概率统计第五章习题课

1.设 $X_1,\cdots,X_n$ 为来自总体 $X\sim N(\mu,\sigma^2)$ 的样本，其中 μ,σ^2 未知.

(1) 求样本的样本空间和联合分布密度.

$$\Omega=\left\{\left(x_1,\cdots,x_n\right)\middle|\,x_i\in R,\ i=1,2,\cdots,n\right\}$$

$$f(x_1,\cdots,x_n)=\prod_{i=1}^{n}f(x_i)=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$=\frac{1}{(2\pi)^{\frac{n}{2}}\sigma^n}\cdot e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

(2) 问下列随机变量中哪些是统计量

$$T_1 = \frac{1}{n-1}\sum_{i=1}^{n} X_i,$$

$$T_2 = X_n - EX_1;$$

$$T_3 = 2X_2 + X_3;$$

$$T_4 = \max(X_1, X_2, \cdots, X_n);$$

$$T_5 = \frac{X_1 - \mu}{\sigma};$$

$$T_6 = \sum_{i=1}^{n}\left(\frac{X_i}{\sigma}\right)^2.$$

2. 设$X_1, X_2, \ldots, X_n$是来自泊松分布$P(\lambda)$的样本，$\bar{X}, S^2$分别为样本均值和样本方差，求

$E(\bar{X}), D(\bar{X}), E(S^2)$

解:由$X \sim P(\lambda)$知$E(X) = \lambda \quad D(X) = \lambda$

$$E(\overline{X}) = E(\frac{1}{n}\sum_{i=1}^{n} x_i) = E(X) = \lambda$$

$$D(\overline{X}) = \frac{1}{n} D(X) = \frac{\lambda}{n}$$

$$E(S^2) = D(X) = \lambda$$

1)关系式 $S^2=\dfrac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2=\dfrac{1}{n-1}(\sum_{i=1}^{n}X_i^2-n\overline{X}^2)$

推导 $\sum_{i=1}^{n}(X_i-\overline{X})^2=\sum_{i=1}^{n}(X_i^{\ 2}-2X_i\overline{X}+\overline{X}^2)$

$$=\sum_{i=1}^{n}X_i^2-2\overline{X}\sum_{i=1}^{n}X_i+\sum_{i=1}^{n}\overline{X}^2=\sum_{i=1}^{n}X_i^2-2n\overline{X}^2+n\overline{X}^2$$

$$=\sum_{i=1}^{n}X_i^2-n\overline{X}^2$$

2) $E(S^2)=D(X)$

推导 $E\left(\overline{X}\right)=E\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)=E(X)$ $D\left(\overline{X}\right)=\frac{1}{n}D(X)$

$$E(S^2)=E\left(\frac{1}{n-1}\left(\sum_{i=1}^{n}X_i^2-n\overline{X}^2\right)\right)=\frac{E\left(\sum_{i=1}^{n}X_i^2\right)-n\left[D\left(\overline{X}\right)+E^2\left(\overline{X}\right)\right]}{n-1}$$

$$=\frac{\sum_{i=1}^{n}E(X_i^2)-DX-nE^2X}{n-1}=\frac{\sum_{i=1}^{n}(DX+E^2X)-DX-nE^2X}{n-1}$$

$$=\frac{nDX+nE^2X-DX-nE^2X}{n-1}=DX$$

3. 在总体$N(10, 4)$中随机抽容量为5的样本

X_1，X_2，X_3，X_4，X_5. 求

(1) $P(|\bar{X}-10|>2)$

(2) $P\{\max(X_1, X_2, X_3, X_4, X_5)>12\}$

(3) $P\{\min(X_1, X_2, X_3, X_4, X_5)>8\}$

$$(1)\ P(|\bar{X}-10|>2)=P\left(\left|\frac{\bar{X}-10}{2/\sqrt{5}}\right|>\sqrt{5}\right)$$

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\sim N(0,1)$$

$$=1-P\left(\left|\frac{\bar{X}-10}{2/\sqrt{5}}\right|\le\sqrt{5}\right)$$

$$=2[1-\Phi(\sqrt{5})]=0.026$$

（2）$P\{\max(X_1, X_2, X_3, X_4, X_5)>12\}$

$$=1-P\{\max(X_1, X_2, X_3, X_4, X_5)\le 12\}$$

$$=1-\prod_{i=1}^{5}P(X_i\le 12)=1-[\Phi(\frac{12-10}{2})]^5=0.5785.$$

（3）$P\{\min(X_1, X_2, X_3, X_4, X_5)>8\}$

$$=\prod_{i=1}^{5}P(X_i>8)=[1-\Phi(\frac{8-10}{2})]^5=0.4215.$$

4.设总体 $X \sim N(0,0.2^2)$,$(X_1,X_2,\cdots, X_8)$为其样本,

求a,使$P(\sum_{1}^{8} X_i^2 < a) = 0.95$.

$$\frac{X-\mu}{\sigma} \sim N(0,1)$$

$$\frac{X_i}{0.2} \sim N(0,1) \qquad \frac{1}{0.2^2}\sum_{i=1}^{8} X_i^2 \sim \chi^2(8)$$

,

$$P(\sum_{1}^{8} X_i^2 < a) = P(\frac{1}{0.2^2}\sum_{i=1}^{8} X_i^2 < \frac{a}{0.2^2})$$

$$= P(\chi^2(8) < \frac{a}{0.2^2}) = 0.95.$$

$$P(\chi^2(8) > \frac{a}{0.2^2}) = 0.05.$$

$$a = 0.62$$

5.某厂灯泡的寿命 $X \sim N(2500, 250^2)$，为使灯泡的平均寿命大于2450的概率超过99%,至少应检查多少灯泡?

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

设应检查n只灯泡,则

$$P(\overline{X} > 2450) = 1 - P(\overline{X} \le 2450)$$

$$= 1 - \Phi\left(\frac{2450 - 2500}{250/\sqrt{n}}\right) = \Phi\left(\frac{\sqrt{n}}{5}\right) > 0.99$$

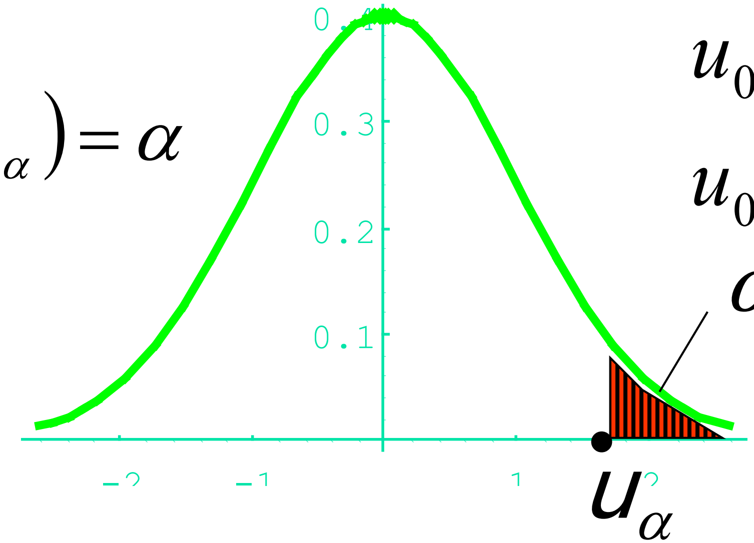
$$\therefore \frac{\sqrt{n}}{5} > 2.33$$

$$n \ge 136$$

7.分位数查表

标准正态分布的上α分位数

$$P(X > u_\alpha) = \alpha$$



$$u_{0.05} = 1.645$$
$$u_{0.025} = 1.96$$

常用数字

$$t_{0.95}(20) = t_{1-0.05}(20) = -t_{0.05}(20) = -1.7247$$

$$F_{0.95}(4,6) = \frac{1}{F_{0.05}(6,4)} = \frac{1}{6.16} = 0.1623$$

8.证明 $F_{1-\alpha}(m,n)=\dfrac{1}{F_{\alpha}(n,m)}$

$P(F(m,n)\geq F_{1-\alpha}(m,n))=1-\alpha$

$P(F(n,m)\geq F_{\alpha}(n,m))=\alpha$

证 $P(F(m,n)\geq F_{1-\alpha}(m,n))=P\left(\dfrac{1}{F(m,n)}\leq\dfrac{1}{F_{1-\alpha}(m,n)}\right)$

$$=1-P\left(\frac{1}{F(m,n)}\geq\frac{1}{F_{1-\alpha}(m,n)}\right)=1-\alpha$$

故 $P\left(\dfrac{1}{F(m,n)}\geq\dfrac{1}{F_{1-\alpha}(m,n)}\right)=\alpha$ 由于 $\dfrac{1}{F(m,n)}\sim F(n,m)$

故 $P\left(F(n,m)\geq\dfrac{1}{F_{1-\alpha}(m,n)}\right)=\alpha$ 因而 $\dfrac{1}{F_{1-\alpha}(m,n)}=F_{\alpha}(n,m)$

9.设 X_1, X_2, X_3, X_4 是来自正态总体 $N(0, 2^2)$ 的样本.

(1) $Y = C[(X_1 - X_2)^2 + (X_3 + X_4)^2] \sim \chi^2(n)$

C, n为多少?

$$X_1 - X_2 \sim N(0,8), \quad X_3 + X_4 \sim N(0,8)$$

$$\frac{X_1 - X_2}{\sqrt{8}} \sim N(0,1), \quad \frac{X_3 + X_4}{\sqrt{8}} \sim N(0,1)$$

$$\left(\frac{X_1 - X_2}{\sqrt{8}}\right)^2 + \left(\frac{X_3 + X_4}{\sqrt{8}}\right)^2 \sim \chi^2(2)$$

$$C = \frac{1}{8}, n = 2.$$

3.5 正态随机变量的结论

如果随机变量 X 与 Y 相互独立，且

$$X \sim N\left(\mu_1,\ \sigma_1^2\right) \quad Y \sim N\left(\mu_2,\ \sigma_2^2\right)$$

$$Z = X + Y,$$

则 $Z \sim N\left(\mu_1 + \mu_2,\ \sigma_1^2 + \sigma_2^2\right)$

χ^2 −分布定义

χ^2分布是由正态分布派生出来的一种分布.

定义: 设 $X_1, X_2, \cdots, X_n$ 相互独立, 都服从正态分布$N(0,1)$, 则称随机变量:

$$\chi^2 = X_1^2 + X_2^2 + \cdots + X_n^2$$

所服从的分布为自由度为 n 的 χ^2 分布.

记为 $\chi^2 \sim \chi^2(n)$

(2) 证明$Z=\dfrac{(X_1-X_2)^2}{(X_3+X_4)^2}\sim F(1,1)$

$$X_1-X_2\sim N(0,8),\quad X_3+X_4\sim N(0,8)$$

$$\frac{(X_1-X_2)^2}{8}\sim\chi^2(1)$$

$$\frac{(X_3+X_4)^2}{8}\sim\chi^2(1)$$

$$\Rightarrow Z=\frac{(X_1-X_2)^2}{(X_3+X_4)^2}\sim F(1,1)$$

F 分布定义

若 $X \sim \chi^2(n_1), Y \sim \chi^2(n_2)$, X, Y 独立，

则称随机变量

$$\mathrm{F} = \frac{X/n_1}{Y/n_2}$$

为服从自由度是n_1, n_2 的 $F-$分布，

记作$F \sim F(n_1, n_2)$.

10. $X_1, X_2, \cdots, X_n$是总体$N(\mu, \sigma^2)$的简单样本,

$$X_{n+1} \sim N(\mu, \sigma^2), Y = \frac{X_{n+1} - \bar{X}}{S}\sqrt{\frac{n}{n+1}} \sim \underline{\qquad\qquad}$$

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}), X_{n+1} \sim N(\mu, \sigma^2),$$

$$X_{n+1} - \bar{X} \sim N(0, \frac{n+1}{n}\sigma^2).$$

$$\frac{X_{n+1} - \bar{X}}{\sigma} \cdot \sqrt{\frac{n}{n+1}} \sim N(0, 1). \qquad \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

$$\Rightarrow \quad Y = \frac{\frac{X_{n+1} - \bar{X}}{\sigma} \cdot \sqrt{\frac{n}{n+1}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2} / (n-1)}} = \frac{X_{n+1} - \bar{X}}{S}\sqrt{\frac{n}{n+1}} \sim t(n-1)$$

t 分布 (Student 分布)

$X \sim N(0,1), Y \sim \chi^2(n), X, Y$独立，则称随机变量

$$\mathrm{T} = \frac{X}{\sqrt{Y/n}}$$

为服从自由度是n的 $t-$分布，记作$T \sim t(n)$.

其密度函数为

$$f(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}} \qquad -\infty < t < \infty$$

11.设$X_1,X_2,\cdots,\ X_9$为总体$N(\mu,\sigma^2)$的样本，

$$Y_1=\frac{1}{6}(X_1+\cdots+X_6),\quad Y_2=\frac{1}{3}(X_7+X_8+X_9),$$

$$S^2=\frac{1}{2}\sum_{i=7}^{9}(X_i-Y_2)^2,\ 证 Z=\frac{\sqrt{2}(Y_1-Y_2)}{S}\sim t(2).$$

证

$$\bar{X}\sim N(\mu,\frac{\sigma^2}{n}) \Rightarrow \begin{matrix} Y_1\sim N(\mu,\frac{\sigma^2}{6}) \\ Y_2\sim N(\mu,\frac{\sigma^2}{3}) \end{matrix} \Rightarrow Y_1-Y_2\sim N(0,\frac{\sigma^2}{2})$$

$$\therefore U=\frac{Y_1-Y_2}{\sigma/\sqrt{2}}\sim N(0,1),$$

$$\because S^2=\frac{1}{2}\sum_{i=7}^{9}(X_i-Y_2)^2,$$

$$\frac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1)$$

$$\therefore V=\frac{2S^2}{\sigma^2}\sim\chi^2(2),$$

$$Z=\frac{U}{\sqrt{V/2}}=\frac{\dfrac{Y_1-Y_2}{\sigma/\sqrt{2}}}{\sqrt{\dfrac{2S^2}{2\sigma^2}}}=\frac{\sqrt{2}(Y_1-Y_2)}{S}\sim t(2)$$

12. 设 $X_1, X_2, \cdots, X_{2n} (n \geq 2)$ 为从正态总体 $X \sim N(\mu, \sigma^2)$ 中抽取的简单随机样本，其样本均值为 $\overline{X} = \frac{1}{2n}\sum_{i=1}^{2n} X_i$，求统计量 $Y = \sum_{i=1}^{n}(X_i + X_{n+i} - 2\overline{X})^2$ 的期望.

令 $Z_i = X_i + X_{n+i}$ 则 $Z_i \sim N(2\mu, 2\sigma^2)$

$$\therefore \overline{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i = \frac{1}{n}\sum_{i=1}^{n}(X_i + X_{n+i}) = 2\overline{X}.$$

则 $Y=\sum_{i=1}^{n}(Z_i-\bar{Z})^2$

$$\frac{1}{n-1}E(Y)=E\left(\frac{1}{n-1}Y\right)=E\left(\frac{1}{n-1}\sum_{i=1}^{n}(Z_i-\bar{Z})^2\right)$$

$$=D(Z_i)=2\sigma^2$$

$$\boxed{E(S^2)=D(Z)}$$

所以 $E(Y)=2(n-1)\sigma^2$

17*.设 $X_1, X_2, \cdots X_n$ 是来自总体 $X \sim N(\mu, \sigma^2)$ 的样本，令

$$Y = \frac{1}{n}\sum_{i=1}^{n}|X_i - \mu|, \text{试证} \quad E(Y) = \sqrt{\frac{2}{\pi}}\sigma, D(Y) = (1-\frac{2}{\pi})\frac{\sigma^2}{n}.$$

解 $\quad Z = \dfrac{X_i - \mu}{\sigma} \sim N(0,1), \qquad f(z) = \dfrac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2}}$

$$\therefore EY = E(\frac{1}{n}\sum_{i=1}^{n}\sigma|Z|) = \frac{\sigma}{n}\sum[E|Z|] = \sqrt{\frac{2}{\pi}}\sigma$$

$$DY = \frac{\sigma^2}{n^2}\sum[D|Z|] = \frac{\sigma^2}{n^2}\sum\left[E(Z^2) - (E|Z|)^2\right]$$

$$= \frac{\sigma^2}{n^2}\sum\left[(E(Z^2) - E^2Z) + E^2Z - (E|Z|)^2\right]$$

$$= \frac{\sigma^2}{n^2}\sum\left[DZ + (EZ)^2 - (E|Z|)^2\right] = (1-\frac{2}{\pi}) \cdot \frac{\sigma^2}{n}$$