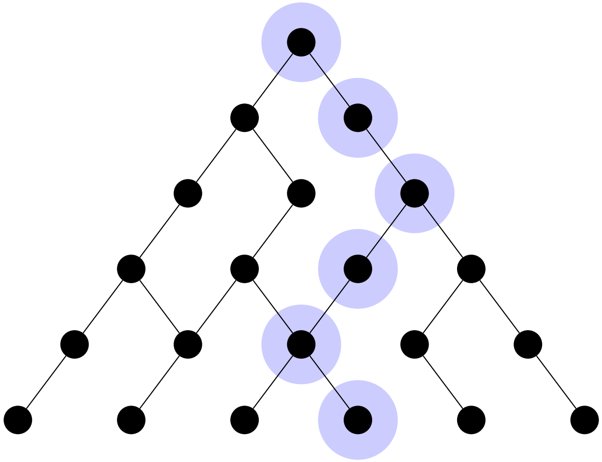
CTL in Pictures



AG ϕ

For all paths, for all states along them, ϕ holds.

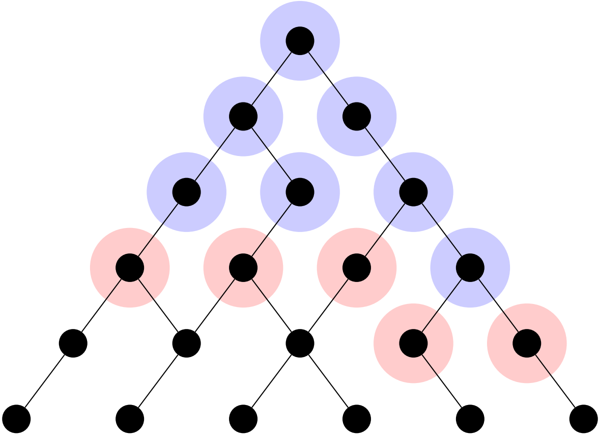
CTL in Pictures



EG ϕ

There exists a path such that, for all states along it, ϕ holds.

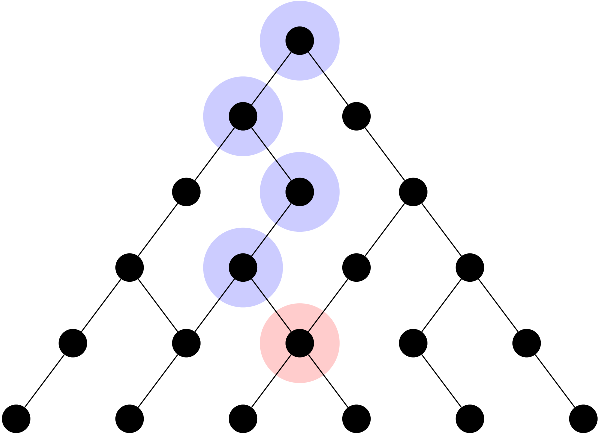
CTL in Pictures



$$\mathbf{A}[\phi \ \mathbf{U} \ \psi]$$

For all paths, ψ eventually holds, and ϕ holds at all states earlier.

CTL in Pictures



$$\mathbf{E}[\phi \ \mathbf{U} \ \psi]$$

There exists a path where ψ eventually holds, and ϕ holds at all states earlier.

Examples of CTL formulas (and their possible readings)

- **EF** ϕ
  it is possible to get to a state where ϕ *is true*

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  A certain process is enabled infinitely often on every computation path

- $\mathbf{AG}\ (\mathit{requested} \to \mathbf{AF}\ \mathit{acknowledged})$
  for any state, if a request ocurs, then it will eventually be acknowledged

- $\mathbf{AG}\ (\phi \to \mathbf{E}[\phi\ \mathbf{U}\ \psi])$
  for any state, if ϕ holds, then there is a future where ψ eventually holds, and ϕ holds for all points in between

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Examples of CTL formulas (and their possible readings)

- **EF** ϕ
  it is possible to get to a state where ϕ *is true*
- **AG AF** *enabled*
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- **AG** $($*requested* $\rightarrow$ **AF** *acknowledged*$)$
  for any state, if a request ocurs, then it will eventually be acknowledged
- **AG** $(\phi \rightarrow \mathbf{E}[\phi\ \mathbf{U}\ \psi])$
  for any state, if ϕ *holds, then there is a future where* ψ *eventually holds, and* ϕ *holds for all points in between*
- **AG** $(\phi \rightarrow \mathbf{EG}\ \psi)$
  for any state, if ϕ *holds then there is a future where* ψ *always holds*
- **EF AG** ϕ
  there exists a possible state in the future, from where ϕ *is always true*

CTL Equivalences

de Morgan dualities for the temporal connectives:

$$\begin{aligned} \neg \mathbf{EX}\,\phi &\equiv \mathbf{AX}\,\neg\phi \\ \neg \mathbf{EF}\,\phi &\equiv \mathbf{AG}\,\neg\phi \\ \neg \mathbf{EG}\,\phi &\equiv \mathbf{AF}\,\neg\phi \end{aligned}$$

Also have

$$\begin{aligned} \mathbf{AF}\,\phi &\equiv \mathbf{A}[\top\ \mathbf{U}\ \phi] \\ \mathbf{EF}\,\phi &\equiv \mathbf{E}[\top\ \mathbf{U}\ \phi] \\ \mathbf{A}[\phi\ \mathbf{U}\ \psi] &\equiv \neg(\mathbf{E}[\neg\psi\ \mathbf{U}\ (\neg\phi \wedge \neg\psi)] \vee \mathbf{EG}\,\neg\psi) \end{aligned}$$

From these, one can show that the sets $\{\mathbf{AU}, \mathbf{EU}, \mathbf{EX}\}$ and $\{\mathbf{EU}, \mathbf{EG}, \mathbf{EX}\}$ are both adequate sets of temporal connectives.