Homework 0: Basic Probability Review Problems
ECE/CS 498 DS Spring 2020
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Problem 1 (Basic Concepts)

(a) **(A1)** For any event A, $P(A) \geq 0$ (probabilities are nonnegative real numbers)
(A2) $P(S) = 1$ (probability of a certain event, an event that must happen is equal 1)
(A3) $P(A \cup B) = P(A) + P(B)$, whenever A and B are mutually exclusive events

(b) pmf is for discrete random variable while pdf is for continuous random variable. A pmf at a point, like $p(a)$, is the probability that a discrete random variable equals the point, i.e. $X = a$. A pdf at a point, like $f(a)$, is the probability desity when a continuous random variable equals the point, i.e. $X = a$. To better illustrate the differences between pmf and pdf, let's refer to the following equations.
for pmf: $p(a) = P\{X = a\}$
for pdf: $P\{X = a\} = \int_a^a f(x)dx = 0$ and $P\{a - \frac{\varepsilon}{2} \leq X \leq a + \frac{\varepsilon}{2}\} = \int_{a-\frac{\varepsilon}{2}}^{a+\frac{\varepsilon}{2}} f(x)dx \approx \varepsilon f(a)$

(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.8 + 0.5 - 0.8 \times 0.5 = 0.9$

(d) $P(A|B,C) \times P(B|C) = \frac{P(A,B,C)}{P(B,C)} \times \frac{P(B,C)}{P(C)} = \frac{P(A,B,C)}{P(C)} = P(A,B|C)$

Problem 4 (Exponential Distributions and Poisson Distributions)

(a) (i)Let $F(x)$ be the cdf of the exponential distribution
For $x \leq 0$, $f(x) = 0 \rightarrow F(x) = \int_{-\infty}^{x} f(x)dx = 0$
For $x > 0$, $f(x) = \lambda e^{-\lambda x} \rightarrow F(x) = \int_0^x f(a)da = [-e^{\lambda a}]_0^x = 1 - e^{-\lambda x}$
$\therefore F(x) = \begin{cases} 1 - e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$
(ii) The memoryless property of the exponential distribution is that the past does not affect its future behavior. In other words, every instant can be considered as the beginning of a new random period.
$P\{X > s + t | X > t\} = \frac{P\{X>s+t \text{ and } X>t\}}{P\{X>t\}}$
If $X > s + t$, $X > t$ is for sure since $s > 0$ in exponential distribution.
$= \frac{P\{X>s+t\}}{P\{X>t\}} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s} = P\{X > s\}$
$\therefore P\{X > s + t \mid X > t\} = P\{X > s\}$, showing that $X > t$ has no effect on the probability
(iii) $E[X] = \int_{-\infty}^{\infty} x\lambda e^{-\lambda x} dx$
$= \int_0^{\infty} x\lambda e^{-\lambda x} dx = \frac{1}{\lambda} \int_0^{\infty} ye^{-y}dy$ (let $y = \lambda x$, we have $dy = \lambda$)
$= \frac{1}{\lambda}[-e^{-y} - ye^{-y}]_0^{\infty} = \frac{1}{\lambda}$

$E[X^2] = \int_{-\infty}^{\infty} x^2 \lambda e^{-\lambda x} dx = \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx$

$= \frac{1}{\lambda^2} \int_0^{\infty} y^2 e^{-y} dy$ (let $y = \lambda x$, we have $dy = \lambda$)

$= \frac{1}{\lambda^2} [-2e^{-y} - 2ye^{-y} - y^2 e^{-y}]_0^{\infty} = \frac{2}{\lambda^2}$

$\mathrm{Var}[X] = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

$\therefore E[X] = \frac{1}{\lambda},\ \mathrm{Var}[X] = \frac{1}{\lambda^2}$

(b) Let $\lambda = np$, we have $p = \frac{\lambda}{n}$

$$\begin{aligned}
\binom{n}{x} p^x (1-p)^{n-x} &= \binom{n}{x} (\frac{\lambda}{n})^x (1 - \frac{\lambda}{n})^{n-x} \\
&= \frac{n!}{x!(n-x)!} (\frac{\lambda}{n})^x (1 - \frac{\lambda}{n})^{n-x} \\
&= \frac{\lambda^x}{x!} \frac{n!}{(n-x)!} \frac{1}{n^x} (1 - \frac{\lambda}{n})^{n-x}
\end{aligned}$$

$$\begin{aligned}
\because \frac{n!}{(n-x)!} \frac{1}{n^x} &= \frac{n(n-1)(n-2)\ldots(n-x+1)(n-x)!}{(n-x)!n^x} \\
&= \frac{n(n-1)(n-2)\ldots(n-x+1)}{n^x} \\
&= \frac{n}{n} \frac{n-1}{n} \frac{n-2}{n} \ldots \frac{n-x+1}{n} \\
&= 1(1 - \frac{1}{n})(1 - \frac{2}{n}) \ldots (1 - \frac{x+1}{n})
\end{aligned}$$

$$\begin{aligned}
\therefore \binom{n}{x} p^x (1-p)^{n-x} &= \frac{\lambda^x}{x!} (1 - \frac{1}{n})(1 - \frac{2}{n}) \ldots (1 - \frac{x+1}{n})(1 - \frac{\lambda}{n})^{n-x} \\
&= \frac{\lambda^x}{x!} (1 - \frac{1}{n})(1 - \frac{2}{n}) \ldots (1 - \frac{x+1}{n})(1 - \frac{\lambda}{n})^{n}(1 - \frac{\lambda}{n})^{-x}
\end{aligned}$$

$$\therefore \lim_{n\to\infty} \binom{n}{x} p^x (1-p)^{n-x} = \frac{\lambda^x}{x!} \lim_{n\to\infty} [(1 - \frac{1}{n}) \ldots (1 - \frac{x+1}{n})] \lim_{n\to\infty} (1 - \frac{\lambda}{n})^n \lim_{n\to\infty} (1 - \frac{\lambda}{n})^{-x}$$

$$\because \lim_{n\to\infty} (1 - \frac{\lambda}{n})^n = e^{-\lambda}$$

$$\therefore \lim_{n\to\infty} \binom{n}{x} p^x (1-p)^{n-x} = \frac{\lambda^x}{x!} \times 1 \times e^{-\lambda} \times 1 = \frac{\lambda^x}{x!} e^{-\lambda} \text{ with } \lambda = np$$

Problem 5 (Marginal/Joint Distributions)

(a) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{\infty} 10e^{-(2x+5y)}\,dy = 10\,[-\frac{1}{5}e^{-(2x+5y)}]_0^{\infty} = 2e^{-2x}$, for $x \geq 0$

$\therefore f_X(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

(b) $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{\infty} 10e^{-(2x+5y)}\,dx = 10\,[-\frac{1}{2}e^{-(2x+5y)}]_0^{\infty} = 5e^{-5y}$, for $y \geq 0$

$\therefore f_Y(y) = \begin{cases} 5e^{-5y}, & y \geq 0 \\ 0, & \text{otherwise} \end{cases}$

(c) X and Y are independent.

$\because f_X(x)f_Y(y) = \begin{cases} 10e^{-(2x+5y)}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise} \end{cases} = f_{X,Y}(x,y)$

$\therefore f_X(x)f_Y(y) = f_{X,Y}(x,y), -\infty < x < \infty, -\infty < y < \infty$

$\therefore$ X and Y are independent.

(d) $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$

$\because f_{X,Y}(x,y) = \begin{cases} 10e^{-(2x+5y)}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$

$\because f_X(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

$\therefore f_{Y|X}(y|x) = \begin{cases} 5e^{-5y}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$

(e)
$$\begin{aligned} P\{Y > X\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{y} f_{X,Y}\,(x,y)\,dxdy \\ &= \int_0^{\infty}\int_0^{y} 10e^{-(2x+5y)}\,dxdy = \int_0^{\infty} [-5e^{-(2x+5y)}]_0^{y}\,dy \\ &= \int_0^{\infty} -5e^{-7y} + 5e^{-5y}\,dy = \lim_{t\to\infty}[\frac{5}{7}e^{-7y} - e^{-5y}]_0^{t} \\ &= -\frac{5}{7} + 1 = \frac{2}{7} \end{aligned}$$

Problem 7 (Covariance and Correlation Coeffcient)

(a)(i) we know that:

$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y)$

$\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$

$\mathrm{Cov}(a_1X+b_1, a_2Y+b_2) = a_1a_2\mathrm{Cov}(X,Y)$

$\therefore \mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y) = 7$

$$\begin{aligned}\mathrm{Var}(2X-2Y) &= \mathrm{Var}(2X) + \mathrm{Var}(-2Y) + 2\mathrm{Cov}(2X,-2Y)\\ &= 4\mathrm{Var}(X) + 4\mathrm{Var}(Y) + 2\mathrm{Cov}(2X,-2Y)\end{aligned}$$

$\because \mathrm{Cov}(2X,-2Y) = -4\mathrm{Cov}(X,Y)$

$\therefore \mathrm{Var}(2X-2Y) = 4\mathrm{Var}(X) + 4\mathrm{Var}(Y) - 8\mathrm{Cov}(X,Y) = 12$

$\therefore \mathrm{Cov}(X,Y) = \frac{7\times4-12}{16} = 1$

(ii) $\because \mathrm{Cov}(X,Y) = 1$ and $\mathrm{Var}(X) = 1$

$\therefore \mathrm{Var}(Y) = 4$

$\therefore \rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = \frac{1}{2}$

(b) $\because X_1, X_2, \ldots, X_{10}$ are uncorrelated

$\therefore \mathrm{Cov}(X_i, X_j) = 0,\ i \neq j$

$\because S_{10} = X_1 + X_2 + \ldots + X_{10}$

$\therefore \mathrm{Var}(\frac{S_{10}}{\sqrt{10}}) = \mathrm{Var}(\frac{X_1}{\sqrt{10}}) + \mathrm{Var}(\frac{X_2}{\sqrt{10}}) + \ldots + \mathrm{Var}(\frac{X_{10}}{\sqrt{10}}) = \frac{5}{10}\times 10 = 5$

Problem 9 (Central Limit Theorem)

(a) The distribution of X should be a binomial distribution.

$\therefore P\{X=x\} = \binom{400}{x}0.98^x(1-0.98)^{400-x} = \binom{400}{x}0.98^x0.02^{400-x},\ x \in \mathbb{N}$ and $x \leq 400$

(b) $\because min\{400\times(0.98),\ 400\times(1-0.98)\} = 8 > 5$

$\therefore$ Central Limit Theorem applies to binomial distribution in this case.

$\because \mu_X = np = 400\times0.98 = 392$ and $\sigma_X = \sqrt{np(1-p)} = \sqrt{400\times0.98\times(1-0.98)} = 2.8$

$\therefore P\{X \geq 390\} \approx P(z \geq \frac{390-392}{2.8}) = P(z \geq -0.714) = 0.7611,$ according to z-table

Problem 10 (Bayes Theorem and Conditional Probabilities)

Define the following events:

A: autonomous vehicles have malfunctions

B: disengagement happens

$\therefore P(B|A) = 0.85,\ P(B|\bar{A}) = 0.002,\ P(A) = 0.0002$

$\therefore P(\bar{A}) = 1 - P(A) = 0.9998$

$\therefore P(B) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A}) \approx 0.00217$

$\therefore P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|\bar{A})P(\bar{A})} = \frac{0.85\times0.0002}{0.85\times0.0002+0.002\times0.9998} \approx 0.0784$

$\therefore$ The probability that a given disengagement is due to a malfunction is about 0.0784.

Problem 12 (Uniform Distribution)

(a) $\therefore f_X(x) = \begin{cases} \frac{1}{10}, & 0 \le x \le 10 \\ 0, & \text{otherwise} \end{cases}$

$f_Y(y) = \begin{cases} \frac{1}{10}, & 0 \le y \le 10 \\ 0, & \text{otherwise} \end{cases}$

$\because$ X and Y are independent to each other

$\therefore f_{X,Y}(x,y) = f_X(x)f_Y(y) = \begin{cases} \frac{1}{100}, & 0 \le x \le 10,\ 0 \le y \le 10 \\ 0, & \text{otherwise} \end{cases}$

$\because P\{X = Y\} = \int_y^y \frac{1}{10}\,dx = 0$

$\therefore$ The probability that bus A and bus B arrive at exactly the same time is 0.

(b) $\therefore P\{X < Y\} = \int_0^{10} \int_0^y \frac{1}{100}\,dxdy = \int_0^{10} [\frac{x}{100}]_0^y\,dy$

$= \int_0^{10} \frac{y}{100}\,dy = [\frac{y^2}{200}]_0^{10} = \frac{1}{2}$

$\therefore$ The probability that bus A arrives earlier than bus B is $\frac{1}{2}$.

(c) $P(Z < z) = P(x < z, y < z)$

$\because$ X and Y are independent of each other.

$\therefore P(Z < z) = P(x < z)P(y < z)$

$= \int_0^z \frac{1}{10}\,dx \times \int_0^z \frac{1}{10}\,dy$

$= \begin{cases} \frac{z^2}{100}, & 0 \le z \le 10 \\ 0, & \text{otherwise} \end{cases}$

$\therefore f_Z(z) = \frac{d}{dz}P(Z < z) = \begin{cases} \frac{z}{50}, & 0 \le z \le 10 \\ 0, & \text{otherwise} \end{cases}$

$\therefore E[Z] = \int_{-\infty}^{\infty} z \cdot \frac{z}{50}\,dz = \int_0^{10} \frac{z^2}{50}\,dz$

$= [\frac{z^3}{150}]_0^{10} = \frac{1000}{150}$ min $= 6.667$ min

(d) $P(w < W) = P(w < x, w < y)$

$\because$ X and Y are independent of each other.

$\therefore P(w < W) = P(w < x)P(w < y)$

$= \int_w^{10} \frac{1}{10}\,dx \times \int_w^{10} \frac{1}{10}\,dy$

$= \begin{cases} 1 - \frac{w}{5} + \frac{w^2}{100}, & 0 \le w \le 10 \\ 0, & \text{otherwise} \end{cases}$

$\therefore P(W < w) = 1 - P(w < W) = \begin{cases} \frac{w}{5} - \frac{w^2}{100}, & 0 \le w \le 10 \\ 0, & \text{otherwise} \end{cases}$

$\therefore f_W(w) = \frac{d}{dw}P(W < w) = \begin{cases} \frac{1}{5} - \frac{w}{50}, & 0 \le w \le 10 \\ 0, & \text{otherwise} \end{cases}$

$\therefore E[W] = \int_{-\infty}^{\infty} w \cdot (\frac{1}{5} - \frac{w}{50})\,dw = \int_0^{10} \frac{w}{5} - \frac{w^2}{50}\,dw$

$= [\frac{w^2}{10} - \frac{w^3}{150}]_0^{10} = \frac{10}{3}$ min $= 3.333$ min

(e) Define T to be the event that bus A and bus B are together at the bus stop.

(i) A arrives first and $x \in [0, 5]$: $P(T_i) = \int_0^5 \int_x^{x+5} \frac{1}{100}\,dydx = \int_0^5 \frac{5}{100}\,dx = \frac{1}{4}$

(ii) A arrives first and $x \in (5, 10]$: $P(T_{ii}) = \int_5^1 0 \int_x^{10} \frac{1}{100}\, dydx = \int_5^{10} \frac{1}{10} - \frac{x}{100}\, dx = \frac{1}{8}$

(iii) B arrives first and $y \in [0, 5]$: $P(T_{iii}) = \int_0^5 \int_y^{y+5} \frac{1}{100}\, dxdy = \int_0^5 \frac{5}{100}\, dy = \frac{1}{4}$

(iv) B arrives first and $y \in (5, 10]$: $P(T_{iv}) = \int_5^1 0 \int_y^{10} \frac{1}{100}\, dxdy = \int_5^{10} \frac{1}{10} - \frac{y}{100}\, dy = \frac{1}{8}$

$\therefore P(T) = \frac{1}{4} + \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{3}{4}$