1. The PDF for X and Y are $f(x) = \frac{\lambda_1^x e^{-\lambda_1}}{x!}$ and $f(y) = \frac{\lambda_2^{y!} e^{-\lambda_2}}{y!}$, respectively.

Let $Z = X + Y$.

$f(z) = P(Z = z)$
$= \Sigma_{k=0}^{z} P(X = k\ and\ Y = y) = P(X = x\ and\ Y = z - y)$
// account for all possible *x* and *y* that sum to the same *z*
$= \Sigma_{k=0}^{z} P(X = k)P(Y = z - k)$ // joint distribution for IID random variables
$= \Sigma_{k=0}^{z} \frac{\lambda_1^x e^{-\lambda_1}}{x!} * \frac{\lambda_2^{z-x} e^{-\lambda_2}}{(z-x)!}$ // definition of PDF for Poisson, as specified earlier
$= \Sigma_{k=0}^{z} \left(\frac{1}{x!(z-x)!}\right) \lambda_1^x \lambda_2^{z-x} e^{-\lambda_1 - \lambda_2}$ // moving terms around and simplifying
$= \Sigma_{k=0}^{z} \left(\frac{z!}{x!(z-x)!}\right) \left(\frac{\lambda_1^x \lambda_2^{z-x} e^{-\lambda_1-\lambda_2}}{z!}\right)$ // multiplying by all powerful 1 as $\frac{z!}{z!}$
$= \frac{e^{-\lambda_z}}{z!} \Sigma_{k=0}^{z} \binom{z}{x} (\lambda_1^x \lambda_2^{z-x})\ where\ \lambda_z = \lambda_1 + \lambda_2$ // more simplification
$= \frac{e^{-\lambda_z}}{z!} (\lambda_1 + \lambda_2)^z = \frac{\lambda_z^z e^{-\lambda_z}}{z!}$ // reversal of binomial expansion
$\therefore X + Y \sim Poisson(\lambda_1 + \lambda_2, \lambda_1 + \lambda_2)$

$P(X = k | X + Y = n) = P(X = k | Z = n) = \frac{P(X=k\ and\ Z=n)}{P(Z=n)}$

For the numerator:
$P(X = k\ and\ Z = n) = P(X = k\ and\ Y = n - k) = P(X = k)P(Y = n - k)$
$= \frac{\lambda_1^k e^{-\lambda_1}}{k!} * \frac{\lambda_2^{(n-k)} e^{-\lambda_2}}{(n-k)!} = \left(\frac{1}{k!(n-k)!}\right) \left(\lambda_1^k e^{-\lambda_1} \lambda_2^{(n-k)} e^{-\lambda_2}\right) = \left(\frac{n!}{k!(n-k)!}\right) \left(\frac{\lambda_1^k \lambda_2^{(n-k)} e^{-\lambda_1-\lambda_2}}{n!}\right)$
$= \frac{e^{-\lambda_z}}{n!} \binom{n}{k} \lambda_1^k \lambda_2^{(n-k)}$

For the denominator:
$P(Z = n) = \frac{\lambda_z^n e^{-\lambda_z}}{n!}$

Together:
$P(X = k | X + Y = n) = \frac{P(X=k\ and\ Z=n)}{P(Z=n)} = \frac{\frac{e^{-\lambda_z}}{n!}\binom{n}{k}\lambda_1^k \lambda_2^{(n-k)}}{\frac{\lambda_z^n e^{-\lambda_z}}{n!}} = \frac{\binom{n}{k}\lambda_1^k \lambda_2^{(n-k)}}{\lambda_z^n} = \frac{\binom{n}{k}\lambda_1^k \lambda_2^{(n-k)}}{\lambda_1+\lambda_2}$

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2. Let $Y = \alpha X + \beta$. X can be re-expressed as $X = \frac{Y-\beta}{\alpha}$.
$f(x) = f\left(\frac{y-\beta}{\alpha}\right) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\left(\frac{y-\beta}{\alpha} - \mu\right)^2}{2\sigma^2}\right) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\left(\frac{y-\beta-\alpha\mu}{\alpha}\right)^2}{2\sigma^2}\right)$
$= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-(\alpha\mu+\beta))^2}{2(\alpha\sigma)^2}\right) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right), where\ \mu_Y = \alpha\mu + \beta\ and\ \sigma_Y^2 = \alpha^2\sigma^2$

$\therefore Y \sim N(\alpha\mu + \beta, \alpha^2\sigma^2)$

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3. For *n* red balls and *m* black balls from the same urn, each red ball is labeled with a unique integer $i = 1,2,\dots,n$. Let there be an indicator variable:

$$X_i = \begin{cases} 1, & \text{if red ball i is taken before any black ball is chosen} \\ 0, & otherwise \end{cases}$$

Beforehand, we need to find the probability of choosing any red ball regardless of number label. To guarantee that, the *n* red balls must be the first *n* chosen balls; otherwise, we chance upon the unfavorable possibility of red ball *i* being chosen after at least 1 black ball being chosen prior. Let $P(Z)$ be the probability of the z^{th} ball being red after all previous chosen balls are red.

$P(Z = 1) = P(first\ ball\ is\ red) = \frac{\#red\ balls\ to\ start\ with}{total\ \#\ of\ balls} = \frac{n}{n+m}$

$P(Z = 2) = P(second\ ball\ is\ red\ |all\ prev\ red) = \frac{\#red\ balls\ left\ after\ first\ ball\ is\ red}{\#balls\ left\ after\ taking\ 1\ ball\ in\ total} = \frac{n-1}{n+m-1}$

$P(Z = 3) = P(third\ ball\ is\ red\ |all\ prev\ red) = \frac{\#red\ balls\ left\ after\ first\ 2\ balls\ are\ red}{\#balls\ left\ after\ taking\ 2\ balls\ in\ total} = \frac{n-2}{n+m-2}$

$\vdots$

$P(Z = i) = P(i^{th}\ ball\ is\ red|Z = i - 1)$ for $i = 1,2,\dots,n$

$\Rightarrow P(Z = n) = \frac{n-n+1}{n+m-n+1} = \frac{1}{m+1}$

Therefore, the probability of choosing any arbitrary red ball before the first black ball is $\frac{1}{m+1}$. Because the red ball is arbitrary (we do not care about the number label) and falls under the condition of being chosen before any black ball, another way of looking at this is compacting all *n* red balls into a single entity of "red ball". Then, we have 1 "red ball" (in reality the set of *n* numbered unique red balls) and *m* black balls. There is a $\frac{1}{m+1}$ probability of selecting "red ball" out of this group of $m + 1$ objects.

$E[X_i] = 0 * P(X_i = 0) + 1 * P(X_i = 1) = P(X_i = 1) = \frac{1}{m+1}$

$E[X] = \Sigma_{i=1}^{n} \left(\frac{1}{m+1}\right) = \frac{n}{m+1}$

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4. (a) See the following indicator variable:

$$Y_i = \begin{cases} 1, & \text{if red ball i is taken between the first and second black balls} \\ 0, & otherwise \end{cases}$$

There are still *m* black balls in total to consider. The group of any number of red balls being chosen between the first and second chosen black balls can be considered together, since we again are not caring about the exact value of *I* (any red ball between the first 2 black balls is all we care about). The first consecutively chosen red balls before the first black ball is not relevant; the sample space relevant to the problem consists of the first black ball, "red ball" right after, second black ball, and rest of black balls. This again leaves us with $m+1$ objects. Choosing the second element in that set out of $(m+1)!$ permutations gives us $P(Y_i = 1) = \frac{1}{m+1}$.

$E[X] = \Sigma_{i=1}^{n}\left(\frac{1}{m+1}\right) = \frac{n}{m+1}$

(b) $E[Y] = E[X]$
(c) The expected values are the same in both scenarios since the sample space for both consists of the same number of items, including one item to represent the group of red balls of interest based on position of being chosen.

5.
$E[h(X)g(Y)] = \int\int h(x)g(y)f(x,y)dxdy$ // definition of expected value
$= \int\int h(x)g(y)f(x)f(y)dxdy$ // property of IID random variables
$= \int\int \big(h(x)f(x)\big)\big(g(y)f(y)\big)dxdy$ // re-grouping
$= \int h(x)f(x)\big[\int g(y)f(x,y)dy\big]dx$ // re-arranging integrand
$= \int h(x)f(x)E[g(Y)]dx$
$= E[h(X)]E[g(Y)]$

6. Measurable spaces are (Ω_1, F_1) and (Ω_2, F_2).
(a) $f: \Omega_1 \rightarrow \Omega_2 \Rightarrow f^{-1}: \Omega_2 \rightarrow \Omega_1$
$S = \{B \subseteq \Omega_2 : \mathrm{f}^{-1}(\mathrm{B}) \in \mathrm{F}_1\}$

Check to ensure properties of σ-field:
(i) Closed under complement.
(ii) Closed under countable union.
(iii) Closed under countable intersection.

7. By definition, a measurable function is defined by the following context. Given two measures, (Ω_1, F_1) and (Ω_2, F_2), a function $f: \Omega_1 \rightarrow \Omega_2$ is measurable if $f^{-1}(B) \in F_1 \forall B \in F_2$. The collection F_2 shown in the question clearly comprises of all elements of the σ-algebra on set Ω_2 (the second parameter in the second measure space). This directly fits the very definition for measurable function.