Bayesian Statistics

Conditioning

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Independence, Conditional Probability with Example Queen of Spades

Before We Begin...

Independence, Conditional Probability

- A, B independent $\Leftrightarrow P(AB) = P(A)P(B)$
- $P(A|B) \stackrel{\text{def}}{=} \frac{P(AB)}{P(B)}$

  $\Rightarrow P(AB) = P(A|B)P(B)$

  by symmetry

  $P(AB) = P(B|A)P(A)$
- A, B independent

  $P(A|B) = P(A)$, or

  $P(B|A) = P(B)$

Queen of Spades

- Deck of 52 cards
- 13 spades, 4 Queens
- One card selected at random
- A – card is spade, B – card is Q

Independent?

AB = Queen of spades, $P(AB) = \frac{1}{52}$

$P(A) = \frac{13}{52}, \quad P(B) = \frac{4}{52}$

$\frac{1}{52} = P(AB) = P(A)P(B) = \frac{13}{52} \times \frac{4}{52} = \frac{52}{52 \times 52} = \frac{1}{52}$ Independent!

- Remove 2 diamond → deck 51 cards

$P(AB) = \frac{1}{51}, P(A) = \frac{13}{51}, P(B) = \frac{4}{51}$

$P(AB) \neq P(A)P(B)$, A, B dependent!

Summary

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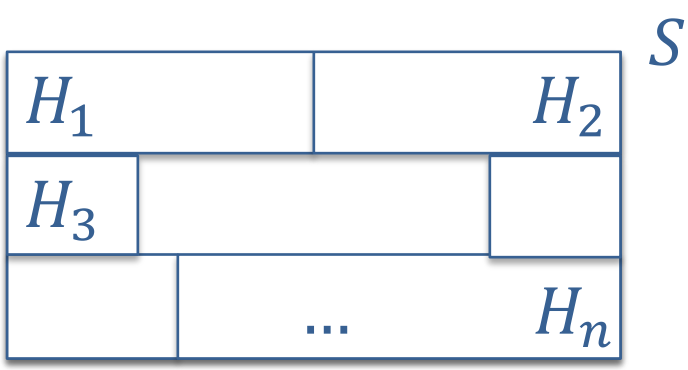
Hypotheses, Total Probability
with Examples of Manufacturing
Bayes & Bridged Circuit

Before We Begin...

Hypotheses and Total Probability



$$S = H_1 \cup H_2 \cup \cdots \cup H_n$$

$$\underbrace{H_i H_j = \emptyset}_{\text{exclusive}}, i \neq j$$

$$A = AS = A(H_1 \cup H_2 \cup \cdots \cup H_n)$$
$$= AH_1 \cup AH_2 \cup \cdots \cup AH_n$$

$$P(A) = P(AH_1) + P(AH_2) + \cdots + P(AH_n)$$

(since $AH_1, \ldots, AH_n$ are exclusive)

$$= P(A|H_1)P(H_1) + \cdots + P(A|H_n)P(H_n)$$

(as probabilities of intersections)

Total Probability

$$P(A) = \sum_{i=1}^{n} P(A|H_i)P(H_i)$$

Example: “Manufacturing Bayes”

Type Machine	Prob. Item Conforming	Production Volume
1	0.94	30%
2	0.95	50%
3	0.97	20%

One item is randomly selected from the production. What is the probability that the item is conforming?

H_i: item is from i^{th} machine

Manufacturing Bayes

$$H_1 \cup H_2 \cup H_3 = S, \quad H_i H_j = \emptyset$$

$$P(H_1) = 0.3, P(H_2) = 0.5, P(H_3) = 0.2$$

Check: $\sum P(H_i) \equiv 1$

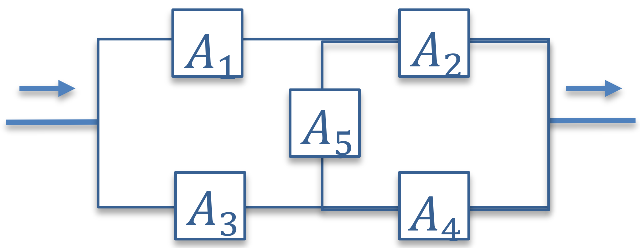
$$A - \text{item is conforming}$$

$$P(A|H_1) = 0.94, P(A|H_2) = 0.95, P(A|H_3) = 0.97$$

By Total Probability:

$$\begin{aligned} P(A) &= P(A|H_1)P(H_1) + P(A|H_2)P(H_2) + P(A|H_3)P(H_3) \\ &= 0.94\times 0.3 + 0.95\times 0.5 + 0.97\times 0.2 \\ &= \boxed{0.957} \end{aligned}$$

Bridged Circuit



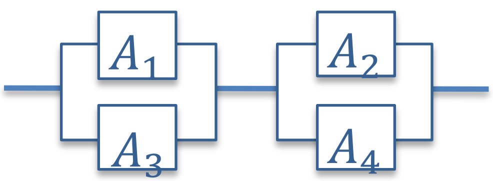
Element	A_1	A_2	A_3	A_4	A_5
Works	0.9	0.8	0.7	0.4	0.6
Fails	0.1	0.2	0.3	0.6	0.4

S – circuit works

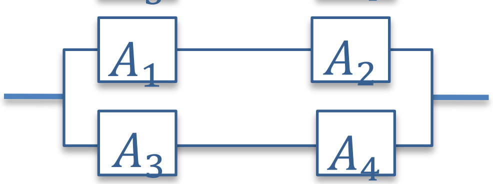
$H_1: A_5$ - works; $H_2: A_5$ - fails

H_1, H_2 are hypotheses

H_1: S



H_2: S



$$
\begin{aligned}
P(S|H_1) &= (1 - q_1 q_3)\,(1 - q_2 q_4) \\
&= (1 - 0.1\times0.3)\,(1 - 0.2\times0.6) \\
&= 0.97\times0.88 = \boxed{0.8536}
\end{aligned}
$$

$$
\begin{aligned}
P(S|H_2) &= 1 - (1 - p_1 p_2)\,(1 - p_3 p_4) \\
&= 1 - (1 - 0.9\times0.8)\,(1 - 0.7\times0.4) \\
&= 1 - 0.28\times0.72 = \boxed{0.7984}
\end{aligned}
$$

Bridged Circuit

$$P(S) = P(S|H_1)\times P(H_1) + P(S|H_2)\times P(H_2)$$

$$= 0.8536\times 0.6 + 0.7984\times 0.4$$

$$= \boxed{0.8315}$$

Summary