Advanced Computational Statistics (STAT 7730)
Homework 5

This homework is due by **11:30 AM on Friday, October 18**. Be sure to explain your reasoning in your solutions, label your figures and attach the code at the end. When you submit your homework, you must bring it (hard copy) to our class.

1. (8 pts) Consider the pmf $p(x)$ defined by $p(x) = c\sqrt{x}\mathbb{1}\big(x \in \{1, 2, \ldots, 50\}\big)$.

   (a) What is the numerical value of c?

   (b) For $n = 10^3$, generate $X_1, \ldots, X_n$ iid with common pmf p. Let $\hat{p}(x) = \#\{k : X_k = x\}/n$ be the *empirical pmf* of $X_1, \ldots, X_n$. Make a graph comparing $p(x)$ and $\hat{p}(x)$. Repeat with $n = 10^6$. Interpret your results.

2. (8 pts) Consider the pdf

$$f(x) = \frac{1}{2\sqrt{x}}\mathbb{1}(0 < x < 1)$$

   (a) How would you convert a uniform$(0, 1)$ random variable U into a random variable X that had pdf f?

   (b) Generate a sample of size 10^3 from f and compare a plot of f to a histogram of your samples. Repeat with a sample of size 10^6.

3. (8 pts) Consider the function

$$g(x) = \big(\sin(10x)\big)^2 \big|x^3 + 2x - 3\big| \mathbb{1}\big(x \in (-1, 0) \cup (1/2, 1)\big)$$

   and the pdf $f(x) = \frac{1}{c}g(x)$ that is the same shape as g. Use rejection sampling to generate a sample of size 10^3 from f and compare a plot of g to a histogram of your samples. Repeat with a sample of size 10^6.

4. (25 pts) This problem is about how to generate uniform random points on a sphere in $\mathbb{R}^d$ or on the intersection of a sphere and a hyperplane in $\mathbb{R}^d$. Let us start with the definition of rotation invarinace. A random vector $\boldsymbol{Y} = (Y_1, Y_2, \cdots, Y_n)^{\mathrm{T}} \in \mathbb{R}^n$ is rotation invariant if $\boldsymbol{Y}$ and $\boldsymbol{RY}$ have the same distribution for any rotation matrix $\boldsymbol{R}$. Note that a rotation matrix is a square matrix whose column vectors are orthonormal, $\boldsymbol{u}^{\mathrm{T}}$ is the transpose of $\boldsymbol{u}$, and the vectors in this problem are column vectors. Let $||\boldsymbol{u}||$ be the L_2 norm of $\boldsymbol{u}$.

   (a) Suppose that $\boldsymbol{Y}$ is a random vector with a pdf f that has the property that $f(\boldsymbol{y}) = g(\|\boldsymbol{y}\|)$ for all $\boldsymbol{y} \in \mathbb{R}^n$ and some function g. Prove that $\boldsymbol{Y}$ is rotation invariant.

   (b) Let $\boldsymbol{Y} = (Y_1, \ldots, Y_n)^{\mathrm{T}}$ be i.i.d. $N(0, 1)$ (standard normal distribution). Show that we can generate a random vector uniformly distributed over $S_n(r)$ by

$$\boldsymbol{X} = \frac{r}{\|\boldsymbol{Y}\|}\boldsymbol{Y}, \tag{1}$$

   where $S_n(r)$ is the surface of the sphere of radius $r > 0$ in $\mathbb{R}^n$

$$S_n(r) \triangleq \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| = r\}.$$

   (Hint: Since the rotations preserve surface area on the sphere, a rotation invariant random vector in $S_n(r)$ must be uniformly distributed over $S_n(r)$.)

(c) In problem (b), if $\boldsymbol{Y}$ is any rotation invariant random vector with $\mathbb{P}(||Y|| = 0) = 0$, is $\boldsymbol{X}$ still uniformly distributed over $S_n(r)$ for each $r > 0$? Prove or disprove the statement.

(d) Let $\boldsymbol{X}^{(n)} = (X_1^{(n)}, \ldots, X_n^{(n)})^{\mathsf{T}}$ be uniformly distributed on $S_n(1)$. Prove that $\sqrt{n}X_1^{(n)} \to \mathrm{N}(0, 1)$ in distribution as $n \to \infty$. (Hint: Use the representation in (b).)

(e) Equation (1) provides a transformation to generate a random vector $\boldsymbol{X}$ uniformly distributed over $S_n(r)$. Construct a transformation to generate a random verctor (say $\tilde{\boldsymbol{X}}$) uniformly distributed over $S_n(r) \cap E$ where $E \triangleq \{\boldsymbol{x} \in \mathbb{R}^n :< \boldsymbol{x}, \boldsymbol{t} >= 0\}$ and $\boldsymbol{t} \in \mathbb{R}^n$ is a fixed nonzero vector.