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# Thesis Title

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# Chapter 1

## Sequences

### 1.1 Sequences and convergence

A *sequence* is an infinite list of numbers. Examples of sequences are

$$\begin{array}{ll} 1, 2, 3, 4, 5, 6, \dots & 1, -1, 1, -1, 1, -1, \dots \\ 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots & 2, 3, 5, 7, 11, 13, \dots \\ 3, 10, 5, 16, 8, 4, 2, \dots & 3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \dots \end{array}$$

The dots indicate that the sequence does not stop after 6 elements, but continues indefinitely. It is the finiteness of paper that forces us to stop at some point. In this regard the mathematical usage of the term ‘sequence’ differs from the colloquial one: in mathematics a finite, ordered list of numbers is a *tuple*. We can speak of a 3-tuple  $(7, 2, 5)$  or an  $n$ -tuple  $(1, 2, \dots, n)$ . A sequence however is an infinite list of numbers. Formally we have the following definition.

**Definition 1.1.** A *sequence* is a map<sup>1</sup>  $x : \mathbb{N} \rightarrow \mathbb{R}$ . We will write  $x_n := x(n)$  for the  $n$ th element of the sequence and denote the whole sequence by  $\{x_n\}_{n=1}^\infty$  or  $\{x_n\}$ .

How do we capture infinitely many elements on a finite sheet of paper? We do this by providing the *general rule* that allows us to calculate any element of the sequence. For example, the first four sequences shown above are defined by the following rules:

$$\begin{array}{ll} x_n = n & x_n = (-1)^n \\ x_n = 2^{2-n} & x_n \text{ is the } n\text{-th prime number.} \end{array}$$

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<sup>1</sup>For the purpose of these notes,  $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ .

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