Θέμα 4: Singular Value Decomposition

$$X = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

1) $$A = X^T X = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$|A - \lambda I] = 0 \Leftrightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (2-\lambda) \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - (-1) \begin{vmatrix} -1 & 0 \\ 1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} -1 & 1-\lambda \\ 1 & 0 \end{vmatrix} = 0$$

$$\Leftrightarrow (2-\lambda)(1-\lambda)^2 - (1-\lambda) - (1-\lambda) = 0$$

$$\Leftrightarrow (2-\lambda)(1-\lambda)^2 - 2(1-\lambda) = 0$$

$$\Leftrightarrow (1-\lambda)\left((2-\lambda)(1-\lambda) - 2\right) = 0$$

$\lambda = 1$ ή $(2-\lambda)(1-\lambda) - 2 = 0$

$$\Leftrightarrow \cancel{2} - 2\lambda - \lambda + \lambda^2 - \cancel{2} = 0$$

$$\Leftrightarrow \lambda^2 - 3\lambda = 0$$

$$\Leftrightarrow \lambda(\lambda - 3) = 0$$

$\lambda = 0$ ή $\lambda = 3$

ιδιοτιμές: $\boxed{\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 1 \\ \lambda_3 &= 3 \end{aligned}}$

$\lambda_1 = 0$:

$$A - 0 \cdot I = A$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{r_1 = \frac{r_1}{2}} \left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \longrightarrow$$

$$\xrightarrow{r_2 = r_2 + r_1} \left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{r_3 = r_3 - r_1} \left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right) \longrightarrow$$

$$\xrightarrow{r_2 = 2r_2} \left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right) \xrightarrow{r_3 = r_3 - \frac{r_2}{2}} \left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow$$

$$\xrightarrow{r_1 = r_1 + \frac{r_2}{2}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \Rightarrow x_1 + 0 + x_3 = 0 \Rightarrow x_1 = -x_3 \\ \Rightarrow 0 + x_2 + x_3 = 0 \Rightarrow x_2 = -x_3 \end{array}$$

$$\boxed{\vec{v}_1 = \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} x_3}$$

$\underline{\lambda_2 = 1:}$

$$A - I = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_2 = r_2 + r_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \longrightarrow$$

$$\xrightarrow{r_3 = r_3 - r_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow{r_2 = -r_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \longrightarrow$$

$$\xrightarrow{r_3 = r_3 - r_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1 = r_1 + r_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \Rightarrow x_1 = 0 \\ \Rightarrow x_2 = x_3 \end{array}$$

$$\boxed{\vec{v}_2 = \begin{pmatrix} 0 \\ x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} x_3}$$

$\underline{\lambda_3 = 3:}$

$$A - 3I = \begin{bmatrix} -1 & -1 & 1 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$\left(\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ -1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right) \xrightarrow{r_1 = -r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right) \longrightarrow$$

$$\xrightarrow{r_2 = r_2 + r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right) \xrightarrow{r_3 = r_3 - r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right) \longrightarrow$$

$$\xrightarrow{r_2 = -r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right) \xrightarrow{r_3 = r_3 + r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow$$

$$\xrightarrow{r_1 = r_1 - r_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \Rightarrow x_1 - 2x_3 = 0 \Rightarrow x_1 = 2x_3 \\ \Rightarrow x_2 = -x_3 \end{array}$$

$$\boxed{\vec{v}_3 = \begin{pmatrix} 2x_3 \\ -x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} x_3}$$

2) μη μηδενικές ιδιοτιμές: $\lambda_3 = 3$ και $\lambda_2 = 1$

άρα $\sigma_1 = \sqrt{\lambda_3} = \sqrt{3}$ $\quad \sigma_2 = \sqrt{\lambda_2} = 1$

$$\boxed{\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}$$

3) $$B = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$|B - I\lambda| = 0 \Leftrightarrow \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \Leftrightarrow$$

$$\Leftrightarrow (2-\lambda)^2 - 1 = 0 \Leftrightarrow 4 - 4\lambda + \lambda^2 - 1 = 0 \Leftrightarrow$$

$$\Leftrightarrow \lambda^2 - 4\lambda + 3 = 0$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16-12}}{2} \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 3 \end{cases}$$

$\underline{\lambda_1 = 1:}$

$$B - I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right) \Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$$

$$\boxed{\vec{v}_1 = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_1}$$

$\underline{\lambda_2 = 3:}$

$$B - 3I = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$\left(\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right) \Rightarrow -x_1 - x_2 = 0 \Rightarrow x_1 = -x_2$$

$$\boxed{\vec{V}_2 = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} x_1}$$

4) $\hat{X} = U \Sigma V^T = \sum_{i=1}^{r} \sigma_i \vec{U}_i \vec{V}_i^T$

όπως λαμβάνουμε υπόψιν μόνο τη μεγαλύτερη ιδιοτιμή
άρα:

$\hat{X} = \sum_{i=1}^{r} \sigma_1 \vec{U}_i \vec{V}_i^T$ όπου $\sigma_1 = \sqrt{3}$

$$V = \begin{bmatrix} \frac{\vec{V}_3}{|\vec{V}_3|} & \frac{\vec{V}_2}{|\vec{V}_2|} & \frac{\vec{V}_1}{|\vec{V}_1|} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

όπου $\vec{V}_1, \vec{V}_2, \vec{V}_3$ από τον A

$$U = \begin{bmatrix} \frac{\vec{V}_2}{|\vec{V}_2|} & \frac{\vec{V}_1}{|\vec{V}_1|} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

άρα $\hat{X} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} \end{bmatrix} \cdot \sqrt{3} =$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{6}\cdot\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{6}}{3} & \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{6}}{6} \end{bmatrix} \cdot \sqrt{3} = \begin{bmatrix} \frac{\sqrt{12}}{6} & -\frac{\sqrt{12}}{12} & \frac{\sqrt{12}}{12} \\ -\frac{\sqrt{12}}{6} & \frac{\sqrt{12}}{12} & -\frac{\sqrt{12}}{12} \end{bmatrix} \cdot \sqrt{3} =$$

$$= \begin{bmatrix} \frac{2\sqrt{3}}{6} & -\frac{2\sqrt{3}}{12} & \frac{2\sqrt{3}}{12} \\ -\frac{2\sqrt{3}}{6} & \frac{2\sqrt{3}}{12} & -\frac{2\sqrt{3}}{12} \end{bmatrix} \cdot \sqrt{3} = \begin{bmatrix} \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{6} \end{bmatrix} \cdot \sqrt{3} \Leftarrow$$

$$\boxed{\hat{X} = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}}$$