IBP

$$F(a_1 a_2 a_3)=\int \frac{d^dk_1\, d^dk_2}{(k_1^2-m_1^2)^{a_1}(k_2^2-m_2^2)^{a_2}[(k_1+k_2)^2-m_3^2]^{a_3}}=\int d^dk_1\, d^dk_2\, f(a_1 a_2 a_3)$$

$$\frac{\partial}{\partial k_{1\mu}}\frac{1}{(k_1^2-m_1^2)^{a_1}}=\frac{-a_1 2k_{1\mu}}{(k_1^2-m_1^2)^{a_1+1}} \qquad \frac{\partial}{\partial k_{2\mu}}\frac{1}{(k_2^2-m_2^2)^{a_2}}=\frac{-a_2 2k_{2\mu}}{(k_2^2-m_2^2)^{a_2+1}}$$

$$\frac{\partial}{\partial k_{1\mu}}\frac{1}{[(k_1+k_2)^2-m_3^2]^{a_3}}=\frac{-a_3 2(k_{1\mu}+k_{2\mu})}{[(k_1+k_2)^2-m_3^2]^{a_3+1}} \qquad \frac{\partial}{\partial k_{2\mu}}\frac{1}{[(k_1+k_2)^2-m_3^2]^{a_3}}=\frac{-a_3 2(k_{1\mu}+k_{2\mu})}{[(k_2+k_3)^2-m_3^2]^{a_3+1}}$$

$$0=\int\frac{\partial}{\partial k_1^\mu}k_1^\mu f(a_1 a_2 a_3)=Df(a_1 a_2 a_3)-2a_1 k_1^2 f(a_1+1\, a_2 a_3)-a_3(2k_1^2+2k_1\cdot k_2)f(a_1 a_2 a_3+1)$$

$$=Df(a_1 a_2 a_3)-2a_1 f(a_1 a_2 a_3)-2a_1 m_1^2 f(a_1+1\, a_2 a_3)-a_3 f(a_1 a_2 a_3)-a_3 f(a_1-1\, a_2 a_3+1)$$
$$+a_3 f(a_1\, a_2-1\, a_3+1)-a_3(m_1^2-m_2^2+m_3^2)f(a_1 a_2\, a_3+1)$$

$$2a_1 m_1^2 f(a_1+1\, a_2 a_3)+a_3(m_1^2-m_2^2+m_3^2)f(a_1\, a_2\, a_3+1)=(D-2a_1-a_3)f(a_1 a_2 a_3)-a_3 f(a_1-1\, a_2\, a_3+1)$$
$$+a_3 f(a_1\, a_2-1\, a_3+1) \quad ①$$

$$0=\frac{\partial}{\partial k_2^\mu}k_2^\mu f(a_1 a_2 a_3)=Df(a_1 a_2 a_3)-2a_2 k_2^2 f(a_1\, a_2+1\, a_3)-a_3(2k_1\cdot k_2+2k_2^2)f(a_1 a_2 a_3+1)$$

$$=Df(a_1 a_2 a_3)-2a_2 f(a_1 a_2 a_3)-2a_2 m_2^2 f(a_1\, a_2+1\, a_3)-a_3 f(a_1 a_2 a_3)-a_3 f(a_1\, a_2-1\, a_3+1)$$
$$+a_3 f(a_1-1\, a_2 a_3+1)-a_3(m_2^2-m_1^2+m_3^2)f(a_1 a_2 a_3+1)$$

$$2a_2 m_2^2 f(a_1\, a_2+1\, a_3)+a_3(m_2^2-m_1^2+m_3^2)f(a_1 a_2 a_3+1)=(D-2a_2-a_3)f(a_1 a_2 a_3)-a_3 f(a_1\, a_2-1\, a_3+1)$$
$$+a_3 f(a_1-1\, a_2 a_3+1) \quad ②$$

$$0=\frac{\partial}{\partial k_{1\mu}}k_{2\mu}f(a_1 a_2 a_3)=-a_1(2k_1\cdot k_2)f(a_1+1\, a_2 a_3)-a_3(2k_1\cdot k_2+2k_2^2)f(a_1 a_2 a_3+1)$$

$$a_1[(k_1+k_2)^2-m_3^2-(k_1^2-m_1^2)-(k_2^2-m_2^2)+m_3^2-m_1^2-m_2^2]f(a_1+1\, a_2 a_3)$$
$$+a_3[(k_1+k_2)^2-m_3^2+k_2^2-m_2^2-(k_1^2-m_1^2)+m_3^2+m_2^2-m_1^2]f(a_1 a_2 a_3+1)=0$$

$$a_1 f(a_1+1\, a_2\, a_3-1)-a_1 f(a_1 a_2 a_3)-a_1 f(a_1+1\, a_2-1\, a_3)+a_1(m_3^2-m_2^2-m_1^2)f(a_1+1\, a_2 a_3)$$
$$+a_3 f(a_1 a_2 a_3)+a_3 f(a_1 a_2-1\, a_3+1)-a_3 f(a_1-1\, a_2 a_3+1)+a_3(m_3^2+m_2^2-m_1^2)f(a_1 a_2 a_3+1)=0$$

$$a_1(m_1^2+m_2^2-m_3^2)f(a_1+1\, a_2 a_3)+a_3(m_1^2-m_2^2-m_3^2)f(a_1\, a_2 a_3+1)=(a_3-a_1)f(a_1 a_2 a_3)$$
$$+a_1 f(a_1+1\, a_2 a_3-1)-a_1 f(a_1+1\, a_2-1\, a_3)+a_3 f(a_1\, a_2-1\, a_3+1)-a_3 f(a_1-1\, a_2\, a_3+1) \quad ③$$

联立 ① ② ③

$$\begin{bmatrix} 2a_1 m_1^2 & 0 & a_3(m_1^2-m_2^2+m_3^2)f(a_1 a_2 a_3+1) \\ 0 & 2a_2 m_2^2 & a_3(m_2^2-m_1^2+m_3^2) \\ a_1(m_1^2+m_2^2-m_3^2) & 0 & +a_3(m_1^2-m_2^2-m_3^2) \end{bmatrix}\begin{bmatrix} f(a_1+1\; a_2\, a_3) \\ f(a_1\; a_2+1\; a_3) \\ f(a_1\; a_2\; a_3+1) \end{bmatrix}$$

记为A

$$=\begin{bmatrix} (D-2a_1-a_3)f(a_1 a_2 a_3)-a_3 f(a_1-1\, a_2\, a_3+1)+a_3 f(a_1 a_2-1\, a_3+1) \\ (D-2a_2-a_3)f(a_1 a_2 a_3)-a_3 f(a_1\, a_2-1\, a_3+1)+a_3 f(a_1-1\, a_2 a_3+1) \\ (a_3-a_1)f(a_1 a_2 a_3)+a_1 f(a_1+1\, a_2 a_3-1)-a_1 f(a_1+1\, a_2-1\, a_3)+a_3 f(a_1\, a_2-1\, a_3+1) \end{bmatrix}$$

$= a_3 f(a_1-1\ a_2 a_3+1)$

$\det A = 2a_1 a_2 a_3 m_2^2 \Delta$ $\qquad \Delta \equiv (m_1+m_2+m_3)\underbrace{(m_1-m_2-m_3)(m_3-m_1-m_2)(m_2-m_1-m_3)}_{\text{Collinear}}$

只要求 $m_2 \neq 0$，$m_1\ m_3$ 任意可以为 0

① $f(a_1+1\, a_2 a_3) = \dfrac{1}{a_1 \Delta}$

```
-2 a3 m3^2 F[-1 + a1, a2, 1 + a3] + 2 a3 m3^2 F[a1, -1 + a2, 1 + a3] +
 (2 a3 (m1^2 - m2^2) + a1 (m1^2 - m2^2 - 3 m3^2) + d (-m1^2 + m2^2 + m3^2)) F[a1, a2, a3] - a1 (m1^2 - m2^2 + m3^2) F[1 + a1, -1 + a2, a3] + a1 (m1^2 - m2^2 + m3^2) F[1 + a1, a2, -1 + a3]
```

② $f(a_1 a_2+1\, a_3) = \dfrac{1}{a_2 m_2^2 \Delta}$

```
a3 m3^2 (m1^2 + m2^2 - m3^2) F[-1 + a1, a2, 1 + a3] + a3 m3^2 (-m1^2 - m2^2 + m3^2) F[a1, -1 + a2, 1 + a3] +
 (-a1 m1^2 m2^2 + d m1^2 m2^2 + a1 m2^4 - d m2^4 + a1 m1^2 m3^2 + d m2^2 m3^2 - a1 m3^4 - a3 (m1^2 - m2^2) (m1^2 + m2^2 - m3^2) + a2 (m1^4 + (m2^2 - m3^2)^2 - 2 m1^2 (m2^2 + m3^2))) F[a1, a2, a3] +
 a1 m1^2 (m1^2 - m2^2 - m3^2) F[1 + a1, -1 + a2, a3] + a1 m1^2 (-m1^2 + m2^2 + m3^2) F[1 + a1, a2, -1 + a3]
```

③ $f(a_1 a_2 a_3+1) = \dfrac{1}{a_3 \Delta}$

```
a3 (m1^2 - m2^2 + m3^2) F[-1 + a1, a2, 1 + a3] - a3 (m1^2 - m2^2 + m3^2) F[a1, -1 + a2, 1 + a3] +
 (d (m1^2 + m2^2 - m3^2) + 2 a1 (-m2^2 + m3^2) + a3 (-3 m1^2 - m2^2 + m3^2)) F[a1, a2, a3] + 2 a1 m1^2 F[1 + a1, -1 + a2, a3] - 2 a1 m1^2 F[1 + a1, a2, -1 + a3]
```

$F(\nu_1\nu_2\nu_3)$ 要作 IBP 其中一个指标 $\nu_i \geq 2$ 另外两个指标 ≥ 1

$\nu_1+\nu_2+\nu_3 = \hat{n} \geq 4$ 才能用 IBP 化简　且每用一次 IBP $\hat{n} \to \hat{n}-1$

\# do $\hat{n} = N \to 4$

对每个 $\hat{n}$ 中 $\nu_1\nu_2\nu_3$ 所有可能组合做 IBP

先对 ν_2 做降阶　$\nu_2 \geq 2$　$\nu_1\nu_3 \geq 1$

\# do $\nu_2 = 2 \to \hat{n}-2$　此时 $\nu_1+\nu_3 = \hat{n}-\nu_2$　$\nu_1\nu_3 \geq 1$

$\nu_1\nu_3$ 取值还有任意性

\# do $\nu_1 = 1 \to \hat{n}-\nu_1-1$

$\mathrm{IBP}[f(a_1 a_2+1\, a_3) \xrightarrow{①}\ ]$

此时对 $\nu_2 \geq 2$　$\nu_1\ \nu_3 \geq 1$ 的所有 $\nu_1\nu_2\nu_3$ 组合都做了一次 IBP

$\hat{n}$ 还剩下的 $\nu_1\nu_2\nu_3$ 的组合是 $\nu_2 = 1$　$\nu_1+\nu_3 = \hat{n}-1$　先对 ν_1 做降阶

\# do $\nu_1 = 2 \to \hat{n}-2$　$\nu_3 = \hat{n}-1-\nu_1$　　$\nu_1 \geq 2$　$\nu_3 \geq 1$

$\mathrm{IBP}[f(a_1+1\, a_2 a_3) \xrightarrow{②}\ ]$

对 $\hat{n}$ 还剩 $\nu_1\nu_2\nu_3$ 的组合是 $\nu_1 = \nu_2 = 1$　$\nu_3 = \hat{n}-2$

$\mathrm{IBP}[f(a_1 a_2 a_3+1) \xrightarrow{③}\ ]$

伪代码　　　$\hat{n} = \nu_1+\nu_2+\nu_3$

```
# do n̂ = N → 4                      (ν1+ν2+ν3 ≥ 4)
  # do ν2 = 2 → n̂-2                  ( ν2≥2  ν1ν3 ≥ 1 )
     # do ν1 = 1 → n̂-ν2-1
          IBP[f(a1 a2+1 a3) —①→ ]
     #enddo
  #enddo
  # do ν1 = 2 → n̂-2              (ν2=1  ν3 ≥ 1)
      IBP[f(a1+1 a2 a3) —②→ ]
  # enddo
  IBP[f(a1 a2 a3+1) —③→ ]        ν1=ν2=1  ν3 ≥ 2
#enddo
```

$F(a_1 a_2 a_3\ m_1 m_2 m_3)$ $F(a_1 a_2 a_3\ m_1 m_2 0)$ $F(a_1 a_2 a_3\ 0\ m_2\ 0)$

$F(a_1 a_2 a_3\ 0\ 0\ 0) = 0$

Collinear case

$(-+++)$

$$B^{\nu_1,\nu_2,\nu_3}(d; m_1^2, m_2^2, m_3^2) \equiv \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \frac{1}{[m_1^2+p^2]^{\nu_1}\,[m_2^2+q^2]^{\nu_2}\,[m_3^2+(p-q)^2]^{\nu_3}}$$

传播子为 p^2+m^2

程序中为 p^2-m^2

$(+---)$

$m_3 = m_1 + m_2$

2.1 General mass case $[m_1, m_2, m_1+m_2]$

Equation (95) of (the journal version of) [24] (with a small correction[2]: $d \to d+2$) provides a recursive 2-loop sunset factorization

$$\boxed{\begin{aligned} B^{\nu_1,\nu_2,\nu_3}_{m_1,m_2,m_3}(d) \;=\; & \frac{-1}{2(d+3-2\Sigma\nu_i)m_1m_2m_3}\Big\{ \\ & +\big[(m_1(d+2-\Sigma\nu_i)+m_2\nu_3-m_3\nu_2)\big]\,\mathbf{1}^- \\ & +\big[(m_1\nu_3+m_2(d+2-\Sigma\nu_i)-m_3\nu_1)\big]\,\mathbf{2}^- \\ & +\big[(m_1\nu_2+m_2\nu_1-m_3(d+2-\Sigma\nu_i))\big]\,\mathbf{3}^-\Big\}\, B^{\nu_1,\nu_2,\nu_3}_{m_1,m_2,m_3}(d)\,, \end{aligned}} \tag{2.4}$$

文献定义

$F(a_1 a_2 a_3)$

$= (-1)^{a_1+a_2+a_3} F(a_1 a_2 a_3)$,

程序中

$n = a_1 + a_2 + a_3$

每次 IBP $n \to n-1$

递推系数与程序中差一个负号

$m_1 = 0 \quad m_2 = m_3 = m \qquad m_3 = m_1 + m_2$

2.2 Special mass case $[0, m, m]$

For the special case with masses $[0, m, m]$, eq. (96) of (the journal version of) [24] gives a recursive 2-loop sunset factorization

$$\boxed{B^{\nu_1,\nu_2,\nu_3}_{0,m,m}(d) \;=\; a_{\nu_1-1} B^{\nu_1-1,\nu_2,\nu_3}_{0,m,m}(d)} \tag{2.15}$$

with rational coefficient function

$$a_{\nu_1} \;=\; -\frac{(d-2\nu_1-2\nu_2)(d-2\nu_1-2\nu_3)(d-\nu_1-\nu_2-\nu_3)}{2m^2(d-2-2\nu_1)(d-1-2\nu_1-\nu_2-\nu_3)(d-2\nu_1-\nu_2-\nu_3)}\,. \tag{2.16}$$

$$F(a_1 m_1\ a_2 m_2\ a_3 m_3) \equiv \int \frac{d^d k_1\, d^d k_2}{(k_1^2-m_1^2)^{a_1}(k_2^2-m_2^2)^{a_2}[(k_1+k_2)^2-m_3^2]^{a_3}}$$

$$F(a_2 m_2\ a_1 m_1\ a_3 m_3) = \int \frac{d^d k_1\, d^d k_2}{(k_1^2-m_2^2)^{a_2}(k_2^2-m_1^2)^{a_1}[(k_1+k_2)^2-m_3^2]^{a_3}}$$

$$F(a_3 m_3 a_2 m_2 a_1 m_1) = \int \frac{d^d k_1\, d^d k_2}{(k_1^2-m_3^2)^{a_3}(k_2^2-m_2^2)^{a_2}[(k_1+k_2)^2-m_1^2]^{a_1}}$$

$k_1 + k_2 = k_1'$

$k_2 = -k_2'$

$$= \int \frac{d^d k_1'\, d^d k_2'}{(k_1'^2-m_1^2)^{a_1}(k_2'^2-m_2^2)^{a_2}[(k_1'+k_2')^2-m_3^2]^{a_3}}$$

$$F(a_2 m_2, a_1 m_1, a_3 m_3) \overset{1\leftrightarrow 2}{=} F(a_1 m_1, a_2 m_2, a_3 m_3) \overset{1\leftrightarrow 3}{=} F(a_3 m_3\ a_2 m_2\ a_1 m_1)$$

$F(\nu_1 \nu_2 \nu_3) \longrightarrow F(111)\ F(011)\ F(101)\ F(110)$

$$F(\nu_1 \nu_2\ 0\ m_1 m_2 m_3) = \int \frac{d^d k_1\, d^d k_2}{(k_1^2-m_1^2)^{\nu_1}(k_2^2-m_2^2)^{\nu_2}} = \int \frac{d^d k_1}{(k_1^2-m_1^2)^{\nu_1}} \int \frac{d^d k_2}{(k_2^2-m_2^2)^{\nu_2}} = F(\nu_1 m_1) F(\nu_2 m_2)$$

$$F(\nu_1\ 0\ \nu_3 m_1 m_2 m_3) = \int \frac{d^d k_1\, d^d k_2}{(k_1^2-m_1^2)^{\nu_1}[(k_1+k_2)^2-m_3^2]^{\nu_3}} \xrightarrow[|J|=1]{k_1\to k_1',\ k_1+k_2\to k_2'} \int \frac{d^d k_1}{(k_1^2-m_1^2)^{\nu_1}} \int \frac{d^d k_2}{(k_2^2-m_3^2)^{\nu_3}} = F(\nu_1 m_1) F(\nu_3 m_3)$$

$$F(0\ \nu_2 \nu_3\ m_1 m_2 m_3) = \int \frac{d^d k_1\, d^d k_2}{(k_2^2-m_2^2)^{\nu_2}[(k_1+k_2)^2-m_3^2]^{\nu_3}} \xrightarrow[|J|=1]{k_2\to k_2',\ k_1+k_2\to k_1'} \int \frac{d^d k_1}{(k_1^2-m_3^2)^{\nu_3}} \int \frac{d^d k_2}{(k_2^2-m_2^2)^{\nu_2}} \quad F(\nu_2 m_2) F(\nu_3 m_3)$$

$$F(a\, m) = \int \frac{d^d k}{(k^2-m^2)^a} = \frac{d-2(a-1)}{2(a-1)m^2} F(a-1\ m)$$

$F(1 m_1) F(1 m_2) = C_1 \quad F(1 m_1) F(1 m_3) = C_2$

$F(1 m_2) F(1 m_3) = C_3 \quad F(1 m_1\ 1 m_2\ 1 m_3) = C_0$