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定理1.4.6　设 $[a_1, a_2] = D_2$，$[D_2, a_3] = D_3$，…，$[D_{n-1}, a_n] = D_n$.

证：$[a_1, \dots, a_n] = D$.

设 $D = [a_1, \dots, a_n]$.

当 $n=2$ 时，$D = D_2$.

设 $n-1$ 时成立，即 $[a_1, \dots, a_{n-1}] = D_{n-1}$

当 n 时，$D_n = [D_{n-1}, a_n]$

$\Rightarrow a_n \mid D_n$，$D_{n-1} \mid D_n \Rightarrow a_i \mid D_n \Rightarrow D_n \geq D$.

由 $a_1 \mid D, \dots, a_n \mid D \Rightarrow a_1 \mid D, \dots, a_{n-1} \mid D \Rightarrow [a_1, \dots, a_{n-1}] \mid D \Rightarrow D_{n-1} \mid D$.

由 $D_{n-1} \mid D$，$a_n \mid D \Rightarrow [D_{n-1}, a_n] \mid D \Rightarrow D_n \mid D \Rightarrow D_n \leq D$

$\Rightarrow D_n = D$.

证：$\sqrt{7}$ 不是有理数.

反证：设 $\sqrt{7}$ 有理，$\sqrt{7} = \frac{p}{q}$，$(p,q)=1$. $\Rightarrow p^2 = 7q^2$

$\Rightarrow$ ~~$q^2 \mid p^2$ 设 $q = p_1^{\alpha_1} \cdots p_s^{\alpha_s} \Rightarrow p^2 = 7 p_1^{2\alpha_1} \cdots p_s^{2\alpha_s}$~~.

$\Rightarrow$ 即 $7 \mid p^2 \Rightarrow$ 则 $7 \mid p$　设 $p = 7m$，则 $q^2 = \frac{p^2}{7} = 7m$

$\Rightarrow 7 \mid q^2 \Rightarrow 7 \mid q \Rightarrow (p,q) \neq 1$，矛盾

$\Rightarrow \sqrt{7}$ 无理.

$\sqrt{17}$ 无理：

设 $\sqrt{17}$ 有理，$\sqrt{17} = \frac{p}{q}$，$(p,q)=1$　$\Rightarrow p^2 = 17q^2$.

$\Rightarrow p^2 \equiv 0 \pmod{17} \Rightarrow 17 \mid p^2$.

若 $17 \nmid p$，则 $17 \nmid p^2 \Rightarrow 17 \mid p \Rightarrow p = 17m$.

$\Rightarrow q^2 = \frac{p^2}{17} = 17m \Rightarrow 17 \mid q^2 \Rightarrow 17 \mid q \Rightarrow (p,q) \neq 17$

$\Rightarrow \sqrt{17}$ 无理.

$ax+by=c$ 有解 $\Longleftrightarrow (a,b)\mid c$.

"$\Rightarrow$" 当 $ax+by=c$ 有解，设 $ax_0+by_0=c$. 设 $d=(a,b)$

$d\mid a,\ d\mid b \Rightarrow d\mid(ax_0+by_0) \Rightarrow d\mid c$.

"$\Leftarrow$" 当 $(a,b)\mid c$，即 $d\mid c$，设 $c=kd$.

由 Bezout：$\exists s,t$： $sa+tb=d \Rightarrow (sk)a+(tk)b=kd=c$.

令 $x=ks,\ y=kt$，即 $ax+by=c$.

$$\begin{cases} ax+by=c \\ ax_0+by_0=c \end{cases} \Rightarrow a(x-x_0)+b(y-y_0)=0 \Rightarrow a\mid b(y-y_0)$$

$$\Rightarrow \frac{a}{(a,b)}\Big|\frac{b}{(a,b)}(y-y_0) \quad 由\left(\frac{a}{(a,b)},\frac{b}{(a,b)}\right)=1 \Rightarrow \frac{a}{(a,b)}\Big|\ y-y_0$$

$$\Rightarrow y-y_0=t\cdot\frac{a}{(a,b)} \Rightarrow y=y_0+t\frac{a}{(a,b)}.$$

代入得，$x=x_0-t\dfrac{b}{(a,b)}$

$7x+4y=100$.

$(a,b)=(7,4)=1\mid 100$.

由 Bezout 等式：$7\times(-1)+4\times(2)=1 \Rightarrow 7\times(-100)+4\times(200)=100$.

$$\begin{cases} x_0=-100 \\ y_0=200 \end{cases} \Rightarrow \begin{cases} x=-100-4t \\ y=200+7t \end{cases}$$

定理 1.6.3. $n=p_1^{\alpha_1}\cdots p_s^{\alpha_s}$，$d$ 是 n 的正因数 $\Longleftrightarrow d=p_1^{\beta_1}\cdots p_s^{\beta_s}$，$0\le\beta_i\le\alpha_i$.

当 $d\mid n$，$d=\prod_{i=1}^{s}p_i^{\beta_i}$. 若 $\exists\beta_j>\alpha_j$，由 $d\mid n$，$p_j^{\beta_j}\mid d$.

$\Rightarrow p_j^{\beta_j}\mid n \Rightarrow p_j^{\beta_j-\alpha_j}\mid p_1^{\alpha_1}\cdots p_{j-1}^{\alpha_{j-1}}p_{j+1}^{\alpha_{j+1}}\cdots p_s^{\alpha_s}$

由于 p_i,p_j 两两互素，上式不可能 $\Rightarrow \beta_j\le\alpha_j$.

若 $d=p_1^{\beta_1}\cdots p_s^{\beta_s}$ 设 $k=\prod_{i=1}^{s}p_i^{\alpha_i-\beta_i}$ 则 $kd=n$.

$\Rightarrow d\mid n$

证毕.

定理 1.6.5.

$d=\prod_{i=1}^{s} p_i^{\gamma_i}$，$\gamma_i=\min(\alpha_{i1},\dots \alpha_{ik})$. 易知 d 满足 gcd 的定义.

同理，$D=\prod_{i=1}^{s} p_i^{\delta_i}$，$\delta_i=\max(\alpha_{i1},\dots,\alpha_{ik})$，满足 lcm 的定义.

定理 1.6.6 设 $a,b\in\mathbb{N}$，则 $\exists\ a',b'\in\mathbb{N}$，$a'|a$，$b'|b$，使 $a'\cdot b'=[a,b]$，$(a',b')=1$.

设 $a=p_1^{\alpha_1}\cdots p_t^{\alpha_t}\,p_{t+1}^{\alpha_{t+1}}\cdots p_s^{\alpha_s}$

$b=p_1^{\beta_1}\cdots p_t^{\beta_t}\,p_{t+1}^{\beta_{t+1}}\cdots p_s^{\beta_s}$

其中 $\alpha_i\geq\beta_i\ (1\leq i\leq t)$，$\alpha_j\leq\beta_j\ (t+1\leq j\leq s)$.

令 $a'=p_1^{\alpha_1}\cdots p_t^{\alpha_t}$，$b'=p_{t+1}^{\beta_{t+1}}\cdots p_s^{\beta_s}$

则 $a'|a$，$b'|b$，$(a',b')=1$，$a'b'=\prod_{i=1}^{s} p_i^{\delta_i}$，$\delta_i=\max(\alpha_i,\beta_i)$

$\Rightarrow a'b'=[a,b]$.