Hidden Markov Models (HMM)

ECE/CS 498 DS U/G

Lecture 18: Hidden Markov Models continued

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Announcements

- Course Timeline
  - Mon 3/30: HMMs Continued, **ICA 4**
  - Wed 4/1: Intro to Factor Graphs
    - **Mid-semester Feedback** (to be completed via Google Form)
    - MP 3 released
- MP 2 Final Checkpoint due **tonight (March 30) @ 11:59 PM on Compass2G**
- Final Project
  - Progress report 2 due **Friday April 17 @ 11:59 PM** on Compass2G
    - There should be *substantial* progress with projects by this point (i.e. meaningful results, ML/AI models)

Markov Model

- Consider a system which can occupy one of *N* discrete *states* or *categories*

$$x_t \in \{1, 2, \ldots, N\} \longrightarrow \text{state at time } t$$

- We are interested in *stochastic* systems, in which state evolution is random
- Any *joint* distribution can be factored into a series of *conditional* distributions:

$$p(x_0, x_1, \ldots, x_T) = p(x_0) \prod_{t=1}^{T} p(x_t \mid x_0, \ldots, x_{t-1})$$

- For a *Markov* process, the next state depends only on the current state:

$$p(x_{t+1} \mid x_0, \ldots, x_t) = p(x_{t+1} \mid x_t)$$

State Transition Matrices

- A *stationary* Markov chain with *N* states is described by an *NxN* *transition matrix:*

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}$$

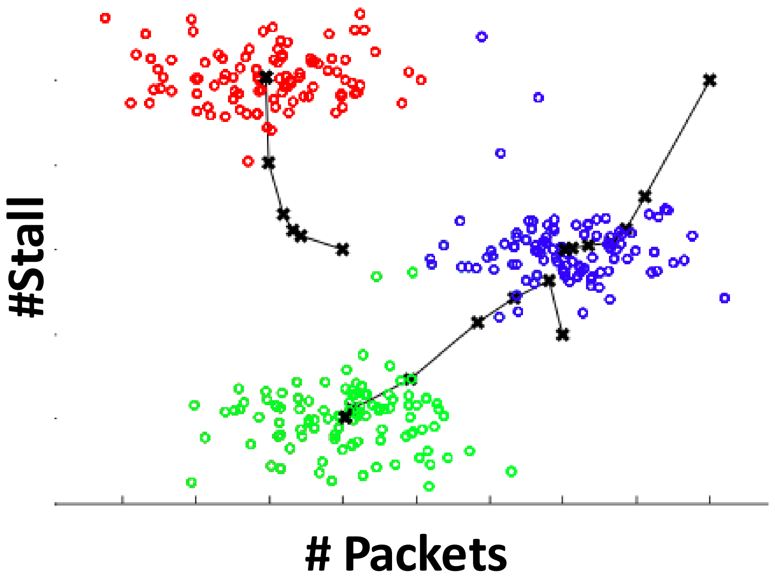
$$q_{ij} \triangleq p(x_{t+1} = i \mid x_t = j)$$

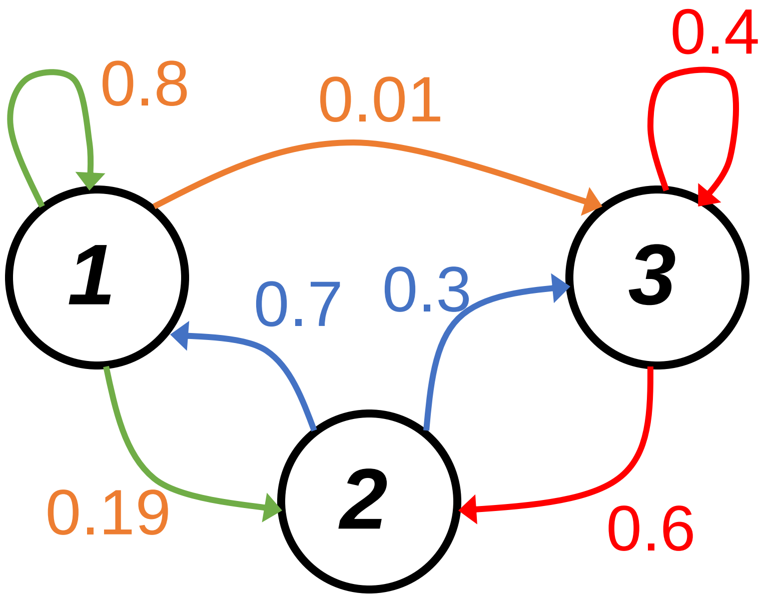
- Constraints on valid transition matrices:

$$q_{ij} \geq 0 \qquad \sum_{i=1}^{N} q_{ij} = 1 \quad \text{for all } j$$

State Transition Diagrams (Packet Stalls in a Network)

Stall experienced by each packet



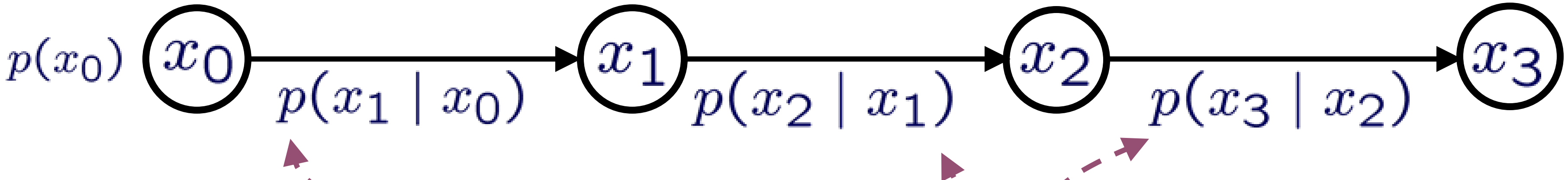


$$q_{ij} \triangleq p(x_{t+1} = i \mid x_t = j)$$

- Think of a particle randomly following an arrow at each discrete time step
- Most useful when *N* (# of States) small, and *Q* (TransProb Matrix) *sparse*

Markov Chains: Graphical Models

$$p(x_0, x_1, \ldots, x_T) = p(x_0) \prod_{t=1}^{T} p(x_t \mid x_{t-1})$$



$$Q = \begin{bmatrix} 0.80 & 0.7 & 0.0 \\ 0.19 & 0.0 & 0.6 \\ 0.01 & 0.3 & 0.4 \end{bmatrix}$$

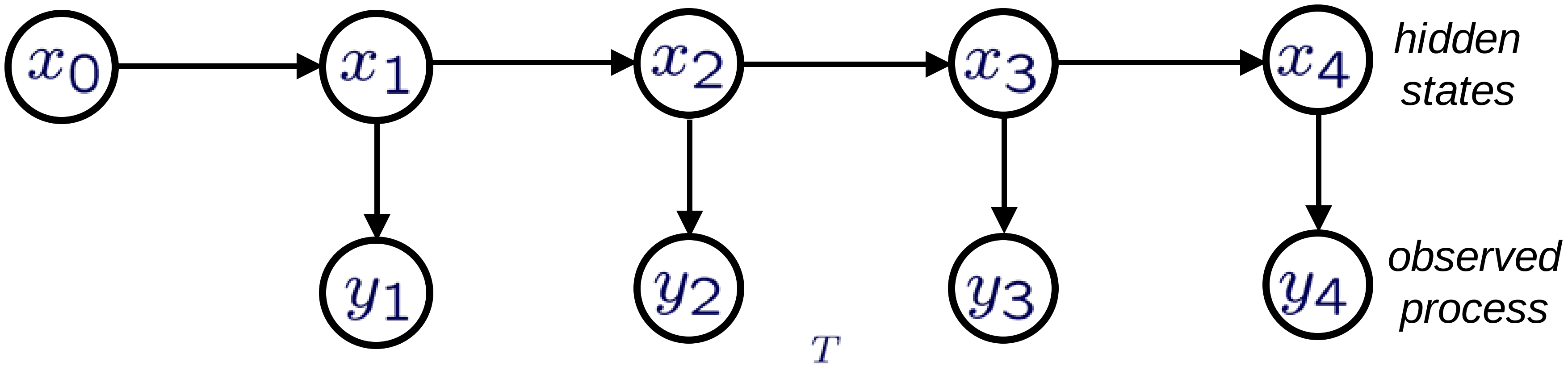
Constraints on valid transition matrices:

$q_{ij} \geq 0$, $\sum_{i=1}^{N} q_{ij} = 1$ for all j

$$q_{ij} \triangleq p(x_{t+1} = i \mid x_t = j)$$

Hidden Markov Models (Packet Stall Example Cont'd)

- Stall exists due to congestion
- Not directly measurable at runtime (hidden)
- Motivates *hidden Markov models (HMM):*

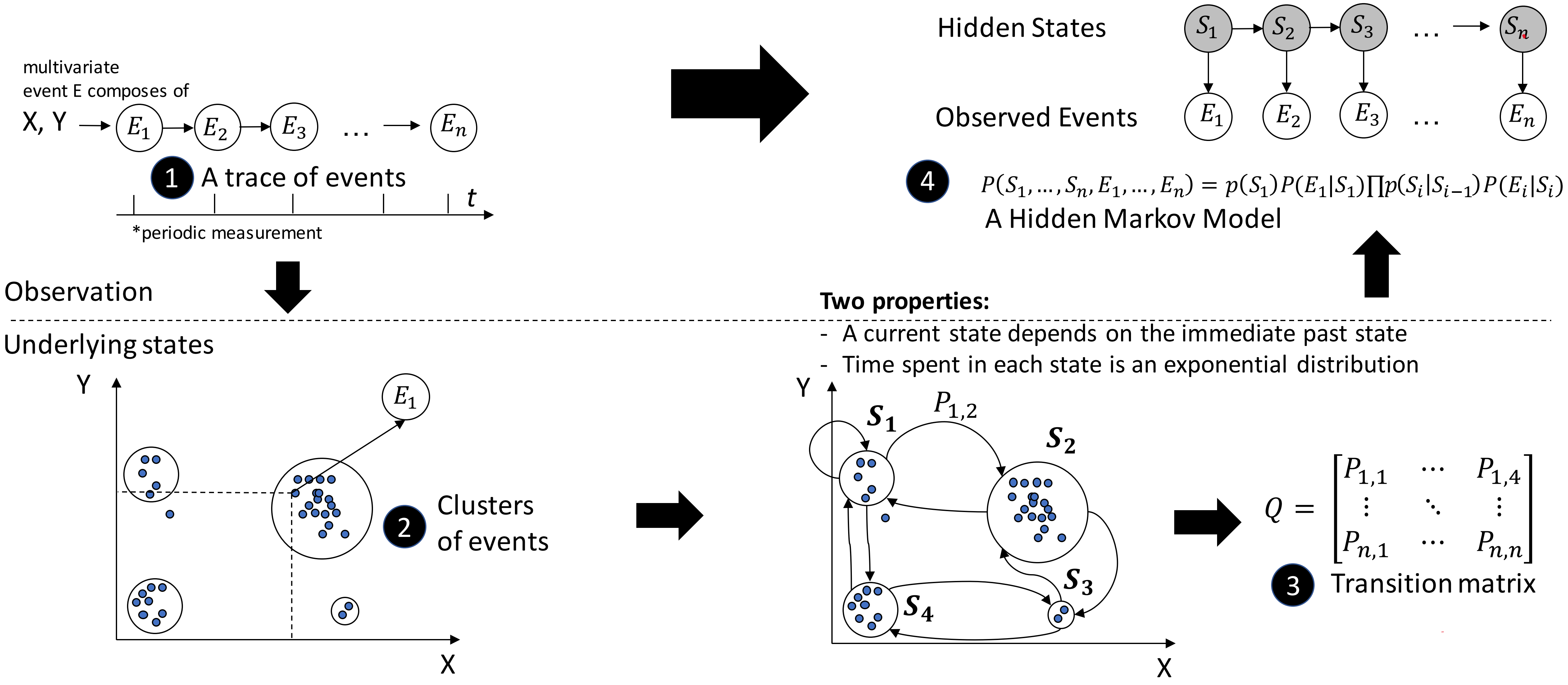


$$p(x_0, x_1, \ldots, x_T) = p(x_0) \prod_{t=1}^{T} p(x_t \mid x_{t-1}) p(y_t \mid x_t)$$

Given x_t, previous observations impact future observations

$$p(y_t, y_{t+1}, \cdots \mid x_t, y_{t-1}, y_{t-2}, \ldots) = p(y_t, y_{t+1}, \cdots \mid x_t)$$

From a trace of events to a Hidden Markov Model



Hidden Markov Models

Model assumptions

An observation depends on its hidden state

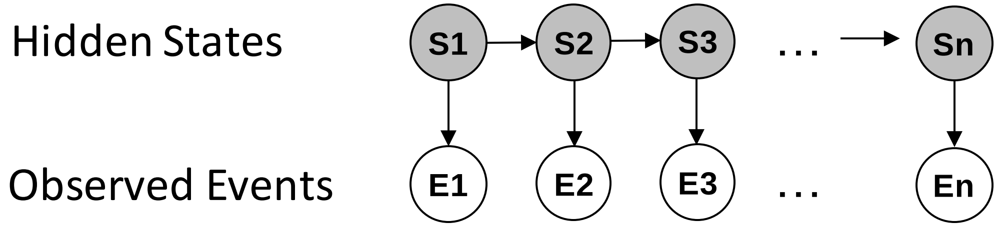
A state variable only depends on the immediate previous state (Markov assumption)

The future observations and the past observations are conditionally independent given the current hidden state

Advantages:

HMM can model sequential nature of input data (future depends on the past)

HMM has a linear-chain structure that clearly separates system state and observed events.

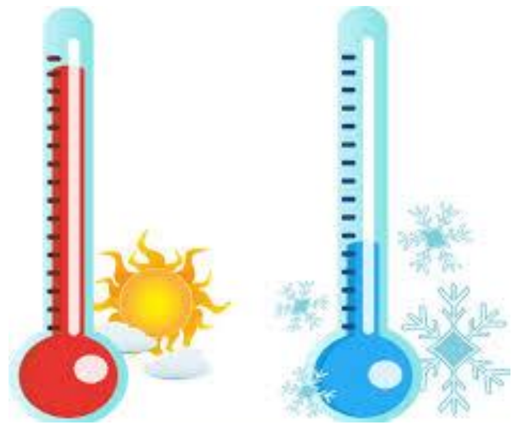


$$P(S_1, \dots, S_n, E_1, \dots, E_n) = p(S_1)P(E_1|S_1)\prod p(S_i|S_{i-1})P(E_i|S_i)$$

A Hidden Markov model on observed events and system states

HMM Motivating Example: Paleontological Temperature Model

- Want to determine the average temperature at a particular place on earth over a sequence of years in the distant past
- Only annual average temperatures -- hot **(H)** and cold (**C**)
    - Probability of a hot year followed by another hot year is 0.7, and the probability of a cold year followed by another cold year is 0.6, independent of the temperature in prior years
- Correlation between the size of tree growth rings and temperature
    - Three different ring sizes, small (**T**), medium (**D**), and large (**L**)
- Assume that probability values from current period held in paleontological period too
- Determine the most likely temperature state in past years
    - Can’t directly observe the temperature in the past
    - We can observe the size of tree rings – can this information be used?



H C

T D L

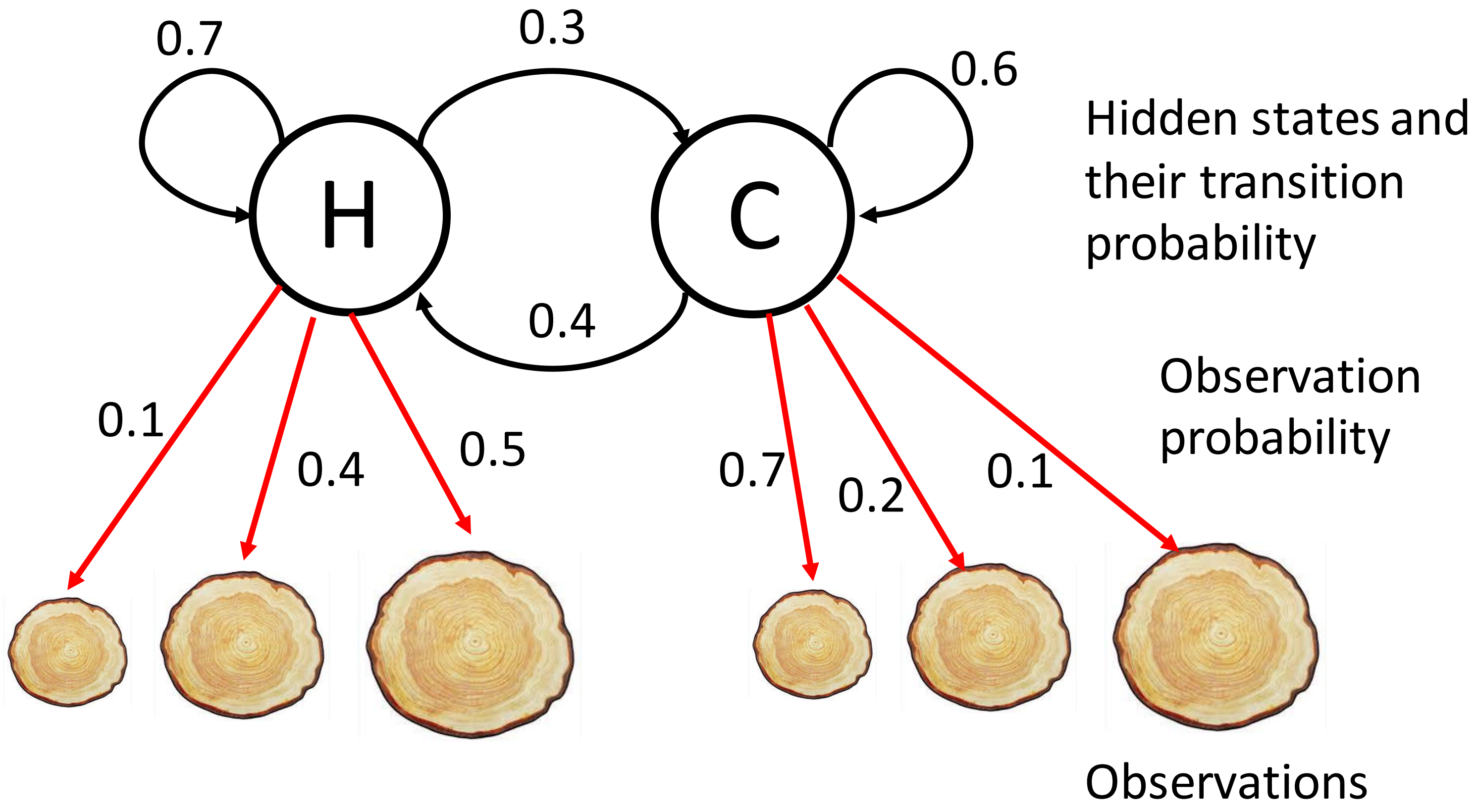
Tree ring size

Source: http://www.math.unl.edu/~sdunbar1/ProbabilityTheory/Lessons/HiddenMarkovModels/Examples/examples.html

Paleontological Temperature Model

- State space of hidden states: $S = \{H, C\}$
- State space of observations: $E = \{T, D, L\}$
- Transition probability matrix: A
- Observation Matrix: B
- Initial distribution for the hidden states: π

Given by an oracle



$$\begin{array}{c c} & \begin{matrix} H & C \end{matrix} \\ \begin{matrix} H \\ C \end{matrix} & \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

A

$$\begin{array}{c c} & \begin{matrix} T & D & L \end{matrix} \\ \begin{matrix} H \\ C \end{matrix} & \begin{bmatrix} 0.1 & 0.4 & 0.5 \\ 0.7 & 0.2 & 0.1 \end{bmatrix} \end{array}$$

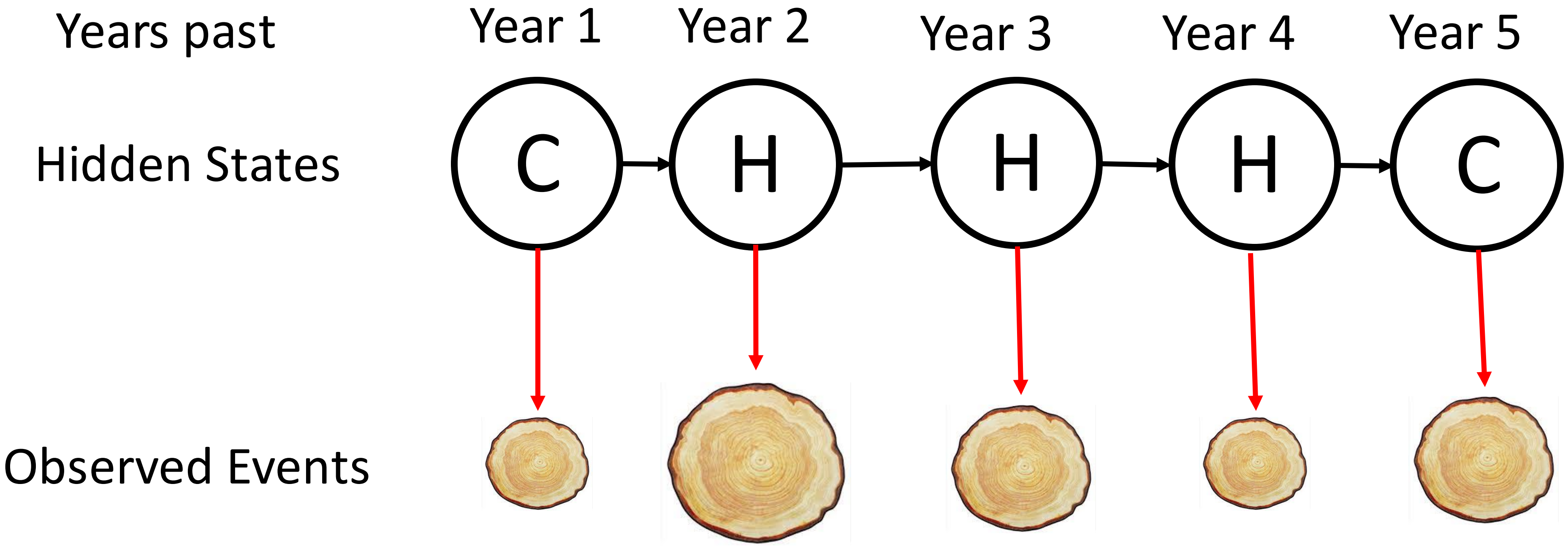
B

$$\begin{array}{c} \begin{matrix} H & C \end{matrix} \\ [0.5\ 0.5] \end{array}$$

π

Paleontological Temperature Model

Example sequence with 5 observations

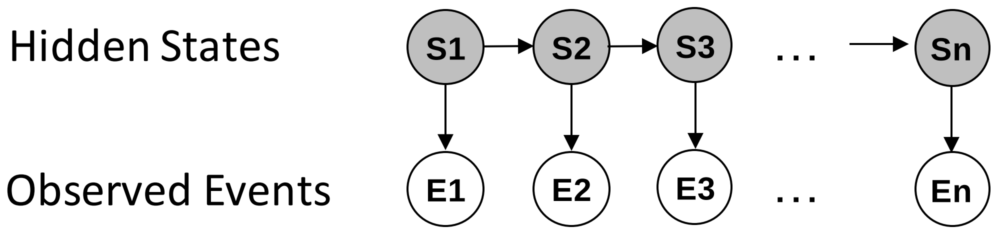


Determine the sequence of hidden states

Hidden Markov Models

Model

- Set of hidden states $\boldsymbol{S} = \{\boldsymbol{\sigma_1}, \dots, \boldsymbol{\sigma_N}\}$
- Set of observable events $\boldsymbol{E} = \{\boldsymbol{\epsilon_1}, \dots, \boldsymbol{\epsilon_M}\}$
- Transition probability matrix $\boldsymbol{A}$
- Observation matrix $\boldsymbol{B}$
- Initial distribution of hidden states $\boldsymbol{\pi}$



A Hidden Markov model on observed events and system states

Model assumptions

- An observation depends on its hidden state
- A state variable only depends on the immediate previous state (Markov assumption)
- The future observations and the past observations are conditionally independent given the current hidden state

$$P(S_1, \dots, S_n, E_1, \dots, E_n) = P(S_1)P(E_1|S_1)\prod_{i=2}^{n} P(S_i|S_{i-1})P(E_i|S_i)$$

Advantages:

- HMM can model sequential nature of input data (future depends on the past)
- HMM has a linear-chain structure that clearly separates system state and observed events.

Inference question – Paleontological Temperature

Given the sequence of 5 observations T, L, D, T, D and the model (A, B, π), how do we choose a corresponding state sequence $S_1, S_2, \ldots, S_n$ which is optimal in some meaningful sense (i.e., best explains the observations) where $S_t \in \{H, C\}$?

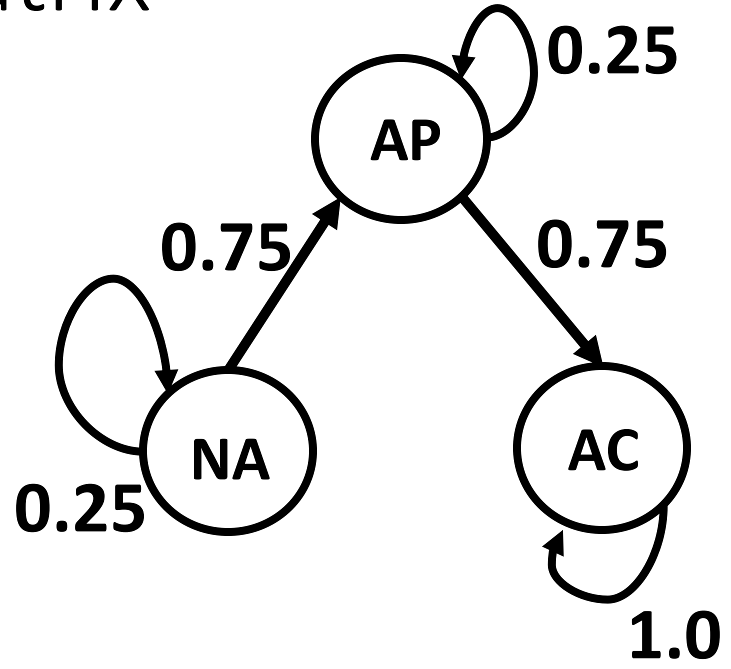
A simpler question: Given the sequence of 5 observations T, L, D, T, D and the model (A, B, π), which of the two is more probable eg., $S_3 = H$ or $S_3 = C$?

HMM Security Example

- Suppose you are a security expert monitoring the NCSA system
- By monitoring the system events, you want to say whether the system is safe or not
  - System's safety is a hidden state
  - Events are observed
  - Events are related to the safety of the system
- Is the system safe?
  - **HMM** to the rescue!

Security Example: Transition Matrix

Transition matrix (A)



The system has three distinct security states –

(a) No Attack **(NA)**,
(b) Attack in Progress **(AP),** and
(c) Attack Complete **(AC).**

- Every hour, the system is being attacked by attackers coordinating together around the world and trying to compromise the system.
- The system states always transition from **NA to AP** and **AP to AC.**
- An attacker is successful in changing the state of the system with probability of 0.75 and fails with a probability of 0.25.
- If the attack fails, the system stays in its current state.
- If the system state reaches **AC** the attack is complete, and the system stays in that state.

$$A = \begin{array}{c} \\ \textbf{NA} \\ \textbf{AP} \\ \textbf{AC} \end{array} \begin{array}{c} \begin{array}{ccc} \textbf{NA} & \textbf{AP} & \textbf{AC} \end{array} \\ \begin{pmatrix} 0.25 & 0.75 & 0 \\ 0 & 0.25 & 0.75 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Transition Probability Matrix

Security Example: Emission matrix and initial distribution

Observation matrix (B)

- Your monitoring system reports two types of events
  - Port Scan **(PS)**
  - Software Installation **(SI)**
- Monitors are always accurate and works. Attackers cannot compromise the monitors. Every hour, we get information from the monitors if the attackers are trying to do **PS or SI.**

$$\mathbf{B} = \begin{array}{c} \\ \textbf{NA} \\ \textbf{AP} \\ \textbf{AC} \end{array} \begin{array}{c} \quad\textbf{PS} \qquad\qquad \textbf{PI} \\ \begin{pmatrix} P_{PS|NA} & P_{PI|NA} \\ P_{PS|AP} & P_{PI|AP} \\ P_{PS|AC} & P_{PI|AC} \end{pmatrix} \end{array}$$

Observation Matrix

Initial distribution (π)

- We have no idea about the initial state of the system.

$$\pi_0 = \begin{array}{c} \textbf{NA} \\ \textbf{AP} \\ \textbf{AC} \end{array} \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

Initial state distribution/prior

General Inference question

Given the sequence of n observations $E_1, E_2, \dots, E_n$, and the model (A, B, π), how do we choose a corresponding state sequence $S_1, S_2, \dots, S_n$ which is optimal in some meaningful sense (i.e., best explains the observations)?

A simpler question: Given the sequence of n observations $E_1, E_2, \dots, E_n$, and the model (A, B, π), what is the most probable state S_t at $t \in \{1, \dots, n\}$?

$$\operatorname*{argmax}_{j \in \{1,\dots,N\}} P(S_t = \sigma_j | E_1, E_2, \dots, E_n)$$

$$S = \{\sigma_1, \dots, \sigma_N\}$$

Breaking down the inference question

$$P(S_t|E_1,E_2,\dots,E_n) = \frac{P(S_t,E_1,\dots,E_n)}{P(E_1,\dots,E_n)} = \frac{P(S_t,E_1,\dots,E_t,E_{t+1},\dots,E_n)}{P(E_1,\dots,E_n)}$$

Bayes rule

$$= \frac{P(E_{t+1},\dots,E_n\,|S_t,E_1,\dots,E_t)\,P(S_t,E_1,\dots,E_t)}{P(E_1,\dots,E_n)}$$

Bayes rule

$$= P(E_{t+1},\dots,E_n\,|S_t,E_1,\dots,E_t)\,P(S_t|E_1,\dots,E_t)\,\frac{P(E_1,\dots,E_t)}{P(E_1,\dots,E_n)}$$

Markov property

$$= \frac{P(E_{t+1},\dots,E_n\,|S_t)\,P(S_t|E_1,\dots,E_t)}{P(E_{t+1},\dots,E_n|E_1,\dots,E_t)}$$

Breaking down the inference question

$$P(S_t|E_1, E_2, \dots, E_n) = \frac{P(E_{t+1}, \dots, E_n \,|S_t)\, P(S_t|E_1, \dots, E_t)}{P(E_{t+1}, \dots, E_n|E_1, \dots, E_t)}$$

$P(S_t|E_1, \dots, E_t)$:
Probability of hidden state at time t given observation up to time t (**Forwards algorithm**)

$P(E_{t+1}, \dots, E_n \,|S_t)$:
Probability of the future observed sequence given the hidden state at time t (**Backwards algorithm**)

$P(E_{t+1}, \dots, E_n|E_1, \dots, E_t)$:
Does not depend on the hidden state (will not affect the maximization because it is just a scaling factor)

Forwards algorithm: Paleontological Temperature

Want to calculate $P(S_t|E_1, \ldots, E_t)$

- Let us calculate it for $t = 2$
- In the example, $E_1 = T, E_2 = L$
- Find $P(S_2 = H|E_1 = T, E_2 = L)$?

$$P(S_2 = H|E_1 = T, E_2 = L) = \frac{P(S_2 = H, E_1 = T, E_2 = L)}{P(E_1 = T, E_2 = L)}$$

Adding hidden state S_1

$$= \frac{\sum_{s \in \{H,C\}} P(S_2 = H, E_1 = T, E_2 = L, S_1 = s)}{P(E_1 = T, E_2 = L)}$$

Forwards algorithm: Paleontological Temperature

$$\frac{\sum_{s\in\{H,C\}} P(S_2 = H, E_1 = T, E_2 = L, S_1 = s)}{P(E_1 = T, E_2 = L)}$$

Bayes rule

$$= \frac{\sum_{s\in\{H,C\}} P(E_2 = L | S_2 = H, E_1 = T, S_1 = s) P(S_2 = H, E_1 = T, S_1 = s)}{P(E_1 = T, E_2 = L)}$$

Markov property

Bayes rule

$$= \frac{\sum_{s\in\{H,C\}} P(E_2 = L | S_2 = H) P(S_2 = H | E_1 = T, S_1 = s) P(S_1 = s | E_1 = T) P(E_1 = T)}{P(E_1 = T, E_2 = L)}$$

Markov property

Bayes rule

$$= \frac{\sum_{s\in\{H,C\}} P(E_2 = L | S_2 = H) P(S_2 = H | S_1 = s) P(S_1 = s | E_1 = T)}{P(E_2 = L | E_1 = T)}$$

Forwards algorithm: Paleontological Temperature

Hidden state given all observations up to that point

Observation probability

Transition probability

Hidden state given all observations up to that point

$$P(S_2 = H | E_1 = T, E_2 = L) = \frac{P(E_2 = L | S_2 = H) \sum_{s \in \{H,C\}} P(S_2 = H | S_1 = s) P(S_1 = s | E_1 = T)}{P(E_2 = L | E_1 = T)}$$

Define: $\alpha_t(i) = P(S_t = \sigma_i | E_1, E_2, \dots, E_t)$ and $Z_t = P(E_t | E_1, \dots E_{t-1})$

Above equation can be written as,

$$\boxed{\alpha_2(H)} = \frac{1}{Z_2} P(E_2 = L | S_2 = H) \sum_{s \in \{H,C\}} P(S_2 = H | S_1 = s) \boxed{\alpha_1(s)}$$

Where, $Z_2 = P(E_2 | E_1)$

Recursion

Forwards algorithm: General Expression

Define: $\alpha_t(j) = P(S_t = \sigma_j | E_1, E_2, \dots, E_t)$ and $Z_t = P(E_t | E_1, \dots E_{t-1})$

In general,

$$\alpha_t(j) = \frac{1}{Z_t} P\big(E_t \big| S_t = \sigma_j\big) \sum_{i=1}^{N} P(S_t = \sigma_j \mid S_{t-1} = \sigma_i)\, \alpha_{t-1}(i) \qquad Z_t = \sum_{j=1}^{N} b_t \odot (A^T \alpha_{t-1})$$

Transition probability a_{ij}

Above equation can be written as a matrix for all j,

$$\begin{bmatrix} \alpha_t(1) \\ \vdots \\ \alpha_t(j) \\ \vdots \\ \alpha_t(N) \end{bmatrix} \propto \begin{bmatrix} P(E_t | S_t = \sigma_1) \\ \vdots \\ P(E_t | S_t = \sigma_j) \\ \vdots \\ P(E_t | S_t = \sigma_N) \end{bmatrix} \odot \begin{bmatrix} a_{11} & \dots & \dots & \dots & a_{N1} \\ \vdots & \ddots & \dots & \dots & \vdots \\ a_{1j} & \dots & a_{ij} & \dots & a_{Nj} \\ \vdots & \dots & \dots & \ddots & \dots \\ a_{1N} & \dots & \dots & \dots & a_{NN} \end{bmatrix} \begin{bmatrix} \alpha_{t-1}(1) \\ \vdots \\ \alpha_{t-1}(i) \\ \vdots \\ \alpha_{t-1}(N) \end{bmatrix}$$

$\odot$ Represents elementwise product (Hadamard product)

$$\alpha_t \propto b_t \odot (A^T \alpha_{t-1})$$

b_t is the column of the observation matrix B corresponding to E_t

Forwards Algorithm: Paleontological Temperature

For observations T, L, D, T, L

$$\begin{array}{ccc} & H & C \\ H & \multicolumn{2}{c}{\multirow{2}{*}{}} \end{array}$$

$$\begin{array}{c|cc} & H & C \\ \hline H & 0.7 & 0.3 \\ C & 0.4 & 0.6 \end{array}$$

Transition probability matrix

$P(S_2 | E_1 = T, E_2 = L)$ is,

$$\begin{bmatrix} \alpha_2(H) \\ \alpha_2(C) \end{bmatrix} \propto \begin{bmatrix} 0.5 \\ 0.1 \end{bmatrix} \odot \left(\begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} \alpha_1(H) \\ \alpha_1(C) \end{bmatrix} \right)$$

$$\begin{array}{cccc} & T & D & L \\ H & 0.1 & \boxed{0.4} & \boxed{0.5} \\ C & 0.7 & \boxed{0.2} & \boxed{0.1} \end{array}$$

Observation matrix

Similarly, $P(S_3 | E_1 = T, E_2 = L,\ E_3 = D)$ is,

$$\begin{bmatrix} \alpha_3(H) \\ \alpha_3(C) \end{bmatrix} \propto \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix} \odot \left(\begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} \alpha_2(H) \\ \alpha_2(C) \end{bmatrix} \right)$$

Forwards Algorithm

1. Input: (A, B, π) and observed sequence $E_1, \dots, E_n$
2. $[\alpha_1, Z_1]$ = normalize($b_1 \odot \pi$)
3. **for** $t = 2 \colon n$ **do**
   $[\alpha_t, Z_t]$ = normalize($b_t \odot (A^T \alpha_{t-1})$)
4. return $\alpha_1, \dots, \alpha_n$ and $\log\big(P(E_1, \dots, E_n)\big) = \sum_t \log(Z_t)$

Note:

Subroutine: [v, Z] = normalize(u): $Z = \sum_j u_j$; $v_j = u_j / Z$;

Breaking down the inference question

$$P(S_t|E_1, E_2, \dots, E_n) = \frac{P(E_{t+1}, \dots, E_n\ |S_t)\ P(S_t|E_1, \dots, E_t)}{P(E_{t+1}, \dots, E_n|E_1, \dots, E_t)}$$

$P(S_t|E_1, \dots, E_t)$:
Probability of hidden state at time t given observation up to time t (**Forwards algorithm**)

$P(E_{t+1}, \dots, E_n\ |S_t)$:
Probability of the future observed sequence given the hidden state at time t (**Backwards algorithm**)

$P(E_{t+1}, \dots, E_n|E_1, \dots, E_t)$:
Does not depend on the hidden state (will not affect the maximization because it is just a scaling factor)

Backwards Algorithm (similar to Forwards Algo.)

Calculate $P(E_{t+1}, \dots, E_n \mid S_t)$

Define: $\beta_t(j) = P(E_{t+1}, \dots, E_n \mid S_t = \sigma_j)$

Include S_t to use information from the one-step future

$$\boxed{\beta_{t-1}(j)} = P(E_t, \dots, E_n \mid S_{t-1} = \sigma_j) = \sum_{i=1}^{N} P(S_t = \sigma_i, E_t, \dots, E_n \mid S_{t-1} = \sigma_j)$$

Chain rule

$$= \sum_{i=1}^{N} P(E_{t+1}, \dots, E_n \mid S_{t-1} = \sigma_j, S_t = \sigma_i, E_t)\, P(E_t \mid S_{t-1} = \sigma_j, S_t = \sigma_i)\, P(S_t = \sigma_i \mid S_{t-1} = \sigma_j)$$

Markov property

$$= \sum_{i=1}^{N} P(E_{t+1}, \dots, E_n \mid S_t = \sigma_i)\, P(E_t \mid S_t = \sigma_i)\, P(S_t = \sigma_i \mid S_{t-1} = \sigma_j)$$

By definition of $\beta_t(j)$

$$= \sum_{i=1}^{N} \boxed{\beta_t(i)}\, P(E_t \mid S_t = \sigma_i)\, P(S_t = \sigma_i \mid S_{t-1} = \sigma_j)$$

Emission probability

Transition probability

In matrix form, we get,

$$\beta_{t-1} = A(b_t \odot \beta_t)$$

$$\beta_t = \begin{bmatrix} \beta_t(1) \\ \vdots \\ \beta_t(N) \end{bmatrix}$$

Backwards Algorithm

1. Input: (A, B, π) and observed sequence $E_1, \ldots, E_n$
2. $\beta_n = 1$; // initialize $\beta_n(j)$ to 1 for all states σ_j
3. **for** $t = n - 1 : 1$ **do**
   $\beta_{t-1} = A(b_t \odot \beta_t)$
4. return $\beta_1, \ldots, \beta_n$

Breaking down the inference question

$$P(S_t|E_1, E_2, \ldots, E_n) = \frac{P(E_{t+1}, \ldots, E_n\,|S_t)\,P(S_t|E_1, \ldots, E_t)}{P(E_{t+1}, \ldots, E_n|E_1, \ldots, E_t)}$$

$P(S_t|E_1, \ldots, E_t)$:
Probability of hidden state at time t given observation up to time t (**Forwards algorithm**)

$P(E_{t+1}, \ldots, E_n\,|S_t)$:
Probability of the future observed sequence given the hidden state at time t (**Backwards algorithm**)

$P(E_{t+1}, \ldots, E_n|E_1, \ldots, E_t)$:
Does not depend on the hidden state (will not affect the maximization because it is just a scaling factor)

Inference – using Forwards-Backwards expressions

$$P(S_t|E_1, E_2, \ldots, E_n) = \frac{P(E_{t+1}, \ldots, E_n\,|S_t)\,P(S_t|E_1, \ldots, E_t)}{P(E_{t+1}, \ldots, E_n|E_1, \ldots, E_t)}$$

For $S_t = \sigma_j$ and $\gamma_t(j) = P\left(S_t = \sigma_j \middle| E_1, E_2, \ldots, E_n\right)$, the above equation is:

$$P\left(S_t = \sigma_j \middle| E_1, E_2, \ldots, E_n\right) = \frac{P(E_{t+1}, \ldots, E_n\,|S_t = \sigma_j)\,P(S_t = \sigma_j|E_1, \ldots, E_t)}{P(E_{t+1}, \ldots, E_n|E_1, \ldots, E_t)}$$

$$\gamma_t(j) = \frac{\beta_t(j)\alpha_t(j)}{P(E_{t+1}, \ldots, E_n|E_1, \ldots, E_t)} = \frac{\beta_t(j)\alpha_t(j)}{\sum_{i=1}^{N} \beta_t(j)\alpha_t(j)}$$

Theorem of total probability

$$\boxed{\gamma_t(j) \propto \beta_t(j)\alpha_t(j)}$$

Inference: Most likely state

- Forwards-backwards algorithm gives $P(S_t = \sigma_j | E_1, \dots, E_n)$ for all j
- Find the individually most likely state at time t given all observations

$$S_t^* = \underset{j \in \{1,..,N\}}{\operatorname{argmax}} \gamma_t(j)$$

Optimality of inference

- In the inference problem we attempt to uncover the hidden part of HMM, i.e., find the "correct" state sequence
- It is impossible to find the "correct" state sequence (solution)
- Use optimality criterion to find the "best" possible solution
- Several reasonable criteria exist and is a strong function of the intended application
    - Most likely state given observations
        - Application in finding average statistics, expected number of correct states
        - Solved using Forwards-Backwards algorithm
    - Single best sequence that maximises probability of observed events
        - Application in continuous speech recognition
        - Solved using Viterbi algorithm

Resources

Rabiner’s (excellent) paper:
https://www.ece.ucsb.edu/Faculty/Rabiner/ece259/Reprints/tutorial%20on%20hmm%20and%20applications.pdf