THE WORST-CASE SENSITIVITY OF OPTIMAL POWER FLOW OPERATOR IN NETWORKED SYSTEMS*

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Abstract. This paper studies that in networked power systems, how the optimal generation induced by the direct current optimal power flow (DC OPF) problem, an linear programming problem, fluctuates as the load changes. Such affect is characterized by the metric of sensitivity of the OPF operator, and the main focus of this this paper is to compute the maximal value of the sensitivity over all the possible combinations of the system parameters. Such optimization in general is a non-convex combinatorial optimization problem and hence difficult to solve. However, under three types of network topologies (tree, uni-cyclic graph and the graph with multiple non-adjacent cycles), the structural properties of the problem make it possible to efficiently calculate the worst-case sensitivity. Our results also provide insights for related applications such as data privacy in power systems, real-time OPF computation, and locational marginal price.

Key words. combinatorial optimization, optimal power flow, sensitivity

AMS subject classifications. XXX

1. Introduction.

Optimal Power Flow, a mathematical program that minimizes some loss function subject to physical laws and other constraints [20], is widely studied and used in power systems, as it plays an important role in the system operation, as well as applications like economic dispatch, environmental dispatch, unit commitment, etc [17, 25]. In this work, we discovered that the changes in power loads can affect the changes in the optimal generations induced by the OPF problem to various degrees among different systems. Though there have been a large amount of literatures [15, 32, 22] studying the OPF sensitivity at certain operation point, to the best of the authors' knowledge, few existing results can be directly applied to quantify the worst-case sensitivity over all the different possible combinations of system parameters and to characterize its relationship to the network topology.

On the other hand, it is noticed that in some recent applications motivated by the wave of big data and cyber physical systems, it is beneficial to make the system less sensitive under all the possible parameters. Three typical applications are provided in subsection 3.2. It is desired to estimate the sensitivity in the worst case and analyze how the power network topology can affect it.

In an abstract and informal way, we consider the DC OPF problem with linear cost function taking the following form:

$$\underset{\mathbf{s}^g}{\text{minimize}} \, \mathbf{f}^\top \mathbf{s}^g \tag{1.1a}$$

$$\text{subject to } \mathbf{A}_{\text{eq}}\mathbf{s}^g = \mathbf{b}_{\text{eq}}(\mathbf{s}^l, \mathbf{b}') \tag{1.1b}$$

$$\mathbf{A}_{\text{in}}\mathbf{s}^g \le \mathbf{b}_{\text{in}} \tag{1.1c}$$

where $\mathbf{b}_{\text{eq}}$ is linear in $\mathbf{s}^l$ and $\mathbf{b}'$. We view $\mathbf{f}$, $\mathbf{b}'$ and $\mathbf{b}_{\text{in}}$ as system parameters, whose values are allowed to change in a set Ω, and $\mathbf{s}^l$ is allowed to change in set $\Omega_{\mathbf{s}^l}$. For now, let us simply consider the concept of the sensitivity of $\mathbf{s}^g$ (the generation vector)

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with respect to $\mathbf{s}^l$ (the load vector) as the Jacobean matrix $\mathbf{J}_{\mathbf{s}^g}(\mathbf{s}^l) = \partial \mathbf{s}^g / \partial \mathbf{s}^l$. Later in section 3, we will give rigorous definitions and guarantee that the optimal solution is unique and differentiable in $\mathbf{s}^l$. Finding the worst-case sensitivity is essentially to solve the optimization problem

$$\underset{(\mathbf{f},\mathbf{b}',\mathbf{b}_{\mathrm{in}})\in\Omega, \mathbf{s}^l\in\Omega_{\mathbf{s}^l}}{\text{maximize}} \left| \frac{\partial \mathbf{s}_i^g}{\partial \mathbf{s}_j^l} \right| \tag{1.2}$$

for some (i, j) pair. Later we will see that (1.2) is equivalent to a combinatorial optimization problem.

There has been considerable research works defining or studying the OPF sensitivity [15, 32, 22]. In [15, 32], the OPF problem is formulated as a parameterized optimization problem where the loads, the upper and lower bounds for generations and branch power flows are all parameterized in ϵ. At the point $\epsilon = 0$, the sensitivity, defined as the derivative of the optimal solution with respect to ϵ, can be computed from the KKT conditions, assuming it is known that which constraints are binding and the values of the optimal solution and Lagrangian multipliers are available at $\epsilon = 0$. Numerical methods are also carried out by those papers. However, in those literatures, the set of binding constraints are always different at different points. As the result, the worst-case sensitivity cannot be efficiently computed as the choice of the binding constraints sets can be exponentially many. In [22], it studied the sensitivity with respect to discrete parameters such as transformer taps. It applies the interior point method proposed in [30], which introduces barrier functions to bypass the unavailability of binding constraints sets. Due to the complexed expressions of the sensitivity in those literatures, it is very difficult to figure out at which operation point the system will be in its worst case in the sense of monotonicity, and we cannot analyze how the network topology and the locations of binding constraints affect the system sensitivity.

Another implication of the high sensitive system is that increasing a load might lead to a significant decrease of some generators while some others increase back. Such counter-intuitive phenomena might cause the negative locational marginal price (LMP), and the latter is a well-known result in the community [19, 4, 24]. The properties of the worst-case sensitivity might provide an alternative perspective to view LMP and provide a way to predict the range of LMP. Interestingly, there are also many other similar and related counter-intuitive phenomena in power networks and other networked systems. In [8, 3], the Kirchhoff-Braess paradox in power networks is studied, where the addition of a conductive line might potentially worsen the power congestion, In [28], they discovered that in the networks controlled by Transmission Control Protocol (TCP), increasing the capacity might negatively affect the aggregate throughout. Inspired by those works, we want to establish the analysis of worst-case sensitivity in this paper and provide insights to guide the design of future power networks.

In the following sections, we start with the derivative of the OPF operator, and obtain its expression as a function of the set of binding constraints and system parameters. Then we focus on three types of network topologies: tree, unicyclic graph and the graph with multiple non-adjacent cycles. Polynomial-time algorithms are provided for those three types of networks based on two structural properties. One is that a proposed metric of unbalance is conserved within each cycle and can help characterize the sensitivity across each cycle. The other is the fact that a complicated graph with multiple non-adjacent cycles can be broken down into smaller unicyclic subgraphs. Each subgraph behaves as one multiplier of the total sensitivity and can thereby simplify the analysis.

2. Background. In this section we define the power network model and the optimal power flow problem we consider in this paper. We introduce some assumptions on allowable parameter sets and show that these assumptions are mild.

Notation. Vectors and matrices are typically written in bold while scalars are not. Given two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $\mathbf{a} \geq \mathbf{b}$ denotes the element-wise partial order $\mathbf{a}_i \geq \mathbf{b}_i$ for $i = 1, \ldots, n$. For a scalar k, we define the projection operator $[k]^- := \min\{0, k\}$. We define $\|\mathbf{x}\|_0$ as the number of non-zero elements of the vector $\mathbf{x}$. Identity and zero matrices are denoted by $\mathbf{I}^n$ and $\mathbf{0}^{n\times m}$ while vectors of all ones are denoted by $\mathbf{1}_n$ where superscripts and subscripts indicate their dimensions. To streamline notation, we omit the dimensions when the context makes it clear. The notation $\mathbb{R}_+$ denotes the nonnegative real set $[0, +\infty)$. For $\mathbf{X} \in \mathbb{R}^{n\times m}$, the restriction $\mathbf{X}_{\{1,3,5\}}$ denotes the $3 \times m$ matrix composed of stacking rows $1, 3$, and 5 on top of each other. We will frequently use a set to describe the rows we wish to form the restriction from, in this case we assume the elements of the set are arranged in increasing order. We will use $\mathbf{e}_m$ to denote the standard base for the m^{th} coordinate, its dimension will be clear from the context. Let $(\cdot)^\dagger$ be the Moore-Penrose inverse. Finally, let $[m] := \{1, 2, \ldots, m\}$ and $[n, m] := \{n, n + 1, \ldots, m\}$.

2.1. System Model. Consider a power network modeled by an undirected connected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} := \mathcal{V}_\mathrm{G} \cup \mathcal{V}_\mathrm{L}$ denotes the set of buses which can be further classified into generators in set $\mathcal{V}_\mathrm{G}$ and loads in set $\mathcal{V}_\mathrm{L}$, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of all branches linking those buses. We will later use the terms (graph, vertex, edge) and (power network, bus, branch) interchangeably. Suppose $\mathcal{V}_\mathrm{G} \cap \mathcal{V}_\mathrm{L} = \emptyset$ and there are $|\mathcal{V}_\mathrm{G}| =: N_\mathrm{G}$ generator and $|\mathcal{V}_\mathrm{L}| =: N_\mathrm{L}$ loads, respectively. For simplicity, let $\mathcal{V}_\mathrm{G} = [N_\mathrm{G}]$, $\mathcal{V}_\mathrm{L} = [N_\mathrm{G} + 1, N_\mathrm{G} + N_\mathrm{L}]$. Let $N = N_\mathrm{G} + N_\mathrm{L}$. Without loss of generality, $\mathcal{G}$ is a connected graph with $|\mathcal{E}| =: E$ edges labelled as $1, 2, \ldots, E$. Let $\mathbf{C} \in \mathbb{R}^{N\times E}$ be the signed incidence matrix. Suppose edge e connects vertices u and v $(u < v)$, then $\mathbf{C}_{u,e} = 1$, $\mathbf{C}_{v,e} = -1$, and all other entries in the e^{th} column of $\mathbf{C}$ are 0. We will us e, (u, v) or (v, u) to denote the same edge. Let $\mathbf{B} = \operatorname{diag}(b_1, b_2, \ldots, b_E)$, where $b_e > 0$ is the susceptance for branch e. As we adopt a DC power flow model, all branches are assumed lossless. Further, we denote the generation and load as $\mathbf{s}^g \in \mathbb{R}^{N_\mathrm{G}}$, $\mathbf{s}^l \in \mathbb{R}^{N_\mathrm{L}}$, respectively. Thus $\mathbf{s}^g_i$ refers to the generation on bus i while $\mathbf{s}^l_i$ refers to the load on bus $N_\mathrm{G} + i$. We will refer to bus $N_\mathrm{G} + i$ simply as load i for simplicity. The power flow on branch $e \in \mathcal{E}$ is denoted as $\mathbf{p}_e$, and $\mathbf{p} := [\mathbf{p}_1, \ldots, \mathbf{p}_E]^\mathsf{T} \in \mathbb{R}^E$ is the vector of all branch power flows. The following assumption is made to simplify the analysis.

Assumption 1. *There are no buses in the network that are both loads and generators. Further more all generator busses have degree* 1. *Formally,* $\mathcal{V}_\mathrm{G} \cap \mathcal{V}_\mathrm{L} = \emptyset$ *and* $\max_i \deg(i) = 1$.

The above assumption is not restrictive. We can always split a bus with both a generator and a load into a bus with only the generator adjacent to another bus with only the load, and connect all the neighbors of the original bus to that load bus, as shown in Figure 1.

2.2. Optimal Power Flow. We focus on the DC OPF problem with a linear cost function [31]. That is to say, the voltage magnitudes are assumed to be fixed and known. Without loss of generality, we assume all the voltage magnitudes to be 1. The decision variables are the voltage angles denoted by vector $\boldsymbol{\theta} \in \mathbb{R}^N$ and power

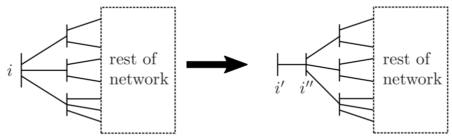

FIG. 1. *Suppose bus i is a generator with load, and is adjacent to 3 other buses, then we can create a new network where i is replaced by i', a pure generator, and i'', a pure load. The original neighbors of i are adjacent to i'' in the new network. The susceptance for branch (i', i'') can be assigned with any positive value. Two networks above are equivalent if we view i' and i'' as a whole.*

generations $\mathbf{s}^g$, given loads $\mathbf{s}^l$. The DC OPF takes the form:

$$\begin{aligned}
&(2.1a) && \underset{\mathbf{s}^g,\boldsymbol{\theta}}{\text{minimize}} \mathbf{f}^\top \mathbf{s}^g \\
&(2.1b) && \text{subject to } \boldsymbol{\theta}_1 = 0 \\
&(2.1c) && \mathbf{C}\mathbf{B}\mathbf{C}^\top \boldsymbol{\theta} = \begin{bmatrix} \mathbf{s}^g \\ -\mathbf{s}^l \end{bmatrix} \\
&(2.1d) && \underline{\mathbf{s}}^g \le \mathbf{s}^g \le \overline{\mathbf{s}}^g \\
&(2.1e) && \underline{\mathbf{p}} \le \mathbf{B}\mathbf{C}^\top \boldsymbol{\theta} \le \overline{\mathbf{p}}.
\end{aligned}$$

Here, $\mathbf{f} \in \mathbb{R}_+^{N_{\mathrm{G}}}$ is the unit cost for each generator, and bus 1 is selected as the slack bus with fixed voltage angle 0. In (2.1c), we let the injections for generators be positive while the injections for loads be the negation of $\mathbf{s}^l$. The upper and lower limits for the generation are set as $\overline{\mathbf{s}}^g$ and $\underline{\mathbf{s}}^g$, respectively, and $\overline{\mathbf{p}}$ and $\underline{\mathbf{p}}$ are the limits for branch power flow. We assume that (2.1) is well posed, i.e. $\overline{\mathbf{s}}^g > \underline{\mathbf{s}}^g$, $\overline{\mathbf{p}} > \underline{\mathbf{p}}$.

Let $\boldsymbol{\tau} \in \mathbb{R}^{N+1}$ be the vector of Lagrangian multipliers associated with equality constraints (2.1b), (2.1c), and $(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_-)$ and $(\boldsymbol{\mu}_+, \boldsymbol{\mu}_-)$ be the Lagrangian multipliers associated with inequalities (2.1d) and (2.1e) respectively. As (2.1) is a linear program [6], the following KKT condition holds at an optimal point when (2.1) is feasible.

$$\begin{aligned}
&(2.2a) && (2.1b) - (2.1e) \\
&(2.2b) && \mathbf{0} = \mathbf{M}^\top \boldsymbol{\tau} + \mathbf{C}\mathbf{B}(\boldsymbol{\mu}_+ - \boldsymbol{\mu}_-) \\
&(2.2c) && -\mathbf{f} = -[\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \cdots, \boldsymbol{\tau}_{N_{\mathrm{G}}}]^\top + \boldsymbol{\lambda}_+ - \boldsymbol{\lambda}_- \\
&(2.2d) && \boldsymbol{\mu}_+, \boldsymbol{\mu}_-, \boldsymbol{\lambda}_+, \boldsymbol{\lambda}_- \ge 0 \\
&(2.2e) && \boldsymbol{\mu}_+^\top (\mathbf{B}\mathbf{C}^\top \boldsymbol{\theta} - \overline{\mathbf{p}}) = \boldsymbol{\mu}_-^\top (\underline{\mathbf{p}} - \mathbf{B}\mathbf{C}^\top \boldsymbol{\theta}) = 0 \\
&(2.2f) && \boldsymbol{\lambda}_+^\top (\mathbf{s}^g - \overline{\mathbf{s}}^g) = \boldsymbol{\lambda}_-^\top (\underline{\mathbf{s}}^g - \mathbf{s}^g) = 0,
\end{aligned}$$

where

$$\mathbf{M} := \begin{bmatrix} \mathbf{C}\mathbf{B}\mathbf{C}^\top \\ 1 \; 0 \; 0 \; \cdots \; 0 \end{bmatrix}$$

is an $(N+1)$-by-N matrix with rank N. Condition (2.2a) corresponds to primal feasibility, condition (2.2d) corresponds to dual feasibility, conditions (2.2e), (2.2f) correpond to complementary slackness, and conditions (2.2b), (2.2c) correspond to stationarity [9].

2.3. OPF Operator. Henceforth we fix the topology and susceptances of the power network. Let $\boldsymbol{\xi} := [(\overline{\mathbf{s}}^g)^\top, (\underline{\mathbf{s}}^g)^\top, \overline{\mathbf{p}}^\top, \underline{\mathbf{p}}^\top]^\top \in \mathbb{R}^{2N_{\mathrm{G}}+2E}$ be the vector of system limits, and let Ω_{para} be the set of all $\boldsymbol{\xi}$ such that:

- $\exists \mathbf{s}^l > 0$, such that (2.1b)-(2.1e) are feasible;
- $\underline{\mathbf{s}}^g \geq 0$.

PROPOSITION 2.1. *The set* Ω_{para} *satisfies* $\text{clos}(\text{int}(\Omega_{\text{para}})) = \text{clos}(\Omega_{\text{para}})$.

Proof. See Appendix A. ∎

For each $\boldsymbol{\xi} \in \Omega_{\text{para}}$, let $\Omega_{\mathbf{s}^l}(\boldsymbol{\xi})$ be the corresponding set of $\mathbf{s}^l$ such that

- (2.1b)-(2.1e) are feasible.
- $\mathbf{s}^l > 0$

Then $\Omega_{\mathbf{s}^l}$ is convex and nonempty. When we fix $\boldsymbol{\xi}$ and there is no confusion, we use $\Omega_{\mathbf{s}^l}$ instead. We now define the operator $\mathcal{OPF}$, which will be used throughout the rest of the paper.

DEFINITION 2.2. *Let the set valued operator* $\mathcal{OPF} : \Omega_{\mathbf{s}^l} \to 2^{\mathbb{R}^{N_\text{G}}}$ *be the mapping such that* $\mathcal{OPF}(\mathbf{x})$ *is the set of optimal solutions to* (2.1) *with parameter* $\mathbf{s}^l = \mathbf{x}$.

Now fix $\mathbf{B}, \mathbf{C}$ and $\boldsymbol{\xi}$, let $\Omega_{\mathbf{f}}$ be the set of $\mathbf{f} \geq 0$ such that $\forall \mathbf{s}^l \in \Omega_{\mathbf{s}^l}$,

- (2.1) has a unique solution;
- all solutions to (2.2) satisfy

$$\|\boldsymbol{\mu}_+\|_0 + \|\boldsymbol{\mu}_-\|_0 + \|\boldsymbol{\lambda}_+\|_0 + \|\boldsymbol{\lambda}_-\|_0 \geq N_\text{G} - 1. \tag{2.3}$$

PROPOSITION 2.3. $\Omega_{\mathbf{f}}$ *is dense in* $\mathbb{R}_+^{N_\text{G}}$.

Proof. See Appendix B. ∎

Proposition 2.3 shows that for a fixed network, it is easy to find an objective vector $\mathbf{f}$ such that (2.1) will always have a unique solution for a reasonable $\mathbf{s}^l$.

ASSUMPTION 2. *The objective vector* $\mathbf{f}$ *is in* $\Omega_{\mathbf{f}}$*, i.e,* $\mathbf{f}$ *always guarantees the uniqueness of the solution to* (2.1) *for all* $\mathbf{s}^l \in \Omega_{\mathbf{s}^l}$.

When Assumption 2 does not hold, Proposition 2.3 implies that we can always perturb $\mathbf{f}$ a little such that the assumption is valid.

Remark 2.4. Under Assumption 2, the value of $\mathcal{OPF}$ is always a singleton, so we can reload $\mathcal{OPF}(\mathbf{x})$ as the function mapping from $\mathbf{x}$ to the unique optimal solution of (2.1) with parameter $\mathbf{s}^l = \mathbf{x}$. [1] Since the solution set to the parametric linear program is both upper and lower hemi-continuous [33], $\mathcal{OPF}$ is continuous as well.

Later analysis is made easier if the set of binding (active) constraints is independent. Here, binding constraints refer to the set of equality constraints (2.1b), (2.1c) *and* those inequality constraints (2.1d), (2.1e) for which either the upper or lower-bounds are active. Grouping the coefficients of these constraints into a single matrix $\mathbf{Z}$ we refer to them as being independent if $\mathbf{Z}$ is full-rank. Let $\widetilde{\Omega}_{\mathbf{s}^l}(\boldsymbol{\xi})$ be the set

$$\{\mathbf{s}^l \in \Omega_{\mathbf{s}^l}(\boldsymbol{\xi}) \mid (2.1) \text{ has exactly } N_\text{G} - 1 \text{ binding inequalities}\}$$

Then we have the following proposition.

PROPOSITION 2.5. *There is a dense set* $\widetilde{\Omega}_{\text{para}} \subseteq \Omega_{\text{para}}$ *satisfying* $\forall \boldsymbol{\xi} \in \widetilde{\Omega}_{\text{para}}$*, we have*

- $\text{clos}(\text{int}(\Omega_{\mathbf{s}^l}(\boldsymbol{\xi}))) = \text{clos}(\Omega_{\mathbf{s}^l}(\boldsymbol{\xi}))$.
- $\widetilde{\Omega}_{\mathbf{s}^l}(\boldsymbol{\xi})$ *is dense in* $\Omega_{\mathbf{s}^l}(\boldsymbol{\xi})$.

[1] Except for Appendix D where $\mathcal{OPF}$ is still viewed as a set valued function, $\mathcal{OPF}$ will be viewed as a vector valued function throughout the paper by default.

Proof. See Appendix C. ∎

Finally, Assumption 3 restricts the value of limits in (2.1) in order to further simplify our analysis.

Assumption 3. *The parameter $\boldsymbol{\xi}$ for the limits of generations and branch power flows is assumed to be in $\widetilde{\Omega}_{\text{para}}$, as defined in Proposition 2.5.*

If Assumption 3 does not hold, Proposition 2.5 implies that we can always perturb $\boldsymbol{\xi}$ such that the assumption holds. The proof of Proposition 2.5 can directly extend to the following two corollaries. Proofs are omitted due to the limited space.

Corollary 2.6. *In Proposition 2.5, $\Omega_{\mathbf{s}^l} \setminus \widetilde{\Omega}_{\mathbf{s}^l}$ can be covered by the union of finitely many affine hyperplanes.*

Corollary 2.7. *For any $\mathbf{s}^l \in \widetilde{\Omega}_{\mathbf{s}^l}$, the $N_{\text{G}} - 1$ tight inequalities in (2.1), along with $N + 1$ equality constraints, are independent.*

3. OPF Sensitivity.

3.1. Definition. To study how the optimal generation reacts to the change in the load, we borrow the concept of sensitivity from [15, 32, 22]. We first define the OPF sensitivity at a fixed point, and then extend to the worst-case sensitivity to take system contingency into consideration.

Definition 3.1. *For fixed $\mathbf{C}$, $\mathbf{B}$, $\mathbf{f} \in \Omega_{\mathbf{f}}$ and $\boldsymbol{\xi} \in \widetilde{\Omega}_{\text{para}}$, consider a generator i and a load j. The pair (i, j) is said to be C-Lipschitz if $\forall \delta > 0$ and $\forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \Omega_{\mathbf{s}^l}$ such that $|\boldsymbol{\alpha}_j - \boldsymbol{\beta}_j| \le \delta$ and $\boldsymbol{\alpha}_k = \boldsymbol{\beta}_k (k \neq j)$, we have $|\mathcal{OPF}_i(\boldsymbol{\alpha}) - \mathcal{OPF}_i(\boldsymbol{\beta})| < C\delta$, where $\mathcal{OPF}_i(\cdot)$ denotes the i^{th} coordinate of the value of $\mathcal{OPF}(\cdot)$.*

Definition 3.2. *For fixed $\mathbf{C}$, $\mathbf{B}$, $\mathbf{f} \in \Omega_{\mathbf{f}}$ and $\boldsymbol{\xi} \in \widetilde{\Omega}_{\text{para}}$, the fixed-point sensitivity of generator i with respective to load j (i.e., bus $N_{\text{G}} + j$) is the minimal nonnegative value, denoted by $C_{i \leftarrow j}$, such that (i, j) is a $C_{i \leftarrow j}$-Lipschitz pair.*

Definition 3.2 assumes that all the network parameters including cost function, generator capacity and line capacity are fixed. In real systems, not only the loads can fluctuate over the time, but also the system parameters might change due to system contingency, maintenance, as well as other unexpected events citation needed. For instance, if generator i is shut down in emergency, then we will revise $\overline{\mathbf{s}}_i^g$ and $\underline{\mathbf{s}}_i^g$ to 0 in the OPF problem in order to avoid misoperation [2]. Therefore, characterizing the worst-case sensitivity over all the possible operation points and system parameters will provide a uniform upper bound for the system sensitivity regardless of the real-time parameter, and also allow the system to tolerate certain contingency.

Definition 3.3. *The worst-case sensitivity of generator i with respective to load j (i.e., bus $N_{\text{G}} + j$) is*

$$C_{i \leftarrow j}^{\text{wc}} := \max_{f \in \Omega_{\mathbf{f}}} \max_{\boldsymbol{\xi} \in \widetilde{\Omega}_{\text{para}}} C_{i \leftarrow j}. \tag{3.1}$$

Remark 3.4. From the definition, both the operation-point or worst-case sensitivity can possibly reach ∞.

In section 4, we will first study the derivative of OPF, and then bridge the gap between OPF derivative and sensitivity.

[2]In fact, to satisfy Assumption 3, $\overline{\mathbf{s}}_i^g$ and $\underline{\mathbf{s}}_i^g$ can be imagined to be set to ϵ_1, ϵ_2 for some small values, respectively.

3.2. Applications. In this subsection, we will show why the metric of the worst-case sensitivity will play an important role in many recent applications. Specifically, we focus on three example problems: power system data privacy, real-time OPF error estimation and the range of LMPs.

3.2.1. Power System Data Privacy. There is always some tension between the utilities and the research community. As the development of the smart grid. More and more smart devices and cyber infrastructure have been playing an important role in today's power systems. While a great amount of electricity usage data become available, the growing concerns about data privacy also arise [27, 2]. On the other hand, the advancement of data science motivates considerable research works driven by data. Both real or synthetic data are desired for data mining, cross validation and result publications [23, 7]. The tradeoff is usually to provide some statistical summary such as the geographical aggregated injections [1]. Here, if we view the load vector $\mathbf{s}^l$ as the private data, and suppose that the statistical summary is some function of all the bus injections $(\mathbf{s}^g, \mathbf{s}^l)$. Then $\mathbf{s}^g = \mathcal{OPF}(\mathbf{s}^l)$ also contain certain amount of the privacy as it serves as a function of $\mathbf{s}^l$. We adopt *differential privacy* [10] as the metric to evaluate the privacy level.

Suppose $\mathcal{M}(\mathbf{s}^g, \mathbf{s}^l)$ is a (randomized) function of $(\mathbf{s}^g, \mathbf{s}^l)$ in $\mathbb{R}^r$. It is reasonable to assume that $\mathbf{s}^g$ is always chosen as the unique optimal solution to the OPF problem, i.e., $\mathbf{s}^g = \mathcal{OPF}(\mathbf{s}^l)$. Thereby $\mathcal{M}(\mathbf{s}^g, \mathbf{s}^l) = \mathcal{M}(\mathcal{OPF}(\mathbf{s}^l), \mathbf{s}^l)$ is actually a function of $\mathbf{s}^l$and we might simply write it as $\mathcal{M}(\mathbf{s}^l)$. We introduced a modified definition of differential privacy in the context of power system data.

DEFINITION 3.5. *For $\Delta, \varrho > 0$, the mechanism $\mathcal{M}$ preserves (Δ, ϱ)-differential privacy if and only if $\forall \mathbf{d}', \mathbf{d}'' \in \Omega_{\mathbf{s}^l}$ such that $\|\mathbf{d}' - \mathbf{d}''\|_0 \leq 1$ and $\|\mathbf{d}' - \mathbf{d}''\|_1 \leq \Delta$, and $\forall \mathcal{W} \subseteq \mathbb{R}^r$, we have*

$$\mathbb{P}\{\mathcal{M}(\mathbf{d}') \in \mathcal{W}\} \leq \exp(\varrho) \cdot \mathbb{P}\{\mathcal{M}(\mathbf{d}'') \in \mathcal{W}\}.$$

Laplace mechanism is one of the most commonly used mechanisms to preserve differential pricy, and in our context, the Laplace mechanism can be characterized by the following lemma.

LEMMA 3.6. *Let $\tilde{\mathcal{M}}(\mathbf{s}^g, \mathbf{s}^l)$ be a deterministic function of $(\mathbf{s}^g, \mathbf{s}^l)$ and we use $\tilde{\mathcal{M}}(\mathbf{s}^l)$ to denote $\tilde{\mathcal{M}}(\mathcal{OPF}(\mathbf{s}^l), \mathbf{s}^l)$ for short. Suppose $\tilde{\mathcal{M}}$ satisfies $\forall \mathbf{d}', \mathbf{d}'' \in \Omega_{\mathbf{s}^l}$ such that $\|\mathbf{d}' - \mathbf{d}''\|_0 \leq 1$ and $\mathbf{d}' - \mathbf{d}''\|_1 \leq \Delta$, we have*

$$\|\tilde{\mathcal{M}}(\mathbf{d}') - \tilde{\mathcal{M}}(\mathbf{d}'')\|_1 \leq \Delta_1.$$

Then the mechanism $\mathcal{M} = \tilde{\mathcal{M}} + \mathbf{Y}$, where all $\mathbf{Y}_i$ are drawn from independent Laplace distributions $\mathscr{L}(\Delta_1/\varrho)$, preserves $(\Delta, \varrho)-$differential privacy.

Suppose $\tilde{\mathcal{M}}$ is smooth and all the elements of the Jacobean matrix $\mathbf{J}_{\tilde{\mathcal{M}}}$ with respect to $\tilde{\mathcal{M}}$ are upper bounded by a constant U. Then by chain's rule, the worst-case sensitivity provides a natural upper bound for Δ_1:

$$\Delta_1 \leq \Big(1 + \max_{j \in [N_{\mathrm{L}}]} \sum_{i \in [N_{\mathrm{G}}]} C^{\mathrm{wc}}_{i \leftarrow j}\Big) Ur\Delta.$$

Plugging the above upper bound in the expression for the parameter of the Laplace noise in Lemma 3.6 will preserve the $(\Delta, \varrho)-$differential privacy. Informally, the sensitivity can be viewed as a metric evaluating how significantly a generator's change can reflect the change in the load. In a system with higher worst-case sensitivity, even a tiny variance in the load can be amplified in the change of the optimal generation and such system might potentially leak more privacy while disclosing the statistical summary involving $\mathbf{s}^g$.

3.2.2. Real-time OPF Error Estimation. Another trend for the future power systems is to incorporate more distributed energy resources and to solve the AC OPF problem at faster timescale [29, 21, 5]. One of the main focuses of the real-time OPF algorithms is to track the optimal solution over a period of time while keeping the error between the optimal solution and the algorithm outcome small. Estimating the solution error of the real-time OPF algorithm can not only provide the guarantee for different algorithms, but also reflect how different power systems will behave under a certain algorithm. We take the state-of-art algorithm proposed by [29] as an example. The authors used

$$\frac{1}{T}\sum_{t=1}^{T}\|\hat{\mathbf{s}^g}(t)-\mathbf{s}^g(t)\| \tag{3.2}$$

as the metric to measure the tracking performance, where $\hat{\mathbf{s}^g}$ and $\mathbf{s}^g$ are the algorithm outcome and the ground truth of the optimal solution. The time period is from $t=1$ to T. For simplicity, we assume the norm is the L2 norm, though in the original paper it is allowed to be weighted by an arbitrary positive definite matrix. The upper bound of (3.2) is also proposed in [29] to be of the form

$$\frac{1}{T}\sum_{t=1}^{T}\|\hat{\mathbf{s}^g}(t)-\mathbf{s}^g(t)\| \le \alpha\frac{1}{T}\sum_{t=1}^{T}\|\mathbf{s}^g(t)-\mathbf{s}^g(t-1)\|+\beta$$

where α and β are the scalars determined by other parameters. We notice that it might be difficult and unnatural to evaluate $\|\mathbf{s}^g(t)-\mathbf{s}^g(t-1)\|$ for 2 reasons. First, the value of the optimal solutions $\mathbf{s}^g(t)$ and $\mathbf{s}^g(t+1)$ are always unknown and so is their difference. Second, the parameters in the OPF problem such as the cost function, system capacities might also change time, so the value of $\|\mathbf{s}^g(t)-\mathbf{s}^g(t-1)\|$ is actually a variable of time. Instead, it might be more natural to assume that $\|\mathbf{s}^l(t)-\mathbf{s}^l(t-1)\|$ is a relatively manageable and constant value as the usage of electricity usually does not change dramatically. Our result of the worst-case sensitivity sheds light on the following relationship between the change in the optimal generation and the change in the load.

$$\begin{aligned}&\|\mathbf{s}^g(t)-\mathbf{s}^g(t-1)\|\\ &\le \max_{j\in[N_{\mathrm{L}}]}\sqrt{\sum_{i\in[N_{\mathrm{G}}]}(C^{\mathrm{wc}}_{i\leftarrow j})^2}\|\mathbf{s}^l(t)-\mathbf{s}^l(t-1)\|.\end{aligned} \tag{3.3}$$

If we view $\|\mathbf{s}^l(t)-\mathbf{s}^l(t-1)\|$ as a known or constant value, then the right hand side of (3.3) is also known and does not change over time even if the cost function and other parameters of the OPF problem change. It can also be seen from (3.3) that in a highly sensitive system, the optimal generation might potentially fluctuate dramatically even if the loads change mildly, and it will make the real-time algorithm more difficult to track the optimal solution. However, it is important to point out that the real-time OPF problem usually adopts the AC model while our result only applies to the DC model, (3.3) should only serve as a rough estimation of the change in $\mathbf{s}^g$ over 2 consecutive time slots in the AC model.

3.2.3. Range of LMPs. Locational marginal price (LMP) is the marginal cost of supplying the next increment of load at a specific bus [26]. LMP is widely used in the control and operation of electricity markets as it characterizes the actual cost to generate and deliver the power. Under the DC model, LMP at a given operating point can be explicitly expressed from KKT conditions [26], and many literatures also propose the methods to forecast LMP in the near future [18]. It is also well known that LMP can be even higher than the price of the most expensive generator or be negative, since lie congestion also plays an important role in LMP, and many papers also focus on the driver of those counter-intuitive LMPs [24]. We found that the worst-case sensitivity can provide a range for the LMP regardless of the operating point. Such result might be an alternative perspective to view LMP since such range can be regarded as the intrinsic LMP range determined by the system topology. Specifically, the absolute value of LMP for load j can be upper bound by

$$\sum_{i\in[N_{\mathrm{G}}]} \mathbf{f}_i C^{\mathrm{wc}}_{i\leftarrow j}.$$

Intuitively, if a system can potentially be highly sensitive, then it will be easier for the system to have extremely high or negative LMPs.

4. On the OPF Derivative.

4.1. The Existence. Before deriving the expressions for the $\mathcal{OPF}$ derivative, it is necessary to guarantee the differentiability. The following lemma proposed in [11] and [12] gives the sufficient condition of the differentiability for parameterized optimization problems. We rephrase the lemma as follows.

LEMMA 4.1. *Consider a generic optimization problem parametrized by θ:*

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} f(x;\theta) \tag{4.1a}$$
$$\text{subject to } g_i(x;\theta)\le 0, i=1,2,\dots,m \tag{4.1b}$$
$$h_j(x;\theta)=0, j=1,2,\dots,l. \tag{4.1c}$$

If (x^*,η^*,ν^*) is the primal-dual optimal solution for some θ_0 and satisfies:
1) x^* is a locally unique primal solution.
2) f, g_i, h_j are twice continuously differentiable in x and differentiable in θ.
3) The gradients $\nabla g_i(x^*)$ for binding inequality constraints and ∇h_j for equality constraints are independent.
4) Strict complementary slackness holds, i.e., $g_i(x^*)=0 \Rightarrow \eta_i>0$.
Then the local derivative $\partial_\theta x^*$ exists at θ_0, and the set of binding constraints is unchanged in a small neighborhood of θ_0.

By checking the above conditions one by one, we have the following theorem for the $\mathcal{OPF}$ derivative.

THEOREM 4.2. *Under Assumption 2 and Assumption 3, for $\mathbf{s}^l\in\widetilde{\Omega}_{\mathbf{s}^l}$, the derivative $\partial_{\mathbf{s}^l}\mathcal{OPF}(\mathbf{s}^l)$ always exists, and the set of binding constraints stay unchanged in some neighborhood of $\mathbf{s}^l$.*

Proof. For condition 1, Assumption 2 guarantees the uniqueness of the optimal solution. As all the constraints and objective functions in (2.1) are linear in the decision variables $(\mathbf{s}^g,\boldsymbol{\theta})$ and parameter $\mathbf{s}^l$, condition 2 is also satisfied. Corollary 2.7 shows that $\forall \mathbf{s}^l\in\widetilde{\Omega}_{\mathbf{s}^l}$, the $N_{\mathrm{G}}-1$ tight inequalities in (2.1), along with $N+1$ equality constraints, are independent, thus condition 3 is also true here. Finally, if condition 4 is not satisfied, then as we always have $N_{\mathrm{G}} - 1$ tight inequality constraints, the non-zero coordinates in the inequality multipliers must be strictly less then $N_{\mathrm{G}} - 1$. On the other hand, under Assumption 2, Proposition 2.3 implies (2.3) always holds, and here comes the contradiction. Therefore, all four conditions in Lemma 4.1 are satisfied, and the derivative $\partial_{\mathbf{s}^l} \mathcal{OPF}(\mathbf{s}^l)$ exists. ∎

4.2. Jacobean Matrix. In this subsection, we will derive the Jacobean matrix

$$\mathbf{J}(\mathbf{s}^l) := \partial_{\mathbf{s}^l} \mathcal{OPF}(\mathbf{s}^l) \tag{4.2}$$

for $\mathbf{s}^l \in \widetilde{\Omega}_{\mathbf{s}^l}$. We might use $\mathbf{J}$ for short when the value of $\mathbf{s}^l$ is clear from the context. Suppose at point $\mathbf{s}^l$, the set of generators corresponding to binding inequalities is $\mathcal{S}_{\mathrm{G}} \subseteq \mathcal{V}_{\mathrm{G}}$, while the set of branches corresponding to binding inequalities is $\mathcal{S}_{\mathrm{B}} \subseteq \mathcal{E}$. By Proposition 2.5 and Assumption 3, we have $|\mathcal{S}_{\mathrm{G}}| + |\mathcal{S}_{\mathrm{B}}| = N_{\mathrm{G}} - 1$. As Lemma 4.1 implies that generators $\mathcal{S}_{\mathrm{G}}$ and branches $\mathcal{S}_{\mathrm{B}}$ still correspond to binding constraints near $\mathbf{s}^l$, there is a local relationship between $\mathcal{OPF}(\mathbf{s}^l)$ and $\mathbf{s}^l$:

$$\mathbf{H} \cdot \left[\begin{array}{c} \mathbf{s}^g \\ \hline \boldsymbol{\theta} \end{array}\right] = \left[\begin{array}{c} \mathbf{0}^{N_{\mathrm{G}} \times 1} \\ \hline -\mathbf{s}^l \\ \gamma^{\mathsf{T}} \cdot \boldsymbol{\xi} \\ 0 \end{array}\right], \mathbf{H} := \left[\begin{array}{c|c} -\mathbf{I}^{N_{\mathrm{G}}} & \mathbf{I}^N_{\mathcal{V}_{\mathrm{G}}} \cdot \mathbf{CBC}^{\mathsf{T}} \\ \hline \mathbf{0}^{N_{\mathrm{L}} \times N_{\mathrm{G}}} & \mathbf{I}^N_{[N_{\mathrm{L}}]+N_{\mathrm{G}}} \cdot \mathbf{CBC}^{\mathsf{T}} \\ \mathbf{I}^{N_{\mathrm{G}}}_{\mathcal{S}_{\mathrm{G}}} & \mathbf{0}^{|\mathcal{S}_{\mathrm{G}}| \times N} \\ \mathbf{0}^{|\mathcal{S}_{\mathrm{B}}| \times N_{\mathrm{G}}} & \mathbf{I}^E_{\mathcal{S}_{\mathrm{B}}} \cdot \mathbf{BC}^{\mathsf{T}} \\ \mathbf{0}^{1 \times N_{\mathrm{G}}} & \mathbf{e}_1^{\mathsf{T}} \end{array}\right]. \tag{4.3}$$

On the right hand side, $\gamma \in \Gamma$ and $\gamma^{\mathsf{T}} \boldsymbol{\xi}$ gives the limits that tight generations and branch power flows hit. By Corollary 2.7, the first $N + N_{\mathrm{G}} - 1$ rows of $\mathbf{H}$ are independent, and clearly the last row $[\mathbf{0}, \mathbf{e}_1^{\mathsf{T}}]$ does not depend on the first $N + N_{\mathrm{G}} - 1$ rows. Hence $\mathbf{H}$ is invertible, and using the block matrix inversion formula, we have

$$\left[\begin{array}{c} \mathbf{s}^g \\ \hline \boldsymbol{\theta} \end{array}\right] = \mathbf{H}^{-1} \left[\begin{array}{c} \mathbf{0}^{N_{\mathrm{G}} \times 1} \\ \hline -\mathbf{s}^l \\ \gamma^{\mathsf{T}} \boldsymbol{\xi} \\ 0 \end{array}\right] = \left[\begin{array}{c|c} * & \mathbf{H}_1 \\ \hline * & * \end{array}\right] \left[\begin{array}{c} \mathbf{0}^{N_{\mathrm{G}} \times 1} \\ \hline -\mathbf{s}^l \\ \gamma^{\mathsf{T}} \boldsymbol{\xi} \\ 0 \end{array}\right] \tag{4.4}$$

with

$$\mathbf{H}_1 = -\mathbf{I}^N_{\mathcal{V}_{\mathrm{G}}} \cdot \mathbf{CBC}^{\mathsf{T}} \cdot \left(\mathbf{R}(\mathcal{S}_{\mathrm{G}}, \mathcal{S}_{\mathrm{B}})^{\mathsf{T}}\right)^{-1}, \mathbf{R}(\mathcal{S}_{\mathrm{G}}, \mathcal{S}_{\mathrm{B}})^{\mathsf{T}} := \begin{bmatrix} \mathbf{I}^N_{\mathcal{V}_{\mathrm{L}}} \cdot \mathbf{CBC}^{\mathsf{T}} \\ \mathbf{I}^N_{\mathcal{S}_{\mathrm{G}}} \cdot \mathbf{CBC}^{\mathsf{T}} \\ \mathbf{I}^E_{\mathcal{S}_{\mathrm{B}}} \cdot \mathbf{BC}^{\mathsf{T}} \\ \mathbf{e}_1^{\mathsf{T}} \end{bmatrix}. \tag{4.5}$$

Recall (4.2) and the fact that the value of $\mathcal{OPF}$ is essentially $\mathbf{s}^g$ in (4.4), so the Jacobean matrix $\mathbf{J}$ is

$$\mathbf{J} = \mathbf{H}_1 \cdot \mathbf{I}^{\mathsf{T}}_{[N_{\mathrm{L}}]}. \tag{4.6}$$

4.3. Relationship to the Worst-Case Sensitivity. It is worth noting that though (4.6) can compute the Jacobean matrix $\mathbf{J}$, it relies on the knowledge of $\mathcal{S}_{\mathrm{G}}$ and $\mathcal{S}_{\mathrm{B}}$. However, the definition of $C^{\mathrm{wc}}_{i \leftarrow j}$ in (3.1) search for the maximal C over all the choices of $\mathbf{f}$ and $\boldsymbol{\xi}$. The next theorem links the Jacobean matrix and the worst-case sensitivity by converting the choice of $\mathbf{f}$ and $\boldsymbol{\xi}$ to the choice of $\mathcal{S}_{\mathrm{G}}$ and $\mathcal{S}_{\mathrm{B}}$.

THEOREM 4.3.

$$C^{\text{wc}}_{i\leftarrow j} = \max_{\substack{\mathcal{S}_{\text{G}}\in\mathcal{V}_{\text{G}},\mathcal{S}_{\text{B}}\in\mathcal{E}\\ |\mathcal{S}_{\text{G}}|+|\mathcal{S}_{\text{B}}|=N_{\text{G}}-1\\ \mathcal{S}_{\text{G}}\perp\mathcal{S}_{\text{B}}}} |\mathbf{J}_{i,j}| \tag{4.7}$$

where $\mathcal{S}_{\text{G}} \perp \mathcal{S}_{\text{B}}$ denotes all the inequality constraints corresponding to $\mathcal{S}_{\text{G}}$ and $\mathcal{S}_{\text{B}}$ are independent to each other and independent to equality constraints.

We first go through the following lemmas in order to build up the final proof for Theorem 4.3.

LEMMA 4.4. *For any $\mathcal{S}_{\text{G}} \in \mathcal{V}_{\text{G}}, \mathcal{S}_{\text{B}} \in \mathcal{E}$ such that $|\mathcal{S}_{\text{G}}| + |\mathcal{S}_{\text{B}}| = N_{\text{G}} - 1$ and $\mathcal{S}_{\text{G}} \perp \mathcal{S}_{\text{B}}$, there exist $\mathbf{f}_* \in \mathbb{R}_+^{N_{\text{G}}}$, $\boldsymbol{\xi}_* \in \Omega_{\text{para}}$ and $\mathbf{s}^l_* \in \Omega_{\mathbf{s}^l}(\boldsymbol{\xi}_*)$ such that (2.1) has unique solution and all the binding constraints at the solution point exactly correspond to $\mathcal{S}_{\text{G}}$ and $\mathcal{S}_{\text{B}}$.*

Proof. See Appendix D. ∎

LEMMA 4.5. *For any $\mathcal{S}_{\text{G}} \in \mathcal{V}_{\text{G}}, \mathcal{S}_{\text{B}} \in \mathcal{E}$ such that $|\mathcal{S}_{\text{G}}| + |\mathcal{S}_{\text{B}}| = N_{\text{G}} - 1$ and $\mathcal{S}_{\text{G}} \perp \mathcal{S}_{\text{B}}$, there exist $\mathbf{f}_{**} \in \Omega_{\mathbf{f}}$, $\boldsymbol{\xi}_{**} \in \widetilde{\Omega}_{\text{para}}$ and an open ball $W \subseteq \widetilde{\Omega}_{\mathbf{s}^l}(\boldsymbol{\xi}_{**})$ such that all the binding constraints exactly correspond to $\mathcal{S}_{\text{G}}$ and $\mathcal{S}_{\text{B}}$ whenever $\mathbf{s}^l \in W$.*

Proof. See Appendix D. ∎

LEMMA 4.6. *For a convex set $\Omega \subseteq \mathbb{R}^n$ and its subset $\Omega_{\text{bad}} \subseteq \Omega$ such that the equality $\text{clos}(\text{int}(\Omega)) = \text{clos}(\Omega)$ holds. Suppose Ω_{bad} can be covered by the union of finitely many affine hyperplanes. For any two points $\mathbf{x}_1, \mathbf{x}_2 \in \Omega$ and their neighborhoods $U(\mathbf{x}_1)$ and $U(\mathbf{x}_2)$, there exist $\mathbf{x}'_1 \in U(\mathbf{x}_1) \cap \Omega$ and $\mathbf{x}'_2 \in U(\mathbf{x}_2) \cap \Omega$ such that the line segment*

$$\mathbf{x}'_1\mathbf{x}'_2 := \{\mathbf{x} | \mathbf{x} = \alpha\mathbf{x}'_1 + (1-\alpha)\mathbf{x}'_2 \text{ for } \alpha \in [0,1]\}$$

satisfies $\mathbf{x}'_1\mathbf{x}'_2 \cap \Omega_{\text{bad}}$ has only finitely many elements.

Proof. See Appendix D. ∎

Now we have all the ingredients for proving Theorem 4.3.

Proof. (Theorem 4.3) For $C^{\text{wc}}_{i\leftarrow j} = C$, then for any $\epsilon > 0$, there must be $\mathbf{f} \in \Omega_{\mathbf{f}}$, $\boldsymbol{\xi} \in \widetilde{\Omega}_{\text{para}}$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Omega_{\mathbf{s}^l}$, $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$ such that $\boldsymbol{\alpha}_k = \boldsymbol{\beta}_k$ for $k \neq j$ and

$$|\mathcal{OPF}_i(\boldsymbol{\alpha}) - \mathcal{OPF}_i(\boldsymbol{\beta})| > (C - \epsilon)|\boldsymbol{\alpha}_j - \boldsymbol{\beta}_j|.$$

Corollary 2.6 shows that $\Omega_{\mathbf{s}^l} \setminus \widetilde{\Omega}_{\mathbf{s}^l}$ can be covered by the union of finitely many affine hyperplanes, so we can let Ω and Ω_{bad} in Lemma 4.6 be $\Omega_{\mathbf{s}^l}$ and $\Omega_{\mathbf{s}^l} \setminus \widetilde{\Omega}_{\mathbf{s}^l}$, respectively, and find sequences $\boldsymbol{\alpha}_{(t)}$ and $\boldsymbol{\beta}_{(t)}$ satisfying $\lim_{t\to\infty} \boldsymbol{\alpha}_{(t)} = \boldsymbol{\alpha}$, $\lim_{t\to\infty} \boldsymbol{\beta}_{(t)} = \boldsymbol{\beta}$, and $\boldsymbol{\alpha}_{(t)}\boldsymbol{\beta}_{(t)} \cap (\Omega_{\mathbf{s}^l} \setminus \widetilde{\Omega}_{\mathbf{s}^l})$ has finite elements for all t. Together with Theorem 4.2, we know there are only finite non-differentiable points along $\boldsymbol{\alpha}_{(t)}\boldsymbol{\beta}_{(t)}$. Since $\mathcal{OPF}$ is continuous in $\mathbf{s}^l$, we have

$$\begin{aligned}
&|\mathcal{OPF}_i(\boldsymbol{\alpha}_{(t)}) - \mathcal{OPF}_i(\boldsymbol{\beta}_{(t)})| \\
=&\left| \int_{\boldsymbol{\alpha}_{(t)}}^{\mathbf{x}_1} \mathbf{J}_{i,j}(\mathbf{x})\mathrm{d}\mathbf{x} + \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{J}_{i,j}(\mathbf{x})\mathrm{d}\mathbf{x} + \cdots + \int_{\mathbf{x}_m}^{\boldsymbol{\beta}_{(t)}} \mathbf{J}_{i,j}(\mathbf{x})\mathrm{d}\mathbf{x} \right| \\
\leq& |\boldsymbol{\alpha}_{(t)}\boldsymbol{\beta}_{(t)}| \max |\mathbf{J}_{i,j}|
\end{aligned}$$

where $\max|\mathbf{J}_{i,j}|$ is short for the right hand side of (4.7). When $t \to \infty$, we have

$$\begin{aligned}
&|\mathcal{OPF}_i(\boldsymbol{\alpha}_{(t)}) - \mathcal{OPF}_i(\boldsymbol{\beta}_{(t)})| \to |\mathcal{OPF}_i(\boldsymbol{\alpha}) - \mathcal{OPF}_i(\boldsymbol{\beta})|,\\
&|\boldsymbol{\alpha}_{(t)}\boldsymbol{\beta}_{(t)}| \to |\boldsymbol{\alpha}_j - \boldsymbol{\beta}_j|.
\end{aligned}$$

Thereby, we obtain $\max|\mathbf{J}_{i,j}| > C - \epsilon$ and when $\epsilon \to 0$, we have $\max|\mathbf{J}_{i,j}| \geq C^{\mathrm{wc}}_{i\leftarrow j}$.

For the other direction, suppose the maximal value of $|\mathbf{J}_{i,j}| = C$ is achieved at $(\mathcal{S}^*_{\mathrm{G}}, \mathcal{S}^*_{\mathrm{B}})$, then Lemma 4.5 provides $\mathbf{f}_{**} \in \Omega_{\mathbf{f}}$, $\boldsymbol{\xi}_{**} \in \widetilde{\Omega}_{\mathrm{para}}$ and $W \subseteq \widetilde{\Omega}_{\mathbf{s}^l}(\boldsymbol{\xi}_{**})$ such that all the binding constraints at the solution point exactly correspond to $\mathcal{S}^*_{\mathrm{G}}$ and $\mathcal{S}^*_{\mathrm{B}}$ whenever $\mathbf{s}^l \in W$. By definition, under $\mathbf{f}_{**}$ and $\boldsymbol{\xi}_{**}$, $|\partial_{\mathbf{s}^l}\mathcal{OPF}(\mathbf{s}^l)| = C$ for all $\mathbf{s}^l \in W$, where W is an open ball. We can pick any two points $\boldsymbol{\alpha}, \boldsymbol{\beta} \in W$ such that $\boldsymbol{\alpha}_k = \boldsymbol{\beta}_k$ for $k \neq j$ and $\boldsymbol{\alpha}_j \neq \boldsymbol{\beta}_j$. Since all the points along $\boldsymbol{\alpha\beta}$ are within W, the value of $\mathbf{J}_{i,j}$ exist and does not change along $\boldsymbol{\alpha\beta}$. As the result,

$$|\mathcal{OPF}_i(\boldsymbol{\alpha} - \mathcal{OPF}_i(\boldsymbol{\beta})| = \left|\int_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}} \mathbf{J}_{i,j}(\mathbf{x})\mathrm{d}\mathbf{x}\right| = \left|\int_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}} (\max|\mathbf{J}_{i,j}|)\mathrm{d}\mathbf{x}\right| = C|\boldsymbol{\alpha}_j - \boldsymbol{\beta}_j|,$$

and thereby $C^{\mathrm{wc}}_{i\leftarrow j} \geq C = \max|\mathbf{J}_{i,j}|$. Above all, (4.3) holds. ◻

In general, in order to figure out $C^{\mathrm{wc}}_{i\leftarrow j}$, it might still take super-exponential time to iterate over all the possible combinations of $\mathcal{S}_{\mathrm{G}}$ and $\mathcal{S}_{\mathrm{B}}$. Due to its computational complexity, we will focus on the network with special topologies and try to simplify the computation in the next subsections, and we will see that under those topologies, the value of $C^{\mathrm{wc}}_{i\leftarrow j}$ presents many interesting structural properties tightly related to the network topology.

5. Network Topology.

In this section, we will mainly investigate three specific types of power network topologies: tree, unicyclic graph and graph with multiple non-adjacent cycles. Under those networks topologies, the value of $\mathbf{J}_{i,j}$ presents interesting structural properties in the sense that some metrics related to $\mathbf{J}_{i,j}$ are conserved within a local structure of the network. Additionally, with the help of that, the computation of the worst-case sensitivity $C^{\mathrm{wc}}_{i\leftarrow j}$ can be significantly simplified as it requires only iterating over very few choices of $(\mathcal{S}_{\mathrm{G}}, \mathcal{S}_{\mathrm{B}})$.

5.1. General Results.

LEMMA 5.1. *Suppose $\mathcal{S}_{\mathrm{B}}$ partitions $\mathcal{G}$ into m disjoint subgraphs $\{\mathcal{G}_i(\mathcal{V}_i, \mathcal{E}_i)\}_{i=1}^m$, where $\cup_i \mathcal{V}_i = \mathcal{V}$ and $(\cup_i \mathcal{E}_i) \cup \mathcal{S}_{\mathrm{B}} = \mathcal{E}$. Then for any i, we have $\mathcal{S}^{\mathrm{c}}_{\mathrm{G}} \cap \mathcal{V}_i \neq \emptyset$ where $\mathcal{S}^{\mathrm{c}}_{\mathrm{G}}$ stands for $\mathcal{V}_{\mathrm{G}} \setminus \mathcal{S}_{\mathrm{G}}$.*

Proof. If not, then all the generators in $\mathcal{V}_i$ are tight. Let

$$\mathbf{T} := \begin{bmatrix} \mathbf{CBC}^\mathsf{T} \\ \mathbf{BC}^\mathsf{T} \end{bmatrix}$$

be the same as in Appendix C. Let $\mathcal{E}_0$ be the subset of $\mathcal{S}_{\mathrm{B}}$ satisfying $\forall e = (u, v) \in \mathcal{E}_0$, $u \notin \mathcal{V}_i$ and $v \in \mathcal{V}_i$. Consider the following constraints:

$$\mathbf{T}_{\{j\}} \cdot \boldsymbol{\theta} = \mathbf{s}^g_j \qquad \text{for } j \in \mathcal{V}_i \cap \mathcal{V}_{\mathrm{G}} \tag{5.1a}$$

$$\mathbf{T}_{\{j\}} \cdot \boldsymbol{\theta} = -\mathbf{s}^l_{j-N_{\mathrm{G}}} \qquad \text{for } j \in \mathcal{V}_i \setminus \mathcal{V}_{\mathrm{G}} \tag{5.1b}$$

$$\mathbf{s}^g_j \in \left\{\overline{\mathbf{s}^g}_j, \underline{\mathbf{s}^g}_j\right\} \qquad \text{for } j \in \mathcal{V}_i \cap \mathcal{V}_{\mathrm{G}} \tag{5.1c}$$

$$\mathbf{T}_{\{N+e\}} \cdot \boldsymbol{\theta} \in \left\{\overline{\mathbf{p}}_e, \underline{\mathbf{p}}_e\right\} \text{for } e \in \mathcal{E}_0 \tag{5.1d}$$

Summing up (5.1a) to (5.1c), all $\mathbf{s}_j^g$ cancel off, and we have

$$\begin{aligned}\sum_{j\in\mathcal{V}_i}\mathbf{T}_{\{j\}} &= \sum_{j\in\mathcal{V}_i}\sum_{e=(j,j')\in\mathcal{E}}(b_e\mathbf{e}_j^\mathsf{T}-b_e\mathbf{e}_{j'}^\mathsf{T})\\ &=\sum_{j\in\mathcal{V}_i}\sum_{\substack{e=(j,j')\\e\in\mathcal{E}_i}}(b_e\mathbf{e}_j^\mathsf{T}-b_e\mathbf{e}_{j'}^\mathsf{T})+\sum_{j\in\mathcal{V}_i}\sum_{\substack{e=(j,j')\\e\in\mathcal{E}_0}}(b_e\mathbf{e}_j^\mathsf{T}-b_e\mathbf{e}_{j'}^\mathsf{T})\\ &=\sum_{\substack{e=(j,j')\\e\in\mathcal{E}_i,j<j'}}(b_e\mathbf{e}_j^\mathsf{T}-b_e\mathbf{e}_{j'}^\mathsf{T})+(b_e\mathbf{e}_{j'}^\mathsf{T}-b_e\mathbf{e}_j^\mathsf{T})+\sum_{j\in\mathcal{V}_i}\sum_{e=(j,j')\in\mathcal{E}_0}(b_e\mathbf{e}_j^\mathsf{T}-b_e\mathbf{e}_{j'}^\mathsf{T})\\ &=\sum_{\substack{e=(u,v)\in\mathcal{E}_0\\u\notin\mathcal{V}_i,v\in\mathcal{V}_i}}(b_e\mathbf{e}_v^\mathsf{T}-b_e\mathbf{e}_u^\mathsf{T}).\end{aligned}$$

As the result, the summation of (5.1a) to (5.1c) is

$$\sum_{\substack{e=(u,v)\in\mathcal{E}_0\\u\notin\mathcal{V}_i,v\in\mathcal{V}_i}}b_e(\mathbf{e}_v-\mathbf{e}_u)^\mathsf{T}\boldsymbol{\theta}=\sum_{j\in\mathcal{V}_i\cap\mathcal{V}_\mathrm{G}}\left(\mathbb{1}_{\mathbf{s}_j^g=\overline{\mathbf{s}}_j^g}\overline{\mathbf{s}}_j^g+\mathbb{1}_{\mathbf{s}_j^g=\underline{\mathbf{s}}_j^g}\underline{\mathbf{s}}_j^g\right)-\sum_{j\in\mathcal{V}_i\setminus\mathcal{V}_\mathrm{G}}\mathbf{s}_{j-N_\mathrm{G}}^l$$

which is linearly dependent to (5.1d), and it contradicts to Corollary 2.7. Thereby, $\mathcal{S}_\mathrm{G}^\mathsf{c}\cap\mathcal{V}_i\neq\emptyset$. ∎

Now for a fixed graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, we can divide the edges into two disjoint categories $\mathcal{E}^\mathrm{I}$ and $\mathcal{E}^\mathrm{II}$, where

$$\begin{aligned}\mathcal{E}^\mathrm{I}&:=\{e\in\mathcal{E}\mid\mathcal{G}(\mathcal{V},\mathcal{E}\setminus\{e\})\text{ is not connected.}\}\\ \mathcal{E}^\mathrm{II}&:=\mathcal{E}\setminus\mathcal{E}^\mathrm{I}.\end{aligned}$$

For set $\mathcal{S}$, we use $\mathcal{S}(n)$ to denote the n^th smallest element in $\mathcal{S}$, and we denote its inverse operation as $\mathcal{S}^{-1}(\cdot)$. We then define a mapping $\Lambda:\mathcal{V}\setminus\mathcal{S}_\mathrm{G}^\mathsf{c}\to[N_\mathrm{L}+|\mathcal{S}_\mathrm{G}|]$ such that

$$\Lambda(\alpha)=\begin{cases}\alpha-N_\mathrm{G}, & \alpha\in\mathcal{V}_\mathrm{L}\\ N_\mathrm{L}+\mathcal{S}_\mathrm{G}^{-1}(\alpha), & \alpha\in\mathcal{S}_\mathrm{G}\end{cases}.$$

Similarly, we let Ξ be the mapping $\Xi:\mathcal{S}_\mathrm{B}\to[N-|\mathcal{S}_\mathrm{B}|,N-1]$ such that $\Xi(e)=N_\mathrm{L}+|\mathcal{S}_\mathrm{G}|+\mathcal{S}_\mathrm{B}^{-1}(e)$. Informaly, Λ maps the index of the load bus and the binding generator to the index of the corresponding column in $\mathbf{R}$ while Ξ maps the index of the binding edge to the index of the corresponding column in $\mathbf{R}$. We propose the following lemma.

For any $e=(u,v)\in\mathcal{E}^\mathrm{I}\cap\mathcal{S}_\mathrm{B}$, e partitions $\mathcal{G}$ into two connected non-empty subgraphs $\mathcal{G}_l(\mathcal{V}_l,\mathcal{E}_l)$ for $l=1,2$. Assume e connects buses $u\in\mathcal{V}_1$ and $v\in\mathcal{V}_2$.

LEMMA 5.2. *Suppose* $\mathbf{R}(\mathcal{S}_\mathrm{G},\mathcal{S}_\mathrm{B})\mathbf{q}=\mathbf{w}$ *for* $\mathbf{q},\mathbf{w}$. *The support of* $\mathbf{w}$ *is the subset of* $\mathcal{V}_1\cup\{v\}$ *and* $\mathbf{1}^\mathsf{T}\mathbf{w}=0$. *Matrix* $\mathbf{R}(\mathcal{S}_\mathrm{G},\mathcal{S}_\mathrm{B})$ *here is not necessarily invertible or even square. Consider*

$$\mathbf{q}^1=\sum_{\alpha\in\mathcal{V}_1\setminus\mathcal{S}_\mathrm{G}^\mathsf{c}}\tilde{\mathbf{q}}_{\Lambda(\alpha)}+\sum_{\beta\in\mathcal{E}_1\cap\mathcal{S}_\mathrm{B}}\tilde{\mathbf{q}}_{\Xi(\beta)} \tag{5.2a}$$

$$\mathbf{q}^2=\sum_{\alpha\in\mathcal{V}_2\setminus\mathcal{S}_\mathrm{G}^\mathsf{c}}\tilde{\mathbf{q}}_{\Lambda(\alpha)}+\sum_{\beta\in\mathcal{E}_2\cap\mathcal{S}_\mathrm{B}}\tilde{\mathbf{q}}_{\Xi(\beta)}, \tag{5.2b}$$

where we use $\tilde{\mathbf{q}}_\alpha$ *to denote the vector* $\mathbf{q}_\alpha \mathbf{e}_\alpha$. *Here* $\mathbf{q}_\alpha$ *is the* α^{th} *coordinate of* $\mathbf{q}$ *and* $\mathbf{e}_\alpha$ *is the base vector. Then*

$$\{t \in [N] \mid \mathbf{e}_t^\mathsf{T} \mathbf{R}(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B}) \mathbf{q}^1 \neq 0\} \subseteq \mathcal{V}_1 \cup \{v\} \tag{5.3a}$$

$$\{t \in [N] \mid \mathbf{e}_t^\mathsf{T} \mathbf{R}(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B}) \mathbf{q}^2 \neq 0\} \subseteq \{u, v\} \tag{5.3b}$$

and there exists $q \in \mathbb{R}$ *such that*

$$\mathbf{R}(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B}) \mathbf{q}^1 + q(\mathbf{e}_e^\mathsf{T} \mathbf{B} \mathbf{C}^\mathsf{T})^\mathsf{T} = \mathbf{R}(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B}) \mathbf{q} \tag{5.4a}$$

$$\mathbf{R}(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B}) \mathbf{q}^2 = q(\mathbf{e}_e^\mathsf{T} \mathbf{B} \mathbf{C}^\mathsf{T})^\mathsf{T}. \tag{5.4b}$$

Specifically, $q = \mathbf{q}_{\Lambda(v)}$ *if* $v < u$ *and* $q = -\mathbf{q}_{\Lambda(v)}$ *otherwise.*

Proof. One necessary condition for $\mathbf{e}_t^\mathsf{T} \mathbf{R}(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B}) \mathbf{q}^1 \neq 0$ is: $\exists \alpha \in \mathcal{V} \setminus \mathcal{S}_\mathrm{G}^\mathsf{c}$ which is adjacent to t such that $\mathbf{q}^1_{\Lambda(\alpha)} \neq 0$, or $\exists \beta \in \mathcal{S}_\mathrm{B}$ which links t such that $\mathbf{q}^1_{\Xi(\beta)} \neq 0$. Since v is the only vertex in $\mathcal{V}_2$ that is adjacent to $\mathcal{G}_1$, so (5.3a) holds. Further, we also have

$$\{t \in [N] \mid \mathbf{w}_t \neq 0\} \subseteq \mathcal{V}_1 \cup \{v\}.$$

It is known that $\mathbf{R}\mathbf{q}^1 + \mathbf{R}\mathbf{q}^2 = \mathbf{R}\mathbf{q} = \mathbf{w}$, so for $t \in \mathcal{V}_2 \setminus \{v\}$, we have $\mathbf{e}_t^\mathsf{T} \mathbf{R}\mathbf{q}^2 = \mathbf{w}_t - \mathbf{e}_t^\mathsf{T} \mathbf{R} \mathbf{q}^1 = 0$. Thus (5.3b) holds. Since each row of $\mathbf{R}(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B})$ sums up to 0, there must exist $q \in \mathbb{R}$ such that $\mathbf{R}(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B}) \mathbf{q}^2 = q(\mathbf{e}_e^\mathsf{T} \mathbf{B} \mathbf{C}^\mathsf{T})^\mathsf{T}$, and thereby (5.4) holds. Further, evaluating the u^{th} coordinate on both side of (5.4b), we obtain $-\mathbf{q}_v b_e = q b_e \operatorname{sgn}(v-u)$ and can solve for q. ∎

A special case in Lemma 5.2 that will be frequently used later is when $\mathbf{w} = -\mathbf{C}\mathbf{B}\mathbf{C}^\mathsf{T} \mathbf{e}_k$ for some $k \in \mathcal{V}_\mathrm{G} \cap \mathcal{V}_1$. Clearly, the support of $-\mathbf{C}\mathbf{B}\mathbf{C}^\mathsf{T} \mathbf{e}_k$ is subset to $\mathcal{V}_1 \cup \{v\}$ and all coordinates sum up to 0.

THEOREM 5.3. *For* $e \in \mathcal{E}^\mathrm{I} \cap \mathcal{S}_\mathrm{B}$, *then* $\forall i \in \mathcal{V}_\mathrm{G}, j \in [N_\mathrm{L}]$, *there exists* $\mathcal{S}'_\mathrm{G} \subseteq \mathcal{V}_\mathrm{G}$ *such that*

$$\mathbf{J}_{i,j}(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B}) = \begin{cases} \mathbf{J}_{i,j}(\mathcal{S}'_\mathrm{G}, \mathcal{S}'_\mathrm{B}), & i \Leftrightarrow j \text{ in } \mathcal{G}(\mathcal{V}, \mathcal{E} \setminus \{e\}) \\ 0, & \text{otherwise} \end{cases} \tag{5.5}$$

where $\mathcal{S}'_\mathrm{B} := \mathcal{S}_\mathrm{B} \setminus \{e\}$ *and* $i \Leftrightarrow j$ *means generator* i *and load* j *(i.e., bus* $N_\mathrm{G} + j$*) are connected by some path.*

Proof. First the cut $\mathcal{S}_\mathrm{B} \setminus \{e\}$ partitions the graph $\mathcal{G}$ into multiple subgraphs and assume e is in a connected subgraph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$. Since $e \in \mathcal{E}^\mathrm{I}$, e also partitions $\mathcal{G}$ into two connected non-empty subgraphs $\mathcal{G}_l(\mathcal{V}_l, \mathcal{E}_l)$, $l \in \{A, B\}$. Clearly e exactly splits $\mathcal{V}'$ into $\mathcal{V}_A \cap \mathcal{V}'$ and $\mathcal{V}_B \cap \mathcal{V}'$. Assume e connects buses $u \in \mathcal{V}_A$ and $v \in \mathcal{V}_B$. Without loss of generality, we will assume $i \in \mathcal{V}_A$. Lemma 5.1 indicates we can always find some $k \in \mathcal{S}_\mathrm{G}^\mathsf{c} \cap \mathcal{V}_B \cap \mathcal{V}'$.

Next we are going to show $\mathbf{C}\mathbf{B}\mathbf{C}^\mathsf{T} \mathbf{e}_k$ is independent to columns $\mathbf{R}(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B}) \mathbf{e}_t$ when $t \neq \Xi(e)$. Let $\mathbf{q}(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B}) = -\mathbf{R}(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B})^{-1} (\mathbf{C}\mathbf{B}\mathbf{C}^\mathsf{T} \mathbf{e}_k)$. By Lemma 5.2, we have

$$\mathbf{R}(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B}) \mathbf{q}^B(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B}) + q_0(\mathbf{e}_e^\mathsf{T} \mathbf{B} \mathbf{C}^\mathsf{T})^\mathsf{T} = -\mathbf{C}\mathbf{B}\mathbf{C}^\mathsf{T} \mathbf{e}_k.$$

Here $\mathbf{q}^B$ corresponds to $\mathbf{q}^1$ in (5.2a). If $\mathbf{C}\mathbf{B}\mathbf{C}^\mathsf{T} \mathbf{e}_k$ is not independent to $\mathbf{R}(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B}) \mathbf{e}_t$ for $t \neq \Xi(e)$, then neither is

$$\mathbf{R}(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B}) \mathbf{e}_{\Xi(e)}.$$

Hence we will have $\mathcal{S}_{\mathrm{G}} \not\perp \mathcal{S}_{\mathrm{B}}$ and the contradiction occurs.

As the result, for $\mathcal{S}'_{\mathrm{G}} = \mathcal{S}_{\mathrm{G}} \cup \{k\}$, we have $\mathcal{S}'_{\mathrm{G}} \perp \mathcal{S}'_{\mathrm{B}}$. Next, we will show that under such $\mathcal{S}'_{\mathrm{G}}$, (5.5) holds.

Consider $\mathbf{q}(\mathcal{S}'_{\mathrm{G}}, \mathcal{S}'_{\mathrm{B}}) = -\mathbf{R}(\mathcal{S}'_{\mathrm{G}}, \mathcal{S}'_{\mathrm{B}})^{-1}(\mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}}\mathbf{e}_i)$. Using Lemma 5.2 again, there must exist $q_1 \in \mathbb{R}$ such that

$$\mathbf{R}(\mathcal{S}'_{\mathrm{G}}, \mathcal{S}'_{\mathrm{B}})\mathbf{q}^A(\mathcal{S}'_{\mathrm{G}}, \mathcal{S}'_{\mathrm{B}}) + q_1(\mathbf{e}_e^{\mathsf{T}}\mathbf{B}\mathbf{C}^{\mathsf{T}})^{\mathsf{T}} = -\mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}}\mathbf{e}_i.$$

Here $\mathbf{q}^B$ corresponds to $\mathbf{q}^1$ in (5.2a).

Now construct $\mathbf{q}^*$ as

$$\mathbf{q}^*_t = \begin{cases} \mathbf{q}^A_t, & t < N_{\mathrm{L}} + (\mathcal{S}'_{\mathrm{G}})^{-1}(k) \\ \mathbf{q}^A_{t+1}, & (\mathcal{S}'_{\mathrm{G}})^{-1}(k) \le t - N_{\mathrm{L}} < |\mathcal{S}_{\mathrm{G}}| + \mathcal{S}_{\mathrm{B}}^{-1}(e) \\ q_1, & t = N_{\mathrm{L}} + |\mathcal{S}_{\mathrm{G}}| + \mathcal{S}_{\mathrm{B}}^{-1}(e) \\ \mathbf{q}^A_t, & t > N_{\mathrm{L}} + |\mathcal{S}_{\mathrm{G}}| + \mathcal{S}_{\mathrm{B}}^{-1}(e) \end{cases}$$

It is easy to check that

$$\mathbf{R}(\mathcal{S}_{\mathrm{G}}, \mathcal{S}_{\mathrm{B}})\mathbf{q}^* = \mathbf{R}(\mathcal{S}'_{\mathrm{G}}, \mathcal{S}'_{\mathrm{B}})\mathbf{q}^A(\mathcal{S}'_{\mathrm{G}}, \mathcal{S}'_{\mathrm{B}}) + q_1(\mathbf{e}_e^{\mathsf{T}}\mathbf{B}\mathbf{C}^{\mathsf{T}})^{\mathsf{T}} = -\mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}}\mathbf{e}_i,$$

hence $\mathbf{q}^*$ is the transpose of the first row of $\mathbf{H}_1$ with respect to $(\mathcal{S}_{\mathrm{G}}, \mathcal{S}_{\mathrm{B}})$, and by (4.6), we can compute the elements in the Jacobean matrix as

$$\mathbf{J}_{1,j}(\mathcal{S}_{\mathrm{G}}, \mathcal{S}_{\mathrm{B}}) = \mathbf{J}_{1,j}(\mathcal{S}'_{\mathrm{G}}, \mathcal{S}'_{\mathrm{B}}), \qquad j \in \mathcal{V}_A \tag{5.6a}$$

$$\mathbf{J}_{1,j}(\mathcal{S}_{\mathrm{G}}, \mathcal{S}_{\mathrm{B}}) = 0, \qquad j \in \mathcal{V}_B \tag{5.6b}$$

The above equations directly imply (5.5). ∎

Corollary 5.4. *For $e \in \mathcal{E}^{\mathrm{I}}$, if $e \in \mathcal{S}_{\mathrm{B}}$, then $\forall i \in \mathcal{V}_{\mathrm{G}}, j \in [N_{\mathrm{L}}]$, there exists $\mathcal{S}'_{\mathrm{G}} \subseteq \mathcal{V}_{\mathrm{G}}$ such that*

$$|\mathbf{J}_{i,j}(\mathcal{S}_{\mathrm{G}}, \mathcal{S}_{\mathrm{B}})| \le |\mathbf{J}_{i,j}(\mathcal{S}'_{\mathrm{G}}, \mathcal{S}'_{\mathrm{B}})| \tag{5.7}$$

where $\mathcal{S}'_{\mathrm{B}} := \mathcal{S}_{\mathrm{B}} \setminus \{e\}$.

Theorem 5.5. *Suppose $\mathcal{S}_{\mathrm{B}}$ partitions $\mathcal{G}$ into n disjoint connected subgraphs $\{\mathcal{G}_l = (\mathcal{V}_l, \mathcal{E}_l)\}_{l=1}^n$. For $i \in \mathcal{V}_{\mathrm{G}}$, if $\exists l$ such that $\mathcal{V}_l \cap \mathcal{S}_{\mathrm{G}}^{\mathsf{c}} = \{i\}$, then for $j \in [N_{\mathrm{L}}]$, we have $\mathbf{J}_{i,j} = \mathbb{1}_{j+N_{\mathrm{G}} \in \mathcal{V}_l}$.*

Proof. See Appendix E. ∎

In the following subsections, as we focus on the value of $\mathbf{J}_{i,j}$ for generator i and load j, we will always assume that $i \in \mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$. Otherwise, Theorem 4.2 implies that generator i will stay at its binding value in the neighborhood of the operating point. Thereby $\mathbf{J}_{i,j} = 0$ for all j if $i \in \mathcal{S}_{\mathrm{G}}$.

5.2. Tree. When $\mathcal{G}$ is a tree, $\mathcal{E}^{\mathrm{I}} = \mathcal{E}$. Combining Theorem 4.3 and Corollary 5.4 gives

$$C^{\mathrm{wc}}_{i \leftarrow j} = \max_{\substack{\mathcal{S}_{\mathrm{B}} = \emptyset \\ |\mathcal{V}_{\mathrm{G}} \setminus \mathcal{S}_{\mathrm{G}}| = 1}} |\mathbf{J}_{i,j}|.$$

We must have $\mathcal{S}_{\mathrm{G}}^{\mathsf{c}} = \{i\}$. Theorem 5.5 shows $\mathbf{J}_{i,j} = 1$ for all $j \in [N_{\mathrm{L}}]$ since $\mathcal{S}_{\mathrm{B}} = \emptyset$ and all buses are in the same connected subgraph. Thereby $C^{\mathrm{wc}}_{i \leftarrow j} \equiv 1$ for tree topology networks.

Additionally, we are now looking at what $\mathbf{J}_{i,j}$ will be for general $(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B})$. For tree topology, an set of m edges will always partition the graph into $m+1$ connected subgraphs. Lemma 5.1 indicates that each subgraph should contain exactly one non-binding generator. Using Theorem 5.5 again and we will obtain

$$\mathbf{J}_{i,j}(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B}) = \begin{cases} 1, & i \Leftrightarrow j \text{ in } \mathcal{G}(\mathcal{V}, \mathcal{E} \setminus \mathcal{S}_\mathrm{B}) \\ 0, & \text{otherwise} \end{cases}.$$

Informally, it means for tree network, $\mathcal{S}_\mathrm{B}$ partitions the network into subgraphs and there will be exactly one non-binding generator in each subgraph. Any change in load will lead to the same amount of change in the generator within the same subgraph as long as $(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B})$ does not alter.

5.3. Unicyclic Graph. A unicyclic graph is a connected graph containing exactly one cycle [16]. Unicyclic graph not only is the first step to explore beyond the tree networks, but also might occur in some simple practical power systems such as IEEE 9-bus case.

Suppose $\mathcal{G}$ is a unicyclic graph containing a cycle of length m. We let $\mathcal{G}^\mathrm{O}(\mathcal{V}^\mathrm{O}, \mathcal{E}^\mathrm{O})$ be the cycle in $\mathcal{G}$. Suppose $\mathcal{V}^\mathrm{O} = \{v_k^\mathrm{O}\}_{k=1}^m$ and $\mathcal{E}^\mathrm{O} = \{e_k^\mathrm{O} = (v_k^\mathrm{O}, v_{k+1}^\mathrm{O})\}_{k=1}^m$ where $v_{m+1}^\mathrm{O} = v_1^\mathrm{O}$. Without loss of generality, we assume $v_{k_1}^\mathrm{O} < v_{k_2}^\mathrm{O}$ for $k_1 < k_2$. Let $\mathcal{V}^\mathrm{nc} = \mathcal{V} \setminus \mathcal{V}^\mathrm{O}$ and $\mathcal{E}^\mathrm{nc} = \mathcal{E} \setminus \mathcal{E}^\mathrm{O}$. By Corollary 5.4, we will assume that $\mathcal{S}_\mathrm{B} \cap \mathcal{E}^\mathrm{I} = \emptyset$, and thereby $\mathcal{S}_\mathrm{B} \subseteq \mathcal{E}^\mathrm{O}$ as in the unicyclic case $\mathcal{E}^\mathrm{I} = \mathcal{E}^\mathrm{nc}$ and $\mathcal{E}^\mathrm{II} = \mathcal{E}^\mathrm{O}$.

To figure out the worst-case sensitivity, we first analyze the value of $\mathbf{J}_{i,j}$ in various situations for fixed generator $i \in \mathcal{V}_\mathrm{G}$ and load $j \in [N_\mathrm{L}]$. If $\mathcal{S}_\mathrm{B} = \emptyset$, then $\mathcal{S}_\mathrm{G} = \mathcal{V}_\mathrm{G} \setminus \{i\}$. By Theorem 5.5, we have $\mathbf{J}_{i,j} = 1$ for all $j \in [N_\mathrm{L}]$. If $|\mathcal{S}_\mathrm{B}| = 1$, then $|\mathcal{S}_\mathrm{G}| = N_\mathrm{G} - 2$ and there are two non-binding generators. By iterating over all the possible choices of $\mathcal{S}_\mathrm{G}$ and $\mathcal{S}_\mathrm{B}$, we can find

$$\max_{\substack{|\mathcal{S}_\mathrm{G}| = N_\mathrm{G} - 2, |\mathcal{S}_\mathrm{B}| = 1 \\ \mathcal{S}_\mathrm{G} \perp \mathcal{S}_\mathrm{B}}} |\mathbf{J}_{i,j}|$$

in polynomial time as there are at most $E\binom{N_\mathrm{G}}{2}$ combinations of $(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B})$.

Next we will focus on the situation that $|\mathcal{S}_\mathrm{B}| = n \geq 2$. In this case, $\mathcal{S}_\mathrm{B}$ serves as n cuts on the cycle and will partition $\mathcal{G}$ into n disjoint connected subgraphs $\{\mathcal{G}_l = (\mathcal{V}_l, \mathcal{E}_l)\}_{l=1}^n$. Assume that $\mathcal{G}_l$ is between $\mathcal{G}_{l-1}$ and $\mathcal{G}_{l+1}$. Each $\mathcal{G}_l$ will be a tree. The cycle $\mathcal{G}^\mathrm{O}$ is also partitioned into n arcs and each $\mathcal{G}_l$ contains exactly one arc. Suppose $\mathcal{V}^\mathrm{O} \cap \mathcal{V}_l = \{v_{a_{l-1}+1}^\mathrm{O}, v_{a_{l-1}+2}^\mathrm{O}, \cdots, v_{a_l}^\mathrm{O}\}$, and subgraphs $\mathcal{G}_l$ and $\mathcal{G}_{l+1}$ are linked by edge $e_{a_l}^\mathrm{O} = (v_{a_l}^\mathrm{O}, v_{a_l+1}^\mathrm{O}) \in \mathcal{S}_\mathrm{B}$. Here, all the subscripts are defined in the sense of wrap-around. Clearly, $\mathcal{S}_\mathrm{B} = \{e_{a_l}^\mathrm{O}\}_{l=1}^n$.

Lemma 5.1 indicates that $|\mathcal{V}_l \cap \mathcal{S}_\mathrm{G}^\mathrm{c}| \geq 1$ for $l = 1, 2, \cdots, n$. Since $|\mathcal{S}_\mathrm{G}^\mathrm{c}| = |\mathcal{S}_\mathrm{B}| + 1 = n + 1$, there must be one $\mathcal{V}_l$ containing two elements in $\mathcal{S}_\mathrm{G}^\mathrm{c}$ while others contain one each. Without loss of generality, we let $|\mathcal{V}_1 \cap \mathcal{S}_\mathrm{G}^\mathrm{c}| = 2$ and $|\mathcal{V}_l \cap \mathcal{S}_\mathrm{G}^\mathrm{c}| = 1$ for $l > 1$. Additionally, this situation can happen only if $N_\mathrm{G} \geq 3$.

Lemma 5.6. *For any bus $\alpha \in \mathcal{V}$, the closest bus to α in $\mathcal{V}^\mathrm{O}$ is unique, and the shortest path linking α and its closest bus in $\mathcal{V}^\mathrm{O}$ is also unique.*

Proof. If the closest bus is not unique, then there are two shortest paths

$$\alpha = u_0 \leftrightarrow u_1 \leftrightarrow u_2 \leftrightarrow \cdots \leftrightarrow u_d$$
$$\alpha = v_0 \leftrightarrow v_1 \leftrightarrow v_2 \leftrightarrow \cdots \leftrightarrow v_d$$

where $u_d, v_d \in \mathcal{V}^{\mathrm{O}}$ but $u_k, v_k \notin \mathcal{V}^{\mathrm{O}}$ when $k < d$. Clearly, $u_0, \cdots, u_d$ are different to each other and so are $v_0, \cdots, v_d$. There must be an arc in $\mathcal{G}^{\mathrm{O}}$ linking u_d and v_d

$$u_d \leftrightarrow w_1 \leftrightarrow w_2 \leftrightarrow \cdots \leftrightarrow w_r \leftrightarrow v_d$$

where $w_k \in \mathcal{V}^{\mathrm{O}}$. Let k^* be the largest k such that $\exists k', u_k = v_{k'}$. As the result, $k^*, k' < d$ and $\{u_{k^*}, u_{k^*+1}, \cdots, u_d\}$ and $\{u_{k'+1}, \cdots, v_d\}$ are disjoint. Then we have a cycle

$$u_{k^*} \leftrightarrow u_{k^*+1} \cdots u_d \leftrightarrow w_1 \cdots w_r \leftrightarrow v_d \cdots v_{k'} = u_{k^*}.$$

Since $u_k \notin \mathcal{V}^{\mathrm{O}}$, it contradicts to the fact that there is only one cycle in the graph.

If the closest bus is unique but the shortest path is not unique, then $u_d = v_d$. We can construct k^* and k' as well. Clearly $(k^*, k') \neq (d-1, d-1)$ since two shortest paths are different. Now we have another cycle

$$u_{k^*} \leftrightarrow u_{k^*+1} \cdots u_d = v_d \leftrightarrow v_{d-1} \cdots v_{k'} = u_{k^*}.$$

Likely, it contradicts to the fact that there is only one cycle in the graph. ◻

Definition 5.7. *For any bus $\alpha \in \mathcal{V}$, we define its root to be the closest bus to α in $\mathcal{V}^{\mathrm{O}}$. The root of α is denoted by $\Psi(\alpha) \in \mathcal{V}^{\mathrm{O}}$. For $\alpha \in \mathcal{V}^{\mathrm{O}}$, $\Psi(\alpha) = \alpha$.*

Let

$$\mathbf{q} = -\mathbf{R}(\mathcal{S}_{\mathrm{G}}, \mathcal{S}_{\mathrm{B}})^{-1}(\mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}}\mathbf{e}_i), \tag{5.8}$$

then we have the following Theorem and its corollaries.

Theorem 5.8. *For any $\alpha \in \mathcal{V} \setminus \mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$, we have $\mathbf{q}_{\Lambda(\alpha)} = \mathbf{q}_{\Lambda(\Psi(\alpha))}$.*

Proof. See Appendix F. ◻

Corollary 5.9. *For any $\alpha \in \mathcal{V} \setminus \mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$, if $\Psi(\alpha) = \Psi(i)$, then $\mathbf{q}_{\Lambda(\alpha)} \equiv 1$.*

Corollary 5.10. *For any $\alpha \in \mathcal{V} \setminus \mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$, if $\Psi(\alpha) = \Psi(u)$ for some $u \in \mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$ and $u \neq i$, we have $\mathbf{q}_{\Lambda(\alpha)} \equiv 0$.*

Proofs of Corollary 5.9 and Corollary 5.10 are in Appendix F.

Definition 5.11. *For an edge $e_k^{\mathrm{O}} = (v_k^{\mathrm{O}}, v_{k+1}^{\mathrm{O}})$ in $\mathcal{E}^{\mathrm{O}}$, its Unbalance is defined as If $v_k^{\mathrm{O}} = \Psi(i)$, then*

$$\langle e_k^{\mathrm{O}} \rangle := \sum_{e=(u,v_k^{\mathrm{O}})\in\mathcal{E}} b_e \mathbf{q}_{\Lambda(v_k^{\mathrm{O}})} - \sum_{\substack{e=(u,v_k^{\mathrm{O}})\in\mathcal{E} \\ u\neq v_{k+1}^{\mathrm{O}}, u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}}} b_e \mathbf{q}_{\Lambda(u)} - b_{e_k^{\mathrm{O}}} \mathbf{q}_{\Lambda(v_k^{\mathrm{O}})} - c_1^k - c_2^k.$$

where

$$c_1^k = \begin{cases} b_{(v_k^{\mathrm{O}},i)}, & (v_k^{\mathrm{O}}, i) \in \mathcal{E} \\ 0, & (v_k^{\mathrm{O}}, i) \notin \mathcal{E} \end{cases}, c_2^k = \begin{cases} b_{e_{k-1}^{\mathrm{O}}} \mathbf{q}_{\Xi(e_{k-1}^{\mathrm{O}})}, & e_{k-1}^{\mathrm{O}} \in \mathcal{S}_{\mathrm{B}} \\ 0, & e_{k-1}^{\mathrm{O}} \notin \mathcal{S}_{\mathrm{B}} \end{cases}.$$

Theorem 5.12. *The value of $\langle e_k^{\mathrm{O}} \rangle$ is a constant $\forall e_k^{\mathrm{O}} \in \mathcal{E}^{\mathrm{O}}$.*

Proof. See Appendix G. ◻

For convenience, we denote the constant for the unbalance as $\langle \mathcal{G} \rangle$ since it is a unique metric for the unicyclic graph. The expression for $\langle e_k^{\mathrm{O}} \rangle$ in Definition 5.11 is quite involved, so we simplify the expression of $\langle e_k^{\mathrm{O}} \rangle$ for those unbinding edges.

COROLLARY 5.13. *For $e^{\mathrm{O}}_k \notin \mathcal{S}_{\mathrm{B}}$, we have*

$$\langle e^{\mathrm{O}}_k \rangle = b_{e^{\mathrm{O}}_k}(\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k+1})} - \mathbf{q}_{\Lambda(v^{\mathrm{O}}_k)}).$$

Proof. See Appendix G. ◻

We have known that $\mathcal{V}_1 \cap \mathcal{S}^{\mathrm{c}}_{\mathrm{G}}$ contains two elements. Assume that they are $x, y (x \neq y)$.

LEMMA 5.14. $\Psi(x) \neq \Psi(y)$.

Proof. If we pick $i = x$, then Corollary 5.9 implies $\mathbf{q}_{\Lambda(\Psi(x))} = 1$, but Corollary 5.10 implies $\mathbf{q}_{\Lambda(\Psi(y))} = 0$. ◻

Let $\Psi(x) = v^{\mathrm{O}}_{k(x)}$ and $\Psi(y) = v^{\mathrm{O}}_{k(y)}$ where $k(x) \neq k(y)$. Since $v^{\mathrm{O}}_{k(x)}$ and $v^{\mathrm{O}}_{k(y)}$ are both in $\mathcal{V}_1$ and on the cycle, there must be an cyclic arc in $\mathcal{G}_1$ connecting them. Without loss of generality, let

$$v^{\mathrm{O}}_{k(x)} \leftrightarrow v^{\mathrm{O}}_{k(x)+1} \leftrightarrow v^{\mathrm{O}}_{k(x)+2} \cdots v^{\mathrm{O}}_{k(y)-1} \leftrightarrow v^{\mathrm{O}}_{k(y)}$$

be such arc, then for any edge e^{O}_k where $k = k(x), k(x)+1, \cdots, k(y)-1$, we have $e^{\mathrm{O}}_k \in \mathcal{E}_l$ and $e^{\mathrm{O}}_k \notin \mathcal{S}_{\mathrm{B}}$.

Depending on the subgraph $\mathcal{G}_l$ the generator i and the load j are within, we have the following cases to discuss.

5.3.1. Case 1. $i \in \mathcal{V}_l$ for some $l > 1$.

In this case, $\mathcal{V}_l \cap \mathcal{S}^{\mathrm{c}}_{\mathrm{G}} = \{i\}$. Theorem 5.5 shows $\mathbf{J}_{i,j} = \mathbb{1}_{j+N_{\mathrm{G}} \in \mathcal{V}_l}$. As $\mathbf{J}_{i,j} = 1$ whenever $\mathcal{S}_{\mathrm{B}} = \emptyset$, this case does not contribute to the value of $\max |\mathbf{J}_{i,j}|$.

5.3.2. Case 2. $i \in \mathcal{V}_1$ and $j + N_{\mathrm{G}} \in \mathcal{V}_1$.

Then i must be either x or y. Corollary 5.13 and Corollary 5.13 directly shed lights on the following result.

THEOREM 5.15. *For fixed $\mathbf{B}$, $\mathbf{C}$, The value of $\langle \mathcal{G} \rangle$ is fully determined by and the values of x, y. Specifically,*

$$\langle \mathcal{G} \rangle = \begin{cases} -\left(\sum\limits_{k=k(x)}^{k(y)-1} \frac{1}{b_{e^{\mathrm{O}}_k}} \right)^{-1}, & i = x \\ \left(\sum\limits_{k=k(x)}^{k(y)-1} \frac{1}{b_{e^{\mathrm{O}}_k}} \right)^{-1}, & i = y \end{cases}. \tag{5.9}$$

Proof. We first have

$$\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k(y)})} - \mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k(x)})} = \sum_{k=k(x)}^{k(y)-1} \frac{1}{b_{e^{\mathrm{O}}_k}} b_{e^{\mathrm{O}}_k}(\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k+1})} - \mathbf{q}_{\Lambda(v^{\mathrm{O}}_k)}) = \sum_{k=k(x)}^{k(y)-1} \frac{1}{b_{e^{\mathrm{O}}_k}} \langle \mathcal{G} \rangle.$$

When $i = x$, $\mathbf{q}_{\Lambda(\Psi(x))} = \mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k(x)})} = 1$ and $\mathbf{q}_{\Lambda(\Psi(y))} = \mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k(y)})} = 0$. When $i = y$, $\mathbf{q}_{\Lambda(\Psi(x))} = \mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k(x)})} = 0$ and $\mathbf{q}_{\Lambda(\Psi(y))} = \mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k(y)})} = 1$. Therefore (5.9) holds. ◻

According to (5.8), $\mathbf{q}^\mathsf{T}$ is the i^{th} row of $\mathbf{H}_1$, so the value of $\mathbf{J}_{i,j}$ is the j^{th} element of $\mathbf{q}$. Assume $\Psi(j + N_{\mathrm{G}})$ is $v^{\mathrm{O}}_{k'} \in \mathcal{V}_1$. Then we have

$$\mathbf{J}_{i,j} = \mathbf{q}_j = \mathbf{q}_{\Lambda(j+N_{\mathrm{G}})} = \mathbf{q}_{\Lambda(\Psi(j+N_{\mathrm{G}}))} = \mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k'})}.$$

Next we will focus on the value of $\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k'})}$. If $k' = k(x)$, then we have $\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k'})} = \mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k(x)})} = \mathbb{1}_{x=i}$. Otherwise, there must be an cyclic arc in $\mathcal{G}_1$ connecting $v^{\mathrm{O}}_{k(x)}$ and $v^{\mathrm{O}}_{k'}$. If the path is

$$v^{\mathrm{O}}_{k(x)} \leftrightarrow v^{\mathrm{O}}_{k(x)+1} \leftrightarrow v^{\mathrm{O}}_{k(x)+2} \cdots v^{\mathrm{O}}_{k'-1} \leftrightarrow v^{\mathrm{O}}_{k'},$$

then

$$\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k'})} = \mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k(x)})} + \sum_{k=k(x)}^{k'-1} \frac{1}{b_{e^{\mathrm{O}}_k}} b_{e^{\mathrm{O}}_k} (\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k+1})} - \mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k})}) = \mathbb{1}_{x=i} + \sum_{k=k(x)}^{k'-1} \frac{1}{b_{e^{\mathrm{O}}_k}} \langle \mathcal{G} \rangle.$$

Similarly, if the path is

$$v^{\mathrm{O}}_{k'} \leftrightarrow v^{\mathrm{O}}_{k'+1} \leftrightarrow v^{\mathrm{O}}_{k'+2} \cdots v^{\mathrm{O}}_{k(x)-1} \leftrightarrow v^{\mathrm{O}}_{k(x)},$$

then

$$\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k'})} = -\mathbb{1}_{x=i} - \sum_{k=k'}^{k(x)-1} \frac{1}{b_{e^{\mathrm{O}}_k}} \langle \mathcal{G} \rangle.$$

Above all, the value of $\mathbf{J}_{i,j}$ is determined and can be easily computed whenever $\mathcal{V}_1 \cap \mathcal{S}^{\mathrm{c}}_{\mathrm{G}} = \{x, y\}$ is known, and other elements in or not in $\mathcal{S}_{\mathrm{G}}$ and $\mathcal{S}_{\mathrm{B}}$ do not matter.

5.3.3. Case 3. $i \in \mathcal{V}_1$ and $j + N_{\mathrm{G}} \in \mathcal{V}_l$ for $l > 1$.

In this case, the value of $\langle \mathcal{G} \rangle$ is still fully determined by $\{x, y\}$. Suppose $\mathcal{V}_l \cap \mathcal{S}^{\mathrm{c}}_{\mathrm{G}} = \{z\}$, and $\Psi(z) = v^{\mathrm{O}}_{k(z)}$. Then $\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k(z)})} = 0$. Similar to the previous case, if $k' = k(z)$ then $\mathbf{J}_{i,j} = \mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k(z)})} = 0$. If not, then there is an cyclic arc in $\mathcal{G}_l$ connecting $v^{\mathrm{O}}_{k(z)}$ and $v^{\mathrm{O}}_{k'}$. If the path is

$$v^{\mathrm{O}}_{k(z)} \leftrightarrow v^{\mathrm{O}}_{k(z)+1} \leftrightarrow v^{\mathrm{O}}_{k(z)+2} \cdots v^{\mathrm{O}}_{k'-1} \leftrightarrow v^{\mathrm{O}}_{k'},$$

then

$$\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k'})} = \sum_{k=k(z)}^{k'-1} \frac{1}{b_{e^{\mathrm{O}}_k}} \langle \mathcal{G} \rangle.$$

If the path is

$$v^{\mathrm{O}}_{k'} \leftrightarrow v^{\mathrm{O}}_{k'+1} \leftrightarrow v^{\mathrm{O}}_{k'+2} \cdots v^{\mathrm{O}}_{k(z)-1} \leftrightarrow v^{\mathrm{O}}_{k(z)},$$

then

$$\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k'})} = -\sum_{k=k'}^{k(z)-1} \frac{1}{b_{e^{\mathrm{O}}_k}} \langle \mathcal{G} \rangle.$$

In this case, the value of $\mathbf{J}_{i,j}$ is determined and can be easily computed whenever $\mathcal{V}_1 \cap \mathcal{S}^{\mathrm{c}}_{\mathrm{G}} = \{x, y\}$ and $\mathcal{V}_l \cap \mathcal{S}^{\mathrm{c}}_{\mathrm{G}} = \{z\}$ are known. Other elements in or not in $\mathcal{S}_{\mathrm{G}}$ and $\mathcal{S}_{\mathrm{B}}$ do not matter.

After discussing over all three cases, we figure out that in order to compute the worst value of $\mathbf{J}_{i,j}$, one does not need to search over all the possible combinations of $\mathcal{S}_{\mathrm{G}}$ and $\mathcal{S}_{\mathrm{B}}$. In fact, the conservation law of edge unbalance implies that the value of

$\mathbf{J}_{i,j}$ only depends on a few elements in $\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$. For two different vertices $v_\alpha^{\mathrm{O}}, v_\beta^{\mathrm{O}} \in \mathcal{V}^{\mathrm{O}}$, define

$$\mathcal{V}^{v_\alpha^{\mathrm{O}} \to v_\beta^{\mathrm{O}}} := \begin{cases} \{v_{\alpha+1}^{\mathrm{O}}, v_{\alpha+2}^{\mathrm{O}}, \cdots, v_{\beta-1}^{\mathrm{O}}\}, & \alpha+1 \neq \beta \\ \emptyset, & \text{otherwise} \end{cases}$$

and

$$\mathcal{E}^{v_\alpha^{\mathrm{O}} \to v_\beta^{\mathrm{O}}} := \{e_\alpha^{\mathrm{O}}, e_{\alpha+1}^{\mathrm{O}}, \cdots, e_{\beta-1}^{\mathrm{O}}\}.$$

We then propose Algorithm 5.1 which can compute $C_{i\leftarrow j}^{\mathrm{wc}} = \max |\mathbf{J}_{i,j}|$ in polynomial time.

5.4. Graph With Multiple Non-Adjacent Cycles. Finally, we study the graphs in which any two cycles have no vertex in common. Intuitively, the macro-structure of such graph is still a tree, but there exist multiple local cycles in the graph. For those graphs, we will see that the worst-case sensitivity $C_{i\leftarrow j}^{\mathrm{wc}}$ of generator i with respect to load j across the graph can be computed as the product of sensitivities for each local cycle in the graph. Thereby, using Algorithm 5.1 in the previous subsection as a module, one can easily compute $C_{i\leftarrow j}^{\mathrm{wc}}$ for arbitrary i, j.

Similar to the previous subsections, we fix generator i and assume $i \in \mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$. Construct $\mathbf{q}$ as in (5.8), so $\mathbf{q}^\mathsf{T}$ is the i^{th} row of $\mathbf{H}_1$.

This section will involve the operations for the subgraphs of $\mathcal{G}$, and in the subgraph, the generator set and load set are no longer $[N_{\mathrm{G}}]$ and $[N_{\mathrm{G}}+1, N]$. If generator $i \in \mathcal{V}_{\mathrm{G}}(\mathcal{G}')$ and load $j + N_{\mathrm{G}} \in \mathcal{V}_{\mathrm{L}}(\mathcal{G}')$, then we use $\mathbf{J}_{\mathcal{G}'}^{i,j+N_{\mathrm{G}}}$ to denote the derivative of the gerator i with respect to the load $j + N_{\mathrm{G}}$. For the original graph $\mathcal{G}$, clearly we have $\mathbf{J}_{\mathcal{G}}^{i,j+N_{\mathrm{G}}} = \mathbf{J}_{i,j}$.

THEOREM 5.16. *For edge $e = (u, v) \in \mathcal{E}^{\mathrm{I}}$, if $u, v \notin \mathcal{V}_{\mathrm{G}}$, then $\mathbf{q}_{\Lambda(u)} = \mathbf{q}_{\Lambda(v)}$.*

Proof. Suppose e cuts the graph into two subgraphs $\mathcal{G}_l(\mathcal{V}_l, \mathcal{E}_l)$ for $l \in \{A, B\}$. Assume $u, i \in \mathcal{V}_A (u \neq i)$ and $v \in \mathcal{V}_B$. Lemma 5.2 decomposed $\mathbf{q} = \mathbf{q}^A + \mathbf{q}^B$ and implies that

$$\mathbf{R}\mathbf{q}^A - \mathbf{q}_{\Lambda(v)} b_e(\mathbf{e}_u - \mathbf{e}_v) = -\mathbf{C}\mathbf{B}\mathbf{C}^\mathsf{T}\mathbf{e}_i \tag{5.10a}$$
$$\mathbf{R}\mathbf{q}^B = -\mathbf{q}_{\Lambda(v)} b_e(\mathbf{e}_u - \mathbf{e}_v) \tag{5.10b}$$

and

$$\{t \in [N] \mid \mathbf{e}_t^\mathsf{T}\mathbf{R}\mathbf{q}^A \neq 0\} \subseteq \mathcal{V}_A \cup \{v\}$$
$$\{t \in [N] \mid \mathbf{e}_t^\mathsf{T}\mathbf{R}\mathbf{q}^B \neq 0\} \subseteq \{u, v\}.$$

Evaluating the v^{th} coordinate on both sides of (5.10a), we have $b_e\mathbf{q}_{\Lambda(u)} - b_e\mathbf{q}_{\Lambda(v)} = 0$. Thereby, $\mathbf{q}_{\Lambda(u)} = \mathbf{q}_{\Lambda(v)}$. ∎

When $u, v \notin \mathcal{V}_{\mathrm{G}}$, the value of $\mathbf{q}_{\Lambda(u)}$ and $\mathbf{q}_{\Lambda(u)}$ are essentially the values of $\mathbf{J}_{\mathcal{G}}^{i,u}$ and $\mathbf{J}_{\mathcal{G}}^{i,v}$. While Theorem 5.12 indicates that the intra-cycle unbalance stays constant, Theorem 5.16 actually tells us that the inter-cycle Jacobean value stays constant.

Consider any subgraph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ of $\mathcal{G}$, where $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E} \subseteq \mathcal{E}'$. Suppose $\mathcal{V}_{\mathrm{G}}(\mathcal{G}') \subseteq \mathcal{V}'$ is the set of *pseudo* generators of $\mathcal{G}'$. Here we use *pseudo* since $\mathcal{V}_{\mathrm{G}}(\mathcal{G}')$ does not necessarily need to be $\mathcal{V}_{\mathrm{G}} \cap \mathcal{V}'$. Similarly, suppose $\mathcal{V}_{\mathrm{L}}(\mathcal{G}') \subseteq \mathcal{V}'$ is the set of *pseudo* loads of $\mathcal{G}'$. For sets $\mathcal{S}_{\mathrm{G}}(\mathcal{G}') \subseteq \mathcal{V}_{\mathrm{G}}$ and $\mathcal{S}_{\mathrm{B}}(\mathcal{G}') \subseteq \mathcal{E}$, we still view them as the binding constraints for $\mathcal{G}'$. Let $\mathbf{C}_{\mathcal{G}'} \in \mathbb{R}^{\mathcal{V}\times\mathcal{E}}$ be the matrix which has the same size as

Algorithm 5.1 Compute $C^{\mathrm{wc}}_{i\leftarrow j}$ for the unicyclic graph.

Input: $\mathbf{B}$, $\mathbf{C}$, N_{G}, N_{L}, $i \in [N_{\mathrm{G}}]$, $j \in [N_{\mathrm{L}}]$
Output: $C^{\mathrm{wc}}_{i\leftarrow j}$

```
C^wc_{i←j} ⇐ 1
if N_G ≥ 2 then
  for x in V_G (x ≠ i) do
    for e in E do
      S_G ⇐ V_G \ {x, i}
      S_B ⇐ E \ {e}
      if S_G ⊥ S_B then
        compute J(S_G, S_B) by (4.6)
        if |J_{i,j}| > C^wc_{i←j} then
          C^wc_{i←j} ⇐ |J_{i,j}|
if N_G < 3 or Ψ(i) = Ψ(j + N_G) then
  return C^wc_{i←j}
else
  for x in V_G (Ψ(i) ≠ Ψ(x) ≠ Ψ(j + N_G)) do
    if Ψ(j + N_G) ∈ V^{Ψ(i)→Ψ(x)} then
      if ∃z ∈ V_G such that Ψ(z) ∈ V^{Ψ(x)→Ψ(i)} then
        ⟨G⟩ ⇐ −( Σ_{e∈E^{Ψ(i)→Ψ(x)}} 1/b_e )^{−1}
        J_{i,j} ⇐ −( Σ_{e∈E^{Ψ(j+N_G)→Ψ(x)}} 1/b_e ) ⟨G⟩
        if |J_{i,j}| > C^wc_{i←j} then
          C^wc_{i←j} ⇐ |J_{i,j}|
      for z ∈ V_G such that Ψ(z) ∈ V^{Ψ(i)→Ψ(x)} do
        ⟨G⟩ ⇐ ( Σ_{e∈E^{Ψ(x)→Ψ(i)}} 1/b_e )^{−1}
        J_{i,j} ⇐ {  ( Σ_{e∈E^{Ψ(z)→Ψ(j+N_G)}} 1/b_e ) ⟨G⟩,     Ψ(z) ∈ V^{Ψ(i)→Ψ(j+N_G)}
                  { −( Σ_{e∈E^{Ψ(j+N_G)→Ψ(z)}} 1/b_e ) ⟨G⟩,    Ψ(z) ∈ V^{Ψ(j+N_G)→Ψ(x)}
        if |J_{i,j}| > C^wc_{i←j} then
          C^wc_{i←j} ⇐ |J_{i,j}|
    else
      if ∃z ∈ V_G such that Ψ(z) ∈ V^{Ψ(i)→Ψ(x)} then
        ⟨G⟩ ⇐ ( Σ_{e∈E^{Ψ(x)→Ψ(i)}} 1/b_e )^{−1}
        J_{i,j} ⇐ −( Σ_{e∈E^{Ψ(x)→Ψ(j+N_G)}} 1/b_e ) ⟨G⟩
        if |J_{i,j}| > C^wc_{i←j} then
          C^wc_{i←j} ⇐ |J_{i,j}|
      for z ∈ V_G such that Ψ(z) ∈ V^{Ψ(x)→Ψ(i)} do
        ⟨G⟩ ⇐ −( Σ_{e∈E^{Ψ(i)→Ψ(x)}} 1/b_e )^{−1}
        J_{i,j} ⇐ {  ( Σ_{e∈E^{Ψ(z)→Ψ(j+N_G)}} 1/b_e ) ⟨G⟩,     Ψ(z) ∈ V^{Ψ(x)→Ψ(j+N_G)}
                  { −( Σ_{e∈E^{Ψ(j+N_G)→Ψ(z)}} 1/b_e ) ⟨G⟩,    Ψ(z) ∈ V^{Ψ(j+N_G)→Ψ(i)}
        if |J_{i,j}| > C^wc_{i←j} then
          C^wc_{i←j} ⇐ |J_{i,j}|
return C^wc_{i←j}
```

$\mathbf{C}$ but whose columns corresponding to edges in $\mathcal{E} \setminus \mathcal{E}'$ are replaced by zero columns. Let

$$\mathbf{R}_{\mathcal{G}'}(\mathcal{S}_{\mathrm{G}}(\mathcal{G}'), \mathcal{S}_{\mathrm{B}}(\mathcal{G}'))^{\mathsf{T}} := \begin{bmatrix} \mathbf{I}^{N}_{\mathcal{V}_{\mathrm{L}}(\mathcal{G}')} \mathbf{C}_{\mathcal{G}'} \mathbf{B} \mathbf{C}^{\mathsf{T}}_{\mathcal{G}'} \\ \mathbf{I}^{N}_{\mathcal{S}_{\mathrm{G}}(\mathcal{G}')} \mathbf{C}_{\mathcal{G}'} \mathbf{B} \mathbf{C}^{\mathsf{T}}_{\mathcal{G}'} \\ \mathbf{I}^{E}_{\mathcal{S}_{\mathrm{B}}(\mathcal{G}')} \mathbf{B} \mathbf{C}^{\mathsf{T}}_{\mathcal{G}'} \\ \mathbf{e}_i^{\mathsf{T}} \end{bmatrix}.$$

Definition 5.17. *If all the columns of $\mathbf{R}_{\mathcal{G}'}$ are independent, then we will denote $\mathcal{S}_{\mathrm{G}}(\mathcal{G}') \perp_{\mathcal{G}'} \mathcal{S}_{\mathrm{B}}(\mathcal{G}')$.*

Remark 5.18. The concept of $\perp_{\mathcal{G}'}$ is the generalization of $\perp$ for $\mathcal{G}$. Informally, $\mathcal{S}_{\mathrm{G}}(\mathcal{G}') \perp_{\mathcal{G}'} \mathcal{S}_{\mathrm{B}}(\mathcal{G}')$ means that in the OPF problem (2.1) on $\mathcal{G}'$ (with generators $\mathcal{V}_{\mathrm{G}}(\mathcal{G}')$ and loads $\mathcal{V}_{\mathrm{L}}(\mathcal{G}')$), all the binding constraints and equality constraints are independent.

Definition 5.19. *For $(\mathcal{G}', \mathcal{V}_{\mathrm{G}}(\mathcal{G}'), \mathcal{V}_{\mathrm{L}}(\mathcal{G}'))$, we say that $\mathcal{S}_{\mathrm{G}}(\mathcal{G}')$ and $\mathcal{S}_{\mathrm{B}}(\mathcal{G}')$ are a perfect pair if $\mathcal{S}_{\mathrm{G}}(\mathcal{G}') \perp \mathcal{S}_{\mathrm{B}}(\mathcal{G}')$ and $|\mathcal{S}_{\mathrm{G}}(\mathcal{G}')| + |\mathcal{S}_{\mathrm{B}}(\mathcal{G}')| = |\mathcal{V}_{\mathrm{G}}(\mathcal{G}')| - 1$.*

Similar to Λ and Ξ, we use $\Lambda_{\mathcal{G}'}$ to map the indices of the load buses $\mathcal{V}'$ and the binding generators $\mathcal{S}_{\mathrm{G}}(\mathcal{G}')$ to the indices of the corresponding columns in $\mathbf{R}_{\mathcal{G}'}$, and use $\Xi_{\mathcal{G}'}$ to map the indices of the binding edges $\mathcal{S}_{\mathrm{B}}$ to the index of the corresponding columns in $\mathbf{R}_{\mathcal{G}'}$. When $\mathcal{S}_{\mathrm{G}}(\mathcal{G}')$ and $\mathcal{S}_{\mathrm{B}}(\mathcal{G}')$ are a *perfect* pair, there exists a unique $\mathbf{a} \in \mathbb{R}^{|\mathcal{V}'|}$ such that $\mathbf{R}_{\mathcal{G}'}\mathbf{a} = -\mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{C}^{\mathsf{T}}_{\mathcal{G}'}\mathbf{e}_i$. Similar to the Jacobean matrix for $\mathcal{G}$, we have $\mathbf{J}^{i,j+N_{\mathrm{G}}}_{\mathcal{G}'} = \mathbf{a}_{\Lambda_{\mathcal{G}'}(j+N_{\mathrm{G}})}$.

Definition 5.20. *We say that $(\mathcal{G}', \mathcal{V}_{\mathrm{G}}(\mathcal{G}'), \mathcal{V}_{\mathrm{L}}(\mathcal{G}'))$ with $i \in \mathcal{V}_{\mathrm{G}}(\mathcal{G}')$ and $j + N_{\mathrm{G}} \in \mathcal{V}_{\mathrm{L}}(\mathcal{G}')$ simulates $(\mathcal{G}, \mathcal{V}_{\mathrm{G}}, \mathcal{V}_{\mathrm{L}})$ if*

- *For any perfect pair $(\mathcal{S}_{\mathrm{G}}, \mathcal{S}_{\mathrm{B}})$ for $\mathcal{G}$ such that $i \in \mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$, there exists a perfect pair $(\mathcal{S}_{\mathrm{G}}(\mathcal{G}'), \mathcal{S}_{\mathrm{B}}(\mathcal{G}'))$ for $\mathcal{G}'$ such that $i \in \mathcal{S}_{\mathrm{G}}(\mathcal{G}')^{\mathsf{c}}$, and $\mathbf{J}^{i,j+N_{\mathrm{G}}}_{\mathcal{G}} = \mathbf{J}^{i,j+N_{\mathrm{G}}}_{\mathcal{G}'}$.*
- *For any perfect pair $(\mathcal{S}_{\mathrm{G}}(\mathcal{G}'), \mathcal{S}_{\mathrm{B}}(\mathcal{G}'))$ for $\mathcal{G}'$ such that $\in \mathcal{S}_{\mathrm{G}}(\mathcal{G}')$, there exists a perfect pair $(\mathcal{S}_{\mathrm{G}}, \mathcal{S}_{\mathrm{B}})$ for $\mathcal{G}$ such that $i \in \mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$, and $\mathbf{J}^{i,j+N_{\mathrm{G}}}_{\mathcal{G}} = \mathbf{J}^{i,j+N_{\mathrm{G}}}_{\mathcal{G}'}$.*

We will then analyze how the edges in $\mathcal{E}^{\mathrm{I}}$ can help partition the computation of $\mathbf{J}^{i,j+N_{\mathrm{G}}}_{\mathcal{G}}$ into the computation in smaller subgraphs. For $e = (u, v) \in \mathcal{E}^{\mathrm{I}}$ such that $u, v \notin \mathcal{V}_{\mathrm{G}}$, it partitions $\mathcal{G}$ into two connected subgraphs $\mathcal{G}_l(\mathcal{V}_l, \mathcal{E}_l)$ for $l = 1, 2$. Suppose $i, u \in \mathcal{V}_1$ and $v \in \mathcal{V}_2$. We first have the following two theorems that can simplify the graph when $j + N_{\mathrm{G}} \in \mathcal{V}_1$.

Theorem 5.21. *When $i, j + N_{\mathrm{G}} \in \mathcal{V}_1$ and $\mathcal{V}_2 \subseteq \mathcal{V}_{\mathrm{L}}$, we can construct a subgraph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ where*

$$\begin{aligned} \mathcal{V}' &= \mathcal{V}_1 \cup \{v\}, & \mathcal{E}' &= \mathcal{E}_1 \\ \mathcal{V}_{\mathrm{G}}(\mathcal{G}') &= \mathcal{V}_{\mathrm{G}} \cap \mathcal{V}_1, & \mathcal{V}_{\mathrm{L}}(\mathcal{G}') &= (\mathcal{V}_{\mathrm{L}} \cap \mathcal{V}_1) \cup \{v\}. \end{aligned}$$

Then $(\mathcal{G}', \mathcal{V}_{\mathrm{G}}(\mathcal{G}'), \mathcal{V}_{\mathrm{L}}(\mathcal{G}'))$ simulates $(\mathcal{G}, \mathcal{V}_{\mathrm{G}}, \mathcal{V}_{\mathrm{L}})$.

Theorem 5.22. *When $i, j + N_{\mathrm{G}} \in \mathcal{V}_1$ and $\mathcal{V}_2 \nsubseteq \mathcal{V}_{\mathrm{L}}$, we can construct a subgraph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ where*

$$\begin{aligned} \mathcal{V}' &= \mathcal{V}_1 \cup \{v\}, & \mathcal{E}' &= \mathcal{E}_1 \\ \mathcal{V}_{\mathrm{G}}(\mathcal{G}') &= (\mathcal{V}_{\mathrm{G}} \cap \mathcal{V}_1) \cup \{v\}, & \mathcal{V}_{\mathrm{L}}(\mathcal{G}') &= \mathcal{V}_{\mathrm{L}} \cap \mathcal{V}_1. \end{aligned}$$

Then $(\mathcal{G}', \mathcal{V}_{\mathrm{G}}(\mathcal{G}'), \mathcal{V}_{\mathrm{L}}(\mathcal{G}'))$ simulates $(\mathcal{G}, \mathcal{V}_{\mathrm{G}}, \mathcal{V}_{\mathrm{L}})$.

The proofs of Theorem 5.21 and Theorem 5.22 are given in Appendix H. Those two theorems informally say that if an edge separate a subgraph from generator i and load j, then we can simply replace the subgraph by a load or a generator depending on whether there is any generator included in the subgraph. Such replacement will not change the worst-case sensitivity, but can simplify the network structure.

We then construct subgraphs $\mathcal{G}'(\mathcal{V}',\mathcal{E}')$ and $\mathcal{G}''(\mathcal{V}'',\mathcal{E}'')$ where

$$\begin{aligned} \mathcal{V}' &= \mathcal{V}_1 \cup \{v\}, & \mathcal{V}'' &= \mathcal{V}_2 \cup \{u\}, \\ \mathcal{E}' &= \mathcal{E}_1 \cup \{e\}, & \mathcal{E}'' &= \mathcal{E}_2 \cup \{e\}. \end{aligned}$$

We further construct

$$\begin{aligned} \mathcal{V}_{\mathrm{G}}(\mathcal{G}') &= \mathcal{V}' \cap \mathcal{V}_{\mathrm{G}}, & \mathcal{V}_{\mathrm{L}}(\mathcal{G}') &= \mathcal{V}' \cap \mathcal{V}_{\mathrm{L}}, \\ \mathcal{V}_{\mathrm{G}}(\mathcal{G}'') &= (\mathcal{V}'' \cap \mathcal{V}_{\mathrm{G}}) \cup \{u\}, & \mathcal{V}_{\mathrm{L}}(\mathcal{G}'') &= \mathcal{V}_2 \cap \mathcal{V}_{\mathrm{L}} \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}_{\mathrm{G}}(\mathcal{G}') &= \mathcal{S}_{\mathrm{G}} \cap \mathcal{V}_1, & \mathcal{S}_{\mathrm{B}}(\mathcal{G}') &= \mathcal{S}_{\mathrm{B}} \cap \mathcal{E}_1, \\ \mathcal{S}_{\mathrm{G}}(\mathcal{G}'') &= \mathcal{S}_{\mathrm{G}} \cap \mathcal{V}_2, & \mathcal{S}_{\mathrm{B}}(\mathcal{G}'') &= \mathcal{S}_{\mathrm{B}} \cap \mathcal{E}_2. \end{aligned}$$

The above construction converts $\mathcal{G}$ into 2 subgraphs with their own pseudo generators and loads. We will cal this process the *proliferation* of $\mathcal{G}$ under e.

Only consider the case that $\mathcal{S}_{\mathrm{B}} \subseteq \mathcal{E}^{\mathrm{II}}$, we have the following theorem.

THEOREM 5.23. *If $\mathcal{S}_{\mathrm{G}}(\mathcal{G}')$ and $\mathcal{S}_{\mathrm{B}}(\mathcal{G}')$ are a perfect pair for $\mathcal{G}'$ and $\mathcal{S}_{\mathrm{G}}(\mathcal{G}'')$ and $\mathcal{S}_{\mathrm{B}}(\mathcal{G}'')$ are a perfect pair for $\mathcal{G}''$, then $\mathcal{S}_{\mathrm{G}}$ and $\mathcal{S}_{\mathrm{B}}$ are a perfect pair for $\mathcal{G}''$.*

Proof. We first have

$$\begin{aligned} N_{\mathrm{G}} &= |\mathcal{V}_{\mathrm{G}}(\mathcal{G}')| + |\mathcal{V}_{\mathrm{G}}(\mathcal{G}'')| - 1 \\ |\mathcal{S}_{\mathrm{G}}| &= |\mathcal{S}_{\mathrm{G}}(\mathcal{G}')| + |\mathcal{S}_{\mathrm{G}}(\mathcal{G}'')| \\ |\mathcal{S}_{\mathrm{B}}| &= |\mathcal{S}_{\mathrm{B}}(\mathcal{G}')| + |\mathcal{S}_{\mathrm{B}}(\mathcal{G}'')|, \end{aligned}$$

hence $|\mathcal{S}_{\mathrm{G}}| + |\mathcal{S}_{\mathrm{B}}| = N_{\mathrm{G}} - 1$. Next we are proving $\mathcal{S}_{\mathrm{G}} \perp \mathcal{S}_{\mathrm{B}}$. If not, then $\mathbf{R}$ is not invertible and there exists $\mathbf{q} \in \mathbb{R}^{N_{\mathrm{G}}}$ such that $\mathbf{q} \neq 0$ and $\mathbf{R}\mathbf{q} = 0$. Consider

$$\mathbf{q}^l = \sum_{\alpha \in \mathcal{V}_l \setminus \mathcal{S}_{\mathrm{G}}^{\mathsf{c}}} \tilde{\mathbf{q}}_{\Lambda(\alpha)} + \sum_{\beta \in \mathcal{E}_l \cap \mathcal{S}_{\mathrm{B}}} \tilde{\mathbf{q}}_{\Xi(\beta)}$$

for $l = 1, 2$. Informally, we disribute the first $N_{\mathrm{G}} - 1$ columns of $\mathbf{R}$ into either $\mathbf{q}^1$ or $\mathbf{q}^2$. Since $\mathbf{R}\mathbf{q} = 0$ and only the last column of $\mathbf{R}$ does not sum up to 0, we have $\mathbf{q} = \mathbf{q}^1 + \mathbf{q}^2$ and thereby at least one of $\mathbf{q}^l$ is non zero. Assume $\mathbf{q}^1 \neq 0$. One necessary condition for $\mathbf{e}_t^{\mathsf{T}}\mathbf{R}\mathbf{q}^1 \neq 0$ is: $\exists \alpha \in \mathcal{V} \setminus \mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$ which is adjacent to t such that $\mathbf{q}^1_{\Lambda(\alpha)} \neq 0$, or $\exists \beta \in \mathcal{S}_{\mathrm{B}}$ which links t such that $\mathbf{q}^1_{\Xi(\beta)} \neq 0$. Since v is the only vertex in $\mathcal{V}_2$ that is adjacent to $\mathcal{G}_1$, so

$$\{t \in [N] \mid \mathbf{e}_t^{\mathsf{T}}\mathbf{R}\mathbf{q}^1 \neq 0\} \subseteq \mathcal{V}_1 \cup \{v\} = \mathcal{V}'.$$

Similarly,

$$\{t \in [N] \mid \mathbf{e}_t^{\mathsf{T}}\mathbf{R}\mathbf{q}^2 \neq 0\} \subseteq \mathcal{V}_2 \cup \{u\} = \mathcal{V}''.$$

Since $\mathbf{R}\mathbf{q}^1 + \mathbf{R}\mathbf{q}^2 = 0$, we have $\forall t \in \mathcal{V} \setminus \mathcal{V}''$, $\mathbf{e}_t^{\mathsf{T}}\mathbf{R}\mathbf{q}^1 = -\mathbf{e}_t^{\mathsf{T}}\mathbf{R}\mathbf{q}^2 = 0$ and $\forall t \in \mathcal{V} \setminus \mathcal{V}'$, $\mathbf{e}_t^{\mathsf{T}}\mathbf{R}\mathbf{q}^2 = -\mathbf{e}_t^{\mathsf{T}}\mathbf{R}\mathbf{q}^1 = 0$. As the result,

$$\{t \in [N] \mid \mathbf{e}_t^{\mathsf{T}}\mathbf{R}\mathbf{q}^l \neq 0\} \subseteq \{u, v\}$$

for $l = 1, 2$. Since $\mathbf{R}\mathbf{q}^1$ is the linear combination of a subset of the first $N_{\mathrm{G}} - 1$ columns of $\mathbf{R}$, all coordinates of $\mathbf{R}\mathbf{q}^1$ sum up to 0. Thus there must be $q \in \mathbb{R}$ such that

$$\mathbf{R}\mathbf{q}^1 + q(\mathbf{e}_v - \mathbf{e}_u) = 0. \tag{5.11}$$

From the construction of $\mathbf{q}^1$, we can see that for t such that $\mathbf{q}^1_t \neq 0$, if $t = \Lambda(\alpha)$ for some $\alpha \in \mathcal{V}_1 \setminus \mathcal{S}^{\mathrm{c}}_{\mathrm{G}}$, then the t^{th} column of $\mathbf{R}$ is $\mathbf{R}\mathbf{e}_t = \mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}}\mathbf{e}_\alpha$, which is the same as $\mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{C}^{\mathsf{T}}_{\mathcal{G}'}\mathbf{e}_\alpha$ and the latter is also a column in $\mathbf{R}_{\mathcal{G}'}$. Likely, if $t = \Xi(\beta)$ for some $\beta \in \mathcal{E}_1 \cap \mathcal{S}_{\mathrm{B}}$, then the t^{th} column of $\mathbf{R}$ is $\mathbf{R}\mathbf{e}_t = \mathbf{C}\mathbf{B}\mathbf{e}_\beta$, which is the same as $\mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{e}_\beta$ and the latter is still a column in $\mathbf{R}_{\mathcal{G}'}$. Moreover, we have $q(\mathbf{e}_v - \mathbf{e}_u) = \mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{e}_e$ is also a column of $\mathbf{R}_{\mathcal{G}'}$ and is different from the columns $\mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{e}_\beta$ for $\beta \in \mathcal{E}_1 \cap \mathcal{S}_{\mathrm{B}}$. Thereby, (5.11) implies that the columns of $\mathbf{R}_{\mathcal{G}'}$ are not independent. It contradicts to $\mathcal{S}_{\mathrm{G}}(\mathcal{G}') \perp \mathcal{S}_{\mathrm{B}}(\mathcal{G}')$, hence we must have $\mathcal{S}_{\mathrm{G}} \perp \mathcal{S}_{\mathrm{B}}$. ∎

THEOREM 5.24. *When $(\mathcal{S}_{\mathrm{G}}, \mathcal{S}_{\mathrm{B}})$ is a perfect pair for $\mathcal{G}$. If $j + N_{\mathrm{G}} \in \mathcal{V}_2$ and $\mathbf{J}^{i,j+N_{\mathrm{G}}}_{\mathcal{G}} \neq 0$, then $(\mathcal{S}_{\mathrm{G}}(\mathcal{G}'), \mathcal{S}_{\mathrm{B}}(\mathcal{G}'))$ is a perfect pair for $\mathcal{G}'$ and $(\mathcal{S}_{\mathrm{G}}(\mathcal{G}''), \mathcal{S}_{\mathrm{B}}(\mathcal{G}''))$ is a perfect pair for $\mathcal{G}''$.*

Proof. First, we are showing each column in $\mathbf{R}_{\mathcal{G}''}$ is also a column in $\mathbf{R}$. For $\alpha \in \mathcal{V}_{\mathrm{L}}(\mathcal{G}'')$, we know that $\alpha \in \mathcal{V}_2 \cap \mathcal{V}_{\mathrm{L}} \subseteq \mathcal{V}''$. Consider that u is the only element in $\mathcal{V}''$ which is adjacent to $\mathcal{V} \setminus \mathcal{V}''$ and $u \notin \mathcal{V}_2$, so all the neighbors of α in $\mathcal{G}$ are also in $\mathcal{G}''$. The column $\mathbf{C}_{\mathcal{G}''}\mathbf{B}\mathbf{C}^{\mathsf{T}}_{\mathcal{G}''}\mathbf{e}_\alpha$ is

$$\sum_{\kappa:(\alpha,\kappa)\in\mathcal{E}''} b_{(\alpha,\kappa)}(\mathbf{e}_\alpha - \mathbf{e}_\kappa) = \sum_{\kappa:(\alpha,\kappa)\in\mathcal{E}} b_{(\alpha,\kappa)}(\mathbf{e}_\alpha - \mathbf{e}_\kappa) = \mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}}\mathbf{e}_\alpha \tag{5.12}$$

which is thereby column α in $\mathbf{R}$. For $\alpha \in \mathcal{S}_{\mathrm{G}}(\mathcal{G}'')$, we have $\alpha \in \mathcal{S}_{\mathrm{G}} \cap \mathcal{V}''$. Clearly $\alpha \neq u$ as well as $u \notin \mathcal{S}_{\mathrm{G}}$. All the neighbors of α in $\mathcal{G}$ are still in $\mathcal{G}''$. Hence (5.12) still holds and the column $\mathbf{C}_{\mathcal{G}''}\mathbf{B}\mathbf{C}^{\mathsf{T}}_{\mathcal{G}''}\mathbf{e}_\alpha$ is also in $\mathbf{R}$. For edge $\beta \in \mathcal{S}_{\mathrm{B}}(\mathcal{G}'')$, the column $\mathbf{C}_{\mathcal{G}''}\mathbf{B}\mathbf{e}_\beta$ is still in $\mathbf{C}\mathbf{B}$ since β is an edge in both $\mathcal{E}$ and $\mathcal{E}''$. Finally, $\mathbf{e}_1$ is trivially a column in $\mathbf{R}$. Therefore, all columns in $\mathbf{R}_{\mathcal{G}''}$ are different columns in $\mathbf{R}$. Matrix $\mathbf{R}$ is invertible implies that all columns of $\mathbf{R}_{\mathcal{G}''}$ are independent of each other.

Second we are showing all the columns of $\mathbf{R}_{\mathcal{G}'}$ are also independent. Similar to $\mathcal{G}''$, we found that for $\alpha \in \mathcal{V}_{\mathrm{L}}(\mathcal{G}') \setminus \{v\}$ or for $\alpha \in \mathcal{S}_{\mathrm{G}}(\mathcal{G}')$, and for $\beta \in \mathcal{S}_{\mathrm{B}}(\mathcal{G}')$, the columns of $\mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{C}^{\mathsf{T}}_{\mathcal{G}'}\mathbf{e}_\alpha$ and $\mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{e}_\beta$ are all different columns in $\mathbf{R}$. It is sufficient to show that the column $\mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{C}^{\mathsf{T}}_{\mathcal{G}'}\mathbf{e}_v$ is also independent of other columns in $\mathbf{R}_{\mathcal{G}'}$.

If not, then there exist coefficients a_α for $\alpha \in (\mathcal{V}_{\mathrm{L}}(\mathcal{G}') \setminus \{v\}) \cup \mathcal{S}_{\mathrm{G}}(\mathcal{G}')$ and b_β for $\beta \in \mathcal{S}_{\mathrm{B}}(\mathcal{G}')$, such that

$$\mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{C}^{\mathsf{T}}_{\mathcal{G}'}\mathbf{e}_v = b_e(\mathbf{e}_v - \mathbf{e}_u) = \sum_{\substack{\alpha\in\mathcal{V}_{\mathrm{L}}(\mathcal{G}')\cup\mathcal{S}_{\mathrm{G}}(\mathcal{G}')\\ \alpha\neq v}} a_\alpha \mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{C}^{\mathsf{T}}_{\mathcal{G}'}\mathbf{e}_\alpha + \sum_{\beta\in\mathcal{S}_{\mathrm{B}}(\mathcal{G}')} b_\beta \mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{e}_\beta. \tag{5.13}$$

We also construct $\mathbf{q}$ as in (5.8), so $\mathbf{q}_j = \mathbf{J}^{i,j+N_{\mathrm{G}}}_{\mathcal{G}} \neq 0$. Using Lemma 5.2, we have

$$\mathbf{R}\mathbf{q}^2 = q b_e(\mathbf{e}_v - \mathbf{e}_u) \tag{5.14}$$

for some $q \in \mathbb{R}$, where

$$\mathbf{q}^2 = \sum_{\alpha\in\mathcal{V}_2\setminus\mathcal{S}^{\mathrm{c}}_{\mathrm{G}}} \tilde{\mathbf{q}}_{\Lambda(\alpha)} + \sum_{\beta\in\mathcal{E}_2\cap\mathcal{S}_{\mathrm{B}}} \tilde{\mathbf{q}}_{\Xi(\beta)}.$$

Since $j+N_{\mathrm{G}} \in \mathcal{V}_2 \setminus \mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$, we have $\mathbf{q}_j^2 = \mathbf{q}_j \neq 0$. Combining (5.13) and (5.14) together, we have

$$\mathbf{R}\mathbf{q}^2 = \sum_{\substack{\alpha \in \mathcal{V}_{\mathrm{L}}(\mathcal{G}') \cup \mathcal{S}_{\mathrm{G}}(\mathcal{G}') \\ \alpha \neq v}} a_\alpha \mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{C}_{\mathcal{G}'}^{\mathsf{T}}\mathbf{e}_\alpha + \sum_{\beta \in \mathcal{S}_{\mathrm{B}}(\mathcal{G}')} b_\beta \mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{e}_\beta. \tag{5.15}$$

Notice that all the columns on the right hand side are also columns in $\mathbf{R}$ as aforementioned. The column $\mathbf{R}\mathbf{e}_j = \mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}}\mathbf{e}_{j+N_{\mathrm{G}}}$ on the left hand side has non-zero coefficient $\mathbf{q}_j^2$. Besides, as $j+N_{\mathrm{G}} \in \mathcal{V}_2$, the neighbors of vertex $j+N_{\mathrm{G}}$ does not include any vertex in $\mathcal{V}_1 \setminus \{u\}$. Thus the support [3] of $\mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}}\mathbf{e}_{j+N_{\mathrm{G}}}$ is disjoint from $\mathcal{V}_1 \setminus \{u\}$. However, for $\alpha \in \mathcal{V}_{\mathrm{L}}(\mathcal{G}') \cup \mathcal{S}_{\mathrm{G}}(\mathcal{G}')$ and $\alpha \neq v$, we have $\alpha \in \mathcal{V}_1$. Vertex α must either be some vertex or be adjacent to some vertex in $\mathcal{V}_1 \setminus \{u\}$ as $\mathcal{V}_1$ is a connected graph consisting of at least two vertices u and i. Therefore the support of $\mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{C}_{\mathcal{G}'}^{\mathsf{T}}\mathbf{e}_\alpha$ is not disjoint from $\mathcal{V}_1 \setminus \{u\}$. Additionally, for $\beta \in \mathcal{S}_{\mathrm{B}}(\mathcal{G}')$, since $e \notin \mathcal{S}_{\mathrm{B}}$ and $\mathcal{S}_{\mathrm{B}}(\mathcal{G}') = \mathcal{S}_{\mathrm{B}} \cap (\mathcal{E}_1 \cup \{e\})$, we have $\beta \in \mathcal{E}_1$. Therefore the support of $\mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{e}_\beta$ is not disjoint from $\mathcal{V}_1 \setminus \{u\}$ as well. Above all, we know that all the columns on the right hand side of (5.15) are columns of $\mathbf{R}$ but are different from the column $\mathbf{R}\mathbf{e}_{j+N_{\mathrm{G}}}$, while the latter is on the left hand side and has a non-zero coefficient. Thereby the columns of $\mathbf{R}$ are not independent of each other and it leads to the contradiction. ∎

THEOREM 5.25. *Suppose $i \in \mathcal{V}_1$ and $j+N_{\mathrm{G}} \in \mathcal{V}_2$. When the binding constraints are all perfect pairs for all $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}''$, we have $\mathbf{J}_{\mathcal{G}}^{i,j+N_{\mathrm{G}}} = \mathbf{J}_{\mathcal{G}'}^{i,v}\mathbf{J}_{\mathcal{G}''}^{u,j+N_{\mathrm{G}}}$.*

Proof. Suppose $\mathbf{x} \in \mathbb{R}^{|\mathcal{V}'|}$ and $\mathbf{y} \in \mathbb{R}^{|\mathcal{V}''|}$ are the unique vectors satisfying

$$\begin{aligned}
\mathbf{R}_{\mathcal{G}'}\mathbf{x} &= -\mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{C}_{\mathcal{G}'}\mathbf{e}_i = -\mathbf{C}\mathbf{B}\mathbf{C}\mathbf{e}_i \\
\mathbf{R}_{\mathcal{G}''}\mathbf{y} &= -\mathbf{C}_{\mathcal{G}''}\mathbf{B}\mathbf{C}_{\mathcal{G}''}\mathbf{e}_u = b_e(\mathbf{e}_v - \mathbf{e}_u).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&-\mathbf{C}\mathbf{B}\mathbf{C}\mathbf{e}_i \\
&= \sum_{\alpha \in \mathcal{V}' \setminus \mathcal{S}_{\mathrm{G}}(\mathcal{G}')^{\mathsf{c}}} \mathbf{x}_{\Lambda_{\mathcal{G}'}(\alpha)} \mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{C}_{\mathcal{G}'}^{\mathsf{T}}\mathbf{e}_\alpha + \sum_{\beta \in \mathcal{S}_{\mathrm{B}}(\mathcal{G}')} \mathbf{x}_{\Xi_{\mathcal{G}'}(\beta)} \mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{e}_\beta \\
&= \sum_{\alpha \in \mathcal{V}_1 \setminus \mathcal{S}_{\mathrm{G}}(\mathcal{G}')^{\mathsf{c}}} \mathbf{x}_{\Lambda_{\mathcal{G}'}(\alpha)} \mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}}\mathbf{e}_\alpha + \sum_{\beta \in \mathcal{S}_{\mathrm{B}}(\mathcal{G}')} \mathbf{x}_{\Xi_{\mathcal{G}'}(\beta)} \mathbf{C}\mathbf{B}\mathbf{e}_\beta + \mathbf{x}_{\Lambda_{\mathcal{G}'}(v)} b_e(\mathbf{e}_v - \mathbf{e}_u) \\
&= \sum_{\alpha \in \mathcal{V}_1 \setminus \mathcal{S}_{\mathrm{G}}(\mathcal{G}')^{\mathsf{c}}} \mathbf{x}_{\Lambda_{\mathcal{G}'}(\alpha)} \mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}}\mathbf{e}_\alpha + \sum_{\beta \in \mathcal{S}_{\mathrm{B}}(\mathcal{G}')} \mathbf{x}_{\Xi_{\mathcal{G}'}(\beta)} \mathbf{C}\mathbf{B}\mathbf{e}_\beta + \mathbf{J}_{\mathcal{G}'}^{i,v}\mathbf{R}_{\mathcal{G}''}\mathbf{y} \\
&= \sum_{\alpha \in \mathcal{V}_1 \setminus \mathcal{S}_{\mathrm{G}}(\mathcal{G}')^{\mathsf{c}}} \mathbf{x}_{\Lambda_{\mathcal{G}'}(\alpha)} \mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}}\mathbf{e}_\alpha + \sum_{\beta \in \mathcal{S}_{\mathrm{B}}(\mathcal{G}')} \mathbf{x}_{\Xi_{\mathcal{G}'}(\beta)} \mathbf{C}\mathbf{B}\mathbf{e}_\beta \\
&\quad + \mathbf{J}_{\mathcal{G}'}^{i,v}\Big(\sum_{\substack{\alpha \in \mathcal{V}_2 \\ \alpha \notin \mathcal{S}_{\mathrm{G}}(\mathcal{G}'')^{\mathsf{c}}}} \mathbf{y}_{\Lambda_{\mathcal{G}''}(\alpha)} \mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}}\mathbf{e}_\alpha + \sum_{\beta \in \mathcal{S}_{\mathrm{B}}(\mathcal{G}'')} \mathbf{y}_{\Xi_{\mathcal{G}''}(\beta)} \mathbf{C}\mathbf{B}\mathbf{e}_\beta \Big).
\end{aligned}$$

We can construct $\mathbf{q} \in \mathbb{R}^N$ as

$$\mathbf{q}_{\Lambda(\alpha)} = \begin{cases} \mathbf{x}_{\Lambda_{\mathcal{G}'}(\alpha)}, & \alpha \in \mathcal{V}_1 \setminus \mathcal{S}_{\mathrm{G}}(\mathcal{G}')^{\mathsf{c}} \\ \mathbf{J}_{\mathcal{G}'}^{i,v}\mathbf{y}_{\Lambda_{\mathcal{G}''}(\alpha)}, & \alpha \in \mathcal{V}_2 \setminus \mathcal{S}_{\mathrm{G}}(\mathcal{G}'')^{\mathsf{c}} \end{cases}$$

[3] The support of a vector is the set of indices whose corresponding coordinates are non zero.

$$\mathbf{q}_{\Xi(\beta)} = \begin{cases} \mathbf{x}_{\Xi_{\mathcal{G}'}(\beta)}, & \beta \in \mathcal{S}_{\mathrm{B}}(\mathcal{G}') \\ \mathbf{J}^{i,v}_{\mathcal{G}'} \mathbf{y}_{\Xi_{\mathcal{G}''}(\beta)}, & \beta \in \mathcal{S}_{\mathrm{B}}(\mathcal{G}'') \end{cases}.$$

Other coordinates of $\mathbf{q}$ are set as 0. We then have $\mathbf{Rq} = -\mathbf{CBC}^{\mathsf{T}}\mathbf{e}_i$, so $\mathbf{J}^{i,j+N_{\mathrm{G}}}_{\mathcal{G}} = \mathbf{J}^{i,v}_{\mathcal{G}'} \mathbf{y}_{\Xi_{\mathcal{G}''}(j+N_{\mathrm{G}})} = \mathbf{J}^{i,v}_{\mathcal{G}'} \mathbf{J}^{u,j+N_{\mathrm{G}}}_{\mathcal{G}''}$. ∎

Theorem 5.23, Theorem 5.24 and Theorem 5.25 sheds lights on the following interesting fact. For edge $e = (u, v) \in \mathcal{E}^{\mathrm{I}}$ where $u, v \notin \mathcal{V}_{\mathrm{G}}$, if e separate generator i and load j in different connected subgraphs, then $\max |\mathbf{J}^{i,j+N_{\mathrm{G}}}_{\mathcal{G}}|$ can be computed as the product of $\max |\mathbf{J}^{i,v}_{\mathcal{G}'}|$ and $\max |\mathbf{J}^{u,j+N_{\mathrm{G}}}_{\mathcal{G}''}|$. In other words, a large graph can proliferate into smaller connected subgraphs, and the product of the worst-case sensitivities of subgraphs will be exactly the worst-case sensitivity of the original graph. Suppose $\mathcal{G}$ has $m + 1$ cycles, which are indicated as $\{\mathcal{G}^{\mathrm{O}_l}(\mathcal{V}^{\mathrm{O}_l}, \mathcal{E}^{\mathrm{O}_l})\}_{l=0}^{m}$. Assume all the cycles are on the shortest path from generator i to load j, otherwise, Theorem 5.21 and Theorem 5.22 implies that we can always replace a subgraph containing that cycle by a individual generator or load without changing the value of $\mathbf{J}_{i,j}$. Without loss of generality, we assume $\mathcal{G}^{\mathrm{O}_l}$ are indexed by their distance to i, and the shortest path connecting i and $\mathcal{G}^{\mathrm{O}_0}$ goes through no other cycles.

Definition 5.26. *The entrance of a cycle $\mathcal{G}^{\mathrm{O}_l}$ with respect to generator i, denoted as $e_l^{\diamond} = (u_l^{\diamond}, v_l^{\diamond})$, is the closest edge to $\mathcal{G}^{\mathrm{O}_l}$ on the shortest path connecting i and $\mathcal{G}^{\mathrm{O}_l}$. When we fix i, we simply call $e_l^{\diamond}$ the entrance of $\mathcal{G}^{\mathrm{O}_l}$.*

The entrances $\{e_l^{\diamond}\}_{l=1}^{m}$ will make $\mathcal{G}$ proliferate into $m+1$ subgraphs $\{\mathcal{G}_l\}_{l=0}^{m}$, and each subgraph contains exactly one cycle. For simplicity, suppose the cycle $\mathcal{G}^{\mathrm{O}_l}$ is contained by the subgraph $\mathcal{G}_l$. Clearly, $i \in \mathcal{V}_0$ and $j + N_{\mathrm{G}} \in \mathcal{V}_m$. Assume $v_{m+1}^{\diamond} = j + N_{\mathrm{G}}$.

Then Algorithm 5.2 is proposed to compute $C^{\mathrm{wc}}_{i\leftarrow j}$ in polynomial time. The illustrative diagram of the algorithm applying to a network with 6 cycles is presented in Figure 2.

Algorithm 5.2 Compute $C^{\mathrm{wc}}_{i\leftarrow j}$ for the graph with multiple non-adjacent cycles.

Input: $\mathbf{B}$, $\mathbf{C}$, N_{G}, N_{L}, $i \in [N_{\mathrm{G}}]$, $j \in [N_{\mathrm{L}}]$
Output: $C^{\mathrm{wc}}_{i\leftarrow j}$
  for $e = (u, v)$ in $\mathcal{E}^{\mathrm{I}}$ **do**
    if $u, v \notin \mathcal{V}_{\mathrm{G}}$ and $i \Leftrightarrow j + N_{\mathrm{G}}$ in $\mathcal{G}(\mathcal{V}, \mathcal{E} \setminus \{e\})$ **then**
      Simplify $\mathcal{G}$ using Theorem 5.21 and Theorem 5.22
  Find a shortest path connecting i and $j + N_{\mathrm{G}}$
  Get $\{\mathcal{G}^{\mathrm{O}_l}\}_{l=0}^{m}$ and $\{\mathcal{G}_l\}_{l=0}^{m}$
  call Algorithm 5.1 to compute $\max \left|\mathbf{J}^{i,v_1^{\diamond}}_{\mathcal{G}_0}\right|$
  for $l = 1$ to m **do**
    call Algorithm 5.1 to compute $\max \left|\mathbf{J}^{u_l^{\diamond},v_{l+1}^{\diamond}}_{\mathcal{G}_l}\right|$
  $C^{\mathrm{wc}}_{i\leftarrow j} \leftarrow \max \left|\mathbf{J}^{i,v_1^{\diamond}}_{\mathcal{G}_0}\right| \prod_{l=1}^{m} \max \left|\mathbf{J}^{u_l^{\diamond},v_{l+1}^{\diamond}}_{\mathcal{G}_l}\right|$
  return $C^{\mathrm{wc}}_{i\leftarrow j}$

Remark 5.27. Recall that Theorem 5.16 implies the inter-cycle sensitivities stay the constant. If we fix the generator i and view the worst-case sensitivity as a function

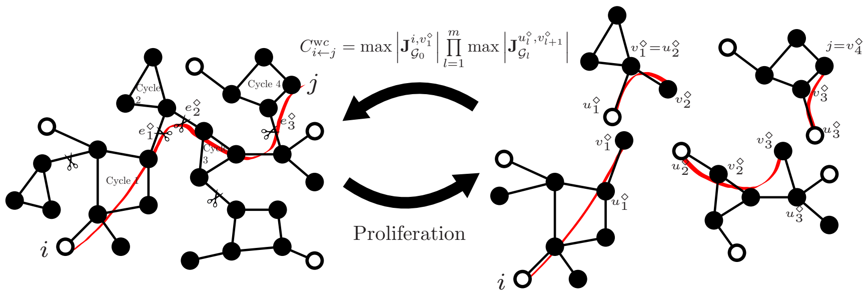

FIG. 2. *Illustrative diagram of Algorithm 5.2. Graph on the left illustrates an example of the original power network with 6 cycles, while the subgraphs on the right are the result after the proliferation. Solid black nodes stand for loads while hollow nodes stand for generators. Two ends of the red ribbon indicate the generator and the load associated with the Jacobean value of interest for each graph or subgraph. It can be seen from the figure that the algorithm first replaces the irrelative cycles which does not lie between generator i and load j by an individual generator or load. Then all four cycles on the shortest path connecting i and j help the network proliferate into 4 smaller subgraphs and each subgraph contains exactly 1 cycle. The worst-case sensitivity $C^{\mathrm{wc}}_{i\leftarrow j}$ across the graph is then broken down into the product of the worst-case sensitivities for each subgraph.*

of load j. Then the above analysis shows that each cycle behaves like a sensitivity amplifier or a sensitivity reducer, while the intra-cycle structure does not affect the sensitivity at all. In practice, a poorly designed network containing a long chain of cycles might potentially amplify the sensitivity to a significant degree from a load on one end to a generator on the other end. On the other hand, if the network is designed in a way such that each cycle can reduce the sensitivity with some contraction rate < 1, then two buses separated by multiple cycles will have less influence on each other. The power branches in $\mathcal{E}^{\mathrm{I}}$ can be arbitrarily designed for other purposes without affecting the sensitivity.

Appendix A. Proof of Proposition 2.1. It is sufficient to show that $\forall \boldsymbol{\xi} \in \Omega_{\text{para}}$, there exists a sequence $(\boldsymbol{\xi}_{(n)})_{n=1}^{\infty}$ such that $\lim\limits_{n\to\infty} \boldsymbol{\xi}_{(n)} = \boldsymbol{\xi}$ and for each $n \in \mathbb{Z}_+$ there is an open neighborhood $U(\boldsymbol{\xi}_{(n)}) \ni \boldsymbol{\xi}_{(n)}$ that $U(\boldsymbol{\xi}_{(n)}) \subseteq \Omega_{\text{para}}$.

First, we observe that (2.1b)-(2.1e) implies the branch power flow $\mathbf{p} := \mathbf{B}\mathbf{C}^{\mathsf{T}}\boldsymbol{\theta}$ satisfies

$$\mathbf{p} = \mathbf{B}\mathbf{C}^{\mathsf{T}}(\mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}})^{\dagger} \begin{bmatrix} \mathbf{s}^{g} \\ -\mathbf{s}^{l} \end{bmatrix}.$$

We use ρ to denote the matrix norm of $\mathbf{B}\mathbf{C}^{\mathsf{T}}(\mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}})^{\dagger}$ induced by the L1 vector norm:

$$\rho := \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{B}\mathbf{C}^{\mathsf{T}}(\mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}})^{\dagger}\mathbf{x}\|_1}{\|\mathbf{x}\|_1}.$$

Now consider any $\hat{\boldsymbol{\xi}} = [(\overline{\mathbf{s}}^{g})^{\mathsf{T}}, (\underline{\mathbf{s}}^{g})^{\mathsf{T}}, \overline{\mathbf{p}}^{\mathsf{T}}, \underline{\mathbf{p}}^{\mathsf{T}}]^{\mathsf{T}} \in \Omega_{\text{para}}$ with $\underline{\mathbf{s}}^{g} \geq 0$, and there exists $(\hat{\mathbf{s}}^{g}, \hat{\mathbf{s}}^{l})$ such that $\hat{\mathbf{s}}^{l} > 0$ and (2.1b)-(2.1e) are satisfied (with associated branch power flow $\hat{\mathbf{p}}$). Then we construct $\boldsymbol{\xi}_{(n)}$ as

$$\boldsymbol{\xi}_{(n)} = \begin{bmatrix} \overline{\mathbf{s}}^{g} + \frac{3}{n}\mathbf{1}_{N_{\mathrm{G}}} \\ \underline{\mathbf{s}}^{g} + \frac{1}{n}\mathbf{1}_{N_{\mathrm{G}}} \\ \overline{\mathbf{p}} + 5\rho\frac{N_{\mathrm{G}}}{n}\mathbf{1}_{E} \\ \underline{\mathbf{p}} - 5\rho\frac{N_{\mathrm{G}}}{n}\mathbf{1}_{E} \end{bmatrix}$$

and its open neighborhood

$$U(\boldsymbol{\xi}_{(n)}) = \left\{\boldsymbol{\xi} \middle| -\begin{bmatrix} \frac{1}{n}\mathbf{1}_{2N_{\mathrm{G}}} \\ \rho\frac{N_{\mathrm{G}}}{n}\mathbf{1}_{2E} \end{bmatrix} < \boldsymbol{\xi} - \boldsymbol{\xi}_{(n)} < \begin{bmatrix} \frac{1}{n}\mathbf{1}_{2N_{\mathrm{G}}} \\ \rho\frac{N_{\mathrm{G}}}{n}\mathbf{1}_{2E} \end{bmatrix} \right\}.$$

Clearly, $\boldsymbol{\xi}_{(n)}$ converges to $\hat{\boldsymbol{\xi}}$ as $n \to \infty$. Next, we are going to prove that for any $\boldsymbol{\xi} \in U(\boldsymbol{\xi}_{(n)})$, we have $\boldsymbol{\xi} \in \Omega_{\mathrm{para}}$. For the convenience of notations, we use $\overline{\mathbf{s}}^g(\boldsymbol{\xi}), \underline{\mathbf{s}}^g(\boldsymbol{\xi}), \overline{\mathbf{p}}(\boldsymbol{\xi}), \underline{\mathbf{p}}(\boldsymbol{\xi})$ to denote the corresponding part in $\boldsymbol{\xi}$. Since

$$\underline{\mathbf{s}}^g(\boldsymbol{\xi}) > \underline{\mathbf{s}}^g(\boldsymbol{\xi}_{(n)}) - \frac{1}{n} = \underline{\mathbf{s}}^g(\hat{\boldsymbol{\xi}}) + \frac{3}{n} - \frac{1}{n} \geq \underline{\mathbf{s}}^g(\hat{\boldsymbol{\xi}}) \geq 0,$$

we only need to check if there exist $(\mathbf{s}^g, \mathbf{s}^l)$ such that (2.1b)-(2.1e) are satisfied and $\mathbf{s}^l > 0$. We construct $\mathbf{s}^g = \hat{\mathbf{s}}^g + \frac{2}{n}\mathbf{1}_{N_{\mathrm{G}}}$ and $\mathbf{s}^l = \hat{\mathbf{s}}^l + \frac{2N_{\mathrm{G}}}{nN_{\mathrm{L}}}\mathbf{1}_{N_{\mathrm{L}}}$, then it is clear that $\mathbf{s}^l \geq \hat{\mathbf{s}}^l > 0$. Since $\mathbf{1}_{N_{\mathrm{G}}}^\top \mathbf{s}^g = \mathbf{1}_{N_{\mathrm{G}}}^\top \hat{\mathbf{s}}^g + \frac{2N_{\mathrm{G}}}{n} = \mathbf{1}_{N_{\mathrm{L}}}^\top \hat{\mathbf{s}}^l + \frac{2N_{\mathrm{G}}}{n} = \mathbf{1}_{N_{\mathrm{L}}}^\top \mathbf{s}^l$, the constructed generation and load are balanced so (2.1c) is satisfied for some $\boldsymbol{\theta}$. Further, we can always shift $\boldsymbol{\theta}$ to make $\boldsymbol{\theta}_1 = 0$ and (2.1b) is thereby satisfied. Next, we can check

$$\mathbf{s}^g = \hat{\mathbf{s}}^g + \frac{1}{n}\mathbf{1}_{N_{\mathrm{G}}} + \frac{1}{n}\mathbf{1}_{N_{\mathrm{G}}} = \underline{\mathbf{s}}^g(\boldsymbol{\xi}_{(n)}) + \frac{1}{n}\mathbf{1}_{N_{\mathrm{G}}} > \underline{\mathbf{s}}^g(\boldsymbol{\xi})$$

$$\mathbf{s}^g = \hat{\mathbf{s}}^g + \frac{3}{n}\mathbf{1}_{N_{\mathrm{G}}} - \frac{1}{n}\mathbf{1}_{N_{\mathrm{G}}} = \overline{\mathbf{s}}^g(\boldsymbol{\xi}_{(n)}) - \frac{1}{n}\mathbf{1}_{N_{\mathrm{G}}} < \overline{\mathbf{s}}^g(\boldsymbol{\xi}),$$

thus (2.1d) is satisfied. Finally,

$$\mathbf{p} = \mathbf{B}\mathbf{C}^\top(\mathbf{C}\mathbf{B}\mathbf{C}^\top)^\dagger \begin{bmatrix} \hat{\mathbf{s}}^g + \frac{2}{n}\mathbf{1}_{N_{\mathrm{G}}} \\ -\hat{\mathbf{s}}^l - \frac{2N_{\mathrm{G}}}{nN_{\mathrm{L}}}\mathbf{1}_{N_{\mathrm{L}}} \end{bmatrix} = \hat{\mathbf{p}} + \mathbf{B}\mathbf{C}^\top(\mathbf{C}\mathbf{B}\mathbf{C}^\top)^\dagger \begin{bmatrix} \frac{2}{n}\mathbf{1}_{N_{\mathrm{G}}} \\ -\frac{2N_{\mathrm{G}}}{nN_{\mathrm{L}}}\mathbf{1}_{N_{\mathrm{L}}} \end{bmatrix},$$

so

$$\begin{aligned} \mathbf{p}_i &\geq \hat{\mathbf{p}}_i - \left\| \mathbf{B}\mathbf{C}^\top(\mathbf{C}\mathbf{B}\mathbf{C}^\top)^\dagger \begin{bmatrix} \frac{2}{n}\mathbf{1}_{N_{\mathrm{G}}} \\ -\frac{2N_{\mathrm{G}}}{nN_{\mathrm{L}}}\mathbf{1}_{N_{\mathrm{L}}} \end{bmatrix} \right\|_1 \geq \underline{\mathbf{p}}_i(\hat{\boldsymbol{\xi}}) - \rho \cdot \left(\frac{2}{n}N_{\mathrm{G}} + \frac{2N_{\mathrm{G}}}{nN_{\mathrm{L}}}N_{\mathrm{L}} \right) \\ &= \underline{\mathbf{p}}_i(\hat{\boldsymbol{\xi}}) - 4\rho\frac{N_{\mathrm{G}}}{n} = \underline{\mathbf{p}}_i(\boldsymbol{\xi}_{(n)}) + \rho\frac{N_{\mathrm{G}}}{n} > \underline{\mathbf{p}}_i(\boldsymbol{\xi}) \end{aligned}$$

and similarly $\mathbf{p}_i < \overline{\mathbf{p}}_i(\boldsymbol{\xi})$. As the result, we have $\underline{\mathbf{p}}(\boldsymbol{\xi}) \leq \mathbf{p} \leq \overline{\mathbf{p}}(\boldsymbol{\xi})$ and (2.1e) is satisfied.

Above all, we have shown that there exist $(\mathbf{s}^g, \mathbf{s}^l)$ such that (2.1b)-(2.1e) are satisfied and $\mathbf{s}^l > 0$. Thus $\boldsymbol{\xi} \in \Omega_{\mathrm{para}}$ and there must be $U(\boldsymbol{\xi}_{(n)}) \subseteq \Omega_{\mathrm{para}}$.

Appendix B. Proof of Proposition 2.3. We first show that for a fixed network $(\mathbf{B}, \mathbf{C}, \boldsymbol{\xi})$ and $\mathbf{s}^l \in \Omega_{\mathbf{s}^l}$, if the primal optimal solution to (2.1) is not unique for some $\mathbf{f}$, then there must exist $\boldsymbol{\tau}, \boldsymbol{\mu}_+, \boldsymbol{\mu}_-, \boldsymbol{\lambda}_+, \boldsymbol{\lambda}_-$ such that (2.2) holds but (2.3) does not.

We first reformulate (2.1) in the matrix form

$$\min_{\mathbf{x}:=[(\mathbf{s}^g)^\top, \boldsymbol{\theta}^\top]^\top} \ \mathbf{f}^\top \mathbf{s}^g \tag{B.1a}$$

$$\text{subject to } \mathbf{A}_{\mathrm{eq}}\mathbf{x} = \mathbf{b}_{\mathrm{eq}} \tag{B.1b}$$

$$\mathbf{A}_{\mathrm{in}}\mathbf{x} \leq \mathbf{b}_{\mathrm{in}} \tag{B.1c}$$

where

$$\mathbf{A}_{\mathrm{eq}} := \begin{bmatrix} \mathbf{0}^{1\times N_{\mathrm{G}}} & \mathbf{e}_1 \\ \begin{matrix} -\mathbf{I}^{N_{\mathrm{G}}} \\ \mathbf{0}^{N_{\mathrm{L}}\times N_{\mathrm{G}}} \end{matrix} & \mathbf{C}\mathbf{B}\mathbf{C}^\top \end{bmatrix}, \mathbf{b}_{\mathrm{eq}} := \begin{bmatrix} 0 \\ -\mathbf{s}^l \end{bmatrix},$$

$$\mathbf{A}_{\text{in}} := \begin{bmatrix} \mathbf{0}^{E\times N_{\text{G}}} & \mathbf{B}\mathbf{C}^{\mathsf{T}} \\ \mathbf{0}^{E\times N_{\text{G}}} & -\mathbf{B}\mathbf{C}^{\mathsf{T}} \\ \mathbf{I}^{N_{\text{G}}} & \mathbf{0}^{N_{\text{G}}\times N} \\ -\mathbf{I}^{N_{\text{G}}} & \mathbf{0}^{N_{\text{G}}\times N} \end{bmatrix}, \mathbf{b}_{\text{in}} := \begin{bmatrix} \overline{\mathbf{p}} \\ -\underline{\mathbf{p}} \\ \overline{\mathbf{s}}^{g} \\ -\underline{\mathbf{s}}^{g} \end{bmatrix}.$$

Geometrically, an LP has multiple optimal solutions if and only if the objective vector is normal to some hyperplane defined by equality constraints and the set of binding inequality constraints (i.e., corresponding rows in $\mathbf{A}_{\text{eq}}$ and $\mathbf{A}_{\text{in}}$). We collect the rows in $\mathbf{A}_{\text{in}}$ which correspond to binding inequality constraints and form a new matrix $\tilde{\mathbf{A}}_{\text{in}}$. Formally, let $\mathcal{X} := \{i | i^{\text{th}} \text{ row of } \mathbf{A}_{\text{in}} \text{ corresponds to a binding constraint.}\}$, then $\tilde{\mathbf{A}}_{\text{in}} = \mathbf{I}_{\mathcal{X}}\mathbf{A}_{\text{in}}$. In our case, the objective vector $[\mathbf{f}^{\mathsf{T}}, \mathbf{0}^{\mathsf{T}}]^{\mathsf{T}}$ is an $N_{\text{G}} + N$ dimensional vector. As there are $N + 1$ linearly independent equality constraints in (B.1b),[4] the rows in $\tilde{\mathbf{A}}_{\text{in}}$ must have rank $\leq N_{\text{G}} - 2$, and $[\mathbf{f}^{\mathsf{T}}, \mathbf{0}^{\mathsf{T}}]$ is in the row space of $[\mathbf{A}_{\text{eq}}^{\mathsf{T}}, \tilde{\mathbf{A}}_{\text{in}}^{\mathsf{T}}]^{\mathsf{T}}$.

Now take any solution $(\mathbf{x}^*, \mathbf{y}^*)$ to (2.2), where $\mathbf{x}^* = [(\mathbf{s}^g)^{\mathsf{T}}, \boldsymbol{\theta}^{\mathsf{T}}]^{\mathsf{T}}$ and $\mathbf{y}^* = [\boldsymbol{\mu}_+^{\mathsf{T}}, \boldsymbol{\mu}_-^{\mathsf{T}}, \boldsymbol{\lambda}_+^{\mathsf{T}}, \boldsymbol{\lambda}_-^{\mathsf{T}}]^{\mathsf{T}}$ are for primal and dual, respectively. Let

$$\mathbf{z} = \begin{bmatrix} -\mathbf{M}^{\mathsf{T}}\boldsymbol{\tau} \\ -\mathbf{f} + [\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \cdots, \boldsymbol{\tau}_{N_{\text{G}}}]^{\mathsf{T}} \end{bmatrix}$$

and then (2.2b) and (2.2c) indicate $\mathbf{z} = \mathbf{A}_{\text{in}}^{\mathsf{T}}\mathbf{y}^*$. Further, (2.2e) and (2.2f) indicate $i \notin \mathcal{X} \Rightarrow \mathbf{y}_i^* = 0$. Therefore, let $\tilde{\mathbf{y}}^* = \mathbf{I}_{\mathcal{X}}\mathbf{y}^*$, and we then have $\mathbf{z} = \tilde{\mathbf{A}}_{\text{in}}^{\mathsf{T}}\tilde{\mathbf{y}}^*$ where $\tilde{\mathbf{y}}^* \geq 0$ element-wisely due to (2.2d). Thus $\mathbf{z}$ is in the convex cone spanned by the rows of $\tilde{\mathbf{A}}_{\text{in}}$. By Carathéodory's Theorem for cones [14], there exists $\underline{\tilde{\mathbf{y}}}^* \geq 0$ such that $\|\underline{\tilde{\mathbf{y}}}^*\|_0 \leq N_{\text{G}} - 2$ and $\mathbf{z} = \tilde{\mathbf{A}}_{\text{in}}^{\mathsf{T}}\underline{\tilde{\mathbf{y}}}^*$. Now let $\underline{\mathbf{y}}^* = \mathbf{I}_{\mathcal{X}}^{\mathsf{T}}\underline{\tilde{\mathbf{y}}}^*$, we will next show that $(\mathbf{x}^*, \underline{\mathbf{y}}^*)$ is also a solution to (2.2).

First, since (2.2a) involves only $\mathbf{x}^*$, (2.2a) still holds for $(\mathbf{x}^*, \underline{\mathbf{y}}^*)$. Second, (2.2b) and (2.2c) also hold because $\mathbf{z} = \tilde{\mathbf{A}}_{\text{in}}^{\mathsf{T}}\underline{\tilde{\mathbf{y}}}^*$ implies $\mathbf{z} = \mathbf{A}_{\text{in}}^{\mathsf{T}}\underline{\mathbf{y}}^*$. Third, (2.2d) holds as $\underline{\tilde{\mathbf{y}}}^* \geq 0$ also implies $\underline{\mathbf{y}}^* \geq 0$. Finally for (2.2e) and (2.2f), $\underline{\mathbf{y}}_i^* = 0$ always holds for $i \notin \mathcal{X}$ since it is constructed from $\mathbf{I}_{\mathcal{X}}^{\mathsf{T}}\underline{\tilde{\mathbf{y}}}^*$ and thereby the complementary conditions are always met. That is to say, in such solution to (2.2) there are at most $N_{\text{G}} - 2$ non-zero coefficients in $\boldsymbol{\mu}_+, \boldsymbol{\mu}_-, \boldsymbol{\lambda}_+, \boldsymbol{\lambda}_-$, i.e.,

$$\|\boldsymbol{\mu}_+\|_0 + \|\boldsymbol{\mu}_-\|_0 + \|\boldsymbol{\lambda}_+\|_0 + \|\boldsymbol{\lambda}_-\|_0 < N_{\text{G}} - 1. \tag{B.4}$$

Thereby, (2.3) implies the uniqueness, and we have

$$\Omega_{\mathbf{f}} = \{\mathbf{f} \mid \forall \mathbf{s}^l \in \Omega_{\mathbf{s}^l}, \text{the solutions of (2.2) satisfy (2.3)}\}. \tag{B.5}$$

For $\mathcal{S} \subseteq [N]$, $\mathcal{T} \subseteq \mathcal{V}_{\text{G}}$ such that $|\mathcal{S}| + |\mathcal{T}| \leq N_{\text{G}} - 2$, we construct $\mathcal{Q}(\mathcal{S}, \mathcal{T})$ to be the set of $\mathbf{f}$ such that $\exists \boldsymbol{\tau} \in \mathbb{R}^{N+1}, \boldsymbol{\mu} \in \mathbb{R}^E, \boldsymbol{\lambda} \in \mathbb{R}^{N_{\text{G}}}$ satisfying:

$$\mathbf{0} = \mathbf{M}^{\mathsf{T}}\boldsymbol{\tau} + \mathbf{C}\mathbf{B}\boldsymbol{\mu} \tag{B.6a}$$

$$-\mathbf{f} = -[\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \cdots, \boldsymbol{\tau}_{N_{\text{G}}}]^{\mathsf{T}} + \boldsymbol{\lambda} \tag{B.6b}$$

$$\boldsymbol{\mu}_i \neq 0 \Rightarrow i \in \mathcal{S} \tag{B.6c}$$

$$\boldsymbol{\lambda}_i \neq 0 \Rightarrow i \in \mathcal{T}. \tag{B.6d}$$

[4] Here, independence means $\mathbf{A}_{\text{eq}}$ has full row rank $N + 1$.

When $\mathcal{S}$ and $\mathcal{T}$ are fixed, the vector $\mathbf{CB}\boldsymbol{\mu}$ takes value in an $|\mathcal{S}|$ dimensional subspace. Since $\operatorname{rank}(\mathbf{M}) = N$, the possible values of $\boldsymbol{\tau}$ must fall within an $|\mathcal{S}|+1$ dimensional subspace. Therefore, (B.6b) implies that $\mathbf{f}$ must be in an $|\mathcal{S}|+1+|\mathcal{T}| \le N_{\mathrm{G}} - 1$ dimensional subspace, and hence $\operatorname{int}(\operatorname{clos}(\mathcal{Q}(\mathcal{S},\mathcal{T})) = \emptyset$. By the Baire Category Theorem [13], we have

$$\left(\bigcup_{\substack{\mathcal{S}\subseteq[N],\mathcal{T}\subseteq\mathcal{V}_{\mathrm{G}} \\ |\mathcal{S}|+|\mathcal{T}|\le N_{\mathrm{G}}-2}} \mathcal{Q}(\mathcal{S},\mathcal{T}) \right) \cap \mathbb{R}_+^{N_{\mathrm{G}}}$$

is of the first category and thereby nowhere dense in $\mathbb{R}_+^{N_{\mathrm{G}}}$. On the other hand, (B.5) and (B.6) imply that

$$\mathbb{R}_+^{N_{\mathrm{G}}} \setminus \left(\bigcup_{\substack{\mathcal{S}\subseteq[N],\mathcal{T}\subseteq\mathcal{V}_{\mathrm{G}} \\ |\mathcal{S}|+|\mathcal{T}|\le N_{\mathrm{G}}-2}} \mathcal{Q}(\mathcal{S},\mathcal{T}) \right) \subseteq \Omega_{\mathbf{f}}$$

and hence $\Omega_{\mathbf{f}}$ is dense in $\mathbb{R}_+^{N_{\mathrm{G}}}$.

Appendix C. Proof of Proposition 2.5.

LEMMA C.1. *Suppose the set $\mathcal{S} \subseteq \mathbb{R}^n$ satisfies the condition that* $\operatorname{clos}(\operatorname{int}(\mathcal{S})) = \operatorname{clos}(\mathcal{S})$*, and $\mathcal{T}$ is an affine hyperplane with dimension strictly less than n. Then $\mathcal{T}$ is nowhere dense in $\mathcal{S}$.*

Proof. If not, then by definition, in the relative topology of $\mathcal{S}$, we have $\operatorname{int}(\mathcal{T}) \neq \emptyset$ since $\mathcal{T}$ is closed. Pick any point $\mathbf{x} \in \operatorname{int}(\mathcal{T})$, there must be an n-dimensional open ball U with radius r centered at $\mathbf{x}$ such that $\mathbf{x} \in U \cap \mathcal{S} \subseteq \mathcal{T}$. In the n-dimensional Euclidean topology, since $\operatorname{clos}(\operatorname{int}(\mathcal{S})) = \operatorname{clos}(\mathcal{S})$, there must be a point $\mathbf{x}_1\mathcal{S}$ such that $|\mathbf{x}-\mathbf{x}_1| \le r/2$ and there is an n-dimensional open ball U_1 centered at $\mathbf{x}_1$ and have radius $< r/2$ satisfying $U_1 \in \mathcal{S}$. Clearly, $U_1 \in U$ as well, and thereby $U_1 \subseteq U \cap \mathcal{S} \subseteq \mathcal{T}$. However, $\mathcal{T}$ is an affine hyperplane with dimension strictly less than n, and there is the contradiction. ∎

Now consider the power equations below:

$$\text{(C.1)} \qquad \mathbf{T}\boldsymbol{\theta} := \begin{bmatrix} \mathbf{CBC}^\intercal \\ \mathbf{BC}^\intercal \end{bmatrix} \cdot \boldsymbol{\theta} = \begin{bmatrix} \mathbf{s}^g \\ -\mathbf{s}^l \\ \mathbf{p} \end{bmatrix}.$$

Proposition 2.3 shows that there will always be at least $N_{\mathrm{G}} - 1$ binding inequality constraints as each non-zero multiplier will force one inequality constraint to be binding. A constraint is binding means some $\mathbf{s}_i^g$ equals either $\overline{\mathbf{s}}_i^g$ or $\underline{\mathbf{s}}_i^g$ (as in the upper N_{G} rows in (C.1)), or some $\mathbf{p}_i$ equals either $\overline{\mathbf{p}}_i$ or $\underline{\mathbf{p}}_i$ (as in the lower E rows in (C.1)). We have $\operatorname{rank}(\mathbf{T}) = N-1$. We will first use the following procedure to construct the set $\widetilde{\Omega}_{\text{para}}$.

I. $\widetilde{\Omega}_{\text{para}} \leftarrow \Omega_{\text{para}}$

II. For each $\mathcal{S} \subseteq \mathcal{V}_{\mathrm{G}} \cup [N+1, N+E]$, construct $\mathbf{T}_{\mathcal{S}}$.

  a) If $\operatorname{rank}(\mathbf{T}_{\mathcal{S}}) = |\mathcal{S}|$, then continue to the next $\mathcal{S}$.

  b) If $\operatorname{rank}(\mathbf{T}_{\mathcal{S}}) < |\mathcal{S}|$, then consider

$$\text{(C.2)} \qquad \Gamma := \prod_{i\in\mathcal{S}\cap\mathcal{V}_{\mathrm{G}}} \{\mathbf{e}_i, \mathbf{e}_{N_{\mathrm{G}}+i}\} \times \prod_{\substack{j\in[E] \\ j+N\in\mathcal{S}}} \{\mathbf{e}_{2N_{\mathrm{G}}+j}, \mathbf{e}_{2N_{\mathrm{G}}+E+j}\}.$$

Now update $\widetilde{\Omega}_{\text{para}}$ as

$$\widetilde{\Omega}_{\text{para}} \leftarrow \widetilde{\Omega}_{\text{para}} \setminus \bigcup_{\gamma\in\Gamma} \{\boldsymbol{\xi} | \exists \boldsymbol{\theta}, \text{s.t. } \gamma^{\mathsf{T}}\boldsymbol{\xi} = \mathbf{T}_{\mathcal{S}}\boldsymbol{\theta}\}. \tag{C.3}$$

III. Return $\widetilde{\Omega}_{\text{para}}$.

In the above procedure, an n-tuple of vectors is also regarded as a matrix of n columns.[5] Since $\gamma \in \Gamma$ is of dimension $|\mathcal{S}|$ and $\mathbf{T}_{\mathcal{S}}\boldsymbol{\theta}$ with $\boldsymbol{\theta} \in \mathbb{R}^N$ defines a subspace of $\leq |\mathcal{S}| - 1$ dimensions, each set of $\{\boldsymbol{\xi} | \exists \boldsymbol{\theta}, \text{s.t. } \gamma^{\mathsf{T}}\boldsymbol{\xi} = \mathbf{T}_{\mathcal{S}}\boldsymbol{\theta}\}$ in (C.3) is a subspace with dimension strictly lower than $2N_{\text{G}} + 2E$, and is thereby nowhere dense in Ω_{para} by Lemma C.1. As the result, we have that $\widetilde{\Omega}_{\text{para}}$ is dense in Ω_{para}. It is sufficient to show that two conditions in Proposition 2.5 are satisfied.

To show $\text{clos}(\text{int}(\Omega_{\mathbf{s}^l}(\boldsymbol{\xi}))) = \text{clos}(\Omega_{\mathbf{s}^l}(\boldsymbol{\xi}))$, it is sufficient to prove that fix $\boldsymbol{\xi} \in \widetilde{\Omega}_{\text{para}}$, $\forall \hat{\mathbf{s}}^l \in \Omega_{\mathbf{s}^l}(\boldsymbol{\xi})$, there exists a sequence $(\mathbf{s}^l_{(n)})_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \mathbf{s}^l_{(n)} = \hat{\mathbf{s}}^l$ and each $\mathbf{s}^l_{(n)}$ has an open neighborhood $U(\mathbf{s}^l_{(n)})$ such that $U(\mathbf{s}^l_{(n)}) \subseteq \Omega_{\mathbf{s}^l}(\boldsymbol{\xi})$. By definition, there exists $\hat{\mathbf{s}}^g$ and $\hat{\boldsymbol{\theta}}$ such that (2.1b)-(2.1e) are satisfied for $\hat{\mathbf{s}}^l$. We also use $\hat{\mathbf{p}}$ to denote the branch power flow associated with $(\hat{\mathbf{s}}^g, \hat{\mathbf{s}}^l)$. Here we overload $\mathcal{S} \subseteq \mathcal{V}_{\text{G}} \cup [N+1, N+E]$ to denote the indices of all the binding inequality constraints for $(\hat{\mathbf{s}}^g, \hat{\mathbf{s}}^l)$. [6] By construction, we have $\text{rank}(\mathbf{T}_{\mathcal{S}}) = |\mathcal{S}| \leq \text{rank}(\mathbf{T}) = N - 1$. There are two situations to discuss.

In the first case, if $|\mathcal{S}| \leq N_{\text{G}} - 1$, then let ρ_1 be the matrix norm of

$$\mathbf{T}_{(\mathcal{V}_{\text{G}} \cup [N+1,N+E]) \setminus \mathcal{S}} \begin{bmatrix} \mathbf{e}_1^{\mathsf{T}} \\ \mathbf{T}_{\mathcal{S}} \\ \mathbf{T}_{\mathcal{V}_{\text{L}}} \end{bmatrix}^{\dagger}$$

induced by the L1 vector norm. Let

$$\epsilon_1 = \min \left\{ \min_{\substack{i\in\mathcal{V}_{\text{G}} \\ \hat{\mathbf{s}}^g_i > \underline{\mathbf{s}}^g_i}} \hat{\mathbf{s}}^g_i - \underline{\mathbf{s}}^g_i, \min_{\substack{i\in\mathcal{V}_{\text{G}} \\ \overline{\mathbf{s}}^g_i > \hat{\mathbf{s}}^g_i}} \overline{\mathbf{s}}^g_i - \hat{\mathbf{s}}^g_i, \min_{\substack{i\in\mathcal{E} \\ \hat{\mathbf{p}}_i > \underline{\mathbf{p}}_i}} \hat{\mathbf{p}}_i - \underline{\mathbf{p}}_i, \min_{\substack{i\in\mathcal{E} \\ \overline{\mathbf{p}}_i > \hat{\mathbf{p}}_i}} \overline{\mathbf{p}}_i - \hat{\mathbf{p}}_i \right\} \tag{C.4}$$

and

$$\epsilon_2 = \min_i \hat{\mathbf{s}}^l_i. \tag{C.5}$$

Now we can construct $\mathbf{s}^l_{(n)} \equiv \hat{\mathbf{s}}^l$, and

$$U(\mathbf{s}^l_{(n)}) = \left\{ \mathbf{s}^l \middle| |\mathbf{s}^l - \mathbf{s}^l_{(n)}| < \frac{1}{2} \min\{\frac{\epsilon_1}{N_{\text{L}}\rho_1}, \epsilon_2\} \mathbf{1}_{N_{\text{L}}} \right\}.$$

It is trivial that $\lim_{n\to\infty} \mathbf{s}^l_{(n)} = \hat{\mathbf{s}}^l$. For any $\mathbf{s}^l \in U(\mathbf{s}^l_{(n)})$, we have

$$\mathbf{s}^l > \mathbf{s}^l_{(n)} - \frac{1}{2}\epsilon_2 = \hat{\mathbf{s}}^l - \frac{1}{2}\min_i \hat{\mathbf{s}}^l_i > 0.$$

[5] Hence, each $\gamma \in \Gamma$ can also be regarded as a $(2N_{\text{G}} + 2E)$-by-N_{G} matrix.

[6] In this section, the index of a constraint associated with generator i (either the upper or lower bounds) is i and the index of a constraint associated with branch i (either the upper or lower bounds) is $i + N_{\text{G}}$. The updatng process in (C.3) guarantees that a generator or branch cannot reach the upper and lower bound at the same time.

Further, we will show that for

$$\boldsymbol{\theta} = \begin{bmatrix} \mathbf{e}_1^\top \\ \mathbf{T}_{\mathcal{S}} \\ \mathbf{T}_{\mathcal{V}_\mathrm{L}} \end{bmatrix}^\dagger \begin{bmatrix} 0 \\ \boldsymbol{\xi}_{\mathcal{S}'} \\ \mathbf{s}^l \end{bmatrix}, \mathbf{s}^g = \mathbf{T}_{\mathcal{V}_\mathrm{G}} \boldsymbol{\theta}, \tag{C.6}$$

(2.1b)-(2.1e) are satisfied. In (C.6), $\mathcal{S}' := \{g(i) | i \in \mathcal{S}\}$ where

$$g(i) := \begin{cases} i, & i \in \mathcal{V}_\mathrm{G},\ \hat{\mathbf{s}}_i^g = \overline{\mathbf{s}}_i^g \\ i + N_\mathrm{G}, & i \in \mathcal{V}_\mathrm{G},\ \hat{\mathbf{s}}_i^g = \underline{\mathbf{s}}_i^g \\ i - N + 2N_\mathrm{G}, & i - N \in \mathcal{E},\ \hat{\mathbf{p}}_{i-N} = \overline{\mathbf{p}}_{i-N} \\ i - N + 2N_\mathrm{G} + E, & i - N \in \mathcal{E},\ \hat{\mathbf{p}}_{i-N} = \underline{\mathbf{p}}_{i-N} \end{cases}.$$

Then (C.6) implies $\mathbf{e}_1^\top \boldsymbol{\theta} = 0$, $\mathbf{T}_{\mathcal{V}_\mathrm{L}} \boldsymbol{\theta} = \mathbf{s}^l$ and $\mathbf{T}_{\mathcal{V}_\mathrm{G}} \boldsymbol{\theta} = \mathbf{s}^g$, which are equivalent to (2.1b), (2.1c). For (2.1d), if a generator $\hat{\mathbf{s}}_i^g$ reaches the upper (or lower) bound, then there must be $i \in \mathcal{S}$, and (C.6) implies $\mathbf{s}_i^g = \mathbf{T}_{\{i\}} \boldsymbol{\theta} = \boldsymbol{\xi}_{\{g(i)\}}$, thus $\mathbf{s}_i^g$ will reach the same bound as $\hat{\mathbf{s}}_i^g$ and stays feasible. If $\hat{\mathbf{s}}_i^g$ reaches neither bound, then

$$|\mathbf{s}_i^g - \hat{\mathbf{s}}_i^g| \le \rho_1 \|\mathbf{s}^l - \hat{\mathbf{s}}^l\|_1 \le \rho_1 \cdot \frac{1}{2} N_\mathrm{L} \frac{\epsilon_1}{N_\mathrm{L} \rho_1} = \frac{1}{2} \epsilon_1,$$

and thereby $\mathbf{s}_i^g$ is still strictly between the bounds and stays feasible. Similarly, the branch flow $\mathbf{p}$ is also within the upper and lower bounds, and (2.1e) is also satisfied. As the result, $\mathbf{s}^l \in \Omega_{\mathbf{s}^l}(\boldsymbol{\xi})$ and thus $U(\hat{\mathbf{s}}^l) \subseteq \Omega_{\mathbf{s}^l}(\boldsymbol{\xi})$.

In the second case, we have $N_\mathrm{G} \le |\mathcal{S}| \le N - 1$, then define

$$\mathbf{T}'(\mathcal{R}) := \begin{bmatrix} \mathbf{e}_1^\top \\ \mathbf{T}_{\mathcal{S}} \\ \mathbf{T}_{\mathcal{R}} \end{bmatrix}, \text{ for } \mathcal{R} \subseteq \mathcal{V}_\mathrm{L}.$$

Let $\mathcal{R}^* = \underset{\mathcal{R}:\mathrm{rank}(\mathbf{T}'(\mathcal{R})) = \mathrm{rank}(\mathbf{T}'(\mathcal{V}_\mathrm{L}))}{\arg\min} |\mathcal{R}|$. If there are multiple such $\mathcal{R}$ to minimize $|\mathcal{R}|$ then pick any one of them. There are two simple observations:

- All rows of matrix $\mathbf{T}'(\mathcal{R}^*)$ are independent.
- All rows of $\mathbf{T}_{\mathcal{V}_\mathrm{L}}$ are in the row space of $\mathbf{T}'(\mathcal{R}^*)$.

We further define

$$\mathbf{T}''(\mathcal{T}) := \begin{bmatrix} \mathbf{T}'(\mathcal{R}^*) \\ \mathbf{T}_{\mathcal{T}} \end{bmatrix},$$

for $\mathcal{T} \subseteq (\mathcal{V}_\mathrm{G} \cap [N+1, N+E]) \backslash \mathcal{S}$. Let

$$\mathcal{T}^* = \underset{\mathcal{T}:\mathrm{rank}(\mathbf{T}''(\mathcal{T})) = \mathrm{rank}(\mathbf{T}''((\mathcal{V}_\mathrm{G} \cap [N+1, N+E]) \backslash \mathcal{S}))}{\arg\min} |\mathcal{T}|.$$

Likely, if there are multiple such $\mathcal{T}$ to minimize $|\mathcal{T}|$ then pick any one of them. There are also two simple observations:

- All rows of the matrix $\mathbf{T}''(\mathcal{T}^*)$ are still independent.
- Now all rows of $\mathbf{T}$ are in the row space of $\mathbf{T}''(\mathcal{T}^*)$.

Let ρ_2 be the matrix norm of

$$\begin{bmatrix} \mathbf{T}_{\mathcal{V}_\mathrm{G} \cup [N+1, N+E]} \\ \mathbf{T}_{\mathcal{V}_\mathrm{L}} \end{bmatrix} (\mathbf{T}''(\mathcal{T}^*))^\dagger$$

induced by the L1 vector norm. Let ϵ_1 and ϵ_2 be the same as in (C.4) and (C.5), and we define the direction vector $\mathbf{v} \in \mathbb{R}^{|\mathcal{S}|}$ as

$$\mathbf{v} := \operatorname{sgn}\left(\mathbf{T}_{\mathcal{S}}\hat{\boldsymbol{\theta}} - \begin{bmatrix} (\overline{\mathbf{s}}^g)_{\mathcal{S}\cap\mathcal{V}_{\mathrm{G}}} \\ (\overline{\mathbf{p}})_{(\mathcal{S}-N)\cap\mathcal{E}} \end{bmatrix}\right) + \operatorname{sgn}\left(\mathbf{T}_{\mathcal{S}}\hat{\boldsymbol{\theta}} - \begin{bmatrix} (\underline{\mathbf{s}}^g)_{\mathcal{S}\cap\mathcal{V}_{\mathrm{G}}} \\ (\underline{\mathbf{p}})_{(\mathcal{S}-N)\cap\mathcal{E}} \end{bmatrix}\right)$$

where sgn applies the sign function to each coordinate of the vector. We then construct

$$\mathbf{s}^l_{(n)} := \mathbf{T}_{\mathcal{V}_{\mathrm{L}}}(\mathbf{T}''(\mathcal{T}^*))^{\dagger} \begin{bmatrix} 0 \\ \mathbf{T}_{\mathcal{S}}\hat{\boldsymbol{\theta}} - \frac{\min\{\epsilon_1,\epsilon_2\}}{2nN\rho_2}\mathbf{v} \\ \mathbf{s}^l_{\mathcal{R}^*} \\ \mathbf{T}_{\mathcal{T}^*}\hat{\boldsymbol{\theta}} \end{bmatrix}.$$

Since all rows of $\mathbf{T}$ are in the row space of $\mathbf{T}''(\mathcal{T}^*)$, we have

$$(\mathbf{T}''(\mathcal{T}^*))^{\dagger} \begin{bmatrix} 0 \\ \mathbf{T}_{\mathcal{S}}\hat{\boldsymbol{\theta}} \\ \mathbf{s}^l_{\mathcal{R}^*} \\ \mathbf{T}_{\mathcal{T}^*}\hat{\boldsymbol{\theta}} \end{bmatrix} - \hat{\boldsymbol{\theta}}$$

is perpendicular to the row space of $\mathbf{T}_{\mathcal{V}_{\mathrm{L}}}$. Therefore,

$$\lim_{n\to\infty} \mathbf{s}^l_{(n)} \to \mathbf{T}_{\mathcal{V}_{\mathrm{L}}}(\mathbf{T}''(\mathcal{T}^*))^{\dagger} \begin{bmatrix} 0 \\ \boldsymbol{\xi}_{\mathcal{S}'} \\ \mathbf{s}^l_{\mathcal{R}^*} \\ \mathbf{T}_{\mathcal{T}^*}\hat{\boldsymbol{\theta}} \end{bmatrix} = \mathbf{T}_{\mathcal{V}_{\mathrm{L}}}\hat{\boldsymbol{\theta}} = \hat{\mathbf{s}}^l.$$

Besides, since

$$\hat{\mathbf{s}}^l - \mathbf{s}^l_{(n)} = \mathbf{T}_{\mathcal{V}_{\mathrm{L}}}(\mathbf{T}''(\mathcal{T}^*))^{\dagger} \begin{bmatrix} 0 \\ \frac{\min\{\epsilon_1,\epsilon_2\}}{2nN\rho_2}\mathbf{v} \\ \mathbf{0}^{(|\mathcal{R}^*|+|\mathcal{T}^*|)\times 1} \end{bmatrix},$$

we have

$$\mathbf{s}^l_{(n)} \geq \hat{\mathbf{s}}^l - \rho_2|\mathcal{S}|\frac{\epsilon_2}{2nN\rho_2}\mathbf{1}_{N_{\mathrm{L}}} \geq \hat{\mathbf{s}}^l - \frac{\epsilon_2}{2n}\mathbf{1}_{N_{\mathrm{L}}} > 0.$$

We then construct the associated $\boldsymbol{\theta}_{(n)}$, $\mathbf{s}^g_{(n)}$ and $\mathbf{p}_{(n)}$ as

$$\begin{aligned} \boldsymbol{\theta}_{(n)} &:= (\mathbf{T}''(\mathcal{T}^*))^{\dagger} \begin{bmatrix} 0 \\ \mathbf{T}_{\mathcal{S}}\hat{\boldsymbol{\theta}} - \frac{\min\{\epsilon_1,\epsilon_2\}}{2nN\rho_2}\mathbf{v} \\ \mathbf{s}^l_{\mathcal{R}^*} \\ \mathbf{T}_{\mathcal{T}^*}\hat{\boldsymbol{\theta}} \end{bmatrix}, \\ \mathbf{s}^g_{(n)} &:= \mathbf{T}_{\mathcal{V}_{\mathrm{G}}}\boldsymbol{\theta}_{(n)}, \\ \mathbf{p}_{(n)} &:= \mathbf{T}_{[N+1,N+E]}\boldsymbol{\theta}_{(n)}. \end{aligned}$$

For $\mathbf{s}^g_{(n)}$, we have

$$|\hat{\mathbf{s}}^g - \mathbf{s}^g_{(n)}| = \left|\mathbf{T}_{\mathcal{V}_{\mathrm{L}}}(\mathbf{T}''(\mathcal{T}^*))^{\dagger} \begin{bmatrix} 0 \\ \frac{\min\{\epsilon_1,\epsilon_2\}}{2nN\rho_2}\mathbf{v} \\ \mathbf{0}^{(|\mathcal{R}^*|+|\mathcal{T}^*|)\times 1} \end{bmatrix}\right| \leq \rho_2|\mathcal{S}|\frac{\epsilon_1}{2nN\rho_2}\mathbf{1}_{N_{\mathrm{G}}} \leq \frac{\epsilon_1}{2n}\mathbf{1}_{N_{\mathrm{G}}},$$

and consider that all the generators that reach the upper or lower bounds in $\hat{\mathbf{s}}^g$ have been moved towards the opposite directions encoded in $\mathbf{v}$. All the coordinates in $\mathbf{s}^g_{(n)}$ will then strictly stay within the limits. The similar argument also applies to $\mathbf{p}_{(n)}$ and implies that all the coordinates in $\mathbf{p}_{(n)}$ also strictly stay within the limits. Thereby,

$\mathbf{s}^l_{(n)} \in \Omega_{\mathbf{s}^l}(\boldsymbol{\xi})$ and there is no binding constraint associated with $(\mathbf{s}^g_{(n)}, \mathbf{s}^l_{(n)})$. We have shown in the first case that when no binding constraints arises, there is always an open neighborhood $U(\mathbf{s}^l_{(n)}) \subseteq \Omega_{\mathbf{s}^l}(\boldsymbol{\xi})$.

Up to now, we have already proved the first result that $\mathrm{clos}(\mathrm{int}(\Omega_{\mathbf{s}^l}(\boldsymbol{\xi}))) = \mathrm{clos}(\Omega_{\mathbf{s}^l}(\boldsymbol{\xi}))$. We will then prove $\widetilde{\Omega}_{\mathbf{s}^l}(\boldsymbol{\xi})$ is dense in $\Omega_{\mathbf{s}^l}(\boldsymbol{\xi})$. In fact, $\forall \boldsymbol{\xi} \in \widetilde{\Omega}_{\mathrm{para}}$, if for some $\mathbf{s}^l \in \Omega_{\mathbf{s}^l}(\boldsymbol{\xi})$, the optimal solution to (2.1) has $\geq N_{\mathrm{G}}$ tight inequality constraints, then we use $\mathcal{S} \subseteq \mathcal{V}_{\mathrm{G}} \cup [N+1, N+E], |\mathcal{S}| = N_{\mathrm{G}}$ again to denote the indices of any N_{G} tight inequality constraints. As those N_{G} inequality constraints are tight, there must exist $\boldsymbol{\theta} \in \mathbb{R}^N$ and $\gamma \in \Gamma$, as defined in (C.2), such that $\gamma^\mathsf{T}\boldsymbol{\xi} = \mathbf{T}_{\mathcal{S}}\boldsymbol{\theta}$. According to (C.3), $\mathrm{rank}(\mathbf{T}_{\mathcal{S}})$ must be exactly N_{G}. Plugging in the optimal $\boldsymbol{\theta}$, as well as the tight limits indexed by some $\gamma \in \Gamma$, into (C.1), we have

$$\gamma \cdot \boldsymbol{\xi} = \mathbf{T}_{\mathcal{S}}\boldsymbol{\theta} \tag{C.7a}$$

$$-\mathbf{s}^l = \mathbf{T}_{\mathcal{V}_{\mathrm{L}}}\boldsymbol{\theta}. \tag{C.7b}$$

For each $\gamma \in \Gamma$, as $\mathrm{rank}(\mathbf{T}_{\mathcal{S}}) = N_{\mathrm{G}}$ but $\mathrm{rank}(\mathbf{T}) = N-1$, the set $\{\mathbf{s}^l \mid \exists\boldsymbol{\theta}, \text{(C.7) holds}\}$ is a subspace in $\mathbb{R}^{N_{\mathrm{L}}}$ and thereby nowhere dense in $\Omega_{\mathbf{s}^l}$ according to Lemma C.1. As the result, we have

$$\widetilde{\Omega}_{\mathbf{s}^l} \supseteq \Omega_{\mathbf{s}^l} \setminus \bigcup_{\gamma\in\Gamma} \{\mathbf{s}^l \mid \exists\boldsymbol{\theta}, \text{(C.7) holds for } \gamma\}$$

must be dense in $\Omega_{\mathbf{s}^l}$.

Appendix D. Proofs of the Lemmas Related to Theorem 4.3.

Proof. (Lemma 4.4) We first construct $\underline{\mathbf{s}}^g \equiv 0$ and $\overline{\mathbf{s}}^g \equiv 2$. Let

$$(\mathbf{s}^g_*)_i = \begin{cases} 0, & i \in \mathcal{S}_{\mathrm{G}} \\ 1, & i \notin \mathcal{S}_{\mathrm{G}} \end{cases} \tag{D.1}$$

and $\mathbf{s}^l_* = \frac{N_{\mathrm{G}} - |\mathcal{S}_{\mathrm{G}}|}{N_{\mathrm{L}}}\mathbf{1}_{N_{\mathrm{L}}}$. Then we let

$$\boldsymbol{\theta}_* = \mathbf{M}^\dagger \begin{bmatrix} \mathbf{s}^g_* \\ -\mathbf{s}^l_* \\ 0 \end{bmatrix}, \ \mathbf{p}_* = \mathbf{B}\mathbf{C}^\mathsf{T}\boldsymbol{\theta}_*$$

and

$$\overline{\mathbf{p}}_i = \begin{cases} (\mathbf{p}_*)_i, & i \in \mathcal{S}_{\mathrm{B}},\ (\mathbf{p}_*)_i \geq 0 \\ \|\mathbf{p}_*\|_\infty + 1, & \text{otherwise} \end{cases},$$

$$\underline{\mathbf{p}}_i = \begin{cases} (\mathbf{p}_*)_i, & i \in \mathcal{S}_{\mathrm{B}},\ (\mathbf{p}_*)_i < 0 \\ -\|\mathbf{p}_*\|_\infty - 1, & \text{otherwise} \end{cases}.$$

Constructing $\boldsymbol{\xi}_* = [(\overline{\mathbf{s}}^g)^\mathsf{T}, (\underline{\mathbf{s}}^g)^\mathsf{T}, \overline{\mathbf{p}}^\mathsf{T}, \underline{\mathbf{p}}^\mathsf{T}]^\mathsf{T}$, it is easy to check that $(\mathbf{s}^g_*, \boldsymbol{\theta}_*)$ is an extreme point of the convex polytope described by (2.1b)-(2.1e) under $(\boldsymbol{\xi}_*, \mathbf{s}^l_*)$ since there are exactly $N + N_{\mathrm{G}}$ equality and binding inequality constraints (corresponding to $\mathcal{S}_{\mathrm{G}}$ and $\mathcal{S}_{\mathrm{B}}$) in total and they are independent. Next, consider the following optimization problem:

$$\underset{\mathbf{x}}{\text{minimize}}\ \ \mathbf{f}^\mathsf{T}\mathbf{s}^g \tag{D.2a}$$

$$\text{subject to } \underline{\mathbf{s}}^g \leq \mathbf{x} \leq \overline{\mathbf{s}}^g \tag{D.2b}$$

$$\underline{\mathbf{p}} \leq \mathbf{B}\mathbf{C}^\mathsf{T}(\mathbf{C}\mathbf{B}\mathbf{C}^\mathsf{T})^\dagger \begin{bmatrix} \mathbf{x} \\ -\mathbf{s}^l \end{bmatrix} \leq \overline{\mathbf{p}}. \tag{D.2c}$$

Here (2.1) and (D.2) are equivalent to each other in the sense that there is a bijection between their feasible points shown as below.

$$(\mathbf{s}^g_{\text{fea}}, \boldsymbol{\theta}_{\text{fea}}) \to \mathbf{x}_{\text{fea}} : \mathbf{x}_{\text{fea}} = \mathbf{s}^g_{\text{fea}}$$
$$\mathbf{x}_{\text{fea}} \to (\mathbf{s}^g_{\text{fea}}, \boldsymbol{\theta}_{\text{fea}}) : \mathbf{s}^g_{\text{fea}} = \mathbf{x}_{\text{fea}}, \boldsymbol{\theta}_{\text{fea}} = \mathbf{M}^\dagger \begin{bmatrix} \mathbf{s}^g_{\text{fea}} \\ -\mathbf{s}^l_* \\ 0 \end{bmatrix}.$$

Since $\boldsymbol{\theta}$ is always linear in $\mathbf{s}^g$, the value of $\mathbf{s}^g_*$ in (D.1) is also an extreme point of the feasible domain in (D.2). Therefore there exists $\mathbf{f}' \in \mathbb{R}^{N_{\text{G}}}$ such that when $\mathbf{f} = \mathbf{f}'$ in (D.2), the optimal solution is uniquely $\mathbf{x} = \mathbf{s}^g$. The equivalence between (2.1) and (D.2) implies that when $\mathbf{f} = \mathbf{f}'$ in (2.1), the optimal solution is uniquely $(\mathbf{s}^g_*, \boldsymbol{\theta}_*)$ as well. Finally, we construct $\mathbf{f}_* = \mathbf{f}' + \|\mathbf{f}\|_\infty \mathbf{1}$, then the optimal solution remains the same as $(\mathbf{s}^g_*, \boldsymbol{\theta}_*)$ and is still unique due to the fact that $\mathbf{1}^\intercal \mathbf{s}^g_* \equiv \mathbf{1}^\intercal \mathbf{s}^l_*$, but we now have $\mathbf{f}_* \geq 0$. ∎

Proof. (Lemma 4.5) We start from $(\mathbf{f}_*, \boldsymbol{\xi}_*, \mathbf{s}^l_*)$ provided in Lemma 4.4, and then perturb the parameters in a specific order to derive the desired $(\mathbf{f}_{**}, \boldsymbol{\xi}_{**}, W)$.

First, [33] shows that to optimal solution set $\mathcal{OPF}(\mathbf{s}^l_*)$ to (2.1) for fixed $\mathbf{s}^l_*$ is both upper hemi-continuous and lower hemi-continuous in $\mathbf{f}$. Now for the convenience of notations, we use $\mathcal{OPF}^{\mathbf{f}}$ to denote $\mathcal{OPF}(\mathbf{s}^l_*)$ under the cost vector vector $\mathbf{f}$. For now, $\boldsymbol{\xi}$ is chosen to be $\boldsymbol{\xi}_*$. Therefore $\mathcal{OPF}^{\mathbf{f}_*} = \{(\mathbf{s}^g_*, \boldsymbol{\theta}_*)\}$. As upper hemi-continuity implies that for any neighborhood U of the point $(\mathbf{s}^g_*, \boldsymbol{\theta}_*)$, there is a neighborhood V of $\mathbf{f}_*$ such that $\forall \mathbf{f} \in V$, $\mathcal{OPF}^{\mathbf{f}} \subseteq U$. Consider that (2.1) is a linear programming problem, so $\mathcal{OPF}^{\mathbf{f}}$ should contain at least a different extreme point $((\mathbf{s}^g)', \boldsymbol{\theta}') \neq (\mathbf{s}^g_*, \boldsymbol{\theta}_*)$ if $\mathcal{OPF}^{\mathbf{f}} \neq \{(\mathbf{s}^g_*, \boldsymbol{\theta}_*)\}$. Since a compact convex polytope has only finite extreme points, we can always choose U to be small enough that $(\mathbf{s}^g_*, \boldsymbol{\theta}_*)$ is the only extreme point in U. Then there must be a neighborhood V of $\mathbf{f}_*$ such that $\forall \mathbf{f} \in V$, $\mathcal{OPF}^{\mathbf{f}} \equiv \{(\mathbf{s}^g_*, \boldsymbol{\theta}_*)\}$. Proposition 2.3 shows that $\Omega_{\mathbf{f}}$ is dense in $\mathbb{R}^{N_{\text{G}}}_+$, so there must be some $\mathbf{f}_{**} \in U \cap \Omega_{\mathbf{f}}$ and under $\mathbf{f}_{**}$, $\mathcal{OPF}^{\mathbf{f}_{**}} = \{(\mathbf{s}^g_*, \boldsymbol{\theta}_*)\}$ and thereby all the binding constraints are the same as the binding constraints under $\mathbf{f}_*$, which exactly correspond to $\mathcal{S}_{\text{G}}$ and $\mathcal{S}_{\text{B}}$. Up to now, the parameters in (2.1) have been updated to $(\mathbf{f}_{**}, \boldsymbol{\xi}_*, \mathbf{s}^l_*)$.

Second, we are going to perturb $\boldsymbol{\xi}_*$ to some point in $\widetilde{\Omega}_{\text{para}}$. We know that

- $(\mathbf{s}^g_*, \boldsymbol{\theta}_*)$ is the unique solution to (2.1).
- All the constraints and the cost function in (2.1) are linear and thereby twice continuously differentiable in $(\mathbf{s}^g, \boldsymbol{\theta})$ and differentiable in $\boldsymbol{\xi}$.
- Since all the binding constraints exactly correspond to $\mathcal{S}_{\text{G}}$ and $\mathcal{S}_{\text{B}}$ where $\mathcal{S}_{\text{G}} \perp \mathcal{S}_{\text{B}}$, the gradients for all the binding inequalities and equality constraints are independent.
- We have $|\mathcal{S}_{\text{G}}| + |\mathcal{S}_{\text{B}}| = N_{\text{G}} - 1$ binding inequality constraints. Together with fact that $\mathbf{f}_{**} \in \Omega_{\mathbf{f}}$ and thus (2.3), strict complementary slackness must hold.

Above all, Lemma 4.1 shows the set of binding constraints do not change in a small neighborhood U of $\boldsymbol{\xi}_*$. Proposition 2.5 shows $\widetilde{\Omega}_{\text{para}}$ is dense in Ω_{para}, so there must be some $\boldsymbol{\xi}_{**} \in U \cap \widetilde{\Omega}_{\text{para}}$ and under $\boldsymbol{\xi}_{**}$, all the binding constraints are the same as the binding constraints under $\boldsymbol{\xi}_*$, which exactly correspond to $\mathcal{S}_{\text{G}}$ and $\mathcal{S}_{\text{B}}$. Up to now, the parameters in (2.1) have been updated to $(\mathbf{f}_{**}, \boldsymbol{\xi}_{**}, \mathbf{s}^l_*)$.

Finally, using the technique similar to the perturbation around $\boldsymbol{\xi}_*$ above, the set of binding constraints do not change as well when $\mathbf{s}^l$ falls within in a small neighborhood U of $\mathbf{s}^l_*$, so it is sufficient to show $U \cap \widetilde{\Omega}_{\mathbf{s}^l}$ contains an open ball W. First, it is easy to find an open ball W' in $U \cap \Omega_{\mathbf{s}^l}$ since $\text{clos}(\text{int}(\Omega_{\mathbf{s}^l})) = \text{clos}(\Omega_{\mathbf{s}^l})$ implies that $\mathbf{s}^l_*$

must be the limit of a sequence of points which are all interior points of $\Omega_{\mathbf{s}^l}$. Thus we can always find an interior point of $\Omega_{\mathbf{s}^l}$ that is strictly within U and take its neighborhood $W' \subseteq U$. Next, as $\Omega_{\mathbf{s}^l} \backslash \widetilde{\Omega}_{\mathbf{s}^l}$ can be covered by the union of finitely many affine hyperplanes, $W' \backslash (\Omega_{\mathbf{s}^l} \backslash \widetilde{\Omega}_{\mathbf{s}^l})$ must contain a smaller open ball W, which is a subset of $U \cap \widetilde{\Omega}_{\mathbf{s}^l}$. ∎

Proof. (Lemma 4.6) Suppose $\Omega_{\mathrm{bad}} \subseteq \cup_{i=1}^m \mathcal{S}_i$ where $\mathcal{S}_i$ is an affine hyperplane. For any point $\mathbf{x} \in \Omega$, we use $\mathcal{S}_i^{\mathbf{x}}$ to denote the affine hyperplane which contains $\mathbf{x}$ and is parallel (or identical) to $\mathcal{S}_i$. Since $\mathrm{clos}(\mathrm{int}(\Omega)) = \mathrm{clos}(\Omega)$, we can always find two n dimensional open balls $V(\mathbf{x}_1) \subseteq U(\mathbf{x}_1)$ and $V(\mathbf{x}_2) \subseteq U(\mathbf{x}_2)$. We then pick any $\mathbf{x}_1' \in V(\mathbf{x}_1)$ and consider the set

$$V(\mathbf{x}_2)' := V(\mathbf{x}_2) \backslash \bigcup_{i=1}^m \mathcal{S}_i^{\mathbf{x}_1'}.$$

As $V(\mathbf{x}_2)$ is of dimension n and each $\mathcal{S}_i^{\mathbf{x}_1'}$ has strictly lower dimension, $V(\mathbf{x}_2)'$ is non-empty, thus we can pick any $\mathbf{x}_2 \in V(\mathbf{x}_2)'$. Clearly $\mathbf{x}_1'\mathbf{x}_2'$ is not parallel to any $\mathcal{S}_i$, otherwise $\mathbf{x}_2'$ must be contained by $\mathcal{S}_i^{\mathbf{x}_1'}$. There are at most 1 intersection between each $\mathcal{S}_i$ and $\mathbf{x}_1'\mathbf{x}_2'$, so $\mathbf{x}_1'\mathbf{x}_2' \cap \Omega_{\mathrm{bad}}$ has at most m elements. ∎

Appendix E. Proof of Theorem 5.5. Let $\mathcal{E}^{\mathrm{bri}} := \{e = (u,v) \in \mathcal{E} \mid u \in \mathcal{V}_l, v \notin \mathcal{V}_l)\}$ be the set of edges bridging $\mathcal{V}_l$ and $\mathcal{V}_l^{\mathsf{c}}$. We have $\mathcal{E}^{\mathrm{bri}} \subseteq \mathcal{S}_{\mathrm{B}}$. [7] Without loss of generality, assume for $e = (u,v) \in \mathcal{E}^{\mathrm{bri}}$, we always have $u < v$. Otherwise we can always reindex the edges. We construct $\mathbf{q} \in \mathbb{R}^N$ as

$$\mathbf{q}_t = \begin{cases} 1, & t \le N_{\mathrm{L}}, t + N_{\mathrm{G}} \in \mathcal{V}_l \\ 1, & N_{\mathrm{L}} < t < N - n, \mathcal{S}_{\mathrm{G}}(t - N_{\mathrm{L}}) \in \mathcal{V}_l \\ -1, & \mathcal{S}_{\mathrm{B}}(t - N_{\mathrm{L}} - |\mathcal{S}_{\mathrm{G}}|) \in \mathcal{E}^{\mathrm{bri}} \\ 0, & \text{otherwise} \end{cases}$$

By Assumption 1, generator i has degree 1, so we let $i' \in \mathcal{V}$ be its only neighbor and $e^* := (i, i')$ be the edge linking i and i'. Clearly $e^* \in \mathcal{E}^{\mathrm{I}}$ and thus (i, i') is not in neither $\mathcal{S}_{\mathrm{B}}$ nor $\mathcal{E}^{\mathrm{bri}}$. In (E.1), it shows that $\mathbf{R}(\mathcal{S}_{\mathrm{G}}, \mathcal{S}_{\mathrm{B}})\mathbf{q} = -\mathbf{CBC}^{\mathsf{T}}\mathbf{e}_i$.

$$\begin{aligned}
&\mathbf{R}(\mathcal{S}_{\mathrm{G}}, \mathcal{S}_{\mathrm{B}})\mathbf{q} \\
&= \sum_{\substack{k \in \mathcal{V}_l \\ k > N_{\mathrm{G}}}} \mathbf{CBC}^{\mathsf{T}}\mathbf{e}_k + \sum_{\substack{k \in \mathcal{V}_l \cap \mathcal{S}_{\mathrm{G}} \\ k \le N_{\mathrm{G}}}} \mathbf{CBC}^{\mathsf{T}}\mathbf{e}_k - \sum_{\substack{e=(u,v) \\ e \in \mathcal{E}^{\mathrm{bri}}, u<v}} \mathbf{BC}^{\mathsf{T}}\mathbf{e}_e = \sum_{k \in \mathcal{V}_l \backslash \{i\}} \mathbf{CBC}^{\mathsf{T}}\mathbf{e}_k - \sum_{\substack{e=(u,v) \\ e \in \mathcal{E}^{\mathrm{bri}}, u<v}} b_e(\mathbf{e}_u - \mathbf{e}_v) \\
&= \sum_{k \in \mathcal{V}_l \backslash \{i\}} \sum_{\substack{k' \in \mathcal{V}: \\ e=(k,k') \in \mathcal{E}}} b_e(\mathbf{e}_k - \mathbf{e}_{k'}) - \sum_{\substack{e=(u,v) \\ u \in \mathcal{V}_l, v \notin \mathcal{V}_l}} b_e(\mathbf{e}_u - \mathbf{e}_v) = \sum_{k \in \mathcal{V}_l} \sum_{\substack{k' \in \mathcal{V}: \\ e=(k,k') \in \mathcal{E}}} b_e(\mathbf{e}_k - \mathbf{e}_{k'}) - \sum_{\substack{e=(u,v) \\ u \in \mathcal{V}_l, v \notin \mathcal{V}_l}} b_e(\mathbf{e}_u - \mathbf{e}_v) - b_{e^*}(\mathbf{e}_i - \mathbf{e}_{i'}) \\
&= \sum_{k \in \mathcal{V}_l} \sum_{\substack{k' \in \mathcal{V}_l: \\ e=(k,k') \in \mathcal{E}}} b_e(\mathbf{e}_k - \mathbf{e}_{k'}) + \sum_{k \in \mathcal{V}_l} \sum_{\substack{k' \in \mathcal{V}_l^{\mathsf{c}}: \\ e=(k,k') \in \mathcal{E}}} b_e(\mathbf{e}_k - \mathbf{e}_{k'}) - \sum_{\substack{e=(u,v) \\ u \in \mathcal{V}_l, v \notin \mathcal{V}_l}} b_e(\mathbf{e}_u - \mathbf{e}_v) - b_{e^*}(\mathbf{e}_i - \mathbf{e}_{i'}) \\
&= \sum_{\substack{e=(k,k') \in \mathcal{E} \\ k<k'}} (b_e(\mathbf{e}_k - \mathbf{e}_{k'}) + b_e(\mathbf{e}_k' - \mathbf{e}_k)) + \sum_{\substack{e=(k,k') \\ k \in \mathcal{V}_l, k' \notin \mathcal{V}_l}} b_e(\mathbf{e}_k - \mathbf{e}_{k'}) - \sum_{\substack{e=(u,v) \\ u \in \mathcal{V}_l, v \notin \mathcal{V}_l}} b_e(\mathbf{e}_u - \mathbf{e}_v) - b_{e^*}(\mathbf{e}_i - \mathbf{e}_{i'}) \\
&\text{(E.1)} \\
&= -\, b_{e^*}(\mathbf{e}_i - \mathbf{e}_{i'}) = -\mathbf{CBC}^{\mathsf{T}}\mathbf{e}_i
\end{aligned}$$

∎

[7] Otherwise, $\mathcal{V}_l$ and $\mathcal{V}_l^{\mathsf{c}}$ will be connected by a path consisting of edges in $\mathcal{S}_{\mathrm{B}}^{\mathsf{c}}$. It contradicts to the fact that $\mathcal{V}_l$ is one of the subgraphs partitioned by $\mathcal{S}_{\mathrm{B}}$.

Thereby, (4.5) implies that $\mathbf{q}$ is the i^{th} row of $\mathbf{H}_1$. Therefore, we have $\mathbf{J}_{i,j} = \mathbf{q}_j = \mathbb{1}_{j+N_{\text{G}} \in \mathcal{V}_l}$ for $j \in [N_{\text{L}}]$.

Appendix F. Proofs of Theorem 5.8 and Its Corollaries. We assume $\alpha \neq \Psi(\alpha)$, otherwise the statement will be trivial. Consider the graph $\mathcal{G}_{\text{tr}}(\mathcal{V}_{\text{tr}}, \mathcal{E}_{\text{tr}})$ where $\mathcal{V}_{\text{tr}} = \{v | \Psi(v) = \Psi(\alpha)\}$ and $\mathcal{E}_{\text{tr}} = \{e = (u,v) \in \mathcal{E} | u, v \in \mathcal{V}_{\text{tr}}\}$. By Lemma 5.6, $\forall v \in \mathcal{V}_{\text{tr}}$ there is a unique shortest path linking v and $\Psi(v)$.

LEMMA F.1. *For any $v \in \mathcal{V}_{\text{tr}} \setminus \{\Psi(\alpha)\}$, all the neighbors of v must be in $\mathcal{V}_{\text{tr}}$ as well.*

Proof. (Lemma F.1) If v has a neighbor v' such that $\Psi(v') \neq \Psi(v)$, then consider the unique path linking v and $\Psi(v)$.

$$v \leftrightarrow u_1 \leftrightarrow u_2 \cdots u_k \leftrightarrow \Psi(v).$$

and the path linking v' and $\Psi(v')$

$$v' \leftrightarrow w_1 \leftrightarrow w_2 \cdots w_r \leftrightarrow \Psi(v').$$

Then there are two un-overlapped paths connecting $\Psi(v)$ and $\Psi(v')$. One is one arc in $\mathcal{G}^{\text{O}}$ that connects $\Psi(v)$ and $\Psi(v')$, and another one is the path

$$\Psi(v) \leftrightarrow u_k \cdots u_1 \leftrightarrow v \leftrightarrow v' \leftrightarrow w_1 \cdots w_r \leftrightarrow \Psi(v')$$

along which no edge is $\mathcal{E}^{\text{O}}$. It means there is another cycle in $\mathcal{G}$ other than $\mathcal{G}^{\text{O}}$, and causes the contradiction. ∎

Proof. (Theorem 5.8) Lemma F.1 implies that for any $v \in \mathcal{V}_{\text{tr}} \setminus \{\Psi(\alpha)\}$, all the edges on the path linking v and $\Psi(v) = \Psi(\alpha)$ are in $\mathcal{E}_{\text{tr}}$ and hence $\mathcal{G}_{\text{tr}}$ is connected. On the hand, $\Psi(\alpha)$ is the only the element in $\mathcal{V}^{\text{O}} \cap \mathcal{V}_{\text{tr}}$ and there is only one cycle in $\mathcal{G}$, so $\mathcal{G}_{\text{tr}}$ contains no cycle. Above all, $\mathcal{G}_{\text{tr}}$ is a tree. Besides, $|\mathcal{V}^{\text{O}} \cap \mathcal{V}_{\text{tr}}| = 1$ also implies $\mathcal{E}_{\text{tr}} \cap \mathcal{E}^{\text{O}} = \emptyset$, so $\mathcal{E}_{\text{tr}} \cap \mathcal{S}_{\text{B}} = \emptyset$. Without loss of generality, we let $\Psi(\alpha)$ be the root of the tree. Define $\text{par}(v)$ be the parent of v for $v \in \mathcal{V}_{\text{tr}} \setminus \{\Psi(\alpha)\}$ and $\text{son}(v)$ be the set of children of v. Recall that Assumption 1 assumes generators always have degree 1, so $\text{par}(v)$ is neither in $\mathcal{V}_{\text{G}}$ nor $\mathcal{S}_{\text{G}}^{\text{c}}$ for all $v \in \mathcal{V}_{\text{tr}} \setminus \{\Psi(\alpha)\}$. To prove Theorem 5.8, it is sufficient to show that $\forall v \in \mathcal{V}_{\text{tr}} \setminus (\{\Psi(\alpha)\} \cup \mathcal{S}_{\text{G}}^{\text{c}})$,

$$\mathbf{q}_{\Lambda(v)} = \mathbf{q}_{\Lambda(\text{par}(v))}. \tag{F.1}$$

We are using mathematical induction to prove (F.1).

When v is a leaf in $\mathcal{G}_{\text{tr}}$, then clearly $v \neq \Psi(\alpha) \in \mathcal{V}^{\text{O}}$. Lemma F.1 implies v must also be a leaf in $\mathcal{G}$. Since $i \in \mathcal{S}_{\text{G}}^{\text{c}}$, v must be different from i. Further, as both v and i are leaf nodes in $\mathcal{G}$, v is not a neighbor of i, neither. [8] The construction of $\mathbf{q}$ in (5.8) is equivalent to

$$\mathbf{R}(\mathcal{S}_{\text{G}}, \mathcal{S}_{\text{B}})\mathbf{q} = -(\mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}}\mathbf{e}_i). \tag{F.2}$$

On the left hand side, the v^{th} coordinate is

$$\sum_{u:(u,v)\in\mathcal{E}} b_e \mathbf{q}_{\Lambda(v)} - \sum_{u:(u,v)\in\mathcal{E}, u\notin\mathcal{S}_{\text{G}}^{\text{c}}} b_e \mathbf{q}_{\Lambda(u)} = b_{(v,\text{par}(v))}(\mathbf{q}_{\Lambda(v)} - \mathbf{q}_{\Lambda(\text{par}(v))}),$$

[8]If two adjacent vertices are both leaves, $\mathcal{G}$ contains no cycles.

while on the right hand side, the v^{th} coordinate of the i^{th} column of $\mathbf{CBC}^\mathsf{T}$ is 0 as i and v are not adjacent. Since all $b_e > 0$, there must be $\mathbf{q}_{\Lambda(v)} = \mathbf{q}_{\Lambda(\text{par}(v))}$.

Suppose (F.1) holds for all $v \in \text{son}(v_0) \setminus \mathcal{S}_{\text{G}}^{\mathsf{c}}$, now we are proving (F.1) also holds for $v = v_0$. When $v = v_0$ and v is not a leaf, there are three situations to discuss.

- If $\text{son}(v) \cap \mathcal{S}_{\text{G}}^{\mathsf{c}} \neq \emptyset$ and $i \notin \text{son}(v) \cap \mathcal{S}_{\text{G}}^{\mathsf{c}}$, we pick any $u \in \text{son}(v) \cap \mathcal{S}_{\text{G}}^{\mathsf{c}}$. Then the u^{th} coordinate on the left hand side of (F.2) is

$$-\sum_{\substack{w:(w,u)\in\mathcal{E} \\ w\notin\mathcal{S}_{\text{G}}^{\mathsf{c}}}} b_e \mathbf{q}_{\Lambda(w)} = -b_{(v,u)} \mathbf{q}_{\Lambda(v)},$$

  while the u^{th} coordinate on the right hand side is 0 as u is neither i nor its neighbor. Thereby $\mathbf{q}_{\Lambda(v)} = 0$. For any $w \in \text{son}(v) \setminus \mathcal{S}_{\text{G}}^{\mathsf{c}}$, we have $v = \text{par}(w)$, so $\mathbf{q}_{\Lambda(w)} = \mathbf{q}_{\Lambda(v)} = 0$. Again, the v^{th} coordinate on the left hand side of (F.2) is

$$\begin{aligned}
&\sum_{w:(w,v)\in\mathcal{E}} b_e \mathbf{q}_{\Lambda(v)} - \sum_{w:(w,v)\in\mathcal{E}, w\notin\mathcal{S}_{\text{G}}^{\mathsf{c}}} b_e \mathbf{q}_{\Lambda(w)} \\
=&0 - b_{(v,\text{par}(v))} \mathbf{q}_{\Lambda(\text{par}(v))} - \sum_{w\in\text{son}(v)\setminus\mathcal{S}_{\text{G}}^{\mathsf{c}}} b_e \mathbf{q}_{\Lambda(w)} \\
=&-b_{(v,\text{par}(v))} \mathbf{q}_{\Lambda(\text{par}(v))}
\end{aligned}$$

  and on the right hand side, the v^{th} coordinate of the i^{th} column of $\mathbf{CBC}^\mathsf{T}$ is still 0 as v is neither i nor its neighbor. Thereby, $\mathbf{q}_{\Lambda(\text{par}(v))} = 0 = \mathbf{q}_{\Lambda(v)}$.

- If $i \in \text{son}(v) \cap \mathcal{S}_{\text{G}}^{\mathsf{c}}$, then the i^{th} coordinate on the left hand side of (F.2) is

$$-\sum_{w:(w,i)\in\mathcal{E}, w\notin\mathcal{S}_{\text{G}}^{\mathsf{c}}} b_e \mathbf{q}_{\Lambda(w)} = -b_{(v,i)} \mathbf{q}_{\Lambda(v)},$$

  and on the right hand side, the i^{th} coordinate of the i^{th} column of $\mathbf{CBC}^\mathsf{T}$ is $b_{(v,i)}$. The equality $-b_{(v,i)}\mathbf{q}_{\Lambda(v)} = -b_{(v,i)}$ gives $\mathbf{q}_{\Lambda(v)} = 1$. This result also tells us that $\text{son}(v) \cap \mathcal{S}_{\text{G}}^{\mathsf{c}} = \{i\}$ since if there were $u \neq i$ such that $u \in \text{son}(v) \cap \mathcal{S}_{\text{G}}^{\mathsf{c}}$, then evaluating the u^{th} coordinate of (F.2) would give $\mathbf{q}_{\Lambda(v)} = 0$ and cause the contradiction. Hence for $w \in \text{son}(v) \setminus \{i\}$, $\mathbf{q}_{\Lambda(w)} = \mathbf{q}_{\Lambda(\text{par}(w))} = \mathbf{q}_{\Lambda(v)} = 1$. Now the v^{th} coordinate on the left hand side of (F.2) is

$$\begin{aligned}
&\sum_{w:(w,v)\in\mathcal{E}} b_e \mathbf{q}_{\Lambda(v)} - \sum_{w:(w,v)\in\mathcal{E}, w\notin\mathcal{S}_{\text{G}}^{\mathsf{c}}} b_e \mathbf{q}_{\Lambda(w)} \\
=&b_{(v,i)} + b_{(v,\text{par}(v))} + \sum_{w\in\text{son}(v)\setminus\{i\}} b_e(\mathbf{q}_{\Lambda(v)} - \mathbf{q}_{\Lambda(w)}) - b_{(v,\text{par}(v))} \mathbf{q}_{\Lambda(\text{par}(v))} \\
=&b_{(v,i)} + b_{(v,\text{par}(v))} - b_{(v,\text{par}(v))} \mathbf{q}_{\Lambda(\text{par}(v))}.
\end{aligned}$$

  On the right hand side, the v^{th} coordinate of the i^{th} column of $\mathbf{CBC}^\mathsf{T}$ is $b_{(v,i)}$. Thereby $\mathbf{q}_{\Lambda(\text{par}(v))} = 1 = \mathbf{q}_{\Lambda(v)}$.

- If $\text{son}(v) \cap \mathcal{S}_{\text{G}}^{\mathsf{c}} = \emptyset$, then for all $w \in \text{son}(v)$, $\mathbf{q}_{\Lambda(w)} = \mathbf{q}_{\Lambda(\text{par}(w))} = \mathbf{q}_{\Lambda(v)}$. The

v^{th} coordinate on the left hand side of (F.2) is

$$\begin{aligned}
&\sum_{w:(w,v)\in\mathcal{E}} b_e \mathbf{q}_{\Lambda(v)} - \sum_{w:(w,v)\in\mathcal{E}} b_e \mathbf{q}_{\Lambda(w)} \\
=& b_{(v,\text{par}(v))}\mathbf{q}_{\Lambda(v)} + \sum_{w\in\text{son}(v)} b_e(\mathbf{q}_{\Lambda(v)} - \mathbf{q}_{\Lambda(w)}) - b_{(v,\text{par}(v))}\mathbf{q}_{\Lambda(\text{par}(v))} \\
=& b_{(v,\text{par}(v))}(\mathbf{q}_{\Lambda(v)} - \mathbf{q}_{\Lambda(\text{par}(v))}).
\end{aligned}$$

On the right hand side, the v^{th} coordinate of the i^{th} column of $\mathbf{CBC}^\mathsf{T}$ is 0 as v is not adjacent to i. Therefore there must be $\mathbf{q}_{\Lambda(v)} = \mathbf{q}_{\Lambda(\text{par}(v))}$.

Above all three situations, (F.1) holds for $v = v_0$. By induction, (F.1) holds $\forall v \in \mathcal{V}_{\text{tr}} \setminus (\{\Psi(\alpha)\} \cup \mathcal{S}^{\mathsf{c}}_{\mathrm{G}})$. It completes the proof of Theorem 5.8. ◻

Proof. (Corollary 5.9 and Corollary 5.10) In the proof of Theorem 5.8, we have shown that $\mathbf{q}_{\Lambda(\text{par}(i))} = 1$. Since i is a leaf vertex, its shortest path to reach $\mathcal{V}^{\mathrm{O}}$ must go through par(i). Lemma F.1 shows that $\Psi(\text{par}(i)) = \Psi(i)$. Then by Theorem 5.8, $\mathbf{q}_{\Lambda(\alpha)} = 1$ whenever $\Psi(\alpha) = \Psi(i)$ for $\alpha \in \mathcal{V} \setminus \mathcal{S}^{\mathsf{c}}_{\mathrm{G}}$.

Similarly, for $u \in \mathcal{S}^{\mathsf{c}}_{\mathrm{G}}$ and $u \neq i$, the proof of Theorem 5.8 shows $\mathbf{q}_{\Lambda(\text{par}(u))} = 0$. We also have $\Psi(\text{par}(u)) = \Psi(u)$. Hence, $\mathbf{q}_{\Lambda(\alpha)} = 0$ for $\alpha \in \mathcal{V} \setminus \mathcal{S}^{\mathsf{c}}_{\mathrm{G}}$ and $\Psi(\alpha) = \Psi(u)$. ◻

Appendix G. Proofs of Theorem 5.12 and Corollary 5.13.

Proof. (Theorem 5.12) We are proving each equality sign in (G.1) holds.

$$\begin{aligned}
\langle e^{\mathrm{O}}_k\rangle =& \sum_{e=(u,v^{\mathrm{O}}_k)\in\mathcal{E}} b_e\mathbf{q}_{\Lambda(v^{\mathrm{O}}_k)} - \sum_{\substack{e=(u,v^{\mathrm{O}}_k)\in\mathcal{E}\\ u\neq v^{\mathrm{O}}_{k+1}, u\notin\mathcal{S}^{\mathsf{c}}_{\mathrm{G}}}} b_e\mathbf{q}_{\Lambda(u)} - b_{e^{\mathrm{O}}_k}\mathbf{q}_{\Lambda(v^{\mathrm{O}}_k)} - c^k_1 - c^k_2 \\
\overset{\text{(a)}}{=}& \sum_{e=(u,v^{\mathrm{O}}_k)\in\mathcal{E}} b_e\mathbf{q}_{\Lambda(v^{\mathrm{O}}_k)} - \sum_{\substack{e=(u,v^{\mathrm{O}}_k)\in\mathcal{E}\\ u\neq v^{\mathrm{O}}_{k+1}, u\notin\mathcal{S}^{\mathsf{c}}_{\mathrm{G}}}} b_e\mathbf{q}_{\Lambda(u)} - b_{e^{\mathrm{O}}_k}\mathbf{q}_{\Lambda(v^{\mathrm{O}}_k)} - c^k_1 - c^k_2 + \sum_{\substack{e=(u,v^{\mathrm{O}}_{k+1})\in\mathcal{E}\\ u\notin\mathcal{S}^{\mathsf{c}}_{\mathrm{G}}}} b_e\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k+1})} - \sum_{\substack{e=(u,v^{\mathrm{O}}_{k+1})\in\mathcal{E}^{\mathrm{nc}}\\ u\notin\mathcal{S}^{\mathsf{c}}_{\mathrm{G}}}} b_e\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k+1})} \\
&- b_{e^{\mathrm{O}}_k}\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k+1})} - b_{e^{\mathrm{O}}_{k+1}}\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k+1})} \\
\overset{\text{(b)}}{=}& \sum_{e=(u,v^{\mathrm{O}}_k)\in\mathcal{E}} b_e\mathbf{q}_{\Lambda(v^{\mathrm{O}}_k)} - \sum_{\substack{e=(u,v^{\mathrm{O}}_k)\in\mathcal{E}\\ u\notin\mathcal{S}^{\mathsf{c}}_{\mathrm{G}}}} b_e\mathbf{q}_{\Lambda(u)} - b_{e^{\mathrm{O}}_k}\mathbf{q}_{\Lambda(v^{\mathrm{O}}_k)} - c^k_1 - c^k_2 + \sum_{\substack{e=(u,v^{\mathrm{O}}_{k+1})\in\mathcal{E}\\ u\notin\mathcal{S}^{\mathsf{c}}_{\mathrm{G}}}} b_e\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k+1})} - \sum_{\substack{e=(u,v^{\mathrm{O}}_{k+1})\in\mathcal{E}^{\mathrm{nc}}\\ u\notin\mathcal{S}^{\mathsf{c}}_{\mathrm{G}}}} b_e\mathbf{q}_{\Lambda(u)} - b_{e^{\mathrm{O}}_{k+1}}\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k+1})} \\
\overset{\text{(c)}}{=}& \sum_{\substack{e=(u,v^{\mathrm{O}}_{k+1})\in\mathcal{E}\\ u\notin\mathcal{S}^{\mathsf{c}}_{\mathrm{G}}}} b_e\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k+1})} - \sum_{\substack{e=(u,v^{\mathrm{O}}_{k+1})\in\mathcal{E}\\ u\neq v^{\mathrm{O}}_{k+2}, u\notin\mathcal{S}^{\mathsf{c}}_{\mathrm{G}}}} b_e\mathbf{q}_{\Lambda(u)} - b_{e^{\mathrm{O}}_{k+1}}\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k+1})} + \sum_{e=(u,v^{\mathrm{O}}_k)\in\mathcal{E}} b_e\mathbf{q}_{\Lambda(v^{\mathrm{O}}_k)} - \sum_{\substack{e=(u,v^{\mathrm{O}}_k)\in\mathcal{E}\\ u\notin\mathcal{S}^{\mathsf{c}}_{\mathrm{G}}}} b_e\mathbf{q}_{\Lambda(u)} - c^k_1 - c^k_2 \\
\overset{\text{(d)}}{=}& \sum_{e=(u,v^{\mathrm{O}}_{k+1})\in\mathcal{E}} b_e\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k+1})} - c^{k+1}_1 - \sum_{\substack{e=(u,v^{\mathrm{O}}_{k+1})\in\mathcal{E}\\ u\neq v^{\mathrm{O}}_{k+2}, u\notin\mathcal{S}^{\mathsf{c}}_{\mathrm{G}}}} b_e\mathbf{q}_{\Lambda(u)} - b_{e^{\mathrm{O}}_{k+1}}\mathbf{q}_{\Lambda(v^{\mathrm{O}}_{k+1})} + \sum_{e=(u,v^{\mathrm{O}}_k)\in\mathcal{E}} b_e\mathbf{q}_{\Lambda(v^{\mathrm{O}}_k)} - \sum_{\substack{e=(u,v^{\mathrm{O}}_k)\in\mathcal{E}\\ u\notin\mathcal{S}^{\mathsf{c}}_{\mathrm{G}}}} b_e\mathbf{q}_{\Lambda(u)} - c^k_1 - c^k_2 \\
\overset{\text{(e)}}{=}& \langle e^{\mathrm{O}}_{k+1}\rangle + \sum_{e=(u,v^{\mathrm{O}}_k)\in\mathcal{E}} b_e\mathbf{q}_{\Lambda(v^{\mathrm{O}}_k)} - \sum_{\substack{e=(u,v^{\mathrm{O}}_k)\in\mathcal{E}\\ u\notin\mathcal{S}^{\mathsf{c}}_{\mathrm{G}}}} b_e\mathbf{q}_{\Lambda(u)} - c^k_2 + c^{k+1}_2 - c^k_1 \\
\overset{\text{(f)}}{=}& \langle e^{\mathrm{O}}_{k+1}\rangle
\end{aligned}$$

(G.1)

■

For (a), we split the neighbors of v^{O}_{k+1} in two categories. One category corresponds to the vertices in $\mathcal{V}^{\mathrm{O}}$, so the edges linking them are in $\mathcal{E}^{\mathrm{O}}$. The other category contains only v_k^{O} and v_{k+2}^{O}, which are the neighbors on the cycle and cannot be in $\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$. Thereby the last four terms cancel off.

For (b), we first use the following equality to merge two terms.

$$\sum_{\substack{e=(u,v_k^{\mathrm{O}})\in\mathcal{E}\\ u\neq v_{k+1}^{\mathrm{O}},u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}}} b_e\mathbf{q}_{\Lambda(u)} + b_{e_{k+1}^{\mathrm{O}}}\mathbf{q}_{\Lambda(v_{k+1}^{\mathrm{O}})} = \sum_{\substack{e=(u,v_k^{\mathrm{O}})\in\mathcal{E}\\ u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}}} b_e\mathbf{q}_{\Lambda(u)}.$$

Then consider that for $u\in\mathcal{V}^{\mathrm{nc}}\setminus\{\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}\}$ such that $(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}$, we have $\Psi(u)=v_{k+1}^{\mathrm{O}}$, hence by Theorem 5.8, we have $\mathbf{q}_{\Lambda(v_{k+1}^{\mathrm{O}})}=\mathbf{q}_{\Lambda(u)}$. Thereby,

$$\sum_{\substack{e=(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}\\ u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}}} b_e\mathbf{q}_{\Lambda(v_{k+1}^{\mathrm{O}})} = \sum_{\substack{e=(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}^{\mathrm{nc}}\\ u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}}} b_e\mathbf{q}_{\Lambda(u)}.$$

For (c), we have

$$\{u|(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}^{\mathrm{nc}},u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}\}\cup\{v_k^{\mathrm{O}}\}=\{(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E},u\neq v_{k+2}^{\mathrm{O}},u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}\}$$

since v_k^{O} and v_{k+2}^{O} are the only neighbors of v_{k+1}^{O} which are not in $\mathcal{V}^{\mathrm{nc}}$ and $v_k^{\mathrm{O}}\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$. As the result, we have

$$b_{e_k^{\mathrm{O}}}\mathbf{q}_{\Lambda(v_k^{\mathrm{O}})} + \sum_{\substack{e=(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}^{\mathrm{nc}}\\ u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}}} b_e\mathbf{q}_{\Lambda(u)} = \sum_{\substack{e=(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}\\ u\neq v_{k+2}^{\mathrm{O}},u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}}} b_e\mathbf{q}_{\Lambda(u)}.$$

By merging the two terms above and reordering the terms, we obtain the right hand side of equality sign (c).

For (d), if $\forall u:(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}$ we have $u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$, then $(i,v_{k+1}^{\mathrm{O}})\notin\mathcal{E}$ since $i\in\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$. Then by definition, $c_1^{k+1}=0$. We directly have

$$\sum_{\substack{e=(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}\\ u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}}} b_e\mathbf{q}_{\Lambda(v_{k+1}^{\mathrm{O}})} = \sum_{e=(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}} b_e\mathbf{q}_{\Lambda(v_{k+1}^{\mathrm{O}})} - c_1^{k+1}. \tag{G.2}$$

If there is some $u\neq i$ such that $(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}$ and $u\in\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$, then u is a generator and $u\in\mathcal{V}^{\mathrm{nc}}$. Thereby $\Psi(u)=v_{k+1}^{\mathrm{O}}$, and by Corollary 5.10, we have $\mathbf{q}_{\Lambda(v_{k+1}^{\mathrm{O}})}=0$. Besides we cannot have $(i,v_{k+1}^{\mathrm{O}})\in\mathcal{E}$ as well, otherwise Corollary 5.9 will imply $\mathbf{q}_{\Lambda(v_{k+1}^{\mathrm{O}})}=1$ and lead to the contradiction. Therefore $c_1^{k+1}=0$ and both sides of (G.2) become 0. Equality (G.2) still holds.

If $(i,v_{k+1}^{\mathrm{O}})\in\mathcal{E}$, then there does not exist $u\neq i$ such that $(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}$ and $u\in\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$. We also have $\mathbf{q}_{\Lambda(v_{k+1}^{\mathrm{O}})}=1$ and $c_1^{k+1}=b_{(v_{k+1}^{\mathrm{O}},i)}$. Therefore,

$$\sum_{\substack{e=(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}\\ u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}}} b_e\mathbf{q}_{\Lambda(v_{k+1}^{\mathrm{O}})} = \sum_{\substack{e=(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}\\ u\neq i}} b_e = \sum_{e=(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}} b_e - b_{(v_{k+1}^{\mathrm{O}},i)} = \sum_{e=(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}} b_e\mathbf{q}_{\Lambda(v_{k+1}^{\mathrm{O}})} - c_1^{k+1}.$$

That is to say, (G.2) holds for all situations. Thus splitting the first term on the right hand side of (c) gives the right hand side of (d).

For (e), by Definition 5.11, the first four terms on the right hand side of (d) is exactly $\langle e_{k+1}^{\mathrm{O}}\rangle + c_2^{k+1}$.

For (f), the value of $\sum_{e=(u,v_k^{\mathrm{O}})\in\mathcal{E}} b_e$ is the $(v_k^{\mathrm{O}},\Lambda(v_k^{\mathrm{O}}))^{\text{th}}$ coordinate of the matrix $\mathbf{R}(\mathcal{S}_{\mathrm{G}},\mathcal{S}_{\mathrm{B}})$. For each u such that $e=(u,v_k^{\mathrm{O}})\in\mathcal{E}$ and $u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$, then the value of $-b_e$ is the $(v_k^{\mathrm{O}},\Lambda(u))^{\text{th}}$ coordinate of $\mathbf{R}(\mathcal{S}_{\mathrm{G}},\mathcal{S}_{\mathrm{B}})$. If $e_{k-1}^{\mathrm{O}}\in\mathcal{S}_{\mathrm{B}}$, then the value of $-c_2^k$ is $-b_{e_{k-1}^{\mathrm{O}}}\mathbf{q}_{\Xi(e_{k-1}^{\mathrm{O}})}$, and the value of $-b_{e_{k-1}^{\mathrm{O}}}$ is exactly the $(v_k^{\mathrm{O}},\Xi(e_{k-1}^{\mathrm{O}}))^{\text{th}}$ coordinate of $\mathbf{R}(\mathcal{S}_{\mathrm{G}},\mathcal{S}_{\mathrm{B}})$. Similarly, if $e_{k+1}^{\mathrm{O}}\in\mathcal{S}_{\mathrm{B}}$, the value of c_2^{k+1} is $b_{e_k^{\mathrm{O}}}\mathbf{q}_{\Xi(e_k^{\mathrm{O}})}$, and $b_{e_k^{\mathrm{O}}}$ is the $(v_k^{\mathrm{O}},\Xi(e_k^{\mathrm{O}}))^{\text{th}}$ coordinate of $\mathbf{R}(\mathcal{S}_{\mathrm{G}},\mathcal{S}_{\mathrm{B}})$. Besides e_{k-1}^{O} and e_k^{O}, v_k^{O} cannot be adjacent to any other binding edges in $\mathcal{S}_{\mathrm{B}}$ as $\mathcal{S}_{\mathrm{B}}\subseteq\mathcal{E}^{\mathrm{O}}$ and e_{k-1}^{O} and e_k^{O} are the only edges in $\mathcal{E}^{\mathrm{O}}$ that are adjacent to v_k^{O}. Above all, we have

$$\sum_{e=(u,v_k^{\mathrm{O}})\in\mathcal{E}} b_e\mathbf{q}_{\Lambda(v_k^{\mathrm{O}})}-\sum_{\substack{e=(u,v_k^{\mathrm{O}})\in\mathcal{E}\\ u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}}} b_e\mathbf{q}_{\Lambda(u)}-c_2^k+c_2^{k+1}$$

is the $v_k^{\mathrm{O}\,\text{th}}$ coordinate of $\mathbf{R}(\mathcal{S}_{\mathrm{G}},\mathcal{S}_{\mathrm{B}})\mathbf{q}$. On the other hand, the value of c_1^k is the $(v_k^{\mathrm{O}},i)^{\text{th}}$ coordinate of $-\mathbf{CBC}^{\mathsf{T}}$. The equality (F.2) gives

$$\sum_{e=(u,v_k^{\mathrm{O}})\in\mathcal{E}} b_e\mathbf{q}_{\Lambda(v_k^{\mathrm{O}})}-\sum_{\substack{e=(u,v_k^{\mathrm{O}})\in\mathcal{E}\\ u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}}} b_e\mathbf{q}_{\Lambda(u)}-c_2^k+c_2^{k+1}=c_1^k$$

and thereby (f) holds. ∎

Proof. (Corollary 5.13) Theorem 5.12 implies

$$\langle e_k^{\mathrm{O}}\rangle=\langle e_{k+1}^{\mathrm{O}}\rangle=\sum_{e=(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}} b_e\mathbf{q}_{\Lambda(v_{k+1}^{\mathrm{O}})}-\sum_{\substack{e=(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}\\ u\neq v_{k+2}^{\mathrm{O}},u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}}} b_e\mathbf{q}_{\Lambda(u)}-b_{e_{k+1}^{\mathrm{O}}}\mathbf{q}_{\Lambda(v_{k+1}^{\mathrm{O}})}-c_1^{k+1}.$$

Here $c_2^{k+1}=0$ since $e_k^{\mathrm{O}}\notin\mathcal{S}_{\mathrm{B}}$. Further, we have

$$\sum_{\substack{e=(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}\\ u\neq v_{k+2}^{\mathrm{O}},u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}}} b_e\mathbf{q}_{\Lambda(u)}=\sum_{\substack{e=(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}^{\mathrm{nc}}\\ u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}}} b_e\mathbf{q}_{\Lambda(u)}+b_{e_k^{\mathrm{O}}}\mathbf{q}_{\Lambda(v_k^{\mathrm{O}})} \tag{G.3}$$

since v_k^{O} and v_{k+2}^{O} are the only neighbors of v_{k+1}^{O} which are not in $\mathcal{V}^{\mathrm{nc}}$. Similar to the proof of (G.2), one can show that

$$\sum_{\substack{e=(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}^{\mathrm{nc}}\\ u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}}} b_e\mathbf{q}_{\Lambda(u)}+c_1^{k+1}=\sum_{e=(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}^{\mathrm{nc}}} b_e\mathbf{q}_{\Lambda(u)}. \tag{G.4}$$

Besides, for u such that $(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}^{\mathrm{nc}}$ and $u\notin\mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$, we know that $\Psi(u)=v_{k+1}^{\mathrm{O}}$ and thereby $\mathbf{q}_{\Lambda(u)}=\mathbf{q}_{\Lambda(v_{k+1}^{\mathrm{O}})}$. Combined with (31) to (G.4), we have

$$\begin{aligned}\langle e_k^{\mathrm{O}}\rangle=&\sum_{e=(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}} b_e\mathbf{q}_{\Lambda(v_{k+1}^{\mathrm{O}})}-\sum_{e=(u,v_{k+1}^{\mathrm{O}})\in\mathcal{E}^{\mathrm{nc}}} b_e\mathbf{q}_{\Lambda(v_{k+1}^{\mathrm{O}})}-b_{e_k^{\mathrm{O}}}\mathbf{q}_{\Lambda(v_k^{\mathrm{O}})}-b_{e_{k+1}^{\mathrm{O}}}\mathbf{q}_{\Lambda(v_{k+1}^{\mathrm{O}})}\\ =&(b_{e_k^{\mathrm{O}}}+b_{e_{k+1}^{\mathrm{O}}})\mathbf{q}_{\Lambda(v_{k+1}^{\mathrm{O}})}-b_{e_k^{\mathrm{O}}}\mathbf{q}_{\Lambda(v_k^{\mathrm{O}})}-b_{e_{k+1}^{\mathrm{O}}}\mathbf{q}_{\Lambda(v_{k+1}^{\mathrm{O}})}\\ =&b_{e_k^{\mathrm{O}}}(\mathbf{q}_{\Lambda(v_{k+1}^{\mathrm{O}})}-\mathbf{q}_{\Lambda(v_k^{\mathrm{O}})}).\end{aligned}$$

∎

Appendix H. Proof of Theorem 5.21 and Theorem 5.22.

Lemma H.1. *For a connected graph $(\mathcal{G}, \mathcal{V}, \mathcal{E})$ and $k \in \mathcal{V}$, then the following $|\mathcal{V}|-1$ columns*

$$\sum_{\alpha':(\alpha,\alpha')\in\mathcal{E}} b_{(\alpha,\alpha')}(\mathbf{e}_\alpha - \mathbf{e}_{\alpha'}) \tag{H.1}$$

for $\alpha \neq k$ are independent of each other, and all of them are perpendicular to $\mathbf{1}$.

Proof. (Lemma H.1) It is known that for connected graphs, $\mathbf{CBC}^\mathsf{T}$ has rank $|\mathcal{V}|-1$ and any $|\mathcal{V}|-1$ columns are independent. The column in (H.1) is actually the α^{th} column of $\mathbf{CBC}^\mathsf{T}$. Thus they are independent. The null space for those $|\mathcal{V}|-1$ columns is spanned by $\mathbf{1}$ as each column sums up to 0. ∎

Lemma H.2. *For the Graph $\mathcal{G}'$ constructed in either Theorem 5.21 or Theorem 5.22. Suppose both $(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B})$ and $(\mathcal{S}_\mathrm{G}(\mathcal{G}'), \mathcal{S}_\mathrm{B}(\mathcal{G}'))$ are perfect pairs. If $\mathcal{S}_\mathrm{G}^\mathsf{c} = \mathcal{S}_\mathrm{G}(\mathcal{G}')^\mathsf{c}$, $\mathcal{S}_\mathrm{B} = \mathcal{S}_\mathrm{B}(\mathcal{G}')$, then $\mathbf{J}_\mathcal{G}^{i,j+N_\mathrm{G}} = \mathbf{J}_{\mathcal{G}'}^{i,j+N_\mathrm{G}}$.*

Proof. (Lemma H.2) Suppose $\mathbf{x} = -\mathbf{R}^{-1}\mathbf{CBC}^\mathsf{T}\mathbf{e}_i \in \mathbb{R}^N$, we can then construct $\mathbf{y} \in \mathbb{R}^{|\mathcal{V}'|}$ such that for $\alpha \in \mathcal{V}' \setminus \mathcal{S}_\mathrm{G}(\mathcal{G}')^\mathsf{c}$, $\mathbf{y}_{\Lambda_{\mathcal{G}'}(\alpha)} = \mathbf{x}_{\Lambda(\alpha)}$, and for $\beta \in \mathcal{E}'$, $\mathbf{y}_{\Xi_{\mathcal{G}'}(\beta)} = \mathbf{x}_{\Xi(\beta)}$. The value of $\mathbf{y}_{|\mathcal{V}'|}$ is set as 0. One can show that

$$\begin{aligned}
\mathbf{R}_{\mathcal{G}'}\mathbf{y} &= \sum_{\substack{\alpha\in\mathcal{V}'\\ \alpha\notin\mathcal{S}_\mathrm{G}(\mathcal{G}')^\mathsf{c}}} \mathbf{y}_{\Lambda_{\mathcal{G}'}(\alpha)}\mathbf{C}_{\mathcal{G}'}\mathbf{BC}_{\mathcal{G}'}^\mathsf{T}\mathbf{e}_\alpha + \sum_{\beta\in\mathcal{S}_\mathrm{B}(\mathcal{G}')} \mathbf{y}_{\Xi_{\mathcal{G}'}(\beta)}\mathbf{C}_{\mathcal{G}'}\mathbf{Be}_\beta\\
&= \sum_{\substack{\alpha\in\mathcal{V}_1\\ \alpha\notin\mathcal{S}_\mathrm{G}^\mathsf{c}}} \mathbf{y}_{\Lambda_{\mathcal{G}'}(\alpha)}\mathbf{CBC}^\mathsf{T}\mathbf{e}_\alpha + \sum_{\beta\in\mathcal{S}_\mathrm{B}} \mathbf{y}_{\Xi_{\mathcal{G}'}(\beta)}\mathbf{CBe}_\beta + \mathbf{y}_{\Lambda_{\mathcal{G}'}(v)}b_e(\mathbf{e}_v - \mathbf{e}_u).
\end{aligned}$$

According to the construction of $\mathbf{y}$, we can replace $\mathbf{y}_{\Lambda_{\mathcal{G}'}(\cdot)}$ and $\mathbf{y}_{\Xi_{\mathcal{G}'}(\cdot)}$ above by $\mathbf{x}_{\Lambda(\cdot)}$ and $\mathbf{x}_{\Xi(\cdot)}$. Additionally, Lemma 5.2 implies

$$\begin{aligned}
&\sum_{\substack{\alpha\in\mathcal{V}_1\\ \alpha\notin\mathcal{S}_\mathrm{G}^\mathsf{c}}} \mathbf{x}_{\Lambda(\alpha)}\mathbf{CBC}^\mathsf{T}\mathbf{e}_\alpha + \sum_{\beta\in\mathcal{S}_\mathrm{B}} \mathbf{x}_{\Xi(\beta)}\mathbf{CBe}_\beta + \mathbf{x}_{\Lambda(v)}b_e(\mathbf{e}_v - \mathbf{e}_u)\\
&= -\mathbf{CBC}^\mathsf{T}\mathbf{e}_i = -\mathbf{C}_{\mathcal{G}'}\mathbf{BC}_{\mathcal{G}'}^\mathsf{T}\mathbf{e}_i.
\end{aligned}$$

Thereby, $\mathbf{y} = -\mathbf{R}_{\mathcal{G}'}^{-1}\mathbf{C}_{\mathcal{G}'}\mathbf{BC}_{\mathcal{G}'}^\mathsf{T}\mathbf{e}_i$, and we have

$$\mathbf{J}_{\mathcal{G}'}^{i,j+N_\mathrm{G}} = \mathbf{y}_{\Lambda_{\mathcal{G}'}(j+N_\mathrm{G})} = \mathbf{x}_{\Lambda(j+N_\mathrm{G})} = \mathbf{J}_\mathcal{G}^{i,j+N_\mathrm{G}}.$$ ∎

Proof. (Theorem 5.21) Consider a perfect pair $(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B})$ for $\mathcal{G}$ with $i \in \mathcal{S}_\mathrm{G}^\mathsf{c}$, we construct

$$\begin{aligned}
\mathcal{S}_\mathrm{G}(\mathcal{G}') &= \mathcal{S}_\mathrm{G} \cap \mathcal{V}_1 = \mathcal{S}_\mathrm{G}\\
\mathcal{S}_\mathrm{B}(\mathcal{G}') &= \mathcal{S}_\mathrm{B} \cap \mathcal{E}_1.
\end{aligned}$$

Then we naturally have $i \in \mathcal{S}_\mathrm{G}(\mathcal{G}')^\mathsf{c}$. If $\mathcal{S}_\mathrm{G}(\mathcal{G}') \not\perp \mathcal{S}_\mathrm{B}(\mathcal{G}')$, then the columns of $\mathbf{R}_{\mathcal{G}'}$ are not independent. We now look at the columns of $\mathbf{R}_{\mathcal{G}'}$, for $\alpha \in \mathcal{V}_\mathrm{L}(\mathcal{G}') \cup \mathcal{S}_\mathrm{G}(\mathcal{G}')$, the support of the column $\mathbf{C}_{\mathcal{G}'}\mathbf{BC}_{\mathcal{G}'}^\mathsf{T}\mathbf{e}_\alpha$ only contains the neighbors of α in $\mathcal{G}'$. In fact, for $\alpha \in \mathcal{V}_1$, the neighbors of α in $\mathcal{G}'$ are equivalent to the neighbors of α in $\mathcal{G}$. Therefore, whenever $\alpha \neq v$ the column $\mathbf{C}_{\mathcal{G}'}\mathbf{BC}_{\mathcal{G}'}^\mathsf{T}\mathbf{e}_\alpha$ is also a column in $\mathbf{R}$. Likely, for $\beta \in \mathcal{S}_\mathrm{B}(\mathcal{G}')$, the support of the column $\mathbf{C}_{\mathcal{G}'}\mathbf{Be}_\beta$ only contains the vertices linked by β, and $\beta \in \mathcal{S}_\mathrm{B}(\mathcal{G}')$ is also an edge in $\mathcal{G}$. Therefore, the column $\mathbf{C}_{\mathcal{G}'}\mathbf{Be}_\beta$ is also a column in $\mathbf{R}$. As the result, the columns of $\mathbf{R}_{\mathcal{G}'}$ are not independent implies that there exist coefficients a_α for $\alpha \in (\mathcal{V}_{\mathrm{L}}(\mathcal{G}') \setminus \{v\}) \cup \mathcal{S}_{\mathrm{G}}(\mathcal{G}')$ and b_β for $\beta \in \mathcal{S}_{\mathrm{B}}(\mathcal{G}')$, such that

$$\begin{aligned}
&\mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{C}_{\mathcal{G}'}^{\mathsf{T}}\mathbf{e}_v = b_e(\mathbf{e}_v - \mathbf{e}_u)\\
&= \sum_{\substack{\alpha \in \mathcal{V}_{\mathrm{L}}(\mathcal{G}') \cup \mathcal{S}_{\mathrm{G}}(\mathcal{G}')\\ \alpha \neq v}} a_\alpha \mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{C}_{\mathcal{G}'}^{\mathsf{T}}\mathbf{e}_\alpha + \sum_{\beta \in \mathcal{S}_{\mathrm{B}}(\mathcal{G}')} b_\beta \mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{e}_\beta\\
&= \sum_{\substack{\alpha \in \mathcal{V}_{\mathrm{L}}(\mathcal{G}') \cup \mathcal{S}_{\mathrm{G}}(\mathcal{G}')\\ \alpha \neq v}} a_\alpha \mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}}\mathbf{e}_\alpha + \sum_{\beta \in \mathcal{S}_{\mathrm{B}}(\mathcal{G}')} b_\beta \mathbf{C}\mathbf{B}\mathbf{e}_\beta. \qquad \text{(H.2)}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\sum_{\alpha \in \mathcal{V}_2} \mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}}\mathbf{e}_\alpha &= \sum_{\alpha \in \mathcal{V}_2} \sum_{\alpha' : (\alpha,\alpha') \in \mathcal{E}} b_{(\alpha,\alpha')}(\mathbf{e}_\alpha - \mathbf{e}_{\alpha'})\\
&= b_e(\mathbf{e}_v - \mathbf{e}_u) + \sum_{\substack{e=(\alpha,\alpha') \in \mathcal{E}_2\\ \alpha < \alpha'}} b_e(\mathbf{e}_\alpha - \mathbf{e}_{\alpha'}) + b_e(\mathbf{e}_{\alpha'} - \mathbf{e}_\alpha)\\
&= b_e(\mathbf{e}_v - \mathbf{e}_u). \qquad \text{(H.3)}
\end{aligned}$$

Combining (H.2) and (H.3), we can see that the columns of $\mathbf{R}$ are not independent. It contradicts to $\mathcal{S}_{\mathrm{G}} \perp \mathcal{S}_{\mathrm{B}}$. Thereby, $\mathcal{S}_{\mathrm{G}}(\mathcal{G}') \perp \mathcal{S}_{\mathrm{B}}(\mathcal{G}')$ is proved.

Next we are showing $|\mathcal{S}_{\mathrm{G}}(\mathcal{G}')| + |\mathcal{S}_{\mathrm{B}}(\mathcal{G}')| = |\mathcal{V}_{\mathrm{G}}(\mathcal{G}')| - 1$. We first apply Lemma H.1 to the graph

$$\mathcal{G}''(\mathcal{V}'' = \mathcal{V}_2 \cup \{u\}, \mathcal{E}'' = \mathcal{E}_2).$$

Then we have all the columns

$$\mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}}\mathbf{e}_\alpha = \sum_{\alpha' : (\alpha,\alpha') \in \mathcal{E}} b_{(\alpha,\alpha')}(\mathbf{e}_\alpha - \mathbf{e}_{\alpha'})$$

for $\alpha \in \mathcal{V}_2$ are independent. For any $\beta \in \mathcal{E}_2$, since $\mathbf{C}\mathbf{B}\mathbf{e}_\beta$ is also independent to $\mathbf{1}$, so $\mathbf{C}\mathbf{B}\mathbf{e}_\beta$ is not independent to the columns $\mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}}\mathbf{e}_\alpha$ in $\mathbf{R}$ where $\alpha \in \mathcal{V}_2$. Therefore, $\beta \notin \mathcal{S}_{\mathrm{B}}$ and $\mathcal{E}_2 \cap \mathcal{S}_{\mathrm{B}} = \emptyset$. As the result, we have

$$\begin{aligned}
|\mathcal{V}_{\mathrm{G}}(\mathcal{G}')| &= |\mathcal{V}_{\mathrm{G}} \cap \mathcal{V}_1| = |\mathcal{V}_{\mathrm{G}}|\\
|\mathcal{S}_{\mathrm{G}}(\mathcal{G}')| &= |\mathcal{S}_{\mathrm{G}} \cap \mathcal{V}_1| = |\mathcal{S}_{\mathrm{G}}|\\
|\mathcal{S}_{\mathrm{B}}(\mathcal{G}')| &= |\mathcal{S}_{\mathrm{B}} \cap \mathcal{E}_1| = |\mathcal{S}_{\mathrm{B}}|.
\end{aligned}$$

Hence we have $|\mathcal{S}_{\mathrm{G}}(\mathcal{G}')| + |\mathcal{S}_{\mathrm{B}}(\mathcal{G}')| = |\mathcal{V}_{\mathrm{G}}(\mathcal{G}')| - 1$, and $(\mathcal{S}_{\mathrm{G}}(\mathcal{G}'), \mathcal{S}_{\mathrm{B}}(\mathcal{G}'))$ forms a perfect pair for $\mathcal{G}'$.

To show the other direction, consider the perfect pair $(\mathcal{S}_{\mathrm{G}}(\mathcal{G}'), \mathcal{S}_{\mathrm{B}}(\mathcal{G}'))$ for $\mathcal{G}'$ with $i \in \mathcal{S}_{\mathrm{G}}(\mathcal{G}')^{\mathsf{c}}$, we construct

$$\mathcal{S}_{\mathrm{G}} = \mathcal{S}_{\mathrm{G}}(\mathcal{G}'), \mathcal{S}_{\mathrm{B}} = \mathcal{S}_{\mathrm{B}}(\mathcal{G}').$$

Clearly $i \in \mathcal{S}_{\mathrm{G}}(\mathcal{G}')^{\mathsf{c}}$, and

$$|\mathcal{S}_{\mathrm{G}}| + |\mathcal{S}_{\mathrm{B}}| = |\mathcal{S}_{\mathrm{G}}(\mathcal{G}')| + |\mathcal{S}_{\mathrm{B}}(\mathcal{G}')| = |\mathcal{V}_{\mathrm{G}}(\mathcal{G}')| - 1 = |\mathcal{V}_{\mathrm{G}}| - 1.$$

To show $\mathcal{S}_{\mathrm{G}} \perp \mathcal{S}_{\mathrm{B}}$, we need to prove $\mathbf{R}$ is invertible. The supports of columns in $\mathbf{R}_{\mathcal{G}'}$ are subsets of $\mathcal{V}'$. By comparison, the columns of $\mathbf{R}$ which are not columns of

$\mathbf{R}_{\mathcal{G}'}$ are $\mathbf{C}\mathbf{B}\mathbf{C}^\mathsf{T}\mathbf{e}_\alpha$ for $\alpha \in \mathcal{V}_2$, whose supports are subsets of $\mathcal{V}''$. By Lemma H.1, those unique columns of $\mathbf{R}$ are independent to each other. It is sufficient to show that any non-zero vector $\mathbf{w}$ spanned by $\mathbf{C}\mathbf{B}\mathbf{C}^\mathsf{T}\mathbf{e}_\alpha$ for $\alpha \in \mathcal{V}_2$ must be independent to the common columns of $\mathbf{R}_{\mathcal{G}'}$ and $\mathbf{R}$. If not, then the support of $\mathbf{w}$ must be $\{u, v\}$, and then $\mathbf{w}$ must be the multiple of $\mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{C}_{\mathcal{G}'}^\mathsf{T}\mathbf{e}_v$, while the latter is a column in $\mathbf{R}_{\mathcal{G}'}$. It leads to the contradiction, so $\mathcal{S}_\mathrm{G} \perp \mathcal{S}_\mathrm{B}$. Above all, $(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B})$ is also a perfect pair for $\mathcal{G}$.

In both directions, our constructions always give $(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B}) = (\mathcal{S}_\mathrm{G}(\mathcal{G}'), \mathcal{S}_\mathrm{B}(\mathcal{G}'))$. Finally Lemma H.2 gives $\mathbf{J}_{\mathcal{G}}^{i,j+N_\mathrm{G}} = \mathbf{J}_{\mathcal{G}'}^{i,j+N_\mathrm{G}}$. ∎

Proof. (Theorem 5.22) If $(\mathcal{S}_\mathrm{G}, \mathcal{S}_\mathrm{B})$ is a perfect pair for $\mathcal{G}$ and $i \in \mathcal{S}_\mathrm{G}^\mathsf{c}$, we first let

$$\mathbf{R}_*^\mathsf{T} := \begin{bmatrix} \mathbf{I}_{\mathcal{V}_\mathrm{L}(\mathcal{G}')}^N \mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{C}_{\mathcal{G}'}^\mathsf{T} \\ \mathbf{I}_{\mathcal{S}_\mathrm{G}\cap\mathcal{V}_1}^N \mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{C}_{\mathcal{G}'}^\mathsf{T} \\ \mathbf{I}_{\mathcal{S}_\mathrm{B}\cap\mathcal{E}_1}^E \mathbf{B}\mathbf{C}_{\mathcal{G}'}^\mathsf{T} \\ \mathbf{e}_i^\mathsf{T} \end{bmatrix}.$$

Then all the columns of $\mathbf{R}_*$ are also columns in $\mathbf{R}$, so they are independent. Consider that $\mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{C}_{\mathcal{G}'}^\mathsf{T}\mathbf{e}_v$ is the multiple of $\mathbf{e}_u - \mathbf{e}_v$, we construct

$$\mathcal{S}_\mathrm{G}(\mathcal{G}') = \begin{cases} \mathcal{S}_\mathrm{G} \cap \mathcal{V}_1, & \mathbf{e}_u - \mathbf{e}_v \in \mathrm{col}(\mathbf{R}_*) \\ (\mathcal{S}_\mathrm{G} \cap \mathcal{V}_1) \cup \{v\}, & \text{otherwise} \end{cases}$$
$$\mathcal{S}_\mathrm{B}(\mathcal{G}') = \mathcal{S}_\mathrm{B} \cap \mathcal{E}_1.$$

By construction, we always have $i \in \mathcal{S}_\mathrm{G}(\mathcal{G}')^\mathsf{c}$ and $\mathcal{S}_\mathrm{G}(\mathcal{G}') \perp \mathcal{S}_\mathrm{B}(\mathcal{G}')$.

Next we will show $|\mathcal{S}_\mathrm{G}(\mathcal{G}')| + |\mathcal{S}_\mathrm{B}(\mathcal{G}')| = |\mathcal{V}_\mathrm{G}(\mathcal{G}')| - 1$. We notice that under such construction, the supports of the columns of $\mathbf{R}_{\mathcal{G}'}$ are the subsets of $\mathcal{V}'$. Hence it is sufficient to show that for any vector $\mathbf{w}$ whose support is the subset of $\mathcal{V}'$, $\mathbf{w}$ is in the column space of $\mathbf{R}_{\mathcal{G}'}$. Suppose $\mathbf{x} = \mathbf{R}^{-1}\mathbf{w}$, and let

$$\mathbf{y} = \sum_{\alpha\in\mathcal{V}_1\setminus\mathcal{S}_\mathrm{G}^\mathsf{c}} \mathbf{x}_{\Lambda(\alpha)}\mathbf{e}_{\Lambda(\alpha)} + \sum_{\beta\in\mathcal{E}_1\cap\mathcal{S}_\mathrm{B}} \mathbf{x}_{\Xi(\beta)}\mathbf{e}_{\Xi(\beta)}. \tag{H.4}$$

Lemma 5.2 implies there exists some $q \in \mathbb{R}$ such that

$$\mathbf{R}\mathbf{y} + q(\mathbf{e}_u - \mathbf{e}_v) = \mathbf{w}. \tag{H.5}$$

The non-zero coordinates of $\mathbf{y}$ exactly correspond to the columns of $\mathbf{R}$ which also show up in $\mathbf{R}_*$. Since by construction, all columns of $\mathbf{R}_*$ are also in $\mathbf{R}_{\mathcal{G}'}$, and $\mathbf{e}_u - \mathbf{e}_v$ is always in the column space of $\mathbf{R}_{\mathcal{G}'}$, we have $\mathbf{w}$ is in the column space of $\mathbf{R}_{\mathcal{G}'}$. Therefore, $\mathcal{S}_\mathrm{G}(\mathcal{G}')$ and $\mathcal{S}_\mathrm{B}(\mathcal{G}')$ are a perfect pair for $\mathcal{G}'$.

Further, we let $\mathbf{w} = -\mathbf{C}\mathbf{B}\mathbf{C}^\mathsf{T}$, then $\mathbf{J}_{\mathcal{G}}^{i,h+N_\mathrm{G}} = \mathbf{x}_{\Lambda(j+N_\mathrm{G})}$. If $\mathbf{e}_u - \mathbf{e}_v \in \mathrm{col}(\mathbf{R}_*)$, then $\mathbf{R}_{\mathcal{G}'} = \mathbf{R}_*$. Suppose $\mathbf{e}_u - \mathbf{e}_v = \mathbf{R}_*\mathbf{h}$ for some $\mathbf{h}$. We construct

$$\mathbf{z} = \sum_{\alpha\in\mathcal{V}_2\setminus\mathcal{S}_\mathrm{G}^\mathsf{c}} \mathbf{x}_{\Lambda(\alpha)}\mathbf{e}_{\Lambda(\alpha)} + \sum_{\beta\in\mathcal{E}_2\cap\mathcal{S}_\mathrm{B}} \mathbf{x}_{\Xi(\beta)}\mathbf{e}_{\Xi(\beta)}. \tag{H.6}$$

Lemma 5.2 implies $\mathbf{R}\mathbf{z} = q(\mathbf{e}_u - \mathbf{e}_v) = q\mathbf{R}_*\mathbf{h}$. Since the columns of $\mathbf{R}$ corresponding to the non-zero coordinates of $\mathbf{z}$ are disjoint to the columns of $\mathbf{R}_*$, we must have $q = 0$, otherwise $\mathbf{R}$ is singular. Therefore, $\mathbf{R}\mathbf{y} = -\mathbf{C}\mathbf{B}\mathbf{C}^\mathsf{T}\mathbf{e}_i$. we then define $\hat{\mathbf{y}} \in \mathbb{R}^{|\mathcal{V}'|}$ such that for $\alpha \in \mathcal{V}_1 \setminus \mathcal{S}_\mathrm{G}^\mathsf{c}$, $\hat{\mathbf{y}}_{\Lambda_{\mathcal{G}'}(\alpha)} = \mathbf{y}_{\Lambda(\alpha)}$, and for $\beta \in \mathcal{E}_1 \cap \mathcal{S}_\mathrm{B}$, $\hat{\mathbf{y}}_{\Xi_{\mathcal{G}'}(\beta)} = \mathbf{y}_{\Xi(\beta)}$. Other coordinates of $\hat{\mathbf{y}}$ are set as 0. If $\mathbf{e}_u - \mathbf{e}_v \notin \mathrm{col}(\mathbf{R}_*)$, then we define $\hat{\mathbf{y}} \in \mathbb{R}^{|\mathcal{V}'|}$

such that for $\alpha \in \mathcal{V}_1 \setminus \mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$, $\hat{\mathbf{y}}_{\Lambda_{\mathcal{G}'}(\alpha)} = \mathbf{y}_{\Lambda(\alpha)}$, and for $\beta \in \mathcal{E}_1 \cap \mathcal{S}_{\mathrm{B}}$, $\hat{\mathbf{y}}_{\Xi_{\mathcal{G}'}(\beta)} = \mathbf{y}_{\Xi(\beta)}$. Let $\hat{\mathbf{y}}_{\Lambda_{\mathcal{G}'}(v)} = -q$, and other coordinates of $\hat{\mathbf{y}}$ are set as 0. In both cases, we have

$$\mathbf{R}_{\mathcal{G}'}\hat{\mathbf{y}} = \mathbf{R}\mathbf{y} = -\mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}}\mathbf{e}_i = -\mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{C}_{\mathcal{G}'}^{\mathsf{T}}\mathbf{e}_i$$

and

$$\mathbf{J}_{\mathcal{G}'}^{i,j+N_{\mathrm{G}}} = \hat{\mathbf{y}}_{\Lambda_{\mathcal{G}'}(j+N_{\mathrm{G}})} = \mathbf{y}_{\Lambda(j+N_{\mathrm{G}})} = \mathbf{x}_{\Lambda(j+N_{\mathrm{G}})} = \mathbf{J}_{\mathcal{G}}^{i,j+N_{\mathrm{G}}}.$$

For the other direction, we assume $(\mathcal{S}_{\mathrm{G}}(\mathcal{G}'), \mathcal{S}_{\mathrm{B}}(\mathcal{G}'))$ is a perfect pair for $\mathcal{G}'$ and $i \in \mathcal{S}_{\mathrm{G}}(\mathcal{G}')^{\mathsf{c}}$. If $v \in \mathcal{S}_{\mathrm{G}}(\mathcal{G}')$, let

$$\mathcal{S}_{\mathrm{G}} = (\mathcal{S}_{\mathrm{G}}(\mathcal{G}') \setminus \{v\}) \cup (\mathcal{V}_2 \cap \mathcal{V}_{\mathrm{G}}), \mathcal{S}_{\mathrm{B}} = \mathcal{S}_{\mathrm{B}}(\mathcal{G}').$$

Then $\mathcal{S}_{\mathrm{G}}^{\mathsf{c}} = (\mathcal{V}_1 \cup \mathcal{V}_{\mathrm{G}} \cup \{v\}) \setminus \mathcal{S}_{\mathrm{G}}(\mathcal{G}') = \mathcal{S}_{\mathrm{G}}(\mathcal{G}')^{\mathsf{c}}$. Clearly $i \in \mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$, and we can compute

$$\begin{aligned}
&|\mathcal{S}_{\mathrm{G}}| + |\mathcal{S}_{\mathrm{B}}| = N_{\mathrm{G}} - |\mathcal{S}_{\mathrm{G}}(\mathcal{G}')^{\mathsf{c}}| + |\mathcal{S}_{\mathrm{B}}(\mathcal{G}')| \\
=&N_{\mathrm{G}} - (|\mathcal{V}_{\mathrm{G}}(\mathcal{G}')| - |\mathcal{S}_{\mathrm{G}}(\mathcal{G}')|) + |\mathcal{S}_{\mathrm{B}}(\mathcal{G}')| \\
=&N_{\mathrm{G}} - |\mathcal{V}_{\mathrm{G}}(\mathcal{G}')| + |\mathcal{S}_{\mathrm{G}}(\mathcal{G}')| + |\mathcal{S}_{\mathrm{B}}(\mathcal{G}')| \\
=&N_{\mathrm{G}} - |\mathcal{V}_{\mathrm{G}}(\mathcal{G}')| + |\mathcal{V}_{\mathrm{G}}(\mathcal{G}')| - 1 = N_{\mathrm{G}} - 1.
\end{aligned}$$

To show $\mathcal{S}_{\mathrm{G}} \perp \mathcal{S}_{\mathrm{B}}$, we need to prove all the columns of $\mathbf{R}$ are independent. If not, suppose $\mathbf{R}\mathbf{x} = 0$ for some vector $\mathbf{x}$. Then by Lemma 5.2, for vectors $\mathbf{y}, \mathbf{z}$ as constructed in (H.4) and (H.6), there exists some $q \in \mathbb{R}$ such that

$$\mathbf{R}\mathbf{y} + q(\mathbf{e}_u - \mathbf{e}_v) = \mathbf{R}\mathbf{x} = 0 \tag{H.7a}$$

$$\mathbf{R}\mathbf{z} = q(\mathbf{e}_u - \mathbf{e}_v). \tag{H.7b}$$

Viewing the subgraph $\mathcal{G}''(\mathcal{V}'' = \mathcal{V}_2 \cup \{u\}, \mathcal{E}'' = \mathcal{E}_2 \cup \{e\})$ as the underlying graph in Lemma H.1, it implies $q \neq 0$, otherwise (H.7b) does not hold. Thereby, (H.7a) implies the columns of $\mathbf{R}_{\mathcal{G}'}$ are not independent and leads to the contradiction. As the result, we have $\mathcal{S}_{\mathrm{G}} \perp \mathcal{S}_{\mathrm{B}}$. Additionally, as we have $\mathcal{S}_{\mathrm{G}}^{\mathsf{c}} = \mathcal{S}_{\mathrm{G}}(\mathcal{G}')^{\mathsf{c}}$ and $\mathcal{S}_{\mathrm{B}} = \mathcal{S}_{\mathrm{B}}(\mathcal{G}')$ in this case, Lemma H.2 guarantees $\mathbf{J}_{\mathcal{G}}^{i,j+N_{\mathrm{G}}} = \mathbf{J}_{\mathcal{G}'}^{i,j+N_{\mathrm{G}}}$.

If $v \notin \mathcal{S}_{\mathrm{G}}(\mathcal{G}')$, pick $k \in \mathcal{V}_2 \cap \mathcal{V}_{\mathrm{G}}$ and let

$$\mathcal{S}_{\mathrm{G}} = (\mathcal{S}_{\mathrm{G}}(\mathcal{G}') \cup (\mathcal{V}_2 \cap \mathcal{V}_{\mathrm{G}})) \setminus \{k\}, \mathcal{S}_{\mathrm{B}} = \mathcal{S}_{\mathrm{B}}(\mathcal{G}').$$

Then $i \in \mathcal{S}_{\mathrm{G}}^{\mathsf{c}}$ and

$$\begin{aligned}
&|\mathcal{S}_{\mathrm{G}}| + |\mathcal{S}_{\mathrm{B}}| = |\mathcal{S}_{\mathrm{G}}(\mathcal{G}')| + |\mathcal{V}_2 \cap \mathcal{V}_{\mathrm{G}}| - 1 + |\mathcal{S}_{\mathrm{B}}(\mathcal{G}')| \\
=&|\mathcal{V}_{\mathrm{G}}(\mathcal{G}')| - 1 + |\mathcal{V}_2 \cap \mathcal{V}_{\mathrm{G}}| - 1 \\
=&|\mathcal{V}_1 \cap \mathcal{V}_{\mathrm{G}}| + 1 - 1 + |\mathcal{V}_2 \cap \mathcal{V}_{\mathrm{G}}| - 1 = N_{\mathrm{G}} - 1.
\end{aligned}$$

To show $\mathcal{S}_{\mathrm{G}} \perp \mathcal{S}_{\mathrm{B}}$, we use the similar argument as in the previous case. suppose $\mathbf{R}\mathbf{x} = 0$ for some vector $\mathbf{x} \neq 0$. Then by Lemma 5.2, for vectors $\mathbf{y}, \mathbf{z}$ as constructed in (H.4) and (H.6), there exists some $q \in \mathbb{R}$ such that (H.7) holds. Apply Lemma H.1 to $\mathcal{G}''$, we have $q = 0$ and $\mathbf{z} = 0$. Then (H.7a) implies $\mathbf{R}\mathbf{y} = 0$. Since $\mathbf{x} \neq 0$, we must have $\mathbf{y} \neq 0$, and then $\mathbf{R}\mathbf{y} = 0$ implies the columns of $\mathbf{R}_{\mathcal{G}'}$ are not independent. It leads to the contradiction. As the result, we have $\mathcal{S}_{\mathrm{G}} \perp \mathcal{S}_{\mathrm{B}}$.

Finally we are showing $\mathbf{J}_{\mathcal{G}'}^{i,j+N_{\mathrm{G}}} = \mathbf{J}_{\mathcal{G}}^{i,j+N_{\mathrm{G}}}$ still holds in this case. Notice that all the columns of $\mathbf{R}_{\mathcal{G}'}$ are also columns of $\mathbf{R}$. Suppose $\mathbf{R}_{\mathcal{G}'}\mathbf{y} = -\mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{C}_{\mathcal{G}'}^{\mathsf{T}}\mathbf{e}_i$, then we can construct $\mathbf{x}$ in the following way. For $\alpha \in \mathcal{V}_1 \setminus \mathcal{S}_{\mathrm{G}}(\mathcal{G}')^{\mathsf{c}}$, let $\mathbf{x}_{\Lambda(\alpha)} = \mathbf{y}_{\Lambda_{\mathcal{G}'}(\alpha)}$, and for $\beta \in \mathcal{E}_1 \cap \mathcal{S}_{\mathrm{B}}(\mathcal{G}')$, let $\mathbf{x}_{\Xi(\beta)} = \mathbf{y}_{\Xi_{\mathcal{G}'}(\beta)}$. Other coordinates of $\mathbf{x}$ are set as 0. Then $\mathbf{R}\mathbf{x} = \mathbf{R}_{\mathcal{G}'}\mathbf{y} = -\mathbf{C}_{\mathcal{G}'}\mathbf{B}\mathbf{C}_{\mathcal{G}'}^{\mathsf{T}}\mathbf{e}_i = -\mathbf{C}\mathbf{B}\mathbf{C}^{\mathsf{T}}\mathbf{e}_i$. Thereby,

$$\mathbf{J}_{\mathcal{G}'}^{i,j+N_{\mathrm{G}}} = \mathbf{y}_{\Lambda_{\mathcal{G}'}(j+N_{\mathrm{G}})} = \mathbf{x}_{\Lambda(j+N_{\mathrm{G}})} = \mathbf{J}_{\mathcal{G}}^{i,j+N_{\mathrm{G}}}. \qquad \square$$

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