NP-completeness

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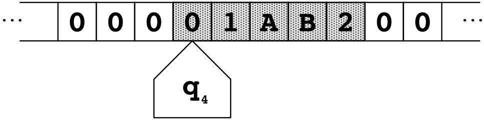
January 29, 2020

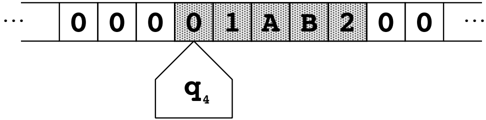
Alan Turing
(1912-1952)



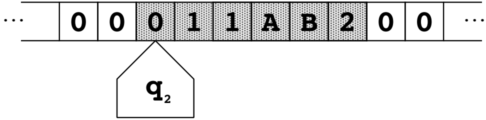
Turing machine (TM)

- TM = (theoretical) device that operates on an infinite tape with cells containing symbols in a finite alphabet (including blank '0')
- The TM has a current state S_i among a finite set of states (including '*Halt*'), and a head pointing to "current" cell in the tape.
- Its transition function describes jumps from state to next state.

- Example: with scanned symbol 0 and state q_4, write 1, move *Left* and go to state q_2. That is:



$$t(0, q_4) = (1, \mathit{Left}, q_2)$$



Decision problems

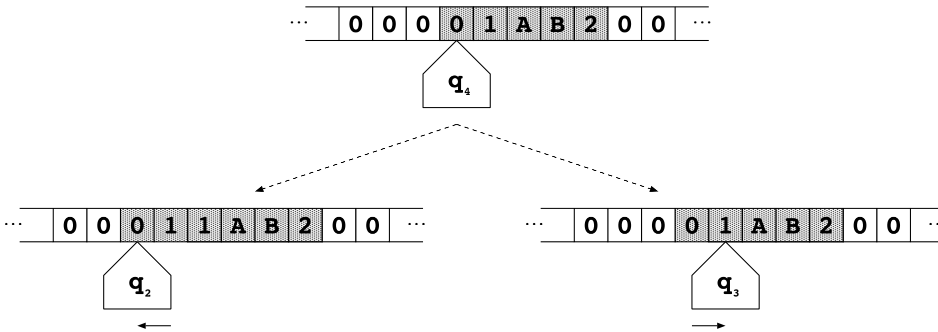
Definition (Decision problem)

A decision problem consists in providing a given tape input and asking the TM for a final output symbol answering *Yes* or *No*.

- Example: *SAT* = given (an encoding of) a propositional formula, does it have at least one model?
- Example: *HALTING* = given another TM plus its input, does it stop or not?
- A decision problem is decidable if the TM stops (answering *Yes* or *No*) in a finite number of steps.
- Examples: *SAT* is decidable. *HALTING* is undecidable.
- A decision problem is in complexity class **P** iff the number of steps carried out by the TM is polynomial on the size n of the input.

Non-deterministic TM

- Now, a non-deterministic Turing Machine (NDTM) is such that the transition function is replaced by a transition relation.
- We may have different possibilities for the next step.
- Example: $t(0, q_4, 1, Left, q_2)$, $t(0, q_4, 0, Right, q_3)$



Non-deterministic TM

- Keypoint: an NDTM provides an affirmative answer to a decision problem when at least one of the executions for the same input answers *Yes*.
- A decision problem is in class **NP** iff the number of steps carried out by the NDTM is polynomial on the size n of the input.
- For *SAT*, we can build an NDTM that performs two steps:
  1. For each atom, generate 1 or 0 nondeterministically. This provides an arbitrary interpretation in linear time.
  2. Test whether the current interpretation is a model or not. Complexity: **ALOGTIME** $\subseteq$ **P**

  The sequence of these two steps takes polynomial time.

P vs NP

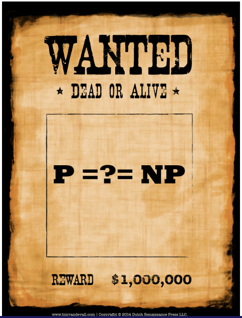
- Any TM is a particular type of NDTM, so **P** $\subseteq$ **NP** trivially, but ...

$$\boxed{\mathbf{P} \stackrel{?}{=} \mathbf{NP}}$$

- Unsolved problem: most accepted conjecture **P** $\subset$ **NP**, but remains unproved.

It is one of the 7 Millenium Prize Problems

`http://www.claymath.org/millennium-problems`



The Clay Mathematics Institute designated $1 million prize for its solution!

Completeness

- A problem *X* is **C**-complete, for some complexity class **C**, iff any problem *Y* in **C** is reducible to *X* in polynomial-time.
- A complete problem is a representative of the class. Example: if an **NP**-complete problem happened to be in **P** then **P** $=$ **NP**.
- *SAT* was the first problem to be identified as **NP**-complete (Cook's theorem, 1971).
- *SAT* is commonly used nowadays for showing that a problem *X* is at least as complex as **NP**. To this aim, just encode *SAT* into *X*.
- The Complexity Zoo
  `https://complexityzoo.uwaterloo.ca/Complexity_Zoo`

Exercise: Turing machine in Prolog

- We use `tape(Ls,S,Rs)` to represent the current symbol `S`, the left fragment of the tape `Ls` (reversed) and the right one `Rs`.

```
compute(Q, T, T) :- final(Q), !.

compute(Q0, tape(Ls0,S,Rs0), T):-
  showmachine(Q0,Ls0,S,Rs0),
  t(Q0,S,  Q1,S1,Action),
  move(Action,tape(Ls0,S1,Rs0),T1),
  compute(Q1,T1,T).

move(l,tape([],     S,Rs), tape([],0,[S|Rs])).
move(l,tape([L|Ls],S,Rs), tape(Ls,L,[S|Rs])).

move(r,tape(Ls,S,[]),      tape([S|Ls],0,[])).
move(r,tape(Ls,S,[R|Rs]), tape([S|Ls],R,Rs)).
```