

Note of TwoLoopSunriseFeynmanIntegrals.jl

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Abstract

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1 Introduction

We provide a `JULIA`¹ package `TwoLoopSunriseFeynmanIntegrals.jl`² for two-loop sunrise Feynman integrals³, which reads [2, Eq. (2.56)]⁴

$$I_{\text{TSI}}^{(\nu_1, \nu_2, \nu_3)}(d, m_1, m_2, m_3) := \int \frac{\widetilde{dq_1} \widetilde{dq_2}}{(-q_1^2 + m_1^2)^{\nu_1} (-q_2^2 + m_2^2)^{\nu_2} (-q_{12}^2 + m_3^2)^{\nu_3}}, \quad (1)$$

where $d = 4 - 2\varepsilon$ is the spacetime dimension, m_i 's are the masses, ν_i 's are the exponents, and q_i are the loop momenta ($q_{12} \equiv q_1 + q_2$). In this note, we take the loop momentum measure as

$$\widetilde{dq} := \frac{d^d q}{i\pi^{d/2}}. \quad (2)$$

One can easily verify that that this integral is invariant under the permutation of (m_1, ν_1) , (m_2, ν_2) , and (m_3, ν_3) , *e.g.*,

$$I_{\text{TSI}}^{(\nu_1, \nu_2, \nu_3)}(d, m_1, m_2, m_3) \equiv I_{\text{TSI}}^{(\nu_2, \nu_3, \nu_1)}(d, m_2, m_3, m_1). \quad (3)$$

If all masses vanish, the integral vanishes as

$$I_{\text{TSI}}^{(\nu_1, \nu_2, \nu_3)}(d, 0, 0, 0) \equiv 0 \quad (4)$$

since the definition of the Feynman integral in the dimensional regularization [2, Sec. 2.4.2].

This note is organized as follows: In Sec. 2, we reduce the two-loop sunrise Feynman integrals to the master integrals (MIs) via integration-by-part (IBP) techniques. In Sec. 3, the MIs are evaluated and expanded into ε -series. Finally, we conclude in Sec. 4.

2 Integration-by-Part Reduction

In this section, we consider the integration-by-part (IBP) reduction for the two-loop sunrise Feynman integrals. The IBP reduction starts from the fact that [2, Eq. (6.2)]

$$\int \widetilde{dq} \frac{\partial}{\partial q^\mu} [k^\mu \cdots] \equiv 0, \quad (5)$$

where k is an arbitrary momentum. We also have

$$\frac{\partial}{\partial q^\mu} \frac{1}{(-q^2 + m^2)^\nu} = \frac{2\nu q_\mu}{(-q^2 + m^2)^{\nu+1}}, \quad (6)$$

$$\frac{\partial}{\partial q_1^\mu} \frac{1}{(-q_{12}^2 + m_3^2)^{\nu_3}} = \frac{2\nu_3 (q_{12})_\mu}{(-q_{12}^2 + m_3^2)^{\nu_3+1}}. \quad (7)$$

Therefore,

$$\begin{aligned} 0 &= \int \widetilde{dq_1} \widetilde{dq_2} \frac{\partial}{\partial q_1^\mu} \frac{q_1^\mu}{(-q_1^2 + m_1^2)^{\nu_1} (-q_2^2 + m_2^2)^{\nu_2} (-q_{12}^2 + m_3^2)^{\nu_3}} \\ &= (d - 2\nu_1 - \nu_3) I_{\text{TSI}}^{(\nu_1, \nu_2, \nu_3)} + 2\nu_1 m_1^2 I_{\text{TSI}}^{(\nu_1+1, \nu_2, \nu_3)} - \nu_3 \left(I_{\text{TSI}}^{(\nu_1-1, \nu_2, \nu_3+1)} - I_{\text{TSI}}^{(\nu_1, \nu_2-1, \nu_3+1)} \right) \\ &\quad + \nu_3 (m_1^2 - m_2^2 + m_3^2) I_{\text{TSI}}^{(\nu_1, \nu_2, \nu_3+1)}. \end{aligned} \quad (8)$$

Or equivalently,

$$\begin{aligned} -2\nu_1 m_1^2 I_{\text{TSI}}^{(\nu_1+1, \nu_2, \nu_3)} - \nu_3 (m_1^2 - m_2^2 + m_3^2) I_{\text{TSI}}^{(\nu_1, \nu_2, \nu_3+1)} &= (d - 2\nu_1 - \nu_3) I_{\text{TSI}}^{(\nu_1, \nu_2, \nu_3)} \\ &\quad - \nu_3 \left(I_{\text{TSI}}^{(\nu_1-1, \nu_2, \nu_3+1)} - I_{\text{TSI}}^{(\nu_1, \nu_2-1, \nu_3+1)} \right). \end{aligned} \quad (9)$$

Similarly, we have

$$\begin{aligned} -2\nu_2 m_2^2 I_{\text{TSI}}^{(\nu_1, \nu_2+1, \nu_3)} - \nu_3 (m_2^2 - m_1^2 + m_3^2) I_{\text{TSI}}^{(\nu_1, \nu_2, \nu_3+1)} &= (d - 2\nu_2 - \nu_3) I_{\text{TSI}}^{(\nu_1, \nu_2, \nu_3)} \\ &\quad - \nu_3 \left(I_{\text{TSI}}^{(\nu_1, \nu_2-1, \nu_3+1)} - I_{\text{TSI}}^{(\nu_1-1, \nu_2, \nu_3+1)} \right). \end{aligned} \quad (10)$$

¹<https://julialang.org>

²<https://github.com/Fenyutanchan/TwoLoopSunriseFeynmanIntegrals.jl.git>

³We suggest Refs. [1, 2] for pedagogical introduction to Feynman integrals.

⁴For simplicity, the prefactor $e^{2\varepsilon\gamma_E}(\mu^2)^{\nu-d}$ with $\nu \equiv \nu_1 + \nu_2 + \nu_3$ for the *modified minimal subtraction scheme* is omitted here.

Now consider

$$\begin{aligned}
0 &= \int \widetilde{dq_1} \widetilde{dq_2} \frac{\partial}{\partial q_2^\mu} \frac{q_1^\mu}{(-q_1^2 + m_1^2)^{\nu_1} (-q_2^2 + m_2^2)^{\nu_2} (-q_{12}^2 + m_3^2)^{\nu_3}} \\
&= (\nu_2 - \nu_3) I_{\text{TSI}}^{(\nu_1, \nu_2, \nu_3)} + \nu_2 I_{\text{TSI}}^{(\nu_1-1, \nu_2+1, \nu_3)} - \nu_2 I_{\text{TSI}}^{(\nu_1, \nu_2+1, \nu_3-1)} + \nu_2 (-m_1^2 - m_2^2 + m_3^2) I_{\text{TSI}}^{(\nu_1, \nu_2+1, \nu_3)} \\
&\quad + \nu_3 I_{\text{TSI}}^{(\nu_1, \nu_2-1, \nu_3+1)} - \nu_3 I_{\text{TSI}}^{(\nu_1-1, \nu_2, \nu_3+1)} + \nu_3 (m_1^2 - m_2^2 + m_3^2) I_{\text{TSI}}^{(\nu_1, \nu_2, \nu_3+1)},
\end{aligned} \tag{11}$$

or equivalently,

$$\begin{aligned}
&\nu_2 (m_1^2 + m_2^2 - m_3^2) I_{\text{TSI}}^{(\nu_1, \nu_2+1, \nu_3)} - \nu_3 (m_1^2 - m_2^2 + m_3^2) I_{\text{TSI}}^{(\nu_1, \nu_2, \nu_3+1)} \\
&= (\nu_2 - \nu_3) I_{\text{TSI}}^{(\nu_1, \nu_2, \nu_3)} + \nu_2 \left(I_{\text{TSI}}^{(\nu_1-1, \nu_2+1, \nu_3)} - I_{\text{TSI}}^{(\nu_1, \nu_2+1, \nu_3-1)} \right) + \nu_3 \left(I_{\text{TSI}}^{(\nu_1, \nu_2-1, \nu_3+1)} - I_{\text{TSI}}^{(\nu_1-1, \nu_2, \nu_3+1)} \right).
\end{aligned} \tag{12}$$

Combining Eqs. (9), (10), and (12), we have

$$\begin{aligned}
&\begin{pmatrix} -2\nu_1 m_1^2 & 0 & -\nu_3 (m_1^2 - m_2^2 + m_3^2) \\ 0 & -2\nu_2 m_2^2 & -\nu_3 (m_2^2 - m_1^2 + m_3^2) \\ 0 & \nu_2 (m_1^2 + m_2^2 - m_3^2) & -\nu_3 (m_1^2 - m_2^2 + m_3^2) \end{pmatrix} \begin{pmatrix} I_{\text{TSI}}^{(\nu_1+1, \nu_2, \nu_3)} \\ I_{\text{TSI}}^{(\nu_1, \nu_2+1, \nu_3)} \\ I_{\text{TSI}}^{(\nu_1, \nu_2, \nu_3+1)} \end{pmatrix} \\
&= \begin{pmatrix} d - 2\nu_1 - \nu_3 & 0 & \nu_3 & 0 & -\nu_3 \\ d - 2\nu_2 - \nu_3 & 0 & -\nu_3 & 0 & \nu_3 \\ \nu_2 - \nu_3 & -\nu_2 & \nu_3 & \nu_2 & -\nu_3 \end{pmatrix} \begin{pmatrix} I_{\text{TSI}}^{(\nu_1 \nu_2 \nu_3)} \\ I_{\text{TSI}}^{(\nu_1, \nu_2+1, \nu_3-1)} \\ I_{\text{TSI}}^{(\nu_1, \nu_2-1, \nu_3+1)} \\ I_{\text{TSI}}^{(\nu_1-1, \nu_2+1, \nu_3)} \\ I_{\text{TSI}}^{(\nu_1-1, \nu_2, \nu_3+1)} \end{pmatrix}.
\end{aligned} \tag{13}$$

Defining the matrices \mathbf{A} and \mathbf{B} as

$$\mathbf{A} := \begin{pmatrix} -2\nu_1 m_1^2 & 0 & -\nu_3 (m_1^2 - m_2^2 + m_3^2) \\ 0 & -2\nu_2 m_2^2 & -\nu_3 (m_2^2 - m_1^2 + m_3^2) \\ 0 & \nu_2 (m_1^2 + m_2^2 - m_3^2) & -\nu_3 (m_1^2 - m_2^2 + m_3^2) \end{pmatrix}, \tag{14}$$

$$\mathbf{B} := \begin{pmatrix} d - 2\nu_1 - \nu_3 & 0 & \nu_3 & 0 & -\nu_3 \\ d - 2\nu_2 - \nu_3 & 0 & -\nu_3 & 0 & \nu_3 \\ \nu_2 - \nu_3 & -\nu_2 & \nu_3 & \nu_2 & -\nu_3 \end{pmatrix}, \tag{15}$$

the solution is given by

$$\begin{pmatrix} I_{\text{TSI}}^{(\nu_1+1, \nu_2, \nu_3)} \\ I_{\text{TSI}}^{(\nu_1, \nu_2+1, \nu_3)} \\ I_{\text{TSI}}^{(\nu_1, \nu_2, \nu_3+1)} \end{pmatrix} = \mathbf{A}^{-1} \mathbf{B} \begin{pmatrix} I_{\text{TSI}}^{(\nu_1 \nu_2 \nu_3)} \\ I_{\text{TSI}}^{(\nu_1, \nu_2+1, \nu_3-1)} \\ I_{\text{TSI}}^{(\nu_1, \nu_2-1, \nu_3+1)} \\ I_{\text{TSI}}^{(\nu_1-1, \nu_2+1, \nu_3)} \\ I_{\text{TSI}}^{(\nu_1-1, \nu_2, \nu_3+1)} \end{pmatrix} \tag{16}$$

if $\det \mathbf{A} \neq 0$. Notice that the determinant of \mathbf{A} is given by

$$\det \mathbf{A} = 2\nu_1 \nu_2 \nu_3 m_1^2 \lambda(m_1^2, m_2^2, m_3^2), \tag{17}$$

where $\lambda(x, y, z)$ is the Källén triangle function [3, Eq. (6.3)–(6.7)]

$$\begin{aligned}
\lambda(x, y, z) &:= x^2 + y^2 + z^2 - 2xy - 2yz - 2zx \\
&= (\sqrt{x} + \sqrt{y} + \sqrt{z})(\sqrt{x} + \sqrt{y} - \sqrt{z})(\sqrt{x} - \sqrt{y} + \sqrt{z})(-\sqrt{x} + \sqrt{y} + \sqrt{z}).
\end{aligned} \tag{18}$$

Hence, there are two cases — non-collinear and collinear — to be considered.

2.1 Non-Collinear Case

In the non-collinear case, the solution in Eq. (16) is valid, which can be used to eliminate the sum of ν_i 's by one from the left-hand-side (LHS) to the right-hand-side (RHS) of Eq. (16), which is the key to reduce the two-loop sunrise Feynman integrals to the master integrals (MIs). The expression of $\mathbf{A}^{-1} \mathbf{B}$ is too lengthy to be presented here, but it could be reproduced by the WOLFRAM MATHEMATICA⁵ notebook [note/Mathematica_notebooks/IBP_NC.nb](https://www.wolfram.com/mathematica). The generated expressions are stored in the directory `ext/ibp_nc/` for further use.

⁵<https://www.wolfram.com/mathematica>

As Eq. (17) shows,

$$\begin{aligned} \det \mathbf{A} &= 0 \\ \Leftrightarrow \quad \nu_1 &= 0 \vee \nu_2 = 0 \vee \nu_3 = 0 \vee \lambda(m_1^2, m_2^2, m_3^2) = 0. \end{aligned} \quad (19)$$

with $m_1 > 0$ ⁶, $m_2 \geq 0$, and $m_3 \geq 0$. The part of $\lambda(m_1^2, m_2^2, m_3^2) = 0$ is so-called collinear condition since it can be factorized as [Eq. (18)]

$$\lambda(m_1^2, m_2^2, m_3^2) = (m_1 + m_2 + m_3)(m_1 + m_2 - m_3)(m_1 - m_2 + m_3)(-m_1 + m_2 + m_3). \quad (20)$$

For the part of $\nu_1 = 0 \vee \nu_2 = 0 \vee \nu_3 = 0$, there are several cases to be considered.

$\nu_1 = \nu_2 = \nu_3 = 0$. The integrals are trivial as

$$I_{\text{TSI}}^{(000)}(d, m_1, m_2, m_3) = \int \widetilde{dq_1} \widetilde{dq_2} \equiv 0. \quad (21)$$

$\nu_2 = \nu_3 = 0$. The integrals are also trivial as

$$I_{\text{TSI}}^{(00\nu_3)} = \int \frac{\widetilde{dq_1} \widetilde{dq_2}}{(-q_1^2 + m_1^2)^{\nu_1}} = \int \widetilde{dq_2} \int \frac{\widetilde{dq_1}}{(-q_1^2 + m_1^2)^{\nu_1}} \equiv 0. \quad (22)$$

For the cases of $\nu_3 = \nu_1 = 0$ or $\nu_1 = \nu_2 = 0$, the permutation of (m_1, ν_1) , (m_2, ν_2) , and (m_3, ν_3) can be applied to obtain the same results.

$\nu_3 = 0$. The integrals are factorized as

$$\begin{aligned} I_{\text{TSI}}^{(\nu_1\nu_20)} &= \int \frac{\widetilde{dq_1} \widetilde{dq_2}}{(-q_1^2 + m_1^2)^{\nu_1} (-q_2^2 + m_2^2)^{\nu_2}} = \int \frac{\widetilde{dq_1}}{(-q_1^2 + m_1^2)^{\nu_1}} \int \frac{\widetilde{dq_2}}{(-q_2^2 + m_2^2)^{\nu_2}} \\ &\equiv I_1^{(\nu_1)}(d, m_1) I_1^{(\nu_2)}(d, m_2), \end{aligned} \quad (23)$$

where $I_1^{(\nu)}(d, m)$ is evaluated to [1, Eq. (10.1)]

$$I_1^{(\nu)}(d, m) = \frac{\Gamma(\nu - \frac{d}{2})}{\Gamma(\nu)} (m^2)^{\frac{d}{2} - \nu}. \quad (24)$$

For the cases of $\nu_1 = 0$ or $\nu_2 = 0$, the permutation of (m_1, ν_1) , (m_2, ν_2) , and (m_3, ν_3) can be applied to obtain the same results.

The integrals with $\nu_1 = 0 \vee \nu_2 = 0 \vee \nu_3 = 0$ are called the boundary cases. For every integral with $\nu_1 \geq 0 \wedge \nu_2 \geq 0 \wedge \nu_3 \geq 0$, the IBP reduction can be applied to reduce the integrals to the boundary cases and $I_{\text{TSI}}^{(111)}$. Actually, there are only two MIs in the non-collinear case — $I_{\text{TSI}}^{(111)}$ and $I_{\text{TSI}}^{(110)}$. However, the boundary cases are easy to evaluate as Eq. (24) shows, so we do not reduce $I_{\text{TSI}}^{(\nu_1\nu_20)}$ to $I_{\text{TSI}}^{(110)}$ in practice.

2.2 Collinear Case

If $\lambda(m_1^2, m_2^2, m_3^2) = 0$, we cannot apply Eq. (16) to reduce the integrals since \mathbf{A} is singular, *i.e.*, \mathbf{A}^{-1} does not exist. Notice that

$$\lambda(m_1^2, m_2^2, m_3^2) = 0 \quad \Leftrightarrow \quad m_1 + m_2 + m_3 = 0 \vee m_1 = m_2 + m_3 \vee m_2 = m_1 + m_3 \vee m_3 = m_1 + m_2. \quad (25)$$

Since $m_i \geq 0$ and $I_{\text{TSI}}^{(\nu_1\nu_2\nu_3)}(d, 0, 0, 0)$ vanishes as Eq. (4) shows, the first condition is excluded. The other three conditions are called the collinear cases. Without loss of generality, the permutation of (m_1, ν_1) , (m_2, ν_2) , and (m_3, ν_3) can be applied to obtain the case $m_1 = m_2 + m_3$.

From Eq. (2.4) of Ref. [4], we have

$$I_{\text{TSI}}^{(\nu_1, \nu_2, \nu_3)}(d, m_1, m_2, m_3) = \frac{1}{2(d+3-2\nu)m_1m_2m_3} \left\{ \begin{aligned} &[m_1(d+2-\nu) - m_2\nu_3 - m_3\nu_2] I_{\text{TSI}}^{(\nu_1-1, \nu_2, \nu_3)} \\ &+ [m_1\nu_3 - m_2(d+2-\nu) - m_3\nu_1] I_{\text{TSI}}^{(\nu_1, \nu_2-1, \nu_3)} \\ &+ [m_1\nu_2 - m_2\nu_1 - m_3(d+2-\nu)] I_{\text{TSI}}^{(\nu_1, \nu_2, \nu_3-1)} \end{aligned} \right\}, \quad (26)$$

⁶If $m_1 = 0$, we can just apply the permutation of (m_1, ν_1) , (m_2, ν_2) , and (m_3, ν_3) to satisfy the condition except for the case of $m_1 = m_2 = m_3 = 0$. However the case of $m_1 = m_2 = m_3 = 0$ is evaluated to 0 as Eq. (4) shows.

where $m_1 = m_2 + m_3$ and $\nu = \nu_1 + \nu_2 + \nu_3$. We can apply this formula to reduce the collinear cases to the boundary cases, *i.e.*, $\nu_i = 0$ for any $i = 1, 2, 3$. The boundary cases of any $\nu_i = 0$ are the same as Eqs. (21), (22), and (23). We also perform the ε -expansion in [note/Mathematica.notebooks/IBP-CL.nb](#), and the results are stored in the directory [ext/ibp-cl/](#) for further use.

Different from the non-collinear case, $I_{\text{TSI}}^{(111)}$ is no longer the boundary case (or MIs) that needs to be evaluated independently. Instead, we have [4, Eq. (2.14)]

$$I_{\text{TSI}}^{(111)}(d, m_1, m_2, m_3) = \frac{d-2}{2(d-3)} \left(\frac{I_{\text{TSI}}^{(011)}}{m_2 m_3} - \frac{I_{\text{TSI}}^{(101)}}{m_3 m_1} - \frac{I_{\text{TSI}}^{(110)}}{m_1 m_2} \right) \quad (27)$$

for $m_1 = m_2 + m_3$.

3 Master Integrals

In this section, we evaluate the MIs and expand them into ε -series.

3.1 Factorization as Products of One-Loop Integrals

As Eq. (23) shows, the factorization of $I_{\text{TSI}}^{(\nu_1 \nu_2 0)} = I_1^{(\nu_1)} I_1^{(\nu_2)}$ makes the evaluation of $I_{\text{TSI}}^{(\nu_1 \nu_2 0)}$ easier. The expansion of $I_1^{(\nu)}(d, m)$ into ε -series is easily obtained as

$$I_1^{(1)}(d, m) = -\frac{m^2}{\varepsilon} - m^2(1 - \gamma_E - \ln m^2) + \mathcal{O}(\varepsilon^1), \quad (28)$$

$$I_1^{(2)}(d, m) = \frac{1}{\varepsilon} - (\gamma_E + \ln m^2) + \mathcal{O}(\varepsilon^1), \quad (29)$$

$$I_1^{(\nu > 2)}(d, m) = \frac{(m^2)^{2-\nu}}{(\nu-1)(\nu-2)} + \mathcal{O}(\varepsilon^1). \quad (30)$$

These expansions are evaluated in [note/Mathematica.notebooks/I.1-loop.nb](#) and stored in the directory [ext/one.loop/](#) for further use.

3.2 Non-Collinear $I_{\text{TSI}}^{(111)}$

All non-vanishing masses. For the master integral $I_{\text{TSI}}^{(111)}$, Eq. (4.9) of Ref. [5] shows

$$I_{\text{TSI}}^{(111)}(d, m_1, m_2, m_3) = \frac{1}{2} (m_1^2)^{1-2\varepsilon} A(\varepsilon) \left[-\frac{1}{\varepsilon^2} (1+x+y) + \frac{2}{\varepsilon} (x \ln x + y \ln y) - x \ln^2 x - y \ln^2 y \right. \\ \left. + (1-x-y) \ln x \ln y - \lambda(1, x, y) \Phi(x, y) \right] \quad (31)$$

where $m_1, m_2, m_3 > 0$,

$$\begin{cases} x := \frac{m_2^2}{m_1^2}, \\ y := \frac{m_3^2}{m_1^2}, \end{cases} \quad (32)$$

with $0 \leq x, y \leq 1$, and

$$A(\varepsilon) := \frac{\Gamma^2(1+\varepsilon)}{(1-\varepsilon)(1-2\varepsilon)}. \quad (33)$$

1. For $\lambda(1, x, y) > 0$, we have $\sqrt{x} + \sqrt{y} < 1$ and

$$\Phi(x, y) := \frac{1}{\sqrt{\lambda(1, x, y)}} \left[2 \ln \left(\frac{1+x-y-\sqrt{\lambda(1, x, y)}}{2} \right) \ln \left(\frac{1-x+y-\sqrt{\lambda(1, x, y)}}{2} \right) \right. \\ \left. - 2 \text{Li}_2 \left(\frac{1-x+y-\sqrt{\lambda(1, x, y)}}{2} \right) - 2 \text{Li}_2 \left(\frac{1+x-y-\sqrt{\lambda(1, x, y)}}{2} \right) \right. \\ \left. - \ln x \ln y + \frac{\pi^2}{3} \right]; \quad (34)$$

2. For $\lambda(1, x, y) < 0$, we have $\sqrt{x} + \sqrt{y} > 1$ and

$$\Phi(x, y) := \frac{2}{\sqrt{-\lambda(1, x, y)}} \left[\text{Cl}_2 \left(2 \arccos \frac{1+x-y}{2\sqrt{x}} \right) + \text{Cl}_2 \left(2 \arccos \frac{1-x+y}{2\sqrt{y}} \right) + \text{Cl}_2 \left(2 \arccos \frac{-1+x+y}{2\sqrt{xy}} \right) \right] \quad (35)$$

The ε -series expansions of $I_{\text{TSI}}^{(111)}$ are evaluated in `note/Mathematica_notebooks/TSI111_NC.nb` and stored in the directory `ext/TSI111_nc/` for further use.

One vanishing mass. For the case of $m_1 > m_2 > m_3 = 0$, Eq. (31) in Ref. [6] gives that

$$\begin{aligned} I_{\text{TSI}}^{(111)}(d, m_1, m_2, 0) &= \frac{\Gamma(3 - \frac{d}{2})\Gamma(\frac{d}{2} - 1)\Gamma^2(2 - \frac{d}{2})}{\Gamma^2(1)\Gamma(\frac{d}{2})\Gamma(4 - d)(m_1^2)^{3-d}} {}_2F_1 \left(\begin{matrix} 3 - d, 2 - \frac{d}{2} \\ 4 - d \end{matrix} \middle| 1 - \frac{m_2^2}{m_1^2} \right) \\ &= \frac{m_1^2 + m_2^2}{\varepsilon} + (1 - \gamma_E)(m_1^2 + m_2^2) - 2(m_1^2 \ln m_1^2 + m_2^2 \ln m_2^2) + \mathcal{O}(\varepsilon), \end{aligned} \quad (36)$$

which is evaluated in `note/Mathematica_notebooks/TSI111_NC_one_vanishing.nb` and stored in the directory `ext/TSI111_nc/` for further use.

4 Conclusion

References

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