## HOMEWORK 2

**P1 (15 points).** Suppose $X$ and $Y$ are independent Poisson random variables with mean $\lambda_1$ and $\lambda_2$. Show that $X + Y$ is also a Poisson random variable with mean $\lambda_1 + \lambda_2$ and then compute $P(X = k|X + Y = n)$.

**P2 (15 points).** Let $X$ be $\mathcal{N}(\mu, \sigma^2)$. Let $Y = \alpha X + \beta$, where $\alpha$ and $\beta$ are constants. Show that $Y$ is $\mathcal{N}(\alpha\mu + \beta, \alpha^2\sigma^2)$.
**Hint:** Change of variable in integration.

**P3 (10 points).** An urn contains $n + m$ balls, of which $n$ are red and $m$ are black. They are withdrawn from the urn, one at a time without replacement. Let $X$ be the number of red balls removed before the first black ball is chosen. Computer $E[X]$.
**Hint:** Use the trick of indicator random variables. Number the red balls $1, 2, \ldots, n$. Define the indicator random variables $X_i$, for $i = 1, 2, \ldots, n$, by

$$X_i = \begin{cases} 1, & \text{if red ball } i \text{ is taken before any black ball is chosen} \\ 0, & \text{otherwise.} \end{cases}$$

Express $X$ in terms of $X_i$.

**P4 (15 points).** Continue with the previous problem setup. Let $Y$ be the number of red balls chosen after the first black ball is chosen but before the second black ball is chosen.
(a) Express $Y$ as the sum of indicator random variables, and compute $E[Y]$.
(b) Compare $E[Y]$ with $E[X]$ obtained in the previous problem.
(c) Can you explain the result obtained in part (b)?
**Hint:** For part (c), let the random variable $R_i$ be the number of red balls between the $(i-1)$-th black ball and the $i$-th black ball in the sequence, where $i = 2, 3 \ldots, m$. Let the random variable $R_1$ be the number of red balls before the first black ball and $R_{m+1}$ be the number of red balls after the last black ball. Argue that all the $R_i$'s have the same distribution.

**P5 (15 points).** Let $X$ and $Y$ be independent continuous random variables. Let $g, h : \mathbb{R} \to \mathbb{R}$ be measurable functions. Show that

$$E[h(X)g(Y)] = E[h(X)]E[g(Y)],$$

whenever the expectations exist. If $X$ and $Y$ have the exponential distribution with parameter 1, find $E[\exp(\frac{1}{2}(X+Y))]$.

**P6 (15 points).** Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces.
(a) Show that, for any function $f : \Omega_1 \to \Omega_2$, the collection $\mathcal{S} = \{B \subseteq \Omega_2 : f^{-1}(B) \in \mathcal{F}_1\}$ is a $\sigma$-field.
(b) Let $\mathcal{A}$ be a collection of subsets on $\Omega_2$. Suppose $f^{-1}(A) \in \mathcal{F}_1$ for each $A \in \mathcal{A}$. Show that $f$ is a measurable function from $(\Omega_1, \mathcal{F}_1)$ to $(\Omega_2, \sigma(\mathcal{A}))$, where $\sigma(\mathcal{A})$ is the $\sigma$-field generated by $\mathcal{A}$, i.e., the smallest $\sigma$-field containing all the sets in $\mathcal{A}$. (Therefore, to check a function is measurable with respect to $\mathcal{F}_1$ and $\sigma(\mathcal{A})$, it is enough to check whether the inverse image is in $\mathcal{F}_1$ for each set in $\mathcal{A}$.)
**Hint:** For (a), apply the definition of a $\sigma$-field.

**P7 (15 points).** Given a set $\Omega_1$ and a measurable space $(\Omega_2, \mathcal{F}_2)$, let $f : \Omega_1 \to \Omega_2$.

$$\mathcal{F}_1 = \{f^{-1}(B) : B \in \mathcal{F}_2\}. \tag{1}$$

Show that $\mathcal{F}_1$ is a $\sigma$-field and that $f$ is measurable with respect to $\mathcal{F}_1$ and $\mathcal{F}_2$. Such $\mathcal{F}_1$ is called the **$\sigma$-field generated by** $f$ and it is denoted by $\sigma(f)$.