Advanced Computational Statistics (STAT 7730)
Homework 1

This homework is due by **11:30 AM on Friday, September 6**. Be sure to provide the details, explain the reasoning in your solutions and attach the code at the end. When you submit your homework, you must bring it (hard copy) to our class.

1. (6 pts) I showed in the first class that the following two statements are false in R.

$$(.3 - .1) == .2$$

$$(.3 - .1)/.2 == (.3/.2 - .1/.2)$$

For each of them, specify the reason for being false and the sources of errors.

2. (6 pts) Let $X \in A \equiv \{x \in \mathbb{Z} : 60 \leq x \leq 100\}$ be a discrete random variable with a probability mass function

$$p_T(x) = \frac{1}{Z} e^{-\frac{1}{T}\sqrt{x}},$$

where parameter $T > 0$ represents the temperature in Kelvin. This problem studies the distribution when the temperature is close to absolute zero.

(1) Let $T = 0.1$. Compute $\max_{x \in A} p_T(x)$.

(2) Let $T = 0.01$. Compute $\max_{x \in A} p_T(x)$.

3. (10 pts) Consider the equation $Ax = b$ where A is an n-by-n invertible matrix and x, b are n-by-1 vectors. Let the exact solution be $\hat{x} = A^{-1}b \neq \mathbf{0}_n$. Assume, due to the errors from the floating point representation, computers solve the following equation instead

$$(A + \epsilon\eta)x = b + \epsilon\xi$$

where η is an n-by-n matrix, and ξ is an n-by-1 vector. Let the solution be $x(\epsilon)$ (note that $x(0) = \hat{x}$ the ideal answer). Then show that

$$\frac{||x(\epsilon) - \hat{x}||}{||\hat{x}||} \leq \epsilon C(A) \left(\frac{||\xi||}{||b||} + \frac{||\eta||}{||A||} \right) + O(\epsilon^2)$$

where $C(A) = ||A^{-1}|| \cdot ||A||$, and $|| \cdot || = || \cdot ||_2$ for both vectors and matrices. (Note that $C(A)$ is sometimes called the condition number associated with the liner equation $Ax = b$.)

4. (10 pts) In my lecture, we discussed solving the least squares problem

$$\hat{\beta} = \underset{b}{\operatorname{argmin}} ||Y - Xb||^2$$

by QR factorization when the rank of $X \in \mathbb{R}^{m \times n}$, rank$(X)$ is equal to n, the number of parameters β_i's. Re-derive the formula under the case when rank$(X) < n$. Do you always have infinitely many solutions when rank$(X) < n$?

5. (10 pts) The optimization problem for the alignment between shapes discussed in class can be formulated as follows:

$$\underset{O \in R^{m \times m}: orthogonal}{\operatorname{argmin}} ||OX - Y||_F,$$

where X and Y are m-by-n matrices. Show that

(1) $\operatorname{argmin}_{O \in R^{m \times m}: orthogonal} ||OX - Y||_F = \operatorname{argmin}_{O \in R^{m \times m}: orthogonal} ||O - YX'||_F$

(2) $\operatorname{argmin}_{O \in R^{m \times m}: orthogonal} ||O - YX'||_F = UV'$, where U and V are orthogonal matrices in the SVD of YX' (i.e. $YX' = U\Sigma V'$).