Amplitude Calculation for Inflaton, Reheaton, and Graviton

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Abstract

In this Git repository [1], the amplitude calculations for the inflaton, reheaton, and graviton are presented, including the reproducible WOLFRAM MATHEMATICA notebooks[1] in the directory `Mathematica_notebooks/` and the technical details in this note. This Git repository is the supplementary material for the paper TBD: arXiv:2503:xxxxx. If it is helpful for your research, please cite the paper[2].

Contents

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[1]All of the WOLFRAM MATHEMATICA notebooks are licensed under the MIT License.

[2]It would be greatly appreciated if the reader cite this Git repository explicitly as well (see `README.md` for the citation information).

1 Introduction

This note is inspired by Refs. [2] and [3]. We consider the inflaton ϕ, the reheaton φ as two real scalar fields, and separate the spacetime metric into two parts as [2, Eq. (2.13)]

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}(x), \tag{1}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric, $\kappa = \sqrt{8\pi G_{\mathrm{N}}}$ is the perturbation parameter, and $h_{\mu\nu}(x)$ is a metric fluctuation such that $\kappa|h_{\mu\nu}| \ll 1$.

Important Note: This note adopts the convention of $(+++)$ instead of $(--+)$ in TBD: arXiv:2503:xxxxx. Please check Appendix A for details.

2 Lagrangians and Feynman Rules

The corresponding action is given by

$$S = \int \mathrm{d}^4x \, \sqrt{-g} \left[\frac{R}{2\kappa^2} + \mathcal{L}_\phi + \mathcal{L}_\varphi + \mathcal{L}_{\phi\varphi^2} \right], \tag{2}$$

where R is the Ricci scalar, and the Lagrangians are given by

$$\mathcal{L}_\phi = -\frac{1}{2} g^{\mu\nu} (\nabla_\mu \phi) \nabla_\nu \phi - \frac{1}{2} m_\phi^2 \phi^2, \tag{3}$$

$$\mathcal{L}_\varphi = -\frac{1}{2} g^{\mu\nu} (\nabla_\mu \varphi) \nabla_\nu \varphi - \frac{1}{2} m_\varphi^2 \varphi^2, \tag{4}$$

$$\mathcal{L}_{\phi\varphi^2} = \frac{\lambda}{2!} \phi \varphi^2. \tag{5}$$

Usually, the inflaton ϕ is considered heavier than the reheaton φ, *i.e.*, $m_\phi > m_\varphi$.

2.1 Vertices

All of the vertices are derived in `Mathematica_notebooks/Feynman-rules-for-Vertices-via-xAct.nb`, which is the reproducible Wolfram Mathematica notebook. The xAct bundle[3] (including xPert [4] and xTras [5]) is applied to derive the Feynman rules for vertices.

2.1.1 Pure Graviton Vertex

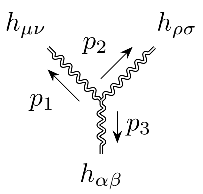
For the pure graviton vertex, the Feynman rule could be derived the Einstein-Hilbert action of

$$S_{\mathrm{EH}} = \frac{1}{2\kappa^2} \int \mathrm{d}^4x \, \sqrt{-g} R, \tag{6}$$

In this note, we only derive the triple-graviton vertex, where we use the package xPert to expand the perturbation of the Einstein-Hilbert action in $\mathcal{O}(\kappa)$ as[4]

```
ExpandPerturbation[Perturbed[LEH, 3] - Perturbed[LEH, 2]]
```

with `LEH` $= \sqrt{-g}\mathcal{L}_{EH} = \sqrt{-g}R/(2\kappa^2)$. After some manipulations, we obtain the Feynman rule for the triple-graviton vertex as



$$= \mathrm{i} V_{(\mu\nu)(\rho\sigma)(\alpha\beta)}(p_1, p_2, p_3). \tag{7}$$

The corresponding expression is too lengthy to be presented here, but it could be found in the reproducible Mathematica notebook `Mathematica_notebooks/Feynman-rules-for-Vertices-via-xAct.nb`.

[3] `https://www.xAct.es`

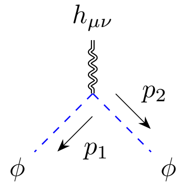
[4] Notice that there are κ^2 terms in the denominator of the Lagrangian.

2.1.2 Vertices of ϕ-ϕ-h and φ-φ-h

For the vertices of ϕ-ϕ-h and φ-φ-h, the action of

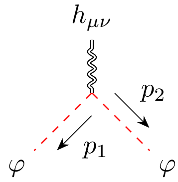
$$S_{\phi+\varphi} = \int \mathrm{d}^4x \sqrt{-g}[\mathcal{L}_\phi + \mathcal{L}_\varphi] \tag{8}$$

is considered. Same as the pure graviton vertex, the Feynman rules for the vertices of ϕ-ϕ-h and φ-φ-h are derived via the package xPERT, which are given by



$$= -\mathrm{i}\kappa\big[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \eta^{\mu\nu}\big(p_1 \cdot p_2 - m_\phi^2\big)\big], \tag{9}$$

and



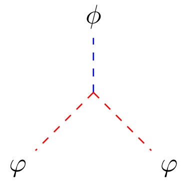
$$= -\mathrm{i}\kappa\big[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \eta^{\mu\nu}\big(p_1 \cdot p_2 - m_\varphi^2\big)\big]. \tag{10}$$

2.1.3 Vertices of ϕ-φ-φ(-h)

For the vertices of ϕ-φ-φ(-h), the action of

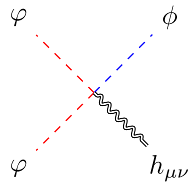
$$S_{\phi\varphi^2} = \int \mathrm{d}^4x \sqrt{-g}\mathcal{L}_{\phi\varphi^2} \tag{11}$$

is considered. Same as the previous vertices, the Feynman rules for the vertices of ϕ-φ-φ(-h) are derived via the package xPERT, which are given by



$$= \mathrm{i}\lambda, \tag{12}$$

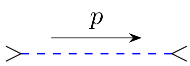
and



$$= \mathrm{i}\kappa\lambda\eta_{\mu\nu}. \tag{13}$$

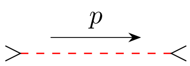
2.2 Propagators and External Legs

In this note, we do not consider the derivation of the Feynman rules for the propagators and external legs. For the scalar fields, the propagators are simply given by [6, Sec. 10]



$$= \frac{-\mathrm{i}}{p^2 + m_\phi^2 - \mathrm{i}0^+}, \tag{14}$$

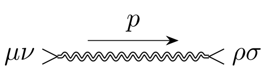
and



$$= \frac{-\mathrm{i}}{p^2 + m_\varphi^2 - \mathrm{i}0^+}. \tag{15}$$

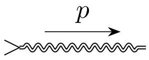
The external legs for scalar fields are always 1 [6, Sec. 10].

For the graviton, the propagator in the *Feynman gauge* is given by [2, Eq. (2.81)]



$$= \frac{1}{2}\frac{-\mathrm{i}}{p^2 - \mathrm{i}0^+}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}). \tag{16}$$

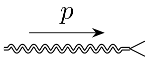
The external legs for the graviton are given by



$$\mu\nu \quad = \varepsilon_{\mu\nu}(\mathbf{p}, \pm 2), \tag{17}$$

and

$$\mu\nu$$



$$= \varepsilon^*_{\mu\nu}(\mathbf{p}, \pm 2), \tag{18}$$

where $\varepsilon_{\mu\nu}(\mathbf{p}, \pm 2)$ and $\varepsilon^*_{\mu\nu}(\mathbf{p}, \pm 2)$ are the polarization tensors of the graviton (± 2 for helicities). We do not discuss the details of the polarization tensors here but refer to Ref. [2, Sec. 2.2] for details. However, we are interested in the summation over the helicities as [7, Eq. (A.6)]

$$\sum_{h=\pm 2} \varepsilon_{\mu\nu}(\mathbf{p}, h)\varepsilon^*_{\rho\sigma}(\mathbf{p}, h) = \frac{1}{2}(\bar{\eta}_{\mu\rho}\bar{\eta}_{\nu\sigma} + \bar{\eta}_{\mu\sigma}\bar{\eta}_{\nu\rho} - \bar{\eta}_{\mu\nu}\bar{\eta}_{\rho\sigma}), \tag{19}$$

where

$$\bar{\eta}_{\mu\nu} = \eta_{\mu\nu} - \frac{p_\mu r_\nu + r_\mu p_\nu}{p \cdot r} \tag{20}$$

with r^μ being an arbitrary null vector for reference.

3 Amplitudes

With the Feynman rules we have derived, we could calculate the amplitudes for the processes of interest, where the package FeynCalc [8–11] is applied.

Important Note: In FeynCalc, the metric signature is -2 instead of $+2$. The Feynman rules derived in the previous section should be modified accordingly:

```
outputRule = {
    ph1[a_] -> FV[ph1, a], ph2[a_] -> FV[ph2, a],
    ph3[a_] -> FV[ph3, a], p\[Phi]1[a_] -> FV[p\[Phi]1, a],
    p\[Phi]2[a_] -> FV[p\[Phi]2, a],
    p\[CurlyPhi]1[a_] -> FV[p\[CurlyPhi]1, a],
    p\[CurlyPhi]2[a_] -> FV[p\[CurlyPhi]2, a], \[Eta][a_, b_] -> -MT[a, b],
    FV[p1_, a_] FV[p2_, -a_] -> -SP[p1, p2]
} // HoldForm
```

A Conventions

Here, we list the conventions used in this note.

A.1 Spacetime: Metric and Curvature

We use the Greek indices $\mu, \nu, \ldots$ for spacetime indices and the Latin indices $i, j, \ldots$ for spatial indices. For D-dimensional spacetime, we have the spacetime indices run from 0 to $D-1$, and the spatial indices run from 1 to $D-1$, where 0 is the time index and other indices are spatial indices. Usually, we consider $D=4$ spacetime dimensions.

In four-dimensional spacetime, the metric signature is chosen to be $+2$, *e.g.*,

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \tag{A.1}$$

for the Minkowski metric.

The covariant and contravariant vectors in spacetime are denoted as A_μ and A^μ, respectively. They are related by the metric $g_{\mu\nu}$ as

$$A_\mu \equiv g_{\mu\nu} A^\nu \quad \text{and} \quad A^\mu \equiv g^{\mu\nu} A_\nu, \tag{A.2}$$

where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, *i.e.*,

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu{}_\rho \quad \text{and} \quad g_{\mu\nu} g^{\nu\rho} = \delta_\mu{}^\rho. \tag{A.3}$$

Here we adopt the Einstein summation convention, where indices appearing twice with one as a superscript and the other as a subscript are summed over. Without specification, the Einstein summation convention is applied throughout this note.

The Riemman curvature tensor for the spacetime is defined as [12, Eq. (3.4)]

$$R^\rho{}_{\sigma\mu\nu} := \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}, \tag{A.4}$$

where $\Gamma^\rho{}_{\mu\nu}$ is the Christoffel symbol defined as [12, Eq. (3.1)]

$$\Gamma^\rho{}_{\mu\nu} := \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}). \tag{A.5}$$

The Ricci tensor and scalar curvature are then given by [12, Eqs. (3.144) and (3.146)]

$$R_{\mu\nu} := R^\rho{}_{\mu\rho\nu}, \tag{A.6}$$

$$R := g^{\mu\nu} R_{\mu\nu}. \tag{A.7}$$

The Einstein tensor is defined as [12, Eq. (3.151)]

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{A.8}$$

and the Einstein field equation is then [12, Eq. (4.100)]

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{A.9}$$

where Λ is the cosmological constant, $\kappa^2 = 8\pi G_\mathrm{N}$ [2, Eq. (2.1)] with $G_\mathrm{N} = 6.70883(15) \times 10^{-39}$ GeV^{-2} being the Newton's constant [13], and $T_{\mu\nu}$ is the energy-momentum tensor.

This convention is also adopted in MTW's *Gravitation* [14], which is known as $(+++)$ convention.

A.2 Natural Units and (reduced) Planck Mass

The natural unit systems are given by

$$c = \hbar = k_\mathrm{B} = \varepsilon_0 = 1, \tag{A.10}$$

where c is the speed of light in vacuum, $\hbar$ is the reduced Planck constant, k_B is the Boltzmann constant, and ε_0 is the vacuum permittivity. We have the vacuum magnetic permeability $\mu_0 = 1$ as well in the natural unit systems since $c^2 = (\varepsilon_0 \mu_0)^{-1}$.

The Planck mass is defined as

$$m_\mathrm{Pl} := \frac{1}{\sqrt{G_\mathrm{N}}}, \tag{A.11}$$

In this note, the reduced Planck mass is widely used, which is defined as

$$M_\mathrm{Pl} := \frac{1}{\sqrt{8\pi G_\mathrm{N}}} \equiv \frac{m_\mathrm{Pl}}{\sqrt{8\pi}}. \tag{A.12}$$

A.3 Kinematics

The four-momentum p is defined as $p^\mu = (E, \mathbf{p})$, where E is the energy and $\mathbf{p}$ is the spatial momentum. The on-shell condition is given by $p^2 = -m^2$ for metric signature $+2$.

In the $n \to m$ scattering process $p_1 + p_2 + \cdots + p_n \to k_1 + k_2 + \cdots + k_m$ with p_i's and k_j's the four-momenta of the incoming and outgoing particles, respectively, the four-momentum conservation is given by

$$p_1 + p_2 + \cdots + p_n = k_1 + k_2 + \cdots + k_m. \tag{A.13}$$

We denote the masses of the particles as m_i's for p_i's and $\mathfrak{m}_j$'s for k_j's, *i.e.*,

$$\begin{cases} p_i^2 = -m_i^2, \\ k_j^2 = -\mathfrak{m}_j^2. \end{cases} \tag{A.14}$$

The Lorentz-invairant phase space element for the four-momentum p is defined as

$$\widetilde{\mathrm{d}p} := \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E}, \tag{A.15}$$

where $p = (E, \mathbf{p})$ and $p^2 = -E^2 + \mathbf{p}^2 = -m^2$.

For the $2 \to 2$ scattering process $p_1 + p_2 \to k_1 + k_2$, the Mandelstam variables are defined as

$$\begin{aligned} s &:= -(p_1 + p_2)^2 \equiv -(k_1 + k_2)^2, \\ t &:= -(p_1 - k_1)^2 \equiv -(p_2 - k_2)^2, \\ u &:= -(p_1 - k_2)^2 \equiv -(p_2 - k_1)^2. \end{aligned} \tag{A.16}$$

For the $1 \to 3$ scattering process $p_1 \to k_1 + k_2 + k_3$, the Mandelstam variables are defined as

$$\begin{aligned} s &:= -(p_1 - k_3)^2 \equiv -(k_1 + k_2)^2, \\ t &:= -(p_1 - k_1)^2 \equiv -(k_2 + k_3)^2, \\ u &:= -(p_1 - k_2)^2 \equiv -(k_1 + k_3)^2. \end{aligned} \tag{A.17}$$

References

[1] Quan-feng Wu. *Amplitude Calculation of Inflaton, Reheaton, and Graviton.* 2025. URL: https://github.com/Fenyutanchan/amplitude-for-inflaton-reheaton-graviton.git.

[2] Ivano Basile et al. *Lectures in Quantum Gravity.* Dec. 2024. arXiv: 2412.08690 [hep-th].

[3] Anna Tokareva. *Graviton Scattering Amplitudes on xAct.* 2024. URL: https://github.com/tokareva90/Graviton-scattering-amplitudes-on-xAct.

[4] David Brizuela, Jose M. Martin-Garcia, and Guillermo A. Mena Marugan. "xPert: Computer algebra for metric perturbation theory". In: *Gen. Rel. Grav.* 41 (2009), pp. 2415–2431. DOI: 10.1007/s10714-009-0773-2. arXiv: 0807.0824 [gr-qc].

[5] Teake Nutma. "xTras : A field-theory inspired xAct package for mathematica". In: *Comput. Phys. Commun.* 185 (2014), pp. 1719–1738. DOI: 10.1016/j.cpc.2014.02.006. arXiv: 1308.3493 [cs.SC].

[6] M. Srednicki. *Quantum field theory.* Cambridge University Press, Jan. 2007. ISBN: 978-0-521-86449-7, 978-0-511-26720-8. DOI: 10.1017/CBO9780511813917.

[7] Basabendu Barman et al. "Gravitational wave from graviton Bremsstrahlung during reheating". In: *JCAP* 05 (2023), p. 019. DOI: 10.1088/1475-7516/2023/05/019. arXiv: 2301.11345 [hep-ph].

[8] R. Mertig, M. Bohm, and Ansgar Denner. "FEYN CALC: Computer algebraic calculation of Feynman amplitudes". In: *Comput. Phys. Commun.* 64 (1991), pp. 345–359. DOI: 10.1016/0010-4655(91)90130-D.

[9] Vladyslav Shtabovenko, Rolf Mertig, and Frederik Orellana. "New Developments in FeynCalc 9.0". In: *Comput. Phys. Commun.* 207 (2016), pp. 432–444. DOI: 10.1016/j.cpc.2016.06.008. arXiv: 1601.01167 [hep-ph].

[10] Vladyslav Shtabovenko, Rolf Mertig, and Frederik Orellana. "FeynCalc 9.3: New features and improvements". In: *Comput. Phys. Commun.* 256 (2020), p. 107478. DOI: 10.1016/j.cpc.2020.107478. arXiv: 2001.04407 [hep-ph].

[11] Vladyslav Shtabovenko, Rolf Mertig, and Frederik Orellana. "FeynCalc 10: Do multiloop integrals dream of computer codes?" In: *Comput. Phys. Commun.* 306 (2025), p. 109357. DOI: 10.1016/j.cpc.2024.109357. arXiv: 2312.14089 [hep-ph].

[12] Sean M. Carroll. *Spacetime and Geometry: An Introduction to General Relativity.* Cambridge University Press, July 2019. ISBN: 978-0-8053-8732-2, 978-1-108-48839-6, 978-1-108-77555-7. DOI: 10.1017/9781108770385.

[13] S. Navas et al. "Review of particle physics". In: *Phys. Rev. D* 110.3 (2024), p. 030001. DOI: 10.1103/PhysRevD.110.030001.

[14] Charles W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation.* San Francisco: W. H. Freeman, 1973. ISBN: 978-0-7167-0344-0, 978-0-691-17779-3.