# Note of `TwoLoopSunriseFeynmanIntegrals.jl`

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**Abstract**

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## Contents



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# 1 Introduction

We provide a Julia[1] package `TwoLoopSunriseFeynmanIntegrals.jl`[2] for two-loop sunrise Feynman integrals[3], which reads [2, Eq. (2.56)][4]

$$I_{\text{TSI}}^{(\nu_1,\nu_2,\nu_3)}(d,m_1,m_2,m_3) := \int \frac{\widetilde{\mathrm{d}q_1}\,\widetilde{\mathrm{d}q_2}}{(-q_1^2+m_1^2)^{\nu_1}(-q_2^2+m_2^2)^{\nu_2}(-q_{12}^2+m_3^2)^{\nu_3}}, \tag{1}$$

where $d = 4-2\varepsilon$ is the spacetime dimension, $m_i$'s are the masses, $\nu_i$'s are the exponents, and $q_i$ are the loop momenta ($q_{12} \equiv q_1+q_2$). In this note, we take the loop momentum measure as

$$\widetilde{\mathrm{d}q} := \frac{\mathrm{d}^d q}{\mathrm{i}\pi^{d/2}}. \tag{2}$$

One can easily verify that that this integral is invariant under the permutation of $(m_1,\nu_1)$, $(m_2,\nu_2)$, and $(m_3,\nu_3)$, *e.g.*,

$$I_{\text{TSI}}^{(\nu_1\nu_2\nu_3)}(d,m_1,m_2,m_3) \equiv I_{\text{TSI}}^{(\nu_2\nu_3\nu_1)}(d,m_2,m_3,m_1). \tag{3}$$

If all masses vanish, the integral vanishes as

$$I_{\text{TSI}}^{(\nu_1\nu_2\nu_3)}(d,0,0,0) \equiv 0 \tag{4}$$

since the definition of the Feynman integral in the dimensional regularization [2, Sec. 2.4.2].

This note is organized as follows: In Sec. 2, we reduce the two-loop sunrise Feynman integrals to the master integrals (MIs) via integration-by-part (IBP) techniques. In Sec. 3, the MIs are evaluated and expanded into $\varepsilon$-series. In Sec. 4, we introduce the implementation of the package. Finally, we conclude in Sec. 5.

# 2 Integration-by-Part Reduction

In this section, we consider the integration-by-part (IBP) reduction for the two-loop sunrise Feynman integrals. The IBP reduction starts from the fact that [2, Eq. (6.2)]

$$\int \widetilde{\mathrm{d}q}\, \frac{\partial}{\partial q^\mu}[k^\mu \cdots] \equiv 0, \tag{5}$$

where $k$ is an arbitrary momentum. We also have

$$\frac{\partial}{\partial q^\mu}\frac{1}{(-q^2+m^2)^\nu} = \frac{2\nu q_\mu}{(-q^2+m^2)^{\nu+1}}, \tag{6}$$

$$\frac{\partial}{\partial q_1^\mu}\frac{1}{(-q_{12}^2+m_3^2)^{\nu_3}} = \frac{2\nu_3 (q_{12})_\mu}{(-q_{12}^2+m_3^2)^{\nu_3+1}}. \tag{7}$$

Therefore,

$$\begin{aligned} 0 = & \int \widetilde{\mathrm{d}q_1}\,\widetilde{\mathrm{d}q_2}\, \frac{\partial}{\partial q_1^\mu} \frac{q_1^\mu}{(-q_1^2+m_1^2)^{\nu_1}(-q_2^2+m_2^2)^{\nu_2}(-q_{12}^2+m_3^2)^{\nu_3}} \\ = & (d-2\nu_1-\nu_3) I_{\text{TSI}}^{(\nu_1\nu_2\nu_3)} + 2\nu_1 m_1^2 I_{\text{TSI}}^{(\nu_1+1,\nu_2,\nu_3)} - \nu_3\Big(I_{\text{TSI}}^{(\nu_1-1,\nu_2,\nu_3+1)} - I_{\text{TSI}}^{(\nu_1,\nu_2-1,\nu_3+1)}\Big) \\ & + \nu_3\big(m_1^2-m_2^2+m_3^2\big) I_{\text{TSI}}^{(\nu_1,\nu_2,\nu_3+1)}. \end{aligned} \tag{8}$$

Or equivalently,

$$\begin{aligned} -2\nu_1 m_1^2 I_{\text{TSI}}^{(\nu_1+1,\nu_2,\nu_3)} - \nu_3\big(m_1^2-m_2^2+m_3^2\big) I_{\text{TSI}}^{(\nu_1,\nu_2,\nu_3+1)} = & (d-2\nu_1-\nu_3) I_{\text{TSI}}^{(\nu_1\nu_2\nu_3)} \\ & - \nu_3\Big(I_{\text{TSI}}^{(\nu_1-1,\nu_2,\nu_3+1)} - I_{\text{TSI}}^{(\nu_1,\nu_2-1,\nu_3+1)}\Big). \end{aligned} \tag{9}$$

Similarly, we have

$$\begin{aligned} -2\nu_2 m_2^2 I_{\text{TSI}}^{(\nu_1,\nu_2+1,\nu_3)} - \nu_3\big(m_2^2-m_1^2+m_3^2\big) I_{\text{TSI}}^{(\nu_1,\nu_2,\nu_3+1)} = & (d-2\nu_2-\nu_3) I_{\text{TSI}}^{(\nu_1\nu_2\nu_3)} \\ & - \nu_3\Big(I_{\text{TSI}}^{(\nu_1,\nu_2-1,\nu_3+1)} - I_{\text{TSI}}^{(\nu_1-1,\nu_2,\nu_3+1)}\Big). \end{aligned} \tag{10}$$

[1] https://julialang.org

[2] https://github.com/Fenyutanchan/TwoLoopSunriseFeynmanIntegrals.jl.git

[3] We suggest Refs. [1, 2] for pedagogical introduction to Feynman integrals.

[4] For simplicity, the prefactor $\mathrm{e}^{2\varepsilon\gamma_{\mathrm{E}}}(\mu^2)^{\nu-d}$ with $\nu \equiv \nu_1+\nu_2+\nu_3$ for the *modified minimal subtraction scheme* is omitted here.

Now consider

$$\begin{aligned} 0 = & \int \widetilde{\mathrm{d}q_1}\, \widetilde{\mathrm{d}q_2}\, \frac{\partial}{\partial q_2^\mu} \frac{q_1^\mu}{(-q_1^2+m_1^2)^{\nu_1}(-q_2^2+m_2^2)^{\nu_2}(-q_{12}^2+m_3^2)^{\nu_3}} \\ = & (\nu_2-\nu_3) I_{\mathrm{TSI}}^{(\nu_1\nu_2\nu_3)} + \nu_2 I_{\mathrm{TSI}}^{(\nu_1-1,\nu_2+1,\nu_3)} - \nu_2 I_{\mathrm{TSI}}^{(\nu_1,\nu_2+1,\nu_3-1)} + \nu_2\big(-m_1^2-m_2^2+m_3^2\big) I_{\mathrm{TSI}}^{(\nu_1,\nu_2+1,\nu_3)} \\ & + \nu_3 I_{\mathrm{TSI}}^{(\nu_1,\nu_2-1,\nu_3+1)} - \nu_3 I_{\mathrm{TSI}}^{(\nu_1-1,\nu_2,\nu_3+1)} + \nu_3\big(m_1^2-m_2^2+m_3^2\big) I_{\mathrm{TSI}}^{(\nu_1,\nu_2,\nu_3+1)}, \end{aligned} \tag{11}$$

or equivalently,

$$\begin{aligned} & \nu_2\big(m_1^2+m_2^2-m_3^2\big) I_{\mathrm{TSI}}^{(\nu_1,\nu_2+1,\nu_3)} - \nu_3\big(m_1^2-m_2^2+m_3^2\big) I_{\mathrm{TSI}}^{(\nu_1,\nu_2,\nu_3+1)} \\ = & (\nu_2-\nu_3) I_{\mathrm{TSI}}^{(\nu_1\nu_2\nu_3)} + \nu_2\Big(I_{\mathrm{TSI}}^{(\nu_1-1,\nu_2+1,\nu_3)} - I_{\mathrm{TSI}}^{(\nu_1,\nu_2+1,\nu_3-1)}\Big) + \nu_3\Big(I_{\mathrm{TSI}}^{(\nu_1,\nu_2-1,\nu_3+1)} - I_{\mathrm{TSI}}^{(\nu_1-1,\nu_2,\nu_3+1)}\Big). \end{aligned} \tag{12}$$

Combining Eqs. (9), (10), and (12), we have

$$\begin{aligned} & \begin{pmatrix} -2\nu_1 m_1^2 & 0 & -\nu_3\big(m_1^2-m_2^2+m_3^2\big) \\ 0 & -2\nu_2 m_2^2 & -\nu_3\big(m_2^2-m_1^2+m_3^2\big) \\ 0 & \nu_2\big(m_1^2+m_2^2-m_3^2\big) & -\nu_3\big(m_1^2-m_2^2+m_3^2\big) \end{pmatrix} \begin{pmatrix} I_{\mathrm{TSI}}^{(\nu_1+1,\nu_2,\nu_3)} \\ I_{\mathrm{TSI}}^{(\nu_1,\nu_2+1,\nu_3)} \\ I_{\mathrm{TSI}}^{(\nu_1,\nu_2,\nu_3+1)} \end{pmatrix} \\ = & \begin{pmatrix} d-2\nu_1-\nu_3 & 0 & \nu_3 & 0 & -\nu_3 \\ d-2\nu_2-\nu_3 & 0 & -\nu_3 & 0 & \nu_3 \\ \nu_2-\nu_3 & -\nu_2 & \nu_3 & \nu_2 & -\nu_3 \end{pmatrix} \begin{pmatrix} I_{\mathrm{TSI}}^{(\nu_1\nu_2\nu_3)} \\ I_{\mathrm{TSI}}^{(\nu_1,\nu_2+1,\nu_3-1)} \\ I_{\mathrm{TSI}}^{(\nu_1,\nu_2-1,\nu_3+1)} \\ I_{\mathrm{TSI}}^{(\nu_1-1,\nu_2+1,\nu_3)} \\ I_{\mathrm{TSI}}^{(\nu_1-1,\nu_2,\nu_3+1)} \end{pmatrix}. \end{aligned} \tag{13}$$

Defining the matrices $\mathbf{A}$ and $\mathbf{B}$ as

$$\mathbf{A} := \begin{pmatrix} -2\nu_1 m_1^2 & 0 & -\nu_3\big(m_1^2-m_2^2+m_3^2\big) \\ 0 & -2\nu_2 m_2^2 & -\nu_3\big(m_2^2-m_1^2+m_3^2\big) \\ 0 & \nu_2\big(m_1^2+m_2^2-m_3^2\big) & -\nu_3\big(m_1^2-m_2^2+m_3^2\big) \end{pmatrix}, \tag{14}$$

$$\mathbf{B} := \begin{pmatrix} d-2\nu_1-\nu_3 & 0 & \nu_3 & 0 & -\nu_3 \\ d-2\nu_2-\nu_3 & 0 & -\nu_3 & 0 & \nu_3 \\ \nu_2-\nu_3 & -\nu_2 & \nu_3 & \nu_2 & -\nu_3 \end{pmatrix}, \tag{15}$$

the solution is given by

$$\begin{pmatrix} I_{\mathrm{TSI}}^{(\nu_1+1,\nu_2,\nu_3)} \\ I_{\mathrm{TSI}}^{(\nu_1,\nu_2+1,\nu_3)} \\ I_{\mathrm{TSI}}^{(\nu_1,\nu_2,\nu_3+1)} \end{pmatrix} = \mathbf{A}^{-1}\mathbf{B} \begin{pmatrix} I_{\mathrm{TSI}}^{(\nu_1\nu_2\nu_3)} \\ I_{\mathrm{TSI}}^{(\nu_1,\nu_2+1,\nu_3-1)} \\ I_{\mathrm{TSI}}^{(\nu_1,\nu_2-1,\nu_3+1)} \\ I_{\mathrm{TSI}}^{(\nu_1-1,\nu_2+1,\nu_3)} \\ I_{\mathrm{TSI}}^{(\nu_1-1,\nu_2,\nu_3+1)} \end{pmatrix} \tag{16}$$

if $\det \mathbf{A} \neq 0$. Notice that the determinant of $\mathbf{A}$ is given by

$$\det \mathbf{A} = 2\nu_1\nu_2\nu_3 m_1^2 \lambda\big(m_1^2, m_2^2, m_3^2\big), \tag{17}$$

where $\lambda(x,y,z)$ is the Källén triangle function [3, Eq. (6.3)–(6.7)]

$$\begin{aligned} \lambda(x,y,z) &:= x^2+y^2+z^2-2xy-2yz-2zx \\ &= \big(\sqrt{x}+\sqrt{y}+\sqrt{z}\big)\big(\sqrt{x}+\sqrt{y}-\sqrt{z}\big)\big(\sqrt{x}-\sqrt{y}+\sqrt{z}\big)\big(-\sqrt{x}+\sqrt{y}+\sqrt{z}\big). \end{aligned} \tag{18}$$

Hence, there are two cases — non-collinear and collinear — to be considered.

## 2.1 Non-Collinear Case

In the non-collinear case, the solution in Eq. (16) is valid, which can be used to eliminate the sum of $\nu_i$'s by one from the left-hand-side (LHS) to the right-hand-side (RHS) of Eq. (16), which is the key to reduce the two-loop sunrise Feynman integrals to the master integrals (MIs). The expression of $\mathbf{A}^{-1}\mathbf{B}$ is too lengthy to be presented here, but it could be reproduced by the Wolfram Mathematica[5] notebook `note/Mathematica_notebooks/IBP_NC.nb`. The generated expressions are stored in the directory `ext/ibp_nc/` for further use.

[5] https://www.wolfram.com/mathematica

As Eq. (17) shows,

$$\begin{aligned} &\det \mathbf{A} = 0 \\ \Leftrightarrow \quad &\nu_1 = 0 \vee \nu_2 = 0 \vee \nu_3 = 0 \vee \lambda\big(m_1^2, m_2^2, m_3^2\big) = 0. \end{aligned} \tag{19}$$

with $m_1 > 0$[6], $m_2 \geq 0$, and $m_3 \geq 0$. The part of $\lambda\big(m_1^2, m_2^2, m_3^2\big) = 0$ is so-called collinear condition since it can be factorized as [Eq. (18)]

$$\lambda\big(m_1^2, m_2^2, m_3^2\big) = (m_1 + m_2 + m_3)(m_1 + m_2 - m_3)(m_1 - m_2 + m_3)(-m_1 + m_2 + m_3). \tag{20}$$

For the part of $\nu_1 = 0 \vee \nu_2 = 0 \vee \nu_3 = 0$, there are several cases to be considered.

$\boldsymbol{\nu_1 = \nu_2 = \nu_3 = 0.}$ The integrals are trivial as

$$I_{\mathrm{TSI}}^{(000)}(d, m_1, m_2, m_3) = \int \widetilde{\mathrm{d}q_1}\, \widetilde{\mathrm{d}q_2} \equiv 0. \tag{21}$$

$\boldsymbol{\nu_2 = \nu_3 = 0.}$ The integrals are also trivial as

$$I_{\mathrm{TSI}}^{(00\nu_3)} = \int \frac{\widetilde{\mathrm{d}q_1}\, \widetilde{\mathrm{d}q_2}}{\big(-q_1^2 + m_1^2\big)^{\nu_1}} = \int \widetilde{\mathrm{d}q_2} \int \frac{\widetilde{\mathrm{d}q_1}}{\big(-q_1^2 + m_1^2\big)^{\nu_1}} \equiv 0. \tag{22}$$

For the cases of $\nu_3 = \nu_1 = 0$ or $\nu_1 = \nu_2 = 0$, the permutation of $(m_1, \nu_1)$, $(m_2, \nu_2)$, and $(m_3, \nu_3)$ can be applied to obtain the same results.

$\boldsymbol{\nu_3 = 0.}$ The integrals are factorized as

$$\begin{aligned} I_{\mathrm{TSI}}^{(\nu_1\nu_2 0)} = \int \frac{\widetilde{\mathrm{d}q_1}\, \widetilde{\mathrm{d}q_2}}{\big(-q_1^2 + m_1^2\big)^{\nu_1}\big(-q_2^2 + m_2^2\big)^{\nu_2}} &= \int \frac{\widetilde{\mathrm{d}q_1}}{\big(-q_1^2 + m_1^2\big)^{\nu_1}} \int \frac{\widetilde{\mathrm{d}q_2}}{\big(-q_2^2 + m_2^2\big)^{\nu_2}} \\ &\equiv I_1^{(\nu_1)}(d, m_1) I_1^{(\nu_2)}(d, m_2), \end{aligned} \tag{23}$$

where $I_1^{(\nu)}(d, m)$ is evaluateed to [1, Eq. (10.1)]

$$I_1^{(\nu)}(d, m) = \frac{\Gamma\big(\nu - \frac{d}{2}\big)}{\Gamma(\nu)} \big(m^2\big)^{\frac{d}{2} - \nu}. \tag{24}$$

For the cases of $\nu_1 = 0$ or $\nu_2 = 0$, the permutation of $(m_1, \nu_1)$, $(m_2, \nu_2)$, and $(m_3, \nu_3)$ can be applied to obtain the same results.

The integrals with $\nu_1 = 0 \vee \nu_2 = 0 \vee \nu_3 = 0$ are called the boundary cases. For every integral with $\nu_1 \geq 0 \wedge \nu_2 \geq 0 \wedge \nu_3 \geq 0$, the IBP reduction can be applied to reduce the integrals to the boundary cases and $I_{\mathrm{TSI}}^{(111)}$. Actually, there are only two MIs in the non-collinear case — $I_{\mathrm{TSI}}^{(111)}$ and $I_{\mathrm{TSI}}^{(110)}$. However, the boundary cases are easy to evaluate as Eq. (24) shows, so we do not reduce $I_{\mathrm{TSI}}^{(\nu_1\nu_2 0)}$ to $I_{\mathrm{TSI}}^{(110)}$ in practice.

### 2.2 Collinear Case

## 3 Master Integrals

In this section, we evaluate the MIs and expand them into $\varepsilon$-series.

### 3.1 Factorization as Products of One-Loop Integrals

As Eq. (23) shows, the factorization of $I_{\mathrm{TSI}}^{(\nu_1\nu_2 0)} = I_1^{(\nu_1)} I_1^{(\nu_2)}$ makes the evaluation of $I_{\mathrm{TSI}}^{(\nu_1\nu_2 0)}$ easier. The expansion of $I_1^{(\nu)}(d, m)$ into $\varepsilon$-series is easily obtained as

$$I_1^{(1)}(d, m) = -\frac{m^2}{\varepsilon} - m^2\big(1 - \gamma_{\mathrm{E}} - \ln m^2\big) + \mathcal{O}\big(\varepsilon^1\big), \tag{25}$$

$$I_1^{(2)}(d, m) = \frac{1}{\varepsilon} - \big(\gamma_{\mathrm{E}} + \ln m^2\big) + \mathcal{O}\big(\varepsilon^1\big), \tag{26}$$

$$I_1^{(\nu > 2)}(d, m) = \frac{\big(m^2\big)^{2-\nu}}{(\nu - 1)(\nu - 2)} + \mathcal{O}\big(\varepsilon^1\big). \tag{27}$$

These expansions are evaluated in `note/Mathematica_notebooks/I_1-loop.nb` and stored in the directory `ext/one_loop/` for further use.

[6] If $m_1 = 0$, we can just apply the permutation of $(m_1, \nu_1)$, $(m_2, \nu_2)$, and $(m_3, \nu_3)$ to satisfy the condition except for the case of $m_1 = m_2 = m_3 = 0$. However the case of $m_1 = m_2 = m_3 = 0$ is evaluated to 0 as Eq. (4) shows.

## 3.2 Non-Collinear $I_{\text{TSI}}^{(111)}$

For the master integral $I_{\text{TSI}}^{(111)}$, Eq. (4.9) of Ref. [4] shows

$$I_{\text{TSI}}^{(111)}(d,m_1,m_2,m_3) = \frac{1}{2}\left(m_1^2\right)^{1-2\varepsilon} A(\varepsilon)\left[-\frac{1}{\varepsilon^2}(1+x+y)+\frac{2}{\varepsilon}(x\ln x+y\ln y)-x\ln^2 x-y\ln^2 y \right. \\ \left. +(1-x-y)\ln x\ln y-\lambda(1,x,y)\Phi(x,y)\right] \tag{28}$$

for

$$\lambda(1,x,y)>0 \quad \text{and} \quad 0<x,y<1, \tag{29}$$

where

$$x := \frac{m_2^2}{m_1^2}, \quad y := \frac{m_3^2}{m_1^2}, \tag{30}$$

and

$$\begin{aligned} A(\varepsilon) &:= \frac{\Gamma^2(1+\varepsilon)}{(1-\varepsilon)(1-2\varepsilon)}, \\ \Phi(x,y) &:= \frac{1}{\sqrt{\lambda(1,x,y)}}\left[\begin{aligned} &2\ln\left(\frac{1+x-y-\sqrt{\lambda(1,x,y)}}{2}\right)\ln\left(\frac{1-x+y-\sqrt{\lambda(1,x,y)}}{2}\right)-\ln x\ln y \\ &-2\mathrm{Li}_2\left(\frac{1-x+y-\sqrt{\lambda(1,x,y)}}{2}\right)-2\mathrm{Li}_2\left(\frac{1+x-y-\sqrt{\lambda(1,x,y)}}{2}\right)+\frac{\pi^2}{3}\end{aligned}\right]. \end{aligned} \tag{31}$$

For $\lambda(1,x,y)<0$ and $0<x,y<1$, Eq. (4.12) of Ref. [4] shows

$$I_{\text{TSI}}^{(111)}(d,m_1,m_2,m_3) = \left(m_1^2\right)^{1-2\varepsilon}\frac{A(\varepsilon)}{2\varepsilon^2}\left[\begin{aligned} &x^{-\varepsilon}y^{-\varepsilon}(1-x-y)\ {}_2F_1\left(\left.\begin{matrix}\varepsilon,\ 1\\ \frac{1}{2}+\varepsilon\end{matrix}\right|-\frac{\lambda(1,\ x,\ y)}{4xy}\right) \\ &-x^{-\varepsilon}(1+x-y)\ {}_2F_1\left(\left.\begin{matrix}\varepsilon,\ 1\\ \frac{1}{2}+\varepsilon\end{matrix}\right|-\frac{\lambda(1,\ x,\ y)}{4x}\right) \\ &-y^{-\varepsilon}(1-x+y)\ {}_2F_1\left(\left.\begin{matrix}\varepsilon,\ 1\\ \frac{1}{2}+\varepsilon\end{matrix}\right|-\frac{\lambda(1,\ x,\ y)}{4y}\right)\end{aligned}\right]. \tag{32}$$

The $\varepsilon$-series expansions of $I_{\text{TSI}}^{(111)}$ are evaluated in `note/Mathematica_notebooks/TSI111_NC.nb` and stored in the directory `ext/TSI111_nc/` for further use.

# 4 Implementation

# 5 Conclusion

# References


[1] Vladimir A. Smirnov. *Analytic tools for Feynman integrals*. Vol. 250. 2012. DOI: 10.1007/978-3-642-34886-0.

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