Note of `TwoLoopSunriseFeynmanIntegrals.jl`

Wen-Di Li*

School of Fundamental Physics and Mathematical Sciences,
Hangzhou Institute for Advanced Study,
University of Chinese Academy of Sciences,
Hangzhou 310024, CHINA

Quan-feng WU†

Institute of High Energy Physics,
Chinese Academy of Sciences,
Beijing 100049, CHINA

June 12, 2025‡

Abstract

TBA.

Contents

*E-mail: liwendi23@mails.ucas.ac.cn

†E-mail: wuquanfeng@ihep.ac.cn

1 Introduction

We provide a JULIA[1] package `TwoLoopSunriseFeynmanIntegrals.jl`[2] for two-loop sunrise Feynman integrals[3], which reads [2, Eq. (2.56)][4]

$$I_{\text{TSI}}^{(\nu_1,\nu_2,\nu_3)}(d,m_1,m_2,m_3) := \int \frac{\widetilde{\mathrm{d}q_1}\,\widetilde{\mathrm{d}q_2}}{(-q_1^2+m_1^2)^{\nu_1}(-q_2^2+m_2^2)^{\nu_2}(-q_{12}^2+m_3^2)^{\nu_3}}, \tag{1}$$

where $d = 4-2\varepsilon$ is the spacetime dimension, m_i's are the masses, ν_i's are the exponents, and q_i are the loop momenta ($q_{12} \equiv q_1 + q_2$). In this note, we take the loop momentum measure as

$$\widetilde{\mathrm{d}q} := \frac{\mathrm{d}^d q}{\mathrm{i}\pi^{d/2}}. \tag{2}$$

One can easily verify that that this integral is invariant under the permutation of (m_1,ν_1), (m_2,ν_2), and (m_3,ν_3), *e.g.*,

$$I_{\text{TSI}}^{(\nu_1\nu_2\nu_3)}(d,m_1,m_2,m_3) \equiv I_{\text{TSI}}^{(\nu_2\nu_3\nu_1)}(d,m_2,m_3,m_1). \tag{3}$$

If all masses vanish, the integral vanishes as

$$I_{\text{TSI}}^{(\nu_1\nu_2\nu_3)}(d,0,0,0) \equiv 0 \tag{4}$$

since the definition of the Feynman integral in the dimensional regularization [2, Sec. 2.4.2].

This note is organized as follows: In Sec. 2, we reduce the two-loop sunrise Feynman integrals to the master integrals (MIs) via integration-by-part (IBP) techniques. In Sec. 3, the MIs are evaluated and expanded into ε-series. Finally, we conclude in Sec. 5.

2 Integration-by-Part Reduction

In this section, we consider the integration-by-part (IBP) reduction for the two-loop sunrise Feynman integrals. The IBP reduction starts from the fact that [2, Eq. (6.2)]

$$\int \widetilde{\mathrm{d}q}\, \frac{\partial}{\partial q^\mu}[k^\mu \cdots] \equiv 0, \tag{5}$$

where k is an arbitrary momentum. We also have

$$\frac{\partial}{\partial q^\mu}\frac{1}{(-q^2+m^2)^\nu} = \frac{2\nu q_\mu}{(-q^2+m^2)^{\nu+1}}, \tag{6}$$

$$\frac{\partial}{\partial q_1^\mu}\frac{1}{(-q_{12}^2+m_3^2)^{\nu_3}} = \frac{2\nu_3 (q_{12})_\mu}{(-q_{12}^2+m_3^2)^{\nu_3+1}}. \tag{7}$$

Therefore,

$$\begin{aligned} 0 = &\int \widetilde{\mathrm{d}q_1}\,\widetilde{\mathrm{d}q_2}\,\frac{\partial}{\partial q_1^\mu}\frac{q_1^\mu}{(-q_1^2+m_1^2)^{\nu_1}(-q_2^2+m_2^2)^{\nu_2}(-q_{12}^2+m_3^2)^{\nu_3}} \\ = &\ (d-2\nu_1-\nu_3) I_{\text{TSI}}^{(\nu_1\nu_2\nu_3)} + 2\nu_1 m_1^2 I_{\text{TSI}}^{(\nu_1+1,\nu_2,\nu_3)} - \nu_3\left(I_{\text{TSI}}^{(\nu_1-1,\nu_2,\nu_3+1)} - I_{\text{TSI}}^{(\nu_1,\nu_2-1,\nu_3+1)}\right) \\ &+ \nu_3\left(m_1^2-m_2^2+m_3^2\right) I_{\text{TSI}}^{(\nu_1,\nu_2,\nu_3+1)}. \end{aligned} \tag{8}$$

Or equivalently,

$$\begin{aligned} -2\nu_1 m_1^2 I_{\text{TSI}}^{(\nu_1+1,\nu_2,\nu_3)} - \nu_3\left(m_1^2-m_2^2+m_3^2\right) I_{\text{TSI}}^{(\nu_1,\nu_2,\nu_3+1)} = &\ (d-2\nu_1-\nu_3) I_{\text{TSI}}^{(\nu_1\nu_2\nu_3)} \\ &- \nu_3\left(I_{\text{TSI}}^{(\nu_1-1,\nu_2,\nu_3+1)} - I_{\text{TSI}}^{(\nu_1,\nu_2-1,\nu_3+1)}\right). \end{aligned} \tag{9}$$

[1] See `https://julialang.org`.
[2] See `https://github.com/Fenyutanchan/TwoLoopSunriseFeynmanIntegrals.jl.git`.
[3] We suggest Refs. [1, 2] for pedagogical introduction to Feynman integrals.
[4] For simplicity, the prefactor $\mathrm{e}^{2\varepsilon\gamma_{\mathrm{E}}}\left(\mu^2\right)^{\nu-d}$ with $\nu \equiv \nu_1+\nu_2+\nu_3$ for the *modified minimal subtraction scheme* is omitted here.

Similarly, we have

$$
\begin{aligned}
-2\nu_2 m_2^2 I_{\mathrm{TSI}}^{(\nu_1,\nu_2+1,\nu_3)} - \nu_3\left(m_2^2 - m_1^2 + m_3^2\right) I_{\mathrm{TSI}}^{(\nu_1,\nu_2,\nu_3+1)} = {} & (d - 2\nu_2 - \nu_3) I_{\mathrm{TSI}}^{(\nu_1\nu_2\nu_3)} \\
& - \nu_3\left(I_{\mathrm{TSI}}^{(\nu_1,\nu_2-1,\nu_3+1)} - I_{\mathrm{TSI}}^{(\nu_1-1,\nu_2,\nu_3+1)}\right).
\end{aligned} \tag{10}
$$

Now consider

$$
\begin{aligned}
0 = {} & \int \widetilde{\mathrm{d}q_1}\, \widetilde{\mathrm{d}q_2}\, \frac{\partial}{\partial q_2^\mu} \frac{q_1^\mu}{(-q_1^2 + m_1^2)^{\nu_1}(-q_2^2 + m_2^2)^{\nu_2}(-q_{12}^2 + m_3^2)^{\nu_3}} \\
= {} & (\nu_2 - \nu_3) I_{\mathrm{TSI}}^{(\nu_1\nu_2\nu_3)} + \nu_2 I_{\mathrm{TSI}}^{(\nu_1-1,\nu_2+1,\nu_3)} - \nu_2 I_{\mathrm{TSI}}^{(\nu_1,\nu_2+1,\nu_3-1)} + \nu_2\left(-m_1^2 - m_2^2 + m_3^2\right) I_{\mathrm{TSI}}^{(\nu_1,\nu_2+1,\nu_3)} \\
& + \nu_3 I_{\mathrm{TSI}}^{(\nu_1,\nu_2-1,\nu_3+1)} - \nu_3 I_{\mathrm{TSI}}^{(\nu_1-1,\nu_2,\nu_3+1)} + \nu_3\left(m_1^2 - m_2^2 + m_3^2\right) I_{\mathrm{TSI}}^{(\nu_1,\nu_2,\nu_3+1)},
\end{aligned} \tag{11}
$$

or equivalently,

$$
\begin{aligned}
& \nu_2\left(m_1^2 + m_2^2 - m_3^2\right) I_{\mathrm{TSI}}^{(\nu_1,\nu_2+1,\nu_3)} - \nu_3\left(m_1^2 - m_2^2 + m_3^2\right) I_{\mathrm{TSI}}^{(\nu_1,\nu_2,\nu_3+1)} \\
= {} & (\nu_2 - \nu_3) I_{\mathrm{TSI}}^{(\nu_1\nu_2\nu_3)} + \nu_2\left(I_{\mathrm{TSI}}^{(\nu_1-1,\nu_2+1,\nu_3)} - I_{\mathrm{TSI}}^{(\nu_1,\nu_2+1,\nu_3-1)}\right) + \nu_3\left(I_{\mathrm{TSI}}^{(\nu_1,\nu_2-1,\nu_3+1)} - I_{\mathrm{TSI}}^{(\nu_1-1,\nu_2,\nu_3+1)}\right).
\end{aligned} \tag{12}
$$

Combining Eqs. (9), (10), and (12), we have

$$
\begin{aligned}
& \begin{pmatrix} -2\nu_1 m_1^2 & 0 & -\nu_3\left(m_1^2 - m_2^2 + m_3^2\right) \\ 0 & -2\nu_2 m_2^2 & -\nu_3\left(m_2^2 - m_1^2 + m_3^2\right) \\ 0 & \nu_2\left(m_1^2 + m_2^2 - m_3^2\right) & -\nu_3\left(m_1^2 - m_2^2 + m_3^2\right) \end{pmatrix} \begin{pmatrix} I_{\mathrm{TSI}}^{(\nu_1+1,\nu_2,\nu_3)} \\ I_{\mathrm{TSI}}^{(\nu_1,\nu_2+1,\nu_3)} \\ I_{\mathrm{TSI}}^{(\nu_1,\nu_2,\nu_3+1)} \end{pmatrix} \\
= {} & \begin{pmatrix} d - 2\nu_1 - \nu_3 & 0 & \nu_3 & 0 & -\nu_3 \\ d - 2\nu_2 - \nu_3 & 0 & -\nu_3 & 0 & \nu_3 \\ \nu_2 - \nu_3 & -\nu_2 & \nu_3 & \nu_2 & -\nu_3 \end{pmatrix} \begin{pmatrix} I_{\mathrm{TSI}}^{(\nu_1\nu_2\nu_3)} \\ I_{\mathrm{TSI}}^{(\nu_1,\nu_2+1,\nu_3-1)} \\ I_{\mathrm{TSI}}^{(\nu_1,\nu_2-1,\nu_3+1)} \\ I_{\mathrm{TSI}}^{(\nu_1-1,\nu_2+1,\nu_3)} \\ I_{\mathrm{TSI}}^{(\nu_1-1,\nu_2,\nu_3+1)} \end{pmatrix}.
\end{aligned} \tag{13}
$$

Defining the matrices $\mathbf{A}$ and $\mathbf{B}$ as

$$
\mathbf{A} \coloneqq \begin{pmatrix} -2\nu_1 m_1^2 & 0 & -\nu_3\left(m_1^2 - m_2^2 + m_3^2\right) \\ 0 & -2\nu_2 m_2^2 & -\nu_3\left(m_2^2 - m_1^2 + m_3^2\right) \\ 0 & \nu_2\left(m_1^2 + m_2^2 - m_3^2\right) & -\nu_3\left(m_1^2 - m_2^2 + m_3^2\right) \end{pmatrix}, \tag{14}
$$

$$
\mathbf{B} \coloneqq \begin{pmatrix} d - 2\nu_1 - \nu_3 & 0 & \nu_3 & 0 & -\nu_3 \\ d - 2\nu_2 - \nu_3 & 0 & -\nu_3 & 0 & \nu_3 \\ \nu_2 - \nu_3 & -\nu_2 & \nu_3 & \nu_2 & -\nu_3 \end{pmatrix}, \tag{15}
$$

the solution is given by

$$
\begin{pmatrix} I_{\mathrm{TSI}}^{(\nu_1+1,\nu_2,\nu_3)} \\ I_{\mathrm{TSI}}^{(\nu_1,\nu_2+1,\nu_3)} \\ I_{\mathrm{TSI}}^{(\nu_1,\nu_2,\nu_3+1)} \end{pmatrix} = \mathbf{A}^{-1}\mathbf{B} \begin{pmatrix} I_{\mathrm{TSI}}^{(\nu_1\nu_2\nu_3)} \\ I_{\mathrm{TSI}}^{(\nu_1,\nu_2+1,\nu_3-1)} \\ I_{\mathrm{TSI}}^{(\nu_1,\nu_2-1,\nu_3+1)} \\ I_{\mathrm{TSI}}^{(\nu_1-1,\nu_2+1,\nu_3)} \\ I_{\mathrm{TSI}}^{(\nu_1-1,\nu_2,\nu_3+1)} \end{pmatrix} \tag{16}
$$

if $\det \mathbf{A} \neq 0$. Notice that the determinant of $\mathbf{A}$ is given by

$$
\det \mathbf{A} = 2\nu_1\nu_2\nu_3 m_1^2 \lambda\left(m_1^2, m_2^2, m_3^2\right), \tag{17}
$$

where $\lambda(x, y, z)$ is the Källén triangle function [3, Eq. (6.3)–(6.7)]

$$
\begin{aligned}
\lambda(x, y, z) & \coloneqq x^2 + y^2 + z^2 - 2xy - 2yz - 2zx \\
& = \left(\sqrt{x} + \sqrt{y} + \sqrt{z}\right)\left(\sqrt{x} + \sqrt{y} - \sqrt{z}\right)\left(\sqrt{x} - \sqrt{y} + \sqrt{z}\right)\left(-\sqrt{x} + \sqrt{y} + \sqrt{z}\right).
\end{aligned} \tag{18}
$$

Hence, there are two cases — non-collinear and collinear — to be considered.

2.1 Non-Collinear Case

In the non-collinear case, the solution in Eq. (16) is valid, which can be used to eliminate the sum of ν_i's by one from the left-hand-side (LHS) to the right-hand-side (RHS) of Eq. (16), which is the key to reduce the two-loop sunrise Feynman integrals to the master integrals (MIs). The expression of $\mathbf{A}^{-1}\mathbf{B}$ is too lengthy to be presented here, but it could be reproduced by the Wolfram Mathematica[5] notebook `note/Mathematica_notebooks/IBP_NC.nb`. The generated expressions are stored in the directory `ext/ibp_nc/` for further use.

As Eq. (17) shows,

$$\begin{aligned} &\det \mathbf{A} = 0 \\ \Leftrightarrow \quad &\nu_1 = 0 \vee \nu_2 = 0 \vee \nu_3 = 0 \vee \lambda\left(m_1^2, m_2^2, m_3^2\right) = 0. \end{aligned} \tag{19}$$

with $m_1 > 0$[6], $m_2 \geq 0$, and $m_3 \geq 0$. The part of $\lambda\left(m_1^2, m_2^2, m_3^2\right) = 0$ is so-called collinear condition since it can be factorized as [Eq. (18)]

$$\lambda\left(m_1^2, m_2^2, m_3^2\right) = (m_1 + m_2 + m_3)(m_1 + m_2 - m_3)(m_1 - m_2 + m_3)(-m_1 + m_2 + m_3). \tag{20}$$

For the part of $\nu_1 = 0 \vee \nu_2 = 0 \vee \nu_3 = 0$, there are several cases to be considered.

$\boldsymbol{\nu_1 = \nu_2 = \nu_3 = 0}$. The integrals are trivial as

$$I_{\mathrm{TSI}}^{(000)}(d, m_1, m_2, m_3) = \int \widetilde{\mathrm{d}q_1}\, \widetilde{\mathrm{d}q_2} \equiv 0. \tag{21}$$

$\boldsymbol{\nu_2 = \nu_3 = 0}$. The integrals are also trivial as

$$I_{\mathrm{TSI}}^{(00\nu_3)} = \int \frac{\widetilde{\mathrm{d}q_1}\, \widetilde{\mathrm{d}q_2}}{\left(-q_1^2 + m_1^2\right)^{\nu_1}} = \int \widetilde{\mathrm{d}q_2} \int \frac{\widetilde{\mathrm{d}q_1}}{\left(-q_1^2 + m_1^2\right)^{\nu_1}} \equiv 0. \tag{22}$$

For the cases of $\nu_3 = \nu_1 = 0$ or $\nu_1 = \nu_2 = 0$, the permutation of (m_1, ν_1), (m_2, ν_2), and (m_3, ν_3) can be applied to obtain the same results.

$\boldsymbol{\nu_3 = 0}$. The integrals are factorized as

$$\begin{aligned} I_{\mathrm{TSI}}^{(\nu_1\nu_2 0)} = \int \frac{\widetilde{\mathrm{d}q_1}\, \widetilde{\mathrm{d}q_2}}{\left(-q_1^2 + m_1^2\right)^{\nu_1}\left(-q_2^2 + m_2^2\right)^{\nu_2}} &= \int \frac{\widetilde{\mathrm{d}q_1}}{\left(-q_1^2 + m_1^2\right)^{\nu_1}} \int \frac{\widetilde{\mathrm{d}q_2}}{\left(-q_2^2 + m_2^2\right)^{\nu_2}} \\ &\equiv I_1^{(\nu_1)}(d, m_1) I_1^{(\nu_2)}(d, m_2), \end{aligned} \tag{23}$$

where $I_1^{(\nu)}(d, m)$ is evaluateed to [1, Eq. (10.1)]

$$I_1^{(\nu)}(d, m) = \frac{\Gamma\left(\nu - \frac{d}{2}\right)}{\Gamma(\nu)} \left(m^2\right)^{\frac{d}{2} - \nu}. \tag{24}$$

For the cases of $\nu_1 = 0$ or $\nu_2 = 0$, the permutation of (m_1, ν_1), (m_2, ν_2), and (m_3, ν_3) can be applied to obtain the same results.

The integrals with $\nu_1 = 0 \vee \nu_2 = 0 \vee \nu_3 = 0$ are called the boundary cases. For every integral with $\nu_1 \geq 0 \wedge \nu_2 \geq 0 \wedge \nu_3 \geq 0$, the IBP reduction can be applied to reduce the integrals to the boundary cases and $I_{\mathrm{TSI}}^{(111)}$. Actually, there are only two MIs in the non-collinear case — $I_{\mathrm{TSI}}^{(111)}$ and $I_{\mathrm{TSI}}^{(110)}$. However, the boundary cases are easy to evaluate as Eq. (24) shows, so we do not reduce $I_{\mathrm{TSI}}^{(\nu_1\nu_2 0)}$ to $I_{\mathrm{TSI}}^{(110)}$ in practice.

2.2 Collinear Case

If $\lambda\left(m_1^2, m_2^2, m_3^2\right) = 0$, we cannot apply Eq. (16) to reduce the integrals since $\mathbf{A}$ is singular, *i.e.*, $\mathbf{A}^{-1}$ does not exist. Notice that

$$\lambda(m_1^2, m_2^2, m_3^2) = 0 \quad \Longleftrightarrow \quad m_1 + m_2 + m_3 = 0 \vee m_1 = m_2 + m_3 \vee m_2 = m_1 + m_3 \vee m_3 = m_1 + m_2. \tag{25}$$

[5] https://www.wolfram.com/mathematica

[6] If $m_1 = 0$, we can just apply the permutation of (m_1, ν_1), (m_2, ν_2), and (m_3, ν_3) to satisfy the condition except for the case of $m_1 = m_2 = m_3 = 0$. However the case of $m_1 = m_2 = m_3 = 0$ is evaluated to 0 as Eq. (4) shows.

Since $m_i \geq 0$ and $I_{\text{TSI}}^{(\nu_1\nu_2\nu_3)}(d,0,0,0)$ vanishes as Eq. (4) shows, the first condition is excluded. The other three conditions are called the collinear cases. Without loss of generality, the permutation of (m_1,ν_1), (m_2,ν_2), and (m_3,ν_3) can be applied to obtain the case $m_1 = m_2 + m_3$.

From Eq. (2.4) of Ref. [4], we have

$$I_{\text{TSI}}^{(\nu_1,\nu_2,\nu_3)}(d,m_1,m_2,m_3) = \frac{1}{2(d+3-2\nu)m_1m_2m_3}\left\{\begin{aligned} &[m_1(d+2-\nu) - m_2\nu_3 - m_3\nu_2]I_{\text{TSI}}^{(\nu_1-1,\nu_2,\nu_3)} \\ +&[m_1\nu_3 - m_2(d+2-\nu) - m_3\nu_1]I_{\text{TSI}}^{(\nu_1,\nu_2-1,\nu_3)} \\ +&[m_1\nu_2 - m_2\nu_1 - m_3(d+2-\nu)]I_{\text{TSI}}^{(\nu_1,\nu_2,\nu_3-1)} \end{aligned}\right\}, \tag{26}$$

where $m_1 = m_2 + m_3$ and $\nu = \nu_1 + \nu_2 + \nu_3$. We can apply this formula to reduce the collinear cases to the boundary cases, *i.e.*, $\nu_i = 0$ for any $i = 1,2,3$. The boundary cases of any $\nu_i = 0$ are the same as Eqs. (21), (22), and (23). We also perform the ε-expansion in `note/Mathematica_notebooks/IBP_CL.nb`, and the results are stored in the directory `ext/ibp_cl/` for further use.

Different from the non-collinear case, $I_{\text{TSI}}^{(111)}$ is no longer the boundary case (or MIs) that needs to be evaluated independently. Instead, we have [4, Eq. (2.14)]

$$I_{\text{TSI}}^{(111)}(d,m_1,m_2,m_3) = \frac{d-2}{2(d-3)}\left(\frac{I_{\text{TSI}}^{(011)}}{m_2m_3} - \frac{I_{\text{TSI}}^{(101)}}{m_3m_1} - \frac{I_{\text{TSI}}^{(110)}}{m_1m_2}\right) \tag{27}$$

for $m_1 = m_2 + m_3$.

3 Master Integrals

In this section, we evaluate the MIs and expand them into ε-series.

3.1 Factorization as Products of One-Loop Integrals

As Eq. (23) shows, the factorization of $I_{\text{TSI}}^{(\nu_1\nu_2 0)} = I_1^{(\nu_1)}I_1^{(\nu_2)}$ makes the evaluation of $I_{\text{TSI}}^{(\nu_1\nu_2 0)}$ easier. The expansion of $I_1^{(\nu)}(d,m)$ into ε-series is easily obtained as

$$I_1^{(1)}(d,m) = -\frac{m^2}{\varepsilon} - m^2\left(1 - \gamma_{\text{E}} - \ln m^2\right) + \mathcal{O}\left(\varepsilon^1\right), \tag{28}$$

$$I_1^{(2)}(d,m) = \frac{1}{\varepsilon} - \left(\gamma_{\text{E}} + \ln m^2\right) + \mathcal{O}\left(\varepsilon^1\right), \tag{29}$$

$$I_1^{(\nu>2)}(d,m) = \frac{\left(m^2\right)^{2-\nu}}{(\nu-1)(\nu-2)} + \mathcal{O}\left(\varepsilon^1\right). \tag{30}$$

These expansions are evaluated in `note/Mathematica_notebooks/I_1-loop.nb` and stored in the directory `ext/one_loop/` for further use.

3.2 Non-Collinear $I_{\text{TSI}}^{(111)}$

All non-vanishing masses. For the master integral $I_{\text{TSI}}^{(111)}$, Eq. (4.9) of Ref. [5] shows

$$I_{\text{TSI}}^{(111)}(d,m_1,m_2,m_3) = \frac{1}{2}\left(m_1^2\right)^{1-2\varepsilon}A(\varepsilon)\left[\begin{aligned} -\frac{1}{\varepsilon^2}(1+x+y) + \frac{2}{\varepsilon}(x\ln x + y\ln y) - x\ln^2 x - y\ln^2 y \\ + (1-x-y)\ln x\ln y - \lambda(1,x,y)\Phi(x,y) \end{aligned}\right] \tag{31}$$

where $m_1, m_2, m_3 > 0$,

$$\begin{cases} x := \dfrac{m_2^2}{m_1^2}, \\ y := \dfrac{m_3^2}{m_1^2}, \end{cases} \tag{32}$$

with $0 \leq x, y \leq 1$, and

$$A(\varepsilon) := \frac{\Gamma^2(1+\varepsilon)}{(1-\varepsilon)(1-2\varepsilon)}. \tag{33}$$

1. For $\lambda(1,x,y) > 0$, we have $\sqrt{x} + \sqrt{y} < 1$ and

$$\Phi(x,y) := \frac{1}{\sqrt{\lambda(1,x,y)}} \left[\begin{aligned} &2\ln\left(\frac{1+x-y-\sqrt{\lambda(1,x,y)}}{2}\right)\ln\left(\frac{1-x+y-\sqrt{\lambda(1,x,y)}}{2}\right) \\ &- 2\mathrm{Li}_2\left(\frac{1-x+y-\sqrt{\lambda(1,x,y)}}{2}\right) - 2\mathrm{Li}_2\left(\frac{1+x-y-\sqrt{\lambda(1,x,y)}}{2}\right) \\ &- \ln x \ln y + \frac{\pi^2}{3} \end{aligned} \right]; \quad (34)$$

2. For $\lambda(1,x,y) < 0$, we have $\sqrt{x} + \sqrt{y} > 1$ and

$$\Phi(x,y) := \frac{2}{\sqrt{-\lambda(1,x,y)}} \left[\begin{aligned} &\mathrm{Cl}_2\left(2\arccos\frac{1+x-y}{2\sqrt{x}}\right) + \mathrm{Cl}_2\left(2\arccos\frac{1-x+y}{2\sqrt{y}}\right) \\ &+ \mathrm{Cl}_2\left(2\arccos\frac{-1+x+y}{2\sqrt{xy}}\right). \end{aligned} \right] \quad (35)$$

The ε-series expansions of $I_{\mathrm{TSI}}^{(111)}$ are evaluated in note/Mathematica_notebooks/TSI111_NC.nb and stored in the directory ext/TSI111_nc/ for further use.

One vanishing mass. For the case of $m_1 > m_2 > m_3 = 0$, Eq. (31) in Ref. [6] gives that

$$\begin{aligned} I_{\mathrm{TSI}}^{(111)}(d, m_1, m_2, 0) &= \frac{\Gamma(3-\frac{d}{2})\Gamma(\frac{d}{2}-1)\Gamma^2(2-\frac{d}{2})}{\Gamma^2(1)\Gamma(\frac{d}{2})\Gamma(4-d)(m_1^2)^{3-d}} {}_2F_1\left(\begin{matrix} 3-d,\ 2-\frac{d}{2} \\ 4-d \end{matrix}\middle| 1-\frac{m_2^2}{m_1^2}\right) \\ &= \frac{m_1^2+m_2^2}{\varepsilon} + (1-\gamma_{\mathrm{E}})(m_1^2+m_2^2) - 2(m_1^2\ln m_1^2 + m_2^2\ln m_2^2) + \mathcal{O}(\varepsilon), \end{aligned} \quad (36)$$

which is evaluated in note/Mathematica_notebooks/TSI111_NC_one_vanishing.nb and stored in the directory ext/TSI111_nc/ for further use.

Two vanishing masses. For the case of $m_1 > m_2 = m_3 = 0$, Eq. (10.39) of Ref. [1] gives that

$$\begin{aligned} I_{\mathrm{TSI}}^{(111)}(d, m_1, 0, 0) &= \frac{\Gamma(3-d)}{(m_1^2)^{3-d}} \frac{\Gamma(2-\frac{d}{2})\Gamma^2(\frac{d}{2}-1)}{\Gamma^3(1)\Gamma(\frac{d}{2})} \\ &= -\frac{m_1^2}{2\varepsilon^2} + \frac{m_1^2}{2\varepsilon}(-3+2\gamma_{\mathrm{E}}+2\ln m_1^2) \\ &\quad + m_1^2\left[-\frac{7}{2} + (3-\gamma_{\mathrm{E}})\gamma_{\mathrm{E}} - \frac{\pi^2}{4} + (3-2\gamma_{\mathrm{E}})\ln m_1^2 - \ln^2 m_1^2\right] + \mathcal{O}(\varepsilon), \end{aligned} \quad (37)$$

which is evaluated in note/Mathematica_notebooks/TSI111_NC_two_vanishing.nb and stored in the directory ext/TSI111_nc/ for further use.

4 Test

The Wolfram Mathematica package AMFlow[7] [7] provides an efficient way to evaluate the Feynman integrals numerically. With the IBP option "FiniteFlow+LiteRed" provided by FiniteFlow[8] [8] and LiteRed[9] [9, 10], we provide several scripts to generate the results via AMFlow and compare them with our results. See the directory ext/test/ for the details.

5 Conclusion

[7]See https://gitlab.com/MultiLoop-PKU/AMFlow.git.
[8]See https://github.com/peraro/FiniteFlow.git and https://github.com/peraro/FiniteFlow-MathTools.git.
[9]https://www.inp.nsk.su/~lee/programs/LiteRed/

References

[1] Vladimir A. Smirnov. *Analytic tools for Feynman integrals*. Vol. 250. 2012. DOI: 10.1007/978-3-642-34886-0.

[2] Stefan Weinzierl. *Feynman Integrals. A Comprehensive Treatment for Students and Researchers*. UNITEXT for Physics. Springer, 2022. ISBN: 978-3-030-99557-7, 978-3-030-99560-7, 978-3-030-99558-4. DOI: 10.1007/978-3-030-99558-4. arXiv: 2201.03593 [hep-th].

[3] E. Byckling and K. Kajantie. *Particle Kinematics*. London: John Wiley & Sons Ltd, 1973.

[4] Andrei I. Davydychev and York Schröder. "Recursion-free solution for two-loop vacuum integrals with "collinear" masses". In: *JHEP* 12 (2022), p. 047. DOI: 10.1007/JHEP12(2022)047. arXiv: 2210.10593 [hep-ph].

[5] Andrei I. Davydychev and J. B. Tausk. "Two loop selfenergy diagrams with different masses and the momentum expansion". In: *Nucl. Phys. B* 397 (1993), pp. 123–142. DOI: 10.1016/0550-3213(93)90338-P.

[6] J. Fleischer and O. V. Tarasov. "Calculation of Feynman diagrams from their small momentum expansion". In: *Z. Phys. C* 64 (1994), pp. 413–426. DOI: 10.1007/BF01560102. arXiv: hep-ph/9403230.

[7] Xiao Liu and Yan-Qing Ma. "AMFlow: A Mathematica package for Feynman integrals computation via auxiliary mass flow". In: *Comput. Phys. Commun.* 283 (2023), p. 108565. DOI: 10.1016/j.cpc.2022.108565. arXiv: 2201.11669 [hep-ph].

[8] Tiziano Peraro. "FiniteFlow: multivariate functional reconstruction using finite fields and dataflow graphs". In: *JHEP* 07 (2019), p. 031. DOI: 10.1007/JHEP07(2019)031. arXiv: 1905.08019 [hep-ph].

[9] R. N. Lee. "Presenting LiteRed: a tool for the Loop InTEgrals REDuction". In: (Dec. 2012). arXiv: 1212.2685 [hep-ph].

[10] Roman N. Lee. "LiteRed 1.4: a powerful tool for reduction of multiloop integrals". In: *J. Phys. Conf. Ser.* 523 (2014). Ed. by Jianxiong Wang, p. 012059. DOI: 10.1088/1742-6596/523/1/012059. arXiv: 1310.1145 [hep-ph].