Multicategory Logit Models

Nominal Responses

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Review

When the response was binary we fit a logistic regression regression model, but a binomial is simply a special case of the multinomial, with more than 2 levels.

Let

$$\pi_j = P(Y = j), \quad j = 1, 2, \ldots, J$$

and consider $J = 2$, and as such, $\pi_1, \pi_2 \ni \pi_1 + \pi_2 = 1$. A simple logistic model was

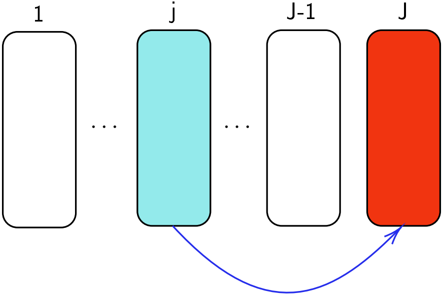
$$\log\left(\frac{\pi_1}{1 - \pi_1}\right) = \log\left(\frac{\pi_1}{\pi_2}\right) = \alpha + \beta x$$

Baseline-category logits

$$\log\left(\frac{\pi_j}{\pi_J}\right) = \alpha_j + \beta_j x, \quad j = 1, \dots, J-1$$

$$\pi_j = \frac{e^{\alpha_j + \beta_j x}}{1 + \sum_{i=1}^{J-1} e^{\alpha_i + \beta_i x}}, \qquad \pi_J = \frac{1}{1 + \sum_{i=1}^{J-1} e^{\alpha_i + \beta_i x}}$$

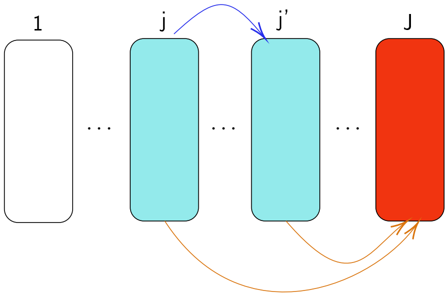
Separate set of parameters (α_j, β_j) for each logit. We compare the probability of being in group j, versus the baseline group J.



Baseline-category logits

Can compare any two groups (that does not include baseline).

$$
\begin{aligned}
\log\left(\frac{\pi_j}{\pi_{j'}}\right) &= \log\left(\frac{\pi_j/\pi_J}{\pi_{j'}/\pi_J}\right) \\
&= \log\left(\frac{\pi_j}{\pi_J}\right) - \log\left(\frac{\pi_{j'}}{\pi_J}\right) \\
&= (\alpha_j - \alpha_{j'}) + (\beta_j - \beta_{j'})x
\end{aligned}
$$



Remarks

- Category used as baseline (i.e., category J) is arbitrary and does not affect model fit, since categories are nominal
- The term e^{β_j} is the multiplicative effect of a 1-unit increase in x on the conditional odds of response j given that response is one of j or J
- Could also use this model with ordinal response variables, but this would ignore information about ordering

Example (Job Satisfaction)

Data from 1991 GSS

Income	Job Satisfaction			
	Dissat	Little	Moderate	Very
< 5k	2	4	13	3
5k-15k	2	6	22	4
15k-25k	0	1	15	8
> 25k	0	3	13	8

Consider $x =$ income scores $(3, 10, 20, 30)$ and define VD=1, LD=2, MS=3, VS=4

Example (continued)

```
> fit.blogit=vglm(cbind(VD,LD,MS,VS)~income,
+  family=multinomial,data=dat)
> summary(fit.blogit)
Coefficients:
               Estimate Std. Error  z value
(Intercept):1  0.563824   0.960138  0.58723
(Intercept):2  0.645091   0.668771  0.96459
(Intercept):3  1.818698   0.528955  3.43828
income:1      -0.198773   0.102096 -1.94693
income:2      -0.070502   0.036954 -1.90785
income:3      -0.046918   0.025519 -1.83858

Residual deviance: 4.17662 on 6 degrees of freedom
Log-likelihood: -16.71316 on 6 degrees of freedom
```

Example (continued)

The prediction equations are

$$\log\left(\frac{\hat{\pi}_1}{\hat{\pi}_4}\right) = 0.564 - 0.199x$$
$$\log\left(\frac{\hat{\pi}_2}{\hat{\pi}_4}\right) = 0.645 - 0.071x$$
$$\log\left(\frac{\hat{\pi}_3}{\hat{\pi}_4}\right) = 1.819 - 0.047x$$

For each logit, the odds of being in a less satisfied category (instead of "very satisfied") decreases as income increases.

Example (continued)

ML estimates determine the effects for all pairs of categories. For example, comparing group 1 and 2, i.e. "dissatisfied" to "little dissatisfied"

$$
\begin{aligned}
\log\left(\frac{\hat{\pi}_1}{\hat{\pi}_2}\right) &= \log\left(\frac{\hat{\pi}_1}{\hat{\pi}_4}\right) - \log\left(\frac{\hat{\pi}_2}{\hat{\pi}_4}\right) \\
&= (0.564 - 0.199x) - (0.645 - 0.071x) \\
&= -0.081 - 0.128x
\end{aligned}
$$

A global test of income effect is $\mathrm{H}_0 : \beta_1 = \beta_2 = \beta_3 = 0$ comparing the model with one with no predictor

$$G^2 = 13.4673 - 4.17662 \quad df = 3 \quad \text{p-value of } 0.0257$$

where 12.4673 is the null deviance obtained by

```
> vglm(cbind(VD,LD,MS,VS)~1,family=multinomial,data=dat)
```

Exercise

For the job satisfaction example, we obtained the logit for comparing "dissatisfied" to "little dissatisfied" to be

$$\log\left(\frac{\hat{\pi}_1}{\hat{\pi}_2}\right) = -0.081 - 0.128x$$

where $\hat{\beta}_1 - \hat{\beta}_2 = -0.128$. Create a 95% confidence interval around $\beta_1 - \beta_2$ and interpret.

We learned

Extended simple logistic regression when responses are (unordered) multinomial.

Remark

Model checking and adequacy procedures such as residuals etc can also be extended.