Formula sheet 3

Matched Pairs

McNemar's

Test: Let $n^\star = n_{12} + n_{21}$

$$n_{12} \sim \text{Bin}(n^\star, 0.5) \Rightarrow z = \frac{n_{12} - n^\star/2}{\sqrt{n^\star \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}} = \frac{n_{12} - n_{21}}{\sqrt{n_{12} + n_{21}}} \overset{\text{approx.}}{\sim} N(0,1)$$

or equivalently $z^2 \sim \chi^2_1$

CI:

$$\underbrace{p_{1+} - p_{+1}}_{\frac{n_{12}-n_{21}}{n}} \mp z_{1-\alpha/2} \frac{1}{n} \sqrt{n_{12} + n_{21} - \frac{(n_{12} - n_{21})^2}{n}}$$

Rater Agreement - Cohen's Kappa

$$\kappa = \frac{\sum_i \pi_{ii} - \sum_i \pi_{i+}\pi_{+i}}{1 - \sum_i \pi_{i+}\pi_{+i}}$$

Test:

$$\frac{\hat{\kappa} - 0}{s_{\hat{\kappa}}} \overset{\text{approx.}}{\sim} N(0,1)$$

CI:

$$\hat{\kappa} \mp z_{a-\alpha/2} s_{\hat{\kappa}}$$

Correlated, Clustered Responses

- GEE is a marginal model allowing for a correlation structure
- GLMM uses a random intercept term

Wald:

- $TS = \frac{\hat{\beta}-\beta_0}{s_{\hat{\beta}}} \overset{\text{H}_0}{\sim} N(0,1)$
- $\hat{\beta} \mp z_{1-\alpha/2} \left(s_{\hat{\beta}}\right)$
- $\sum_{i=1}^k c_i \hat{\beta}_i \mp z_{1-\alpha/2} \sqrt{\hat{V}\left(\sum_{i=1}^k c_i \hat{\beta}_i\right)}$,where

$$V\left(\sum_{i=1}^k c_i \hat{\beta}_i\right) = \sum_{i=1}^k c_i^2 V(\hat{\beta}_i) + 2 \sum_{i<j}\sum c_i c_j \text{Cov}(\hat{\beta}_i, \hat{\beta}_j)$$

LRT-Goodness of fit:

$$D(y;\hat{\mu}) := G^2 = -2[L(\hat{\mu};y) - L(y;y)] \xrightarrow[\text{H}_0]{d} \chi^2_{df}$$

LRT:

$$G^2 = D(y;\hat{\mu}_0) - D(y;\hat{\mu}_1) \xrightarrow[\text{H}_0]{d} \chi^2_{df}$$

Loglinear Models

2 Way

Independence model:

$$\log(\mu_{ij}) = \lambda + \lambda_i^X + \lambda_j^Y$$

Saturated model:

$$\log(\mu_{ij}) = \lambda + \lambda_i^X + \lambda_j^Y + \lambda_{ij}^{XY}$$

$$df = \underbrace{\text{number of Poisson counts}}_{\text{number of cells in table}} - \text{number of parameters}$$

Odds ratio (for saturated model):

$$\log\left(\frac{\mu_{ij}\mu_{i'j'}}{\mu_{ij'}\mu_{i'j}}\right) = \log(\mu_{ij}) + \log(\mu_{i'j'}) - \log(\mu_{ij'}) - \log(\mu_{i'j}) = \cdots = \lambda_{ij}^{XY} + \lambda_{i'j'}^{XY} - \lambda_{ij'}^{XY} - \lambda_{i'j}^{XY}$$

3 Way

- X, Y, Z are *mutual independent*, (X, Y, Z) if $\pi_{ijk} = \pi_{i++}\pi_{+j+}\pi_{++k}$

$$\log(\mu_{ijk}) = \lambda + \lambda_i^X + \lambda_j^Y + \lambda_k^Z$$

- Y is *jointly independent* of X and Z, (XZ, Y) if $\pi_{ijk} = \pi_{+j+}\pi_{i+k}$

$$\log(\mu_{ijk}) = \lambda + \lambda_i^X + \lambda_j^Y + \lambda_k^Z + \lambda_{ik}^{XZ}$$

- X and Y are *conditionally independent* given Z, (XZ, YZ) if $\pi_{ij|k} = \pi_{i+|k}\pi_{+j|k}$

$$\log(\mu_{ijk}) = \lambda + \lambda_i^X + \lambda_j^Y + \lambda_k^Z + \lambda_{ik}^{XZ} + \lambda_{jk}^{YZ}$$

- *Homogeneous association*, (XZ, XY, YZ) if two variables have the same association for all levels of the third, e.g. $\pi_{ij|k} = \pi_{ij|k'}$ same $\forall k, k'$

$$\log(\mu_{ijk}) = \lambda + \lambda_i^X + \lambda_j^Y + \lambda_k^Z + \lambda_{ik}^{XZ} + \lambda_{ij}^{YZ} + \lambda_{ij}^{XY}$$

- *Non restricted association*, (saturated model) (XYZ)

$$\begin{aligned}\log(\mu_{ijk}) =& \lambda + \lambda_i^X + \lambda_j^Y + \lambda_k^Z \\ &+ \lambda_{ik}^{XZ} + \lambda_{ij}^{YZ} + \lambda_{ij}^{XY} + \lambda_{ijk}^{XYZ}\end{aligned}$$

Loglinear-Logit Connection

$$\begin{aligned}\log\left(\frac{P(Y=1)}{1-P(Y=1)}\right) =& \underbrace{(\lambda_1^Y - \cancelto{0}{\lambda_2^Y})}_{\alpha} \\ &+ \underbrace{(\lambda_{i1}^{XY} - \cancelto{0}{\lambda_{i2}^{XY}})}_{\beta_i^X} \\ =& \alpha + \beta_i^X\end{aligned}$$

For (XY, XZ, YZ),

$$\log(\mu_{ijk}) = \lambda + \lambda_i^X + \lambda_j^Y + \lambda_k^Z + \lambda_{ik}^{XZ} + \lambda_{ij}^{YZ} + \lambda_{ij}^{XY}$$

Suppose Y is binary, treated as the response, and let

$$\pi_{ik} = P(Y = 1 | X = i, Z = k)$$

then

$$\begin{aligned}\text{logit}(\pi_{ik}) =& \log(\mu_{i1k}) - \log(\mu_{i2k}) \\ =& \cdots \\ =& \underbrace{(\lambda_1^Y - \cancelto{0}{\lambda_2^Y})}_{\alpha} + \underbrace{(\lambda_{i1}^{XY} - \cancelto{0}{\lambda_{i2}^{XY}})}_{\beta_i^X} \\ &+ \underbrace{(\lambda_{1k}^{YZ} - \cancelto{0}{\lambda_{2k}^{YZ}})}_{\beta_k^Z} \\ =& \alpha + \beta_i^X + \beta_k^Z\end{aligned}$$