COMP 3200
Artificial Intelligence

Lecture 12
Two-Player Games
Mini-Max Search
Alpha-Beta Pruning

Multi-Player Games

- So far we have only looked at problems with a single agent (single agent search)
- Heuristic search can also be applied to environments (and games) with multiple agents (or players)
- Games can have a number of properties, similar to single agent environments

Fully vs. Partially Observable

Deterministic vs. Stochastic

Dynamic vs. Static

Game Properties: Players

- How many players are in the game?
- Examples of Game Players:
  - Chess, Go: 2 Players
  - Baseball: 18 Players, 2 Teams
  - Starcraft: 1v1, 2v2, 8 Person FFA
  - Poker: 10 People at a table

Game Properties: Payoffs

- What does each player hope to achieve?
- Game Payoff Examples:
  - Prisoner's Dilemma: Maximize Reward
  - Poker: Maximize Profits
  - Chess: Win the game
- Zero Sum Game:
  - Each player's gain or loss of payoff/utility is equally balanced by the utility of the other players
  - Win / Lose games are zero sum

Most Traditional Board Games

- Two-Player
- Zero Sum
- Alternating Move
- Perfect Information
- Deterministic
- Discrete

The Chess-Playing Computer

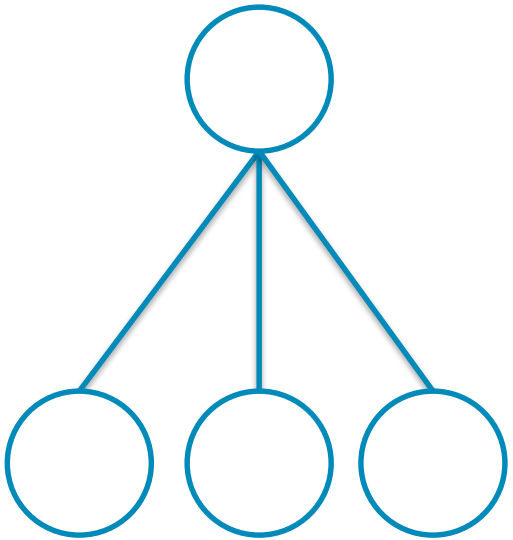
Game-Playing Computer

- How can we design an algorithm to play a two-player alternating move game?

1. Analysis / Strategy / Tactics
2. Thousands of If-Then Statements
3. Look-Ahead and Evaluate

Look-Ahead and Evaluate

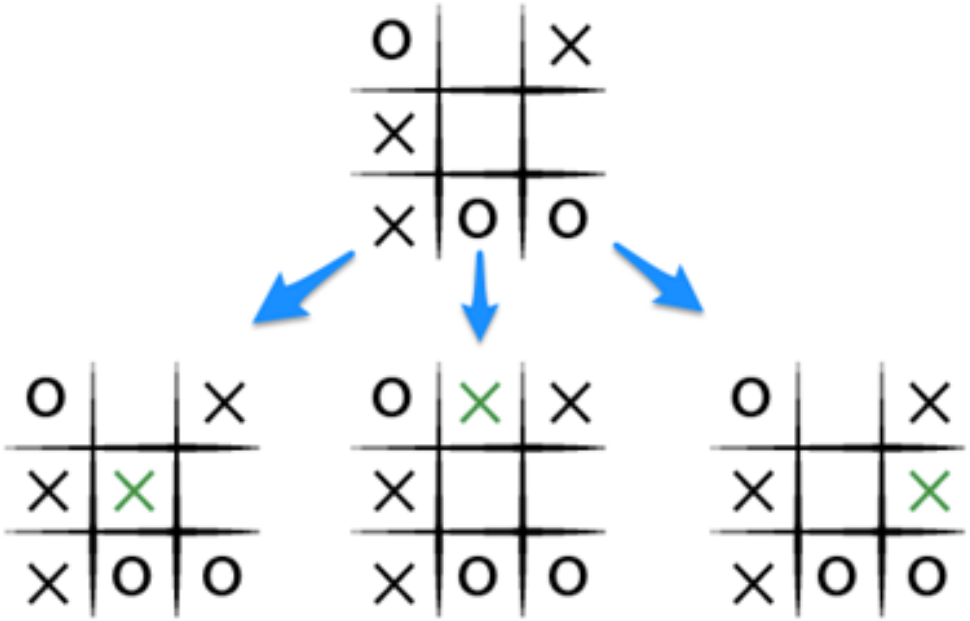
- Generate a list of actions from a given state
- Evaluate those actions based on features of the resulting states
- Do the action which has the highest evaluation



Lookahead and Evaluate

```
1.  Function LookaheadAndEvaluate(state)
2.      maxVal = -infinity
3.      bestAction = null
4.      for (action : state.getLegalActions())
5.          child = state.doAction(action)
6.          val = eval(child)
7.          if (val > maxVal)
8.              maxVal = val
9.              bestAction = action
10.     return bestAction
```

Tic-Tac-Toe Example



Game-Playing Computer

- How can we design an algorithm to play a two-player alternating move game?

1. Analysis / Strategy / Tactics
2. Thousands of If-Then Statements
3. Look-Ahead and Evaluate
4. Search the Entire Game Tree

How big is the game tree?

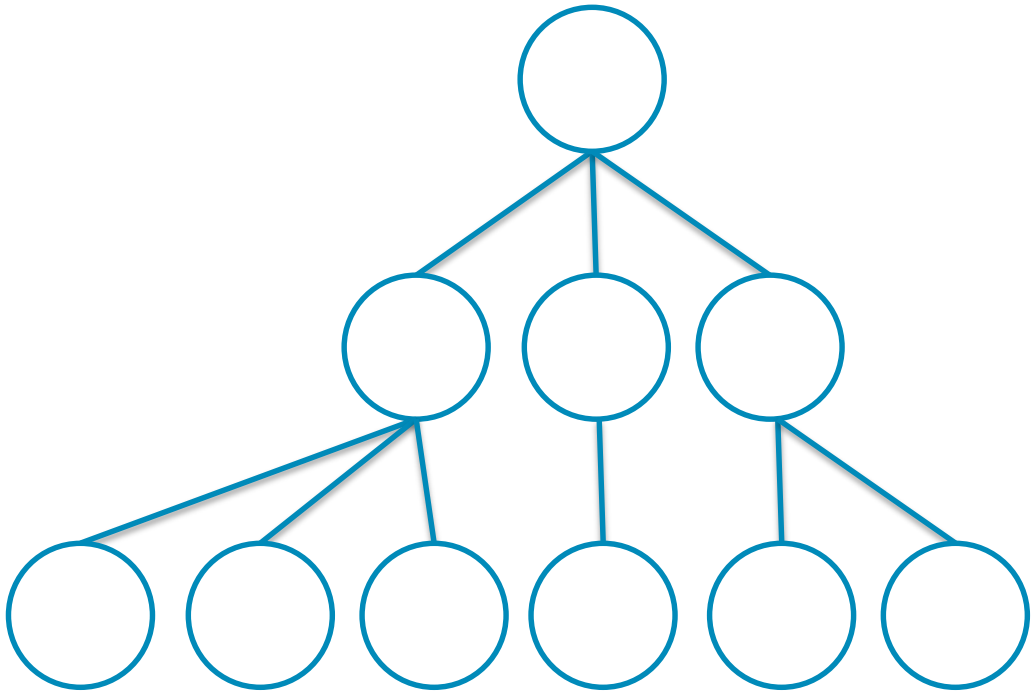
- Some number of actions at each state
  - b = “Branching Factor”
- Game ends after some number of moves
  - d = Search Depth
- Game tree = b^d
- Chess = 10^{120}

Game-Playing Computer

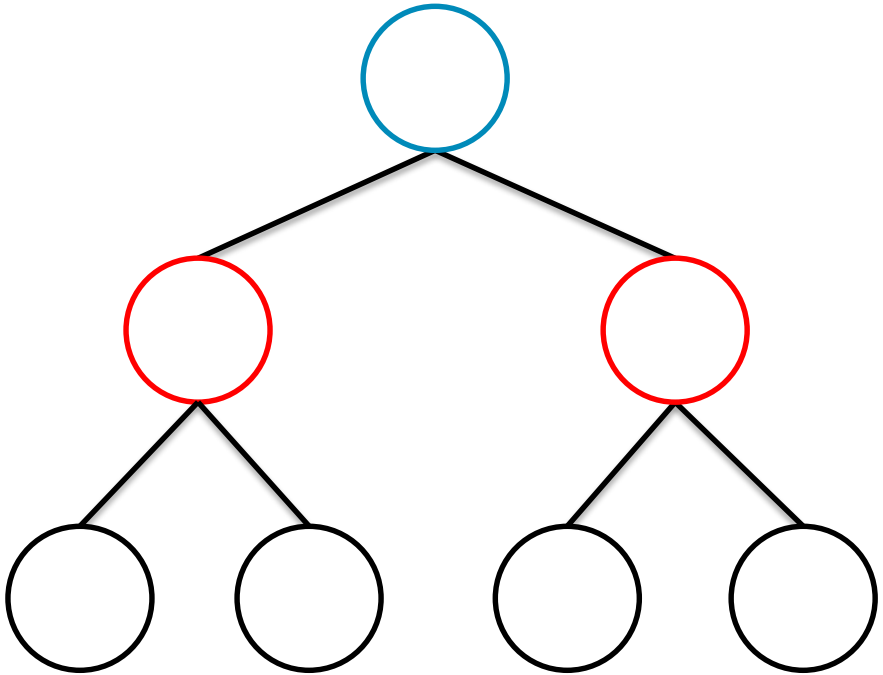
- How can we design an algorithm to play a two-player alternating move game?

1. Analysis / Strategy / Tactics
2. Thousands of If-Then Statements
3. Look-Ahead and Evaluate
4. Search the Entire Game Tree
5. Look-Ahead as Far as Possible

Look Ahead as Far as Possible



Competing Players



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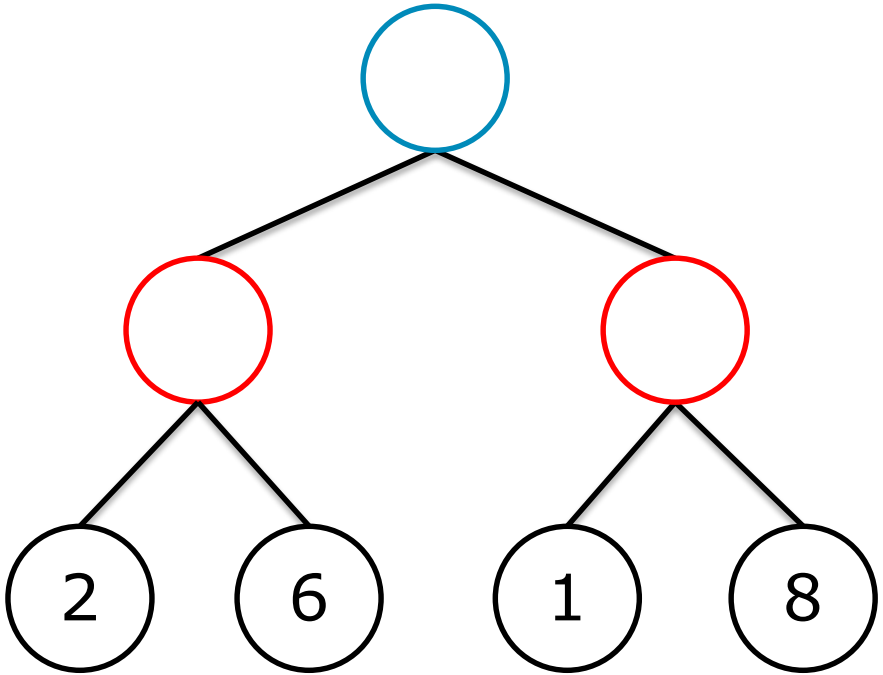
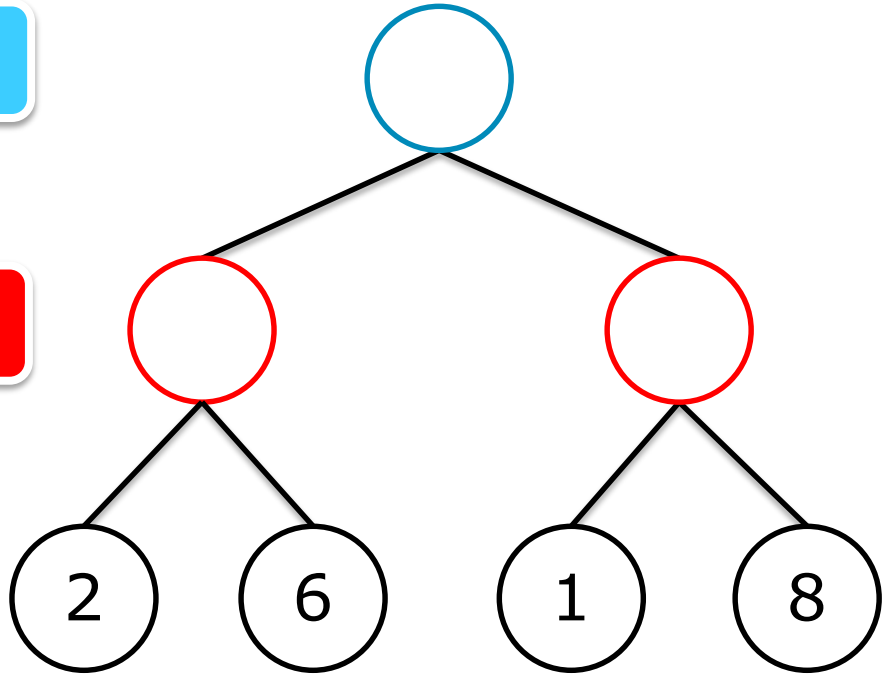


Competing Players

Max Player

Min Player



Competing Players

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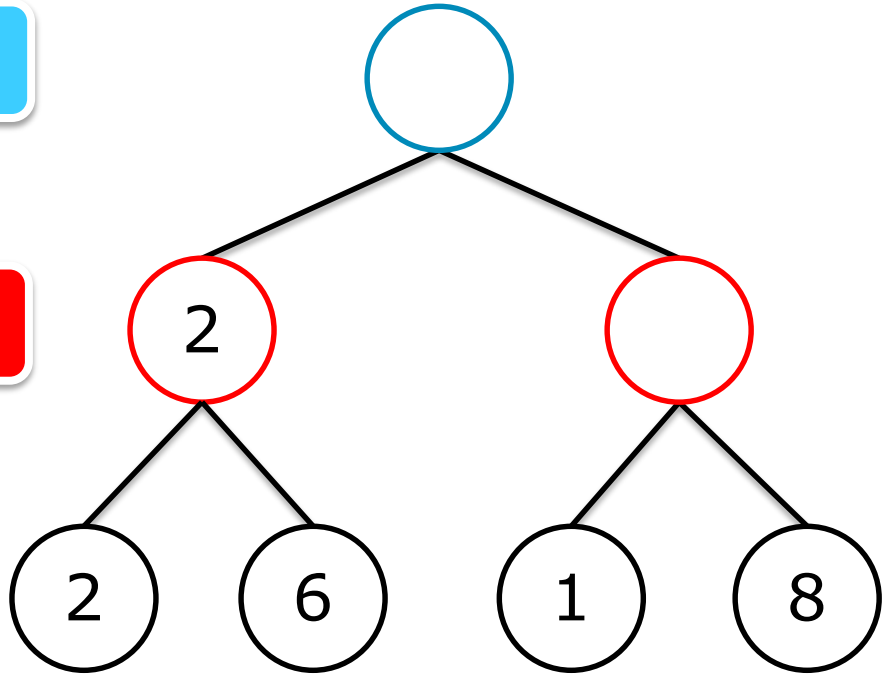

Competing Players

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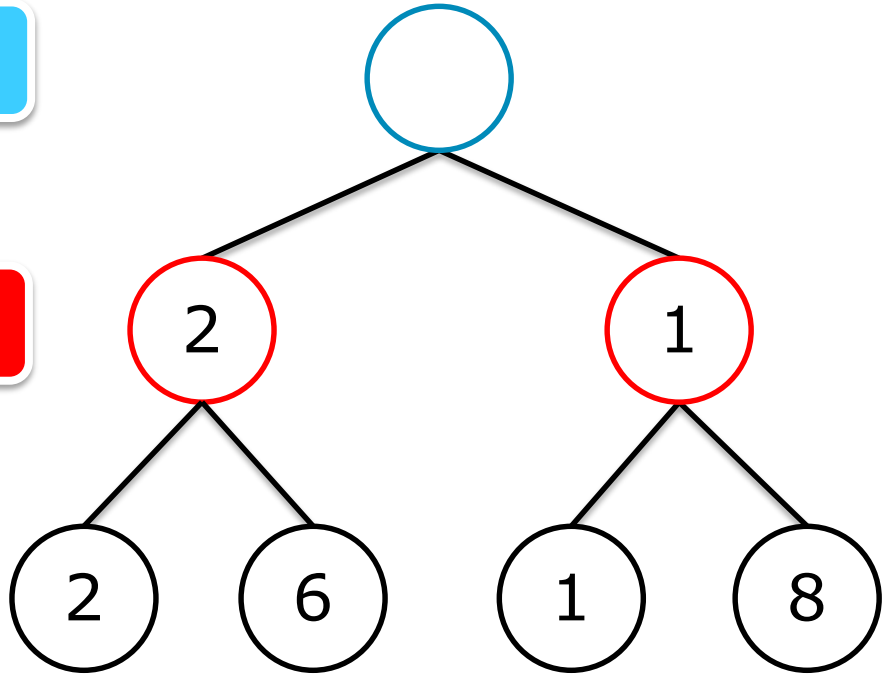

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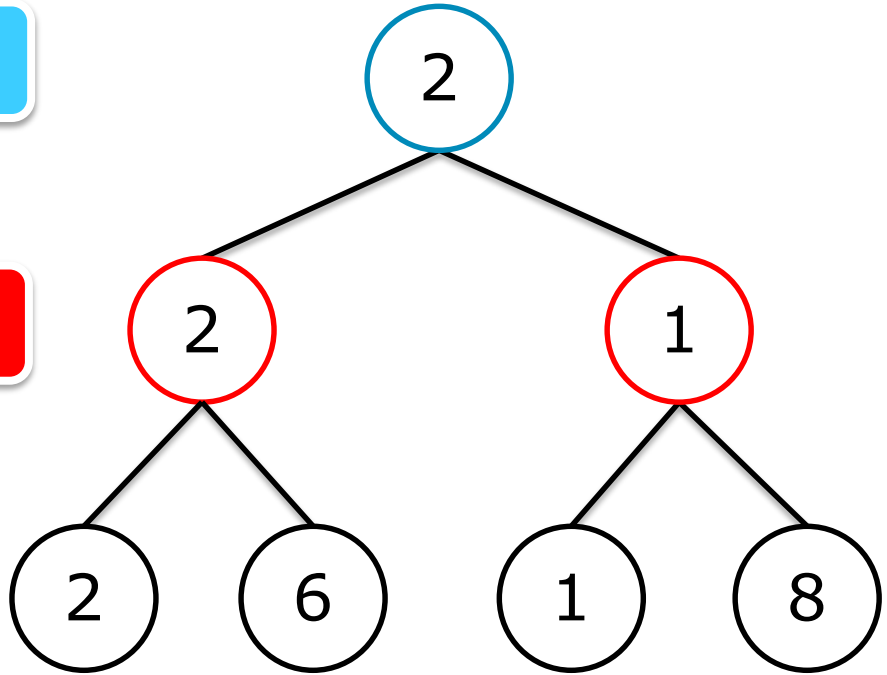

Lookahead and Evaluate

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3.      bestAction = null
4.      for (action : state.getLegalActions())
5.          child = state.doAction(action)
6.          val = eval(child)
7.          if (val > maxVal)
8.              maxVal = val
9.              bestAction = action
10.     return bestAction
```

MaxValue (single depth)

```
1. Function MaxValue(s)
2.     v = -infinity
3.     for (c in children(s))
4.         v' = eval(c)
5.         if (v' > v) v = v'
6.     return v
```

MaxValue

```
1. Function MaxValue(s)
2.     if terminal(s)
3.         return eval(s)
4.     v = -infinity
5.     for (c in children(s))
6.         v'= MinValue(c)
7.         if (v' > v) v = v'
8.     return v
```

MaxValue and MinValue (full tree)

```
1.  Function MaxValue(s)
2.      if (terminal(s))
3.          return eval(s)
4.      v = -infinity
5.      for (c in children(s))
6.          v'= MinValue(c)
7.          if (v' > v) v = v'
8.      return v
```

```
1.  Function MinValue(s)
2.      if (terminal(s))
3.          return eval(s)
4.      v = +infinity
5.      for (c in children(s))
6.          v'= MaxValue(c)
7.          if (v' < v) v = v'
8.      return v
```

eval(s) returns score w.r.t. the maximizing player

MaxValue and MinValue (depth limit)

```
1.  Function MaxValue(s, d)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      v = -infinity
5.      for (c in children(s))
6.          v'= MinValue(c, d+1)
7.          if (v' > v) v = v'
8.      return v
```

```
1.  Function MinValue(s, d)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      v = +infinity
5.      for (c in children(s))
6.          v'= MaxValue(c, d+1)
7.          if (v' < v) v = v'
8.      return v
```

eval(s) returns score w.r.t. the maximizing player

Minimax Algorithm

```
1.  Function MiniMax(s, d, max)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      if (max)        // maximizing player
5.          v = -infinity
6.          for (c in children(s))
7.              v = max(v, MiniMax(c, d+1, false))
8.      else            // minimizing player
9.          v = +infinity
10.         for (c in children(s))
11.             v = min(v, MiniMax(c, d+1, true))
```

Initial Call: MiniMax(startState, 0, true)

Negamax Algorithm

```
1.  Function NegaMax(s, d, player)
2.      if (terminal(s) d > maxD)
3.          return eval(s, player)
4.      v = -infinity
5.      for (c in children(s))
6.          v = max(v, -NegaMax(child, d+1, !player))
```

Relies on: max(a, b) = -max(-a, -b)

Initial Call: NegaMax(startState, 0, startState.player)

Mini-Max Properties

- Complete and Optimal: Will find the optimal solution to a given max depth
- Each player plays a best response to the possible actions of the other player
- Mini-Max plays the Nash Equilibrium

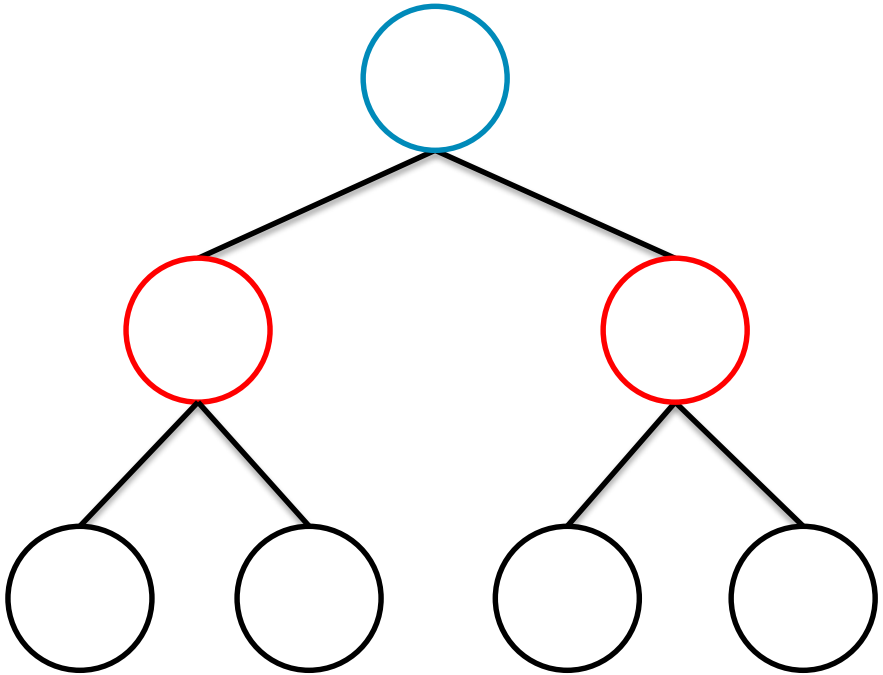
Nash Equilibrium (NE)

- For two-player, finite, zero-sum games, a Nash Equilibrium exists
- Recall that in a NE:
  - Each player is best responding to the other
  - Neither player can gain by deviating
  - Neither player has any regrets
- Playing a NE is a very strong strategy

Improving MiniMax

Max

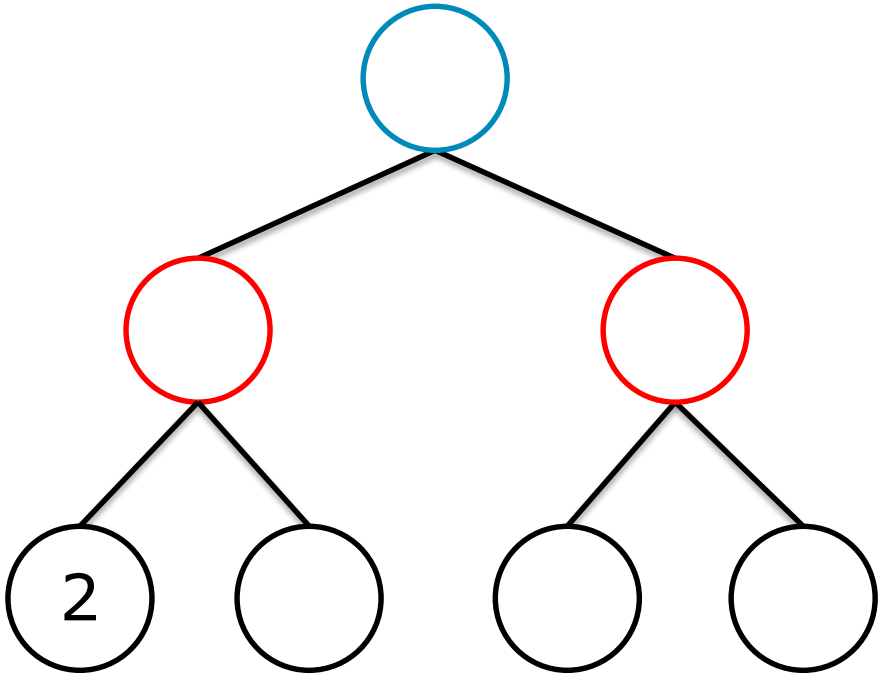
Min



Improving MiniMax

Max

Min



Improving MiniMax

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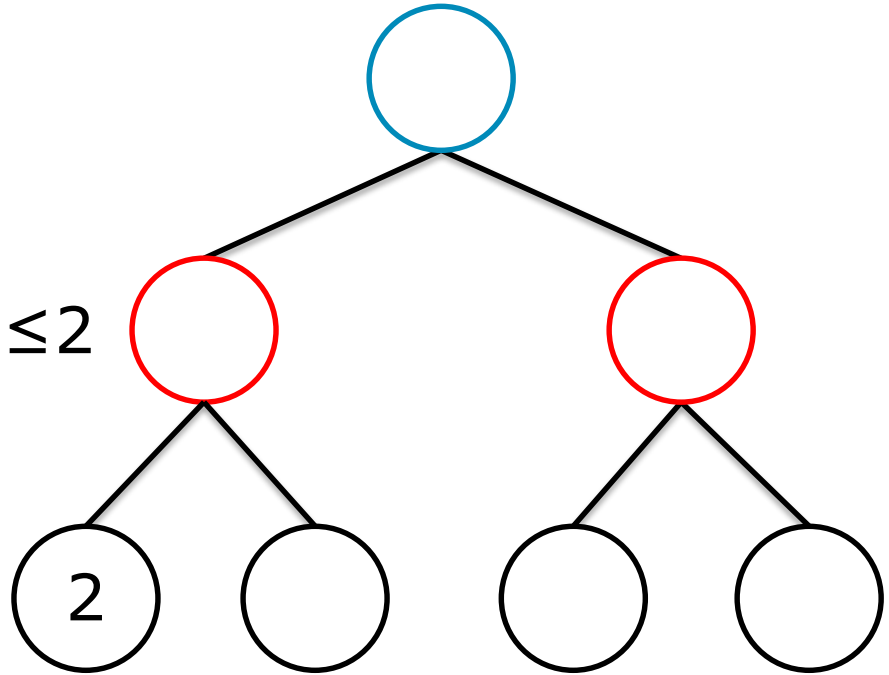

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Max

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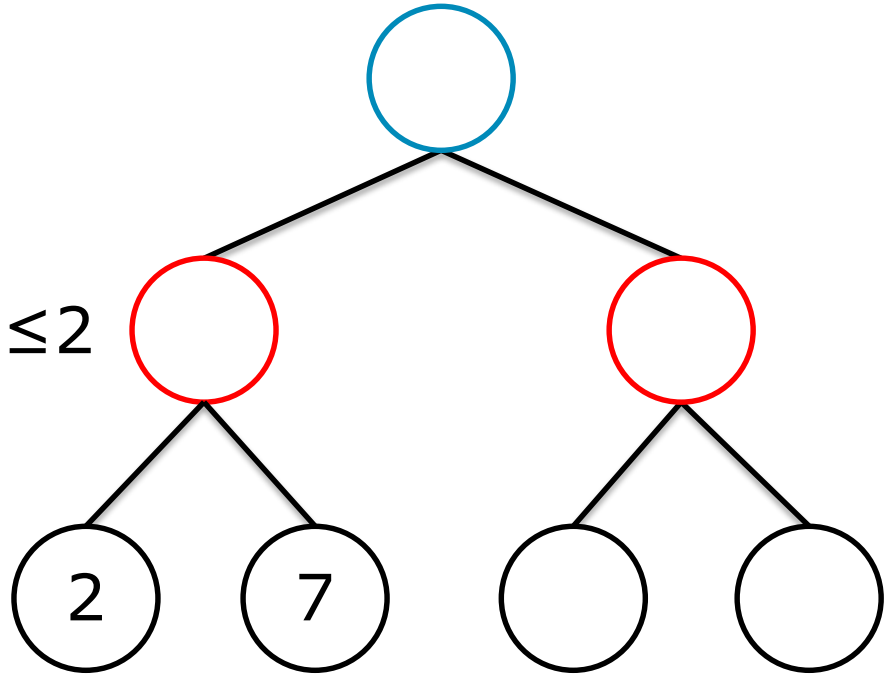

Improving MiniMax

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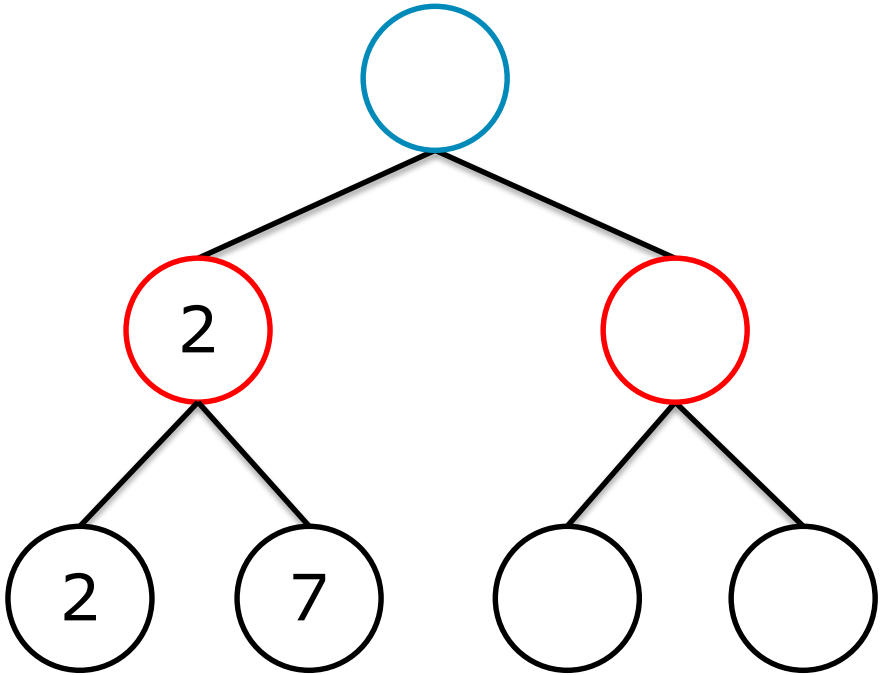

Improving MiniMax

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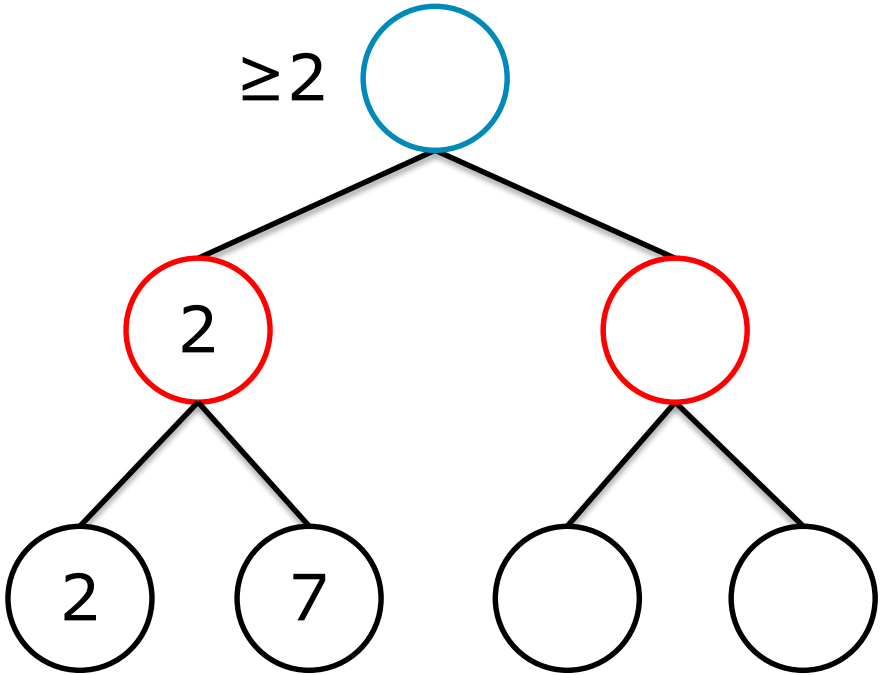
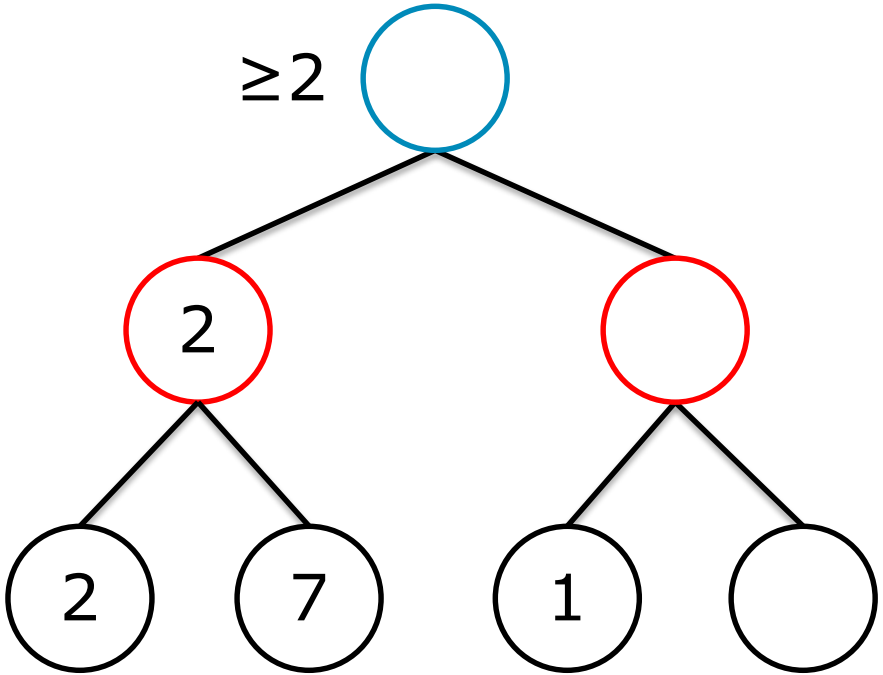
Min



Improving MiniMax

Max

Min



Improving MiniMax

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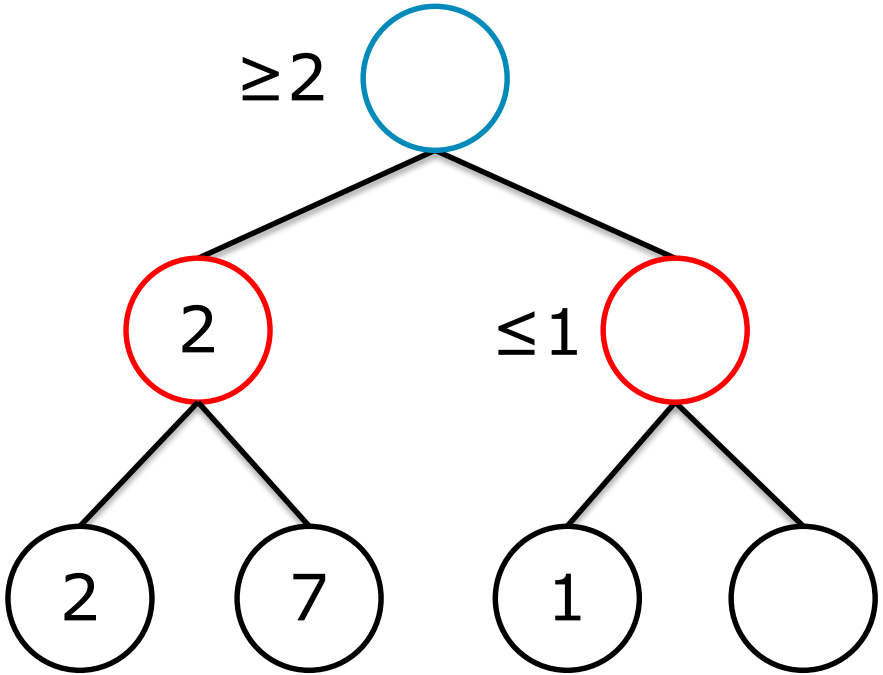
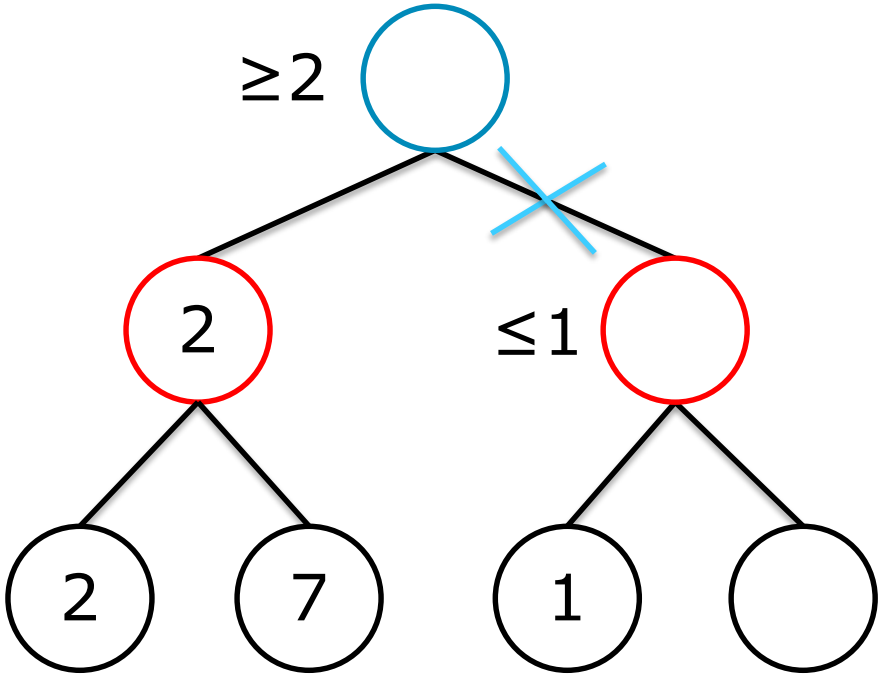


Improving MiniMax

Max

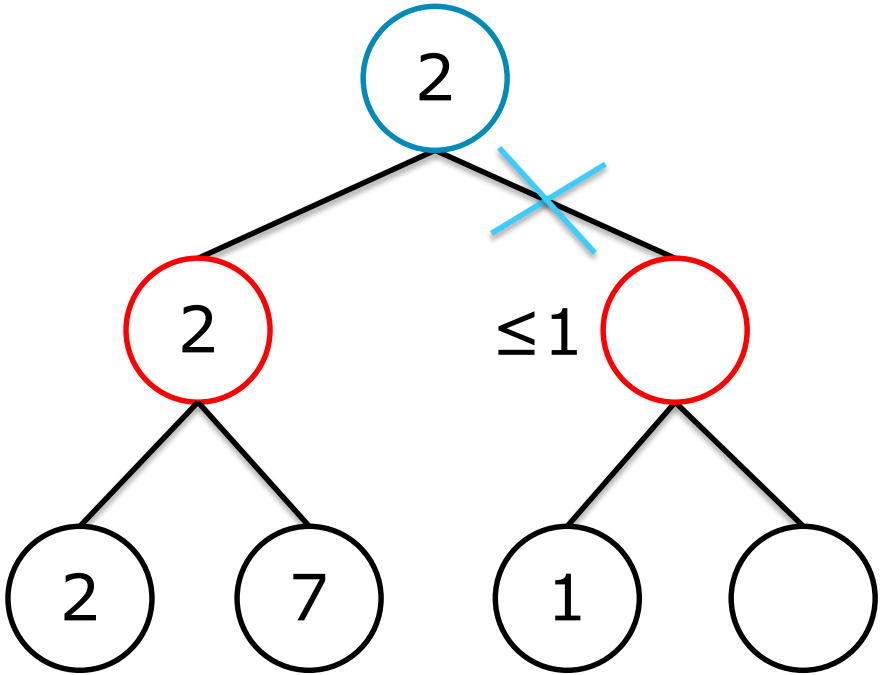
Min



Improving MiniMax

Max

Min

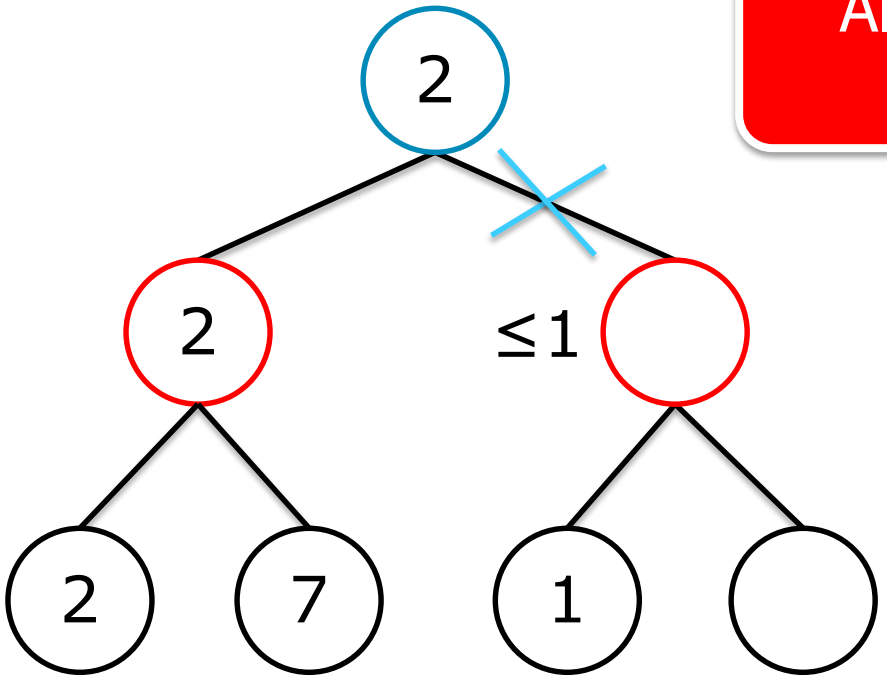


Improving MiniMax

Max

Min

Alpha-Beta Pruning



Alpha-Beta Pruning

- Alpha-Beta is not a different algorithm than MiniMax, it is an optimization
  - “Minimax with Alpha-Beta Pruning”
- Maintains all of the properties of minimax but is strictly better
- Cutting off branches of the search tree yields exponential savings

Alpha-Beta Pruning

- Uses the MiniMax algorithm
- Introduce 2 new variables
- Alpha (α) – Best alternative for Max on this particular path through the tree
- Beta (β) – Best alternative for Min on this particular path through the tree
- α, β are an ever narrowing window which 'good' paths pass through, and other paths are pruned

MaxValue and MinValue (minimax)

```
1.  Function MaxValue(s, d)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      v = -infinity
5.      for (c in children(s))
6.          v'= MinValue(c, d+1)
7.          if (v' > v) v = v'
8.      return v
```

```
1.  Function MinValue(s, d)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      v = +infinity
5.      for (c in children(s))
6.          v'= MaxValue(c, d+1)
7.          if (v' < v) v = v'
8.      return v
```

eval(s) returns score w.r.t. the maximizing player

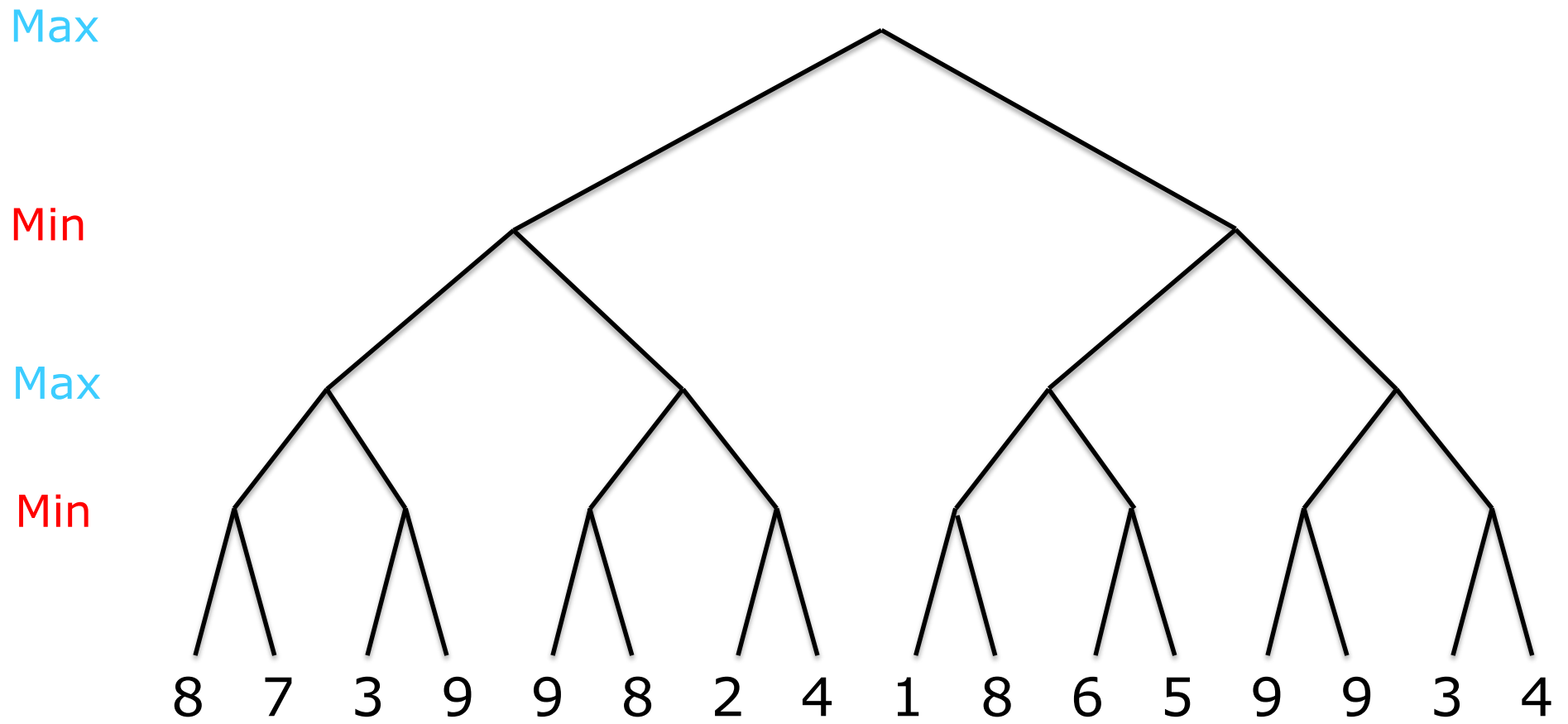
MaxValue and MinValue (alpha beta)

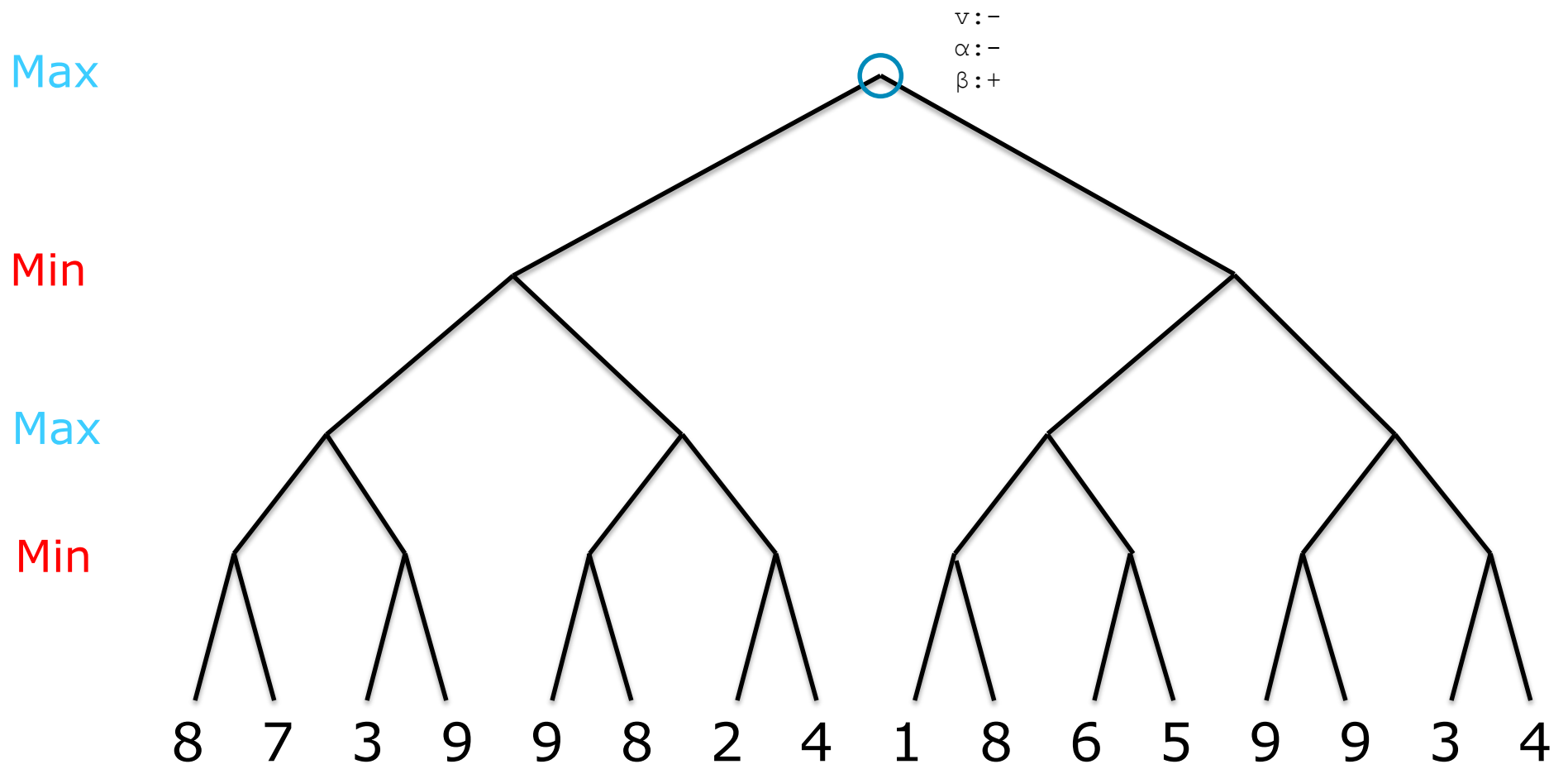
```
1.  Function MaxValue(s, α, β, d)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      v = -infinity
5.      for (c in children(s))
6.          v'= MinValue(c, α, β, d+1)
7.          if (v' > v) v = v'
8.          if (v' ≥ β) return v
9.          if (v' > α) α = v'
10.     return v
```

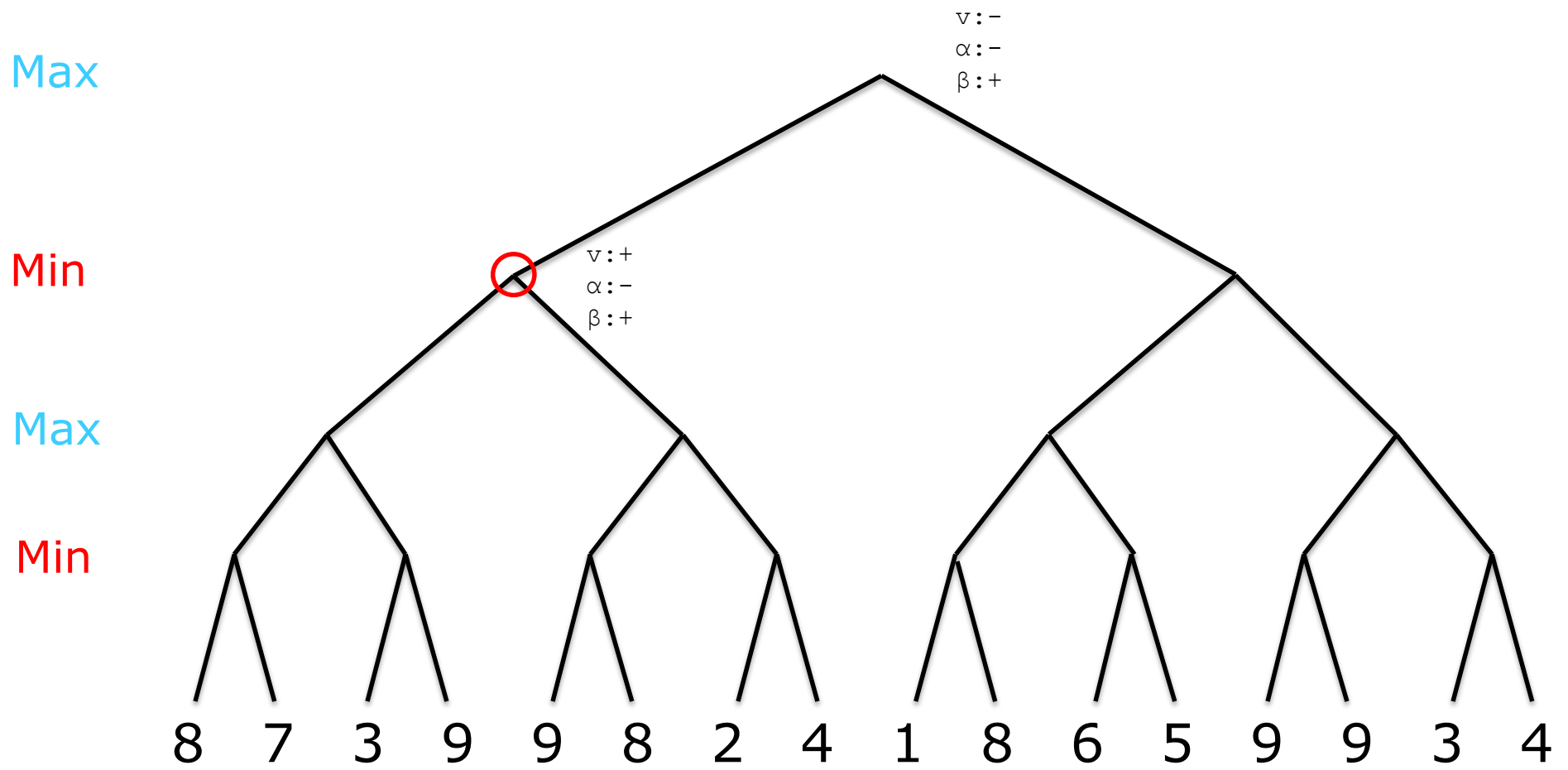
```
1.  Function MinValue(s, α, β, d)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      v = +infinity
5.      for (c in children(s))
6.          v'= MaxValue(c, α, β, d+1)
7.          if (v' < v) v = v'
8.          if (v' ≤ α) return v
9.          if (v' < β) β = v'
10.     return v
```

Initial Call: MaxValue(startState, -infinity, +infinity, 0)

MaxValue(startState, -infinity, +infinity, 0)




v : −
α : −
β : +
Max
Min
Max
Min
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4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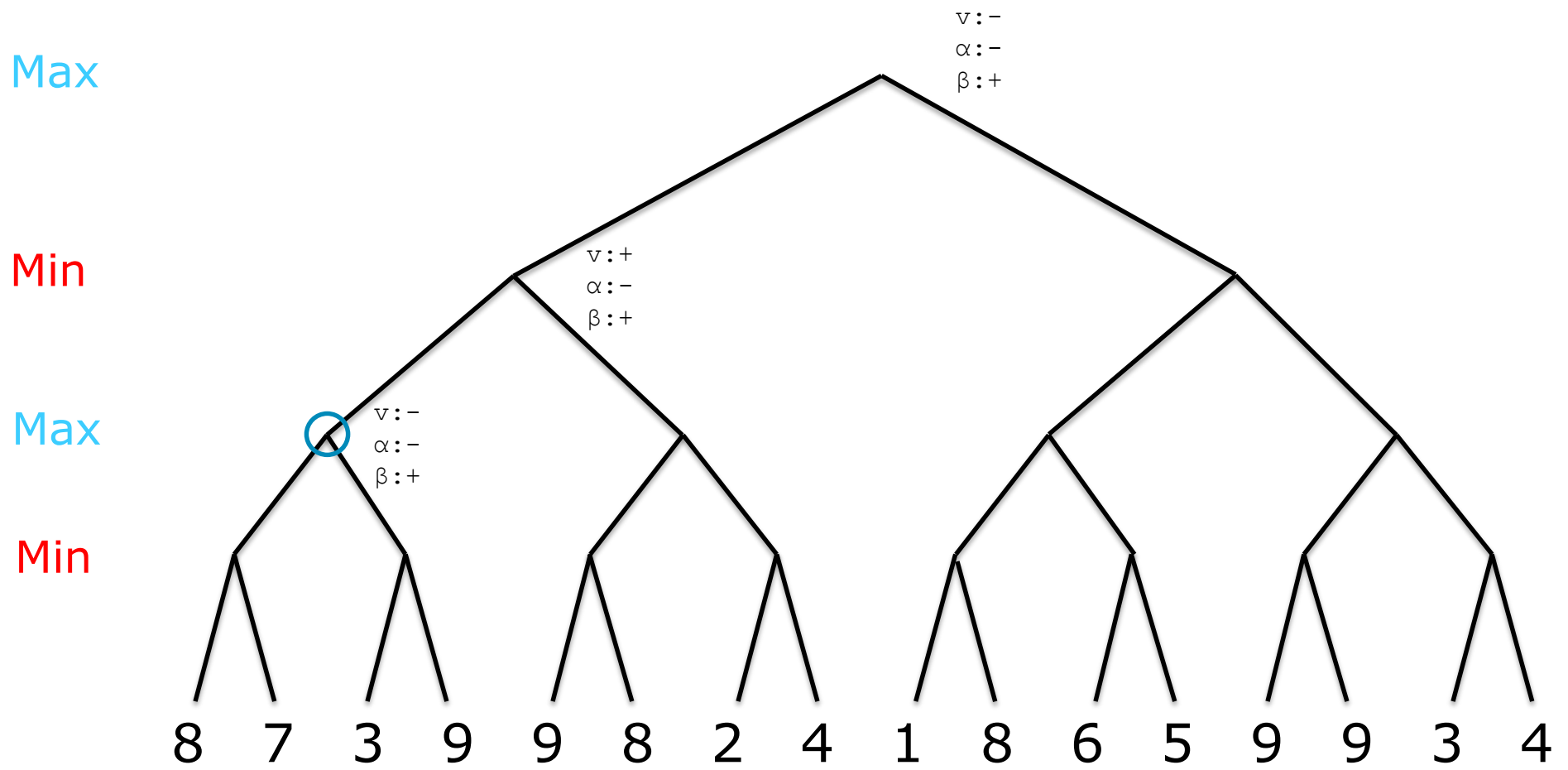
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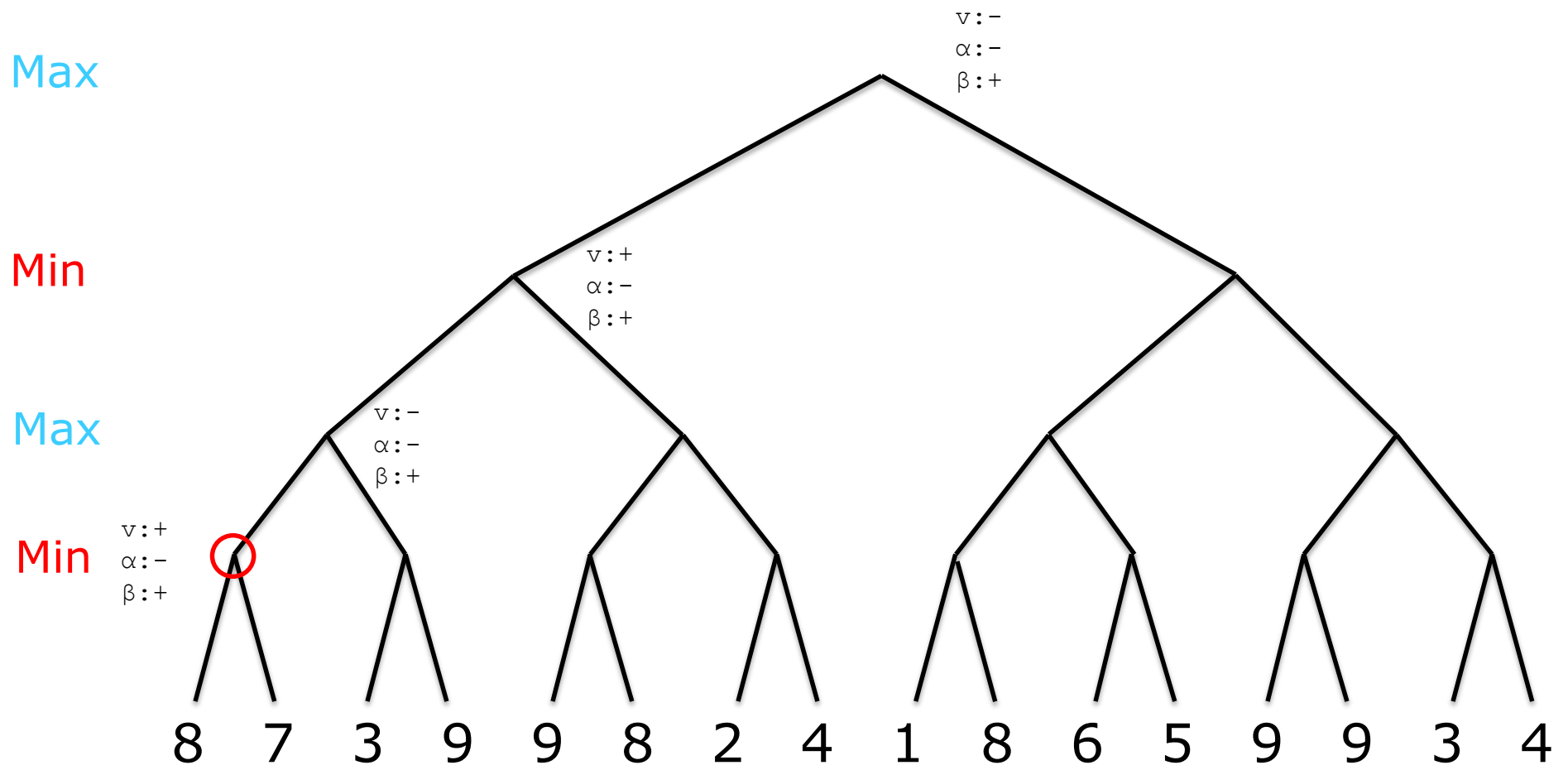
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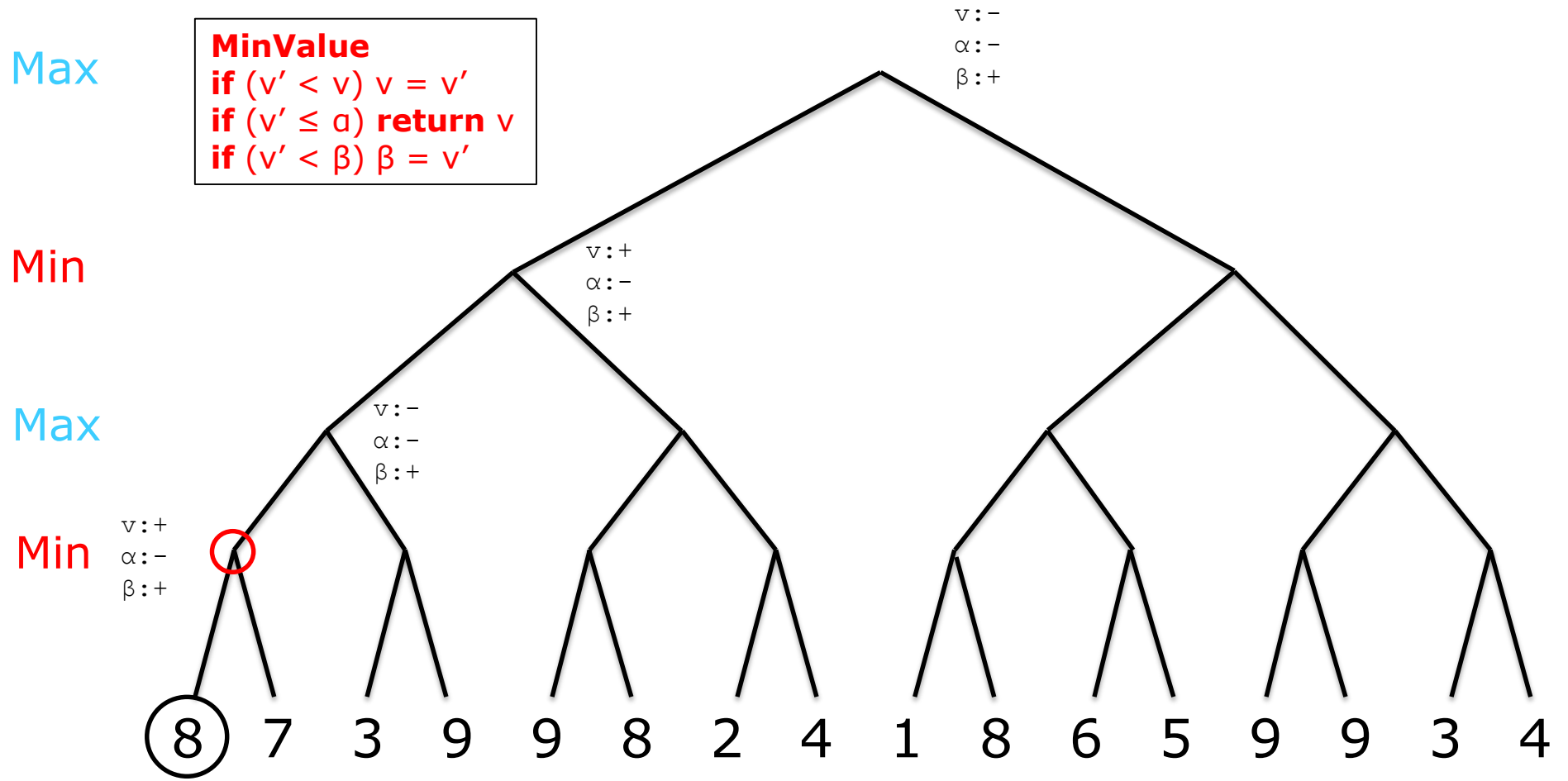
MinValue
if (v′ < v) v = v′
if (v′ ≤ α) return v
if (v′ < β) β = v′
Max
Min
Max
Min
v : −
α : −
β : +
v : +
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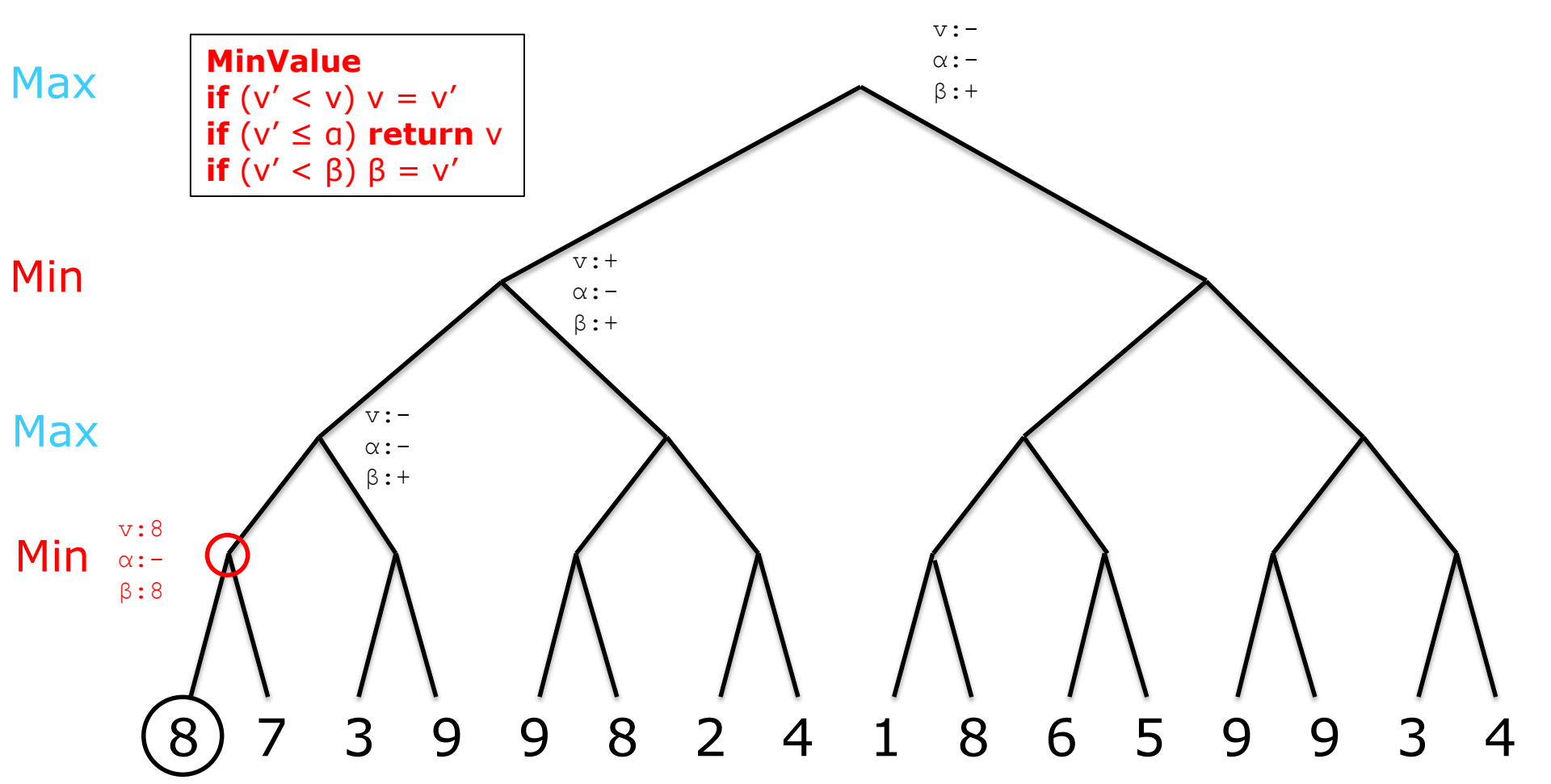
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β : +
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α : −
β : 8
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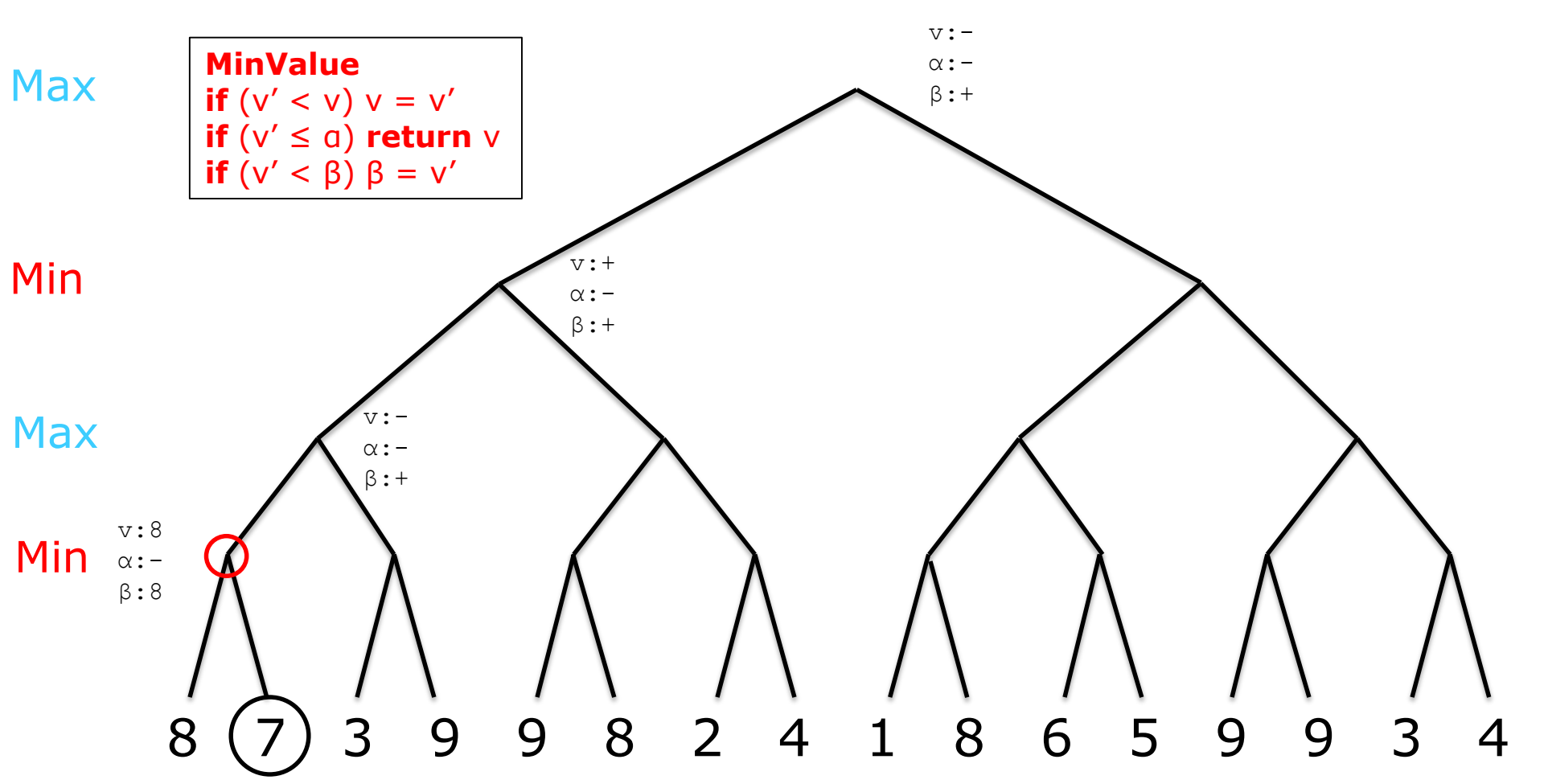
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α : −
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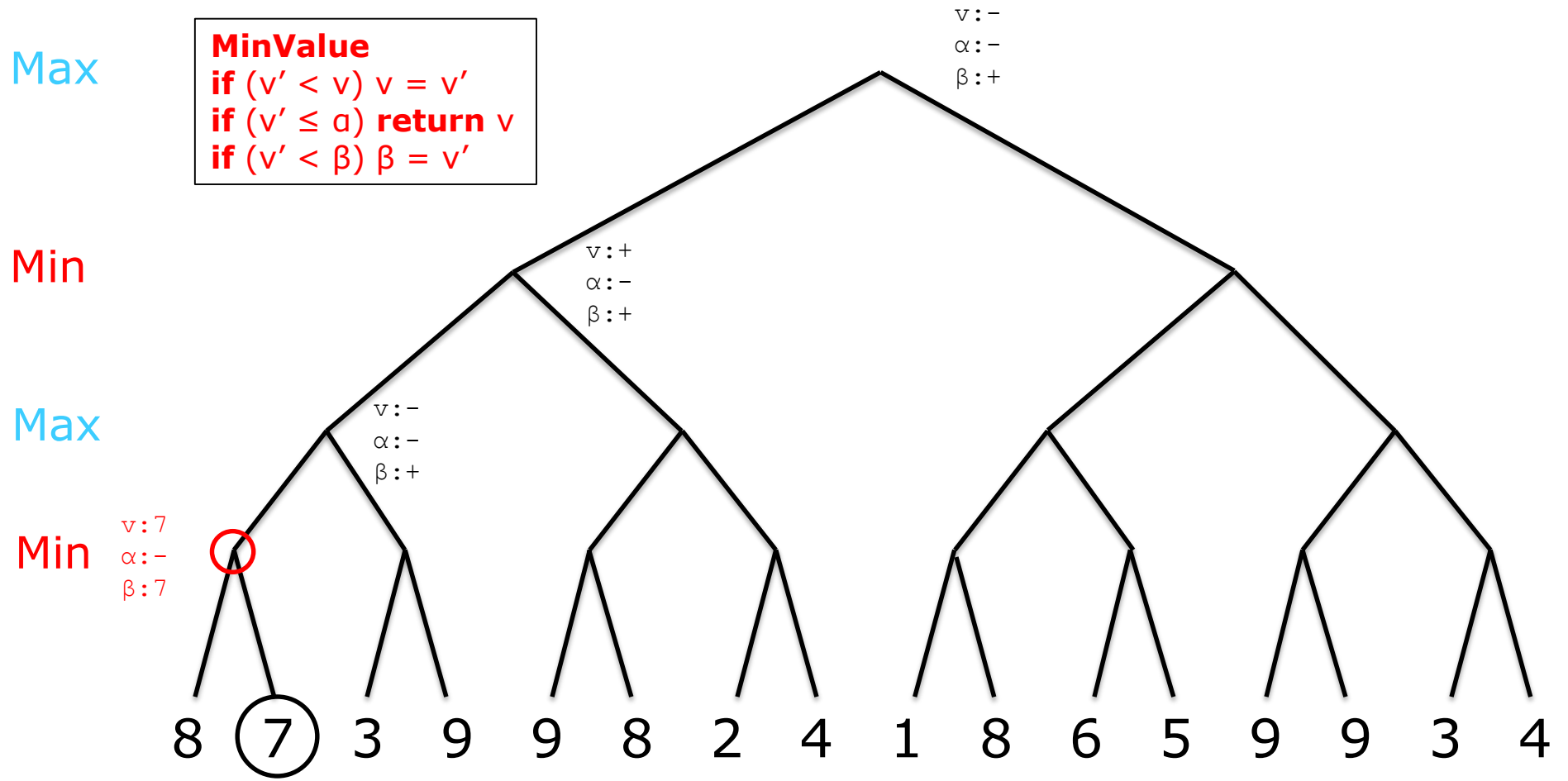
α : −
β : 8
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4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Max
Min
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α : −
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v : +
α : −
β : +
v : −
α : −
β : +
v : 7
α : −
β : 7
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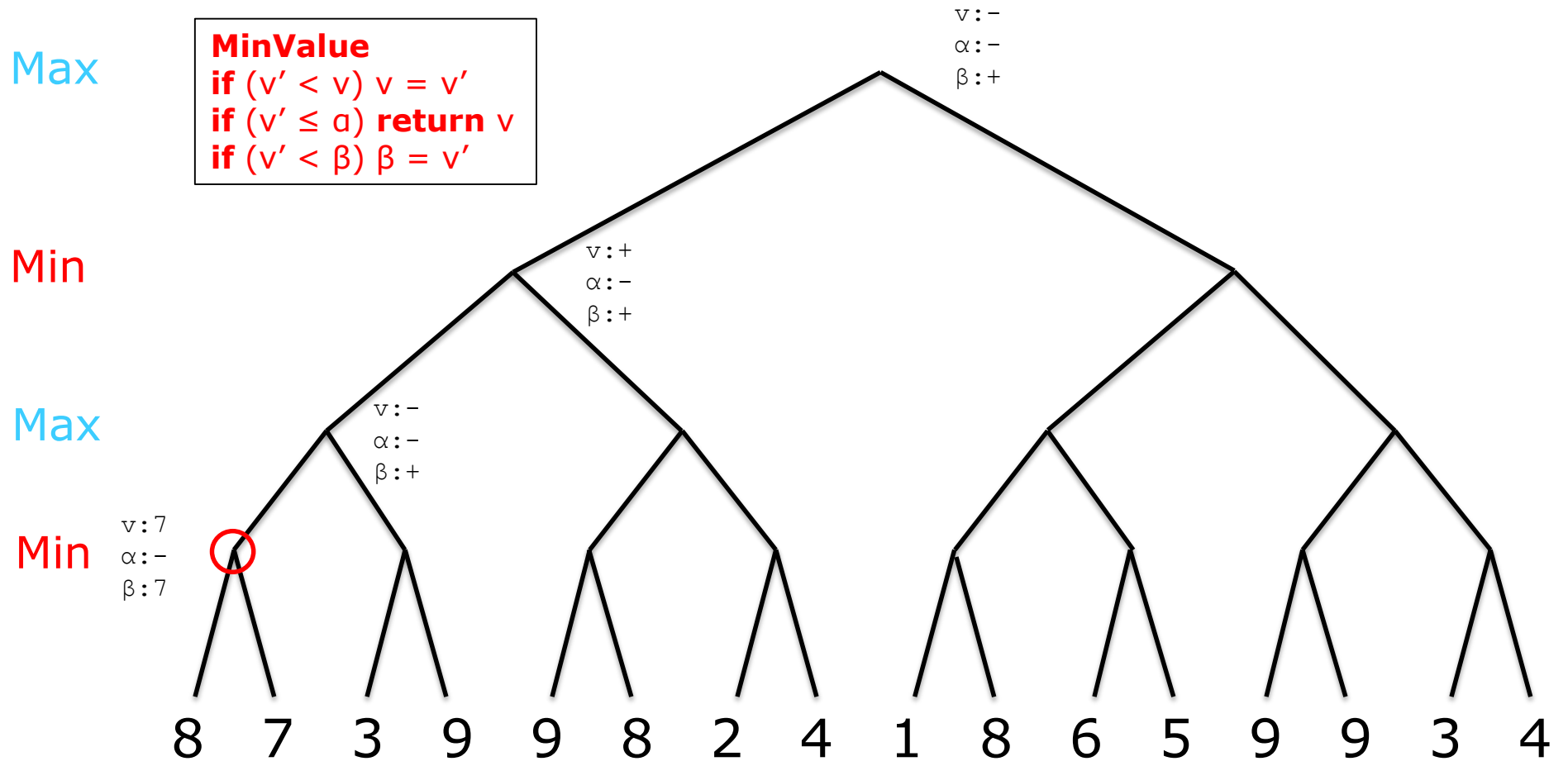
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Min
Max
Min
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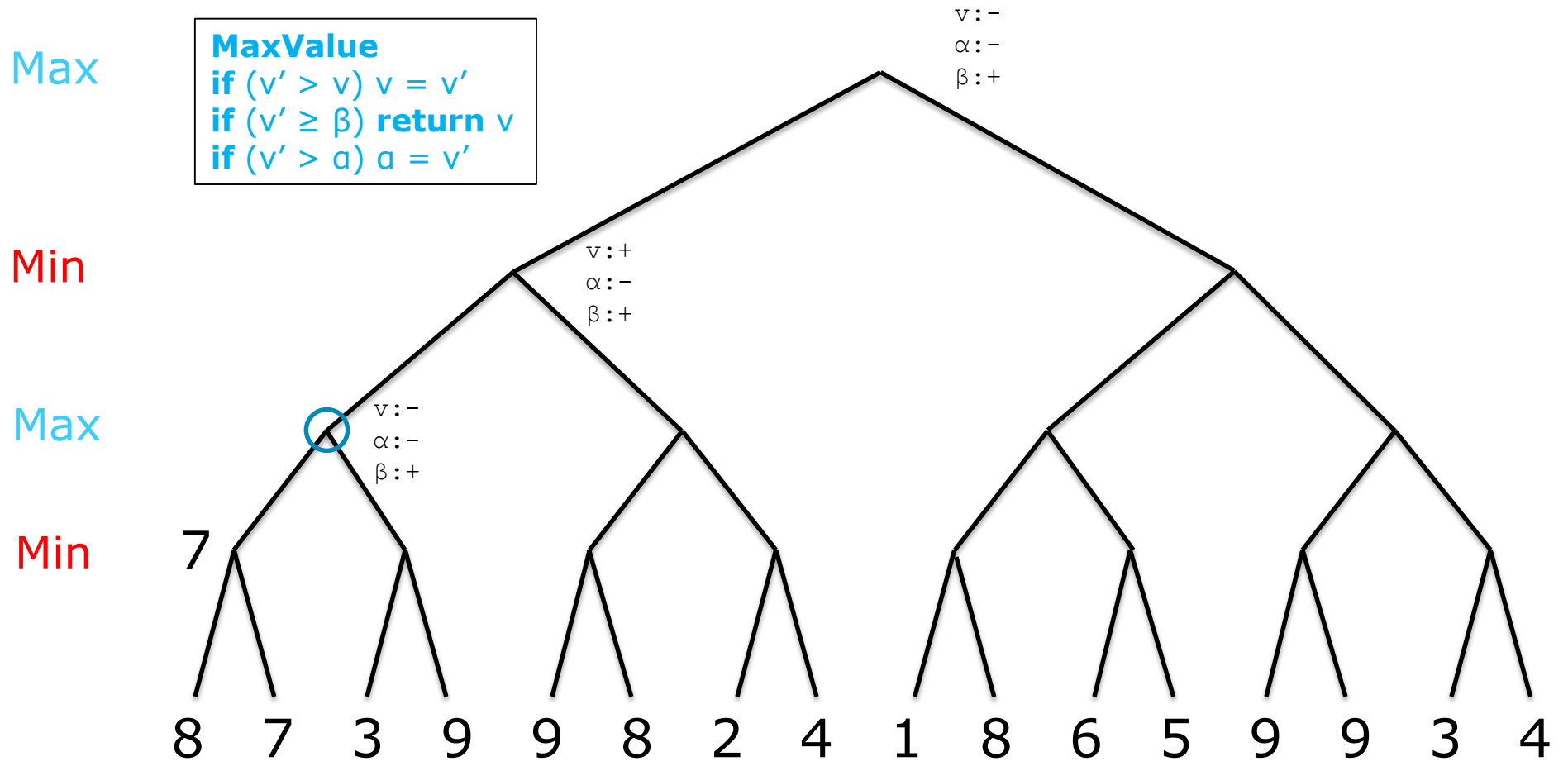
α : −
β : +
v : +
α : −
β : +
v : −
α : −
β : +
v : 7
α : −
β : 7
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4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Min
Max
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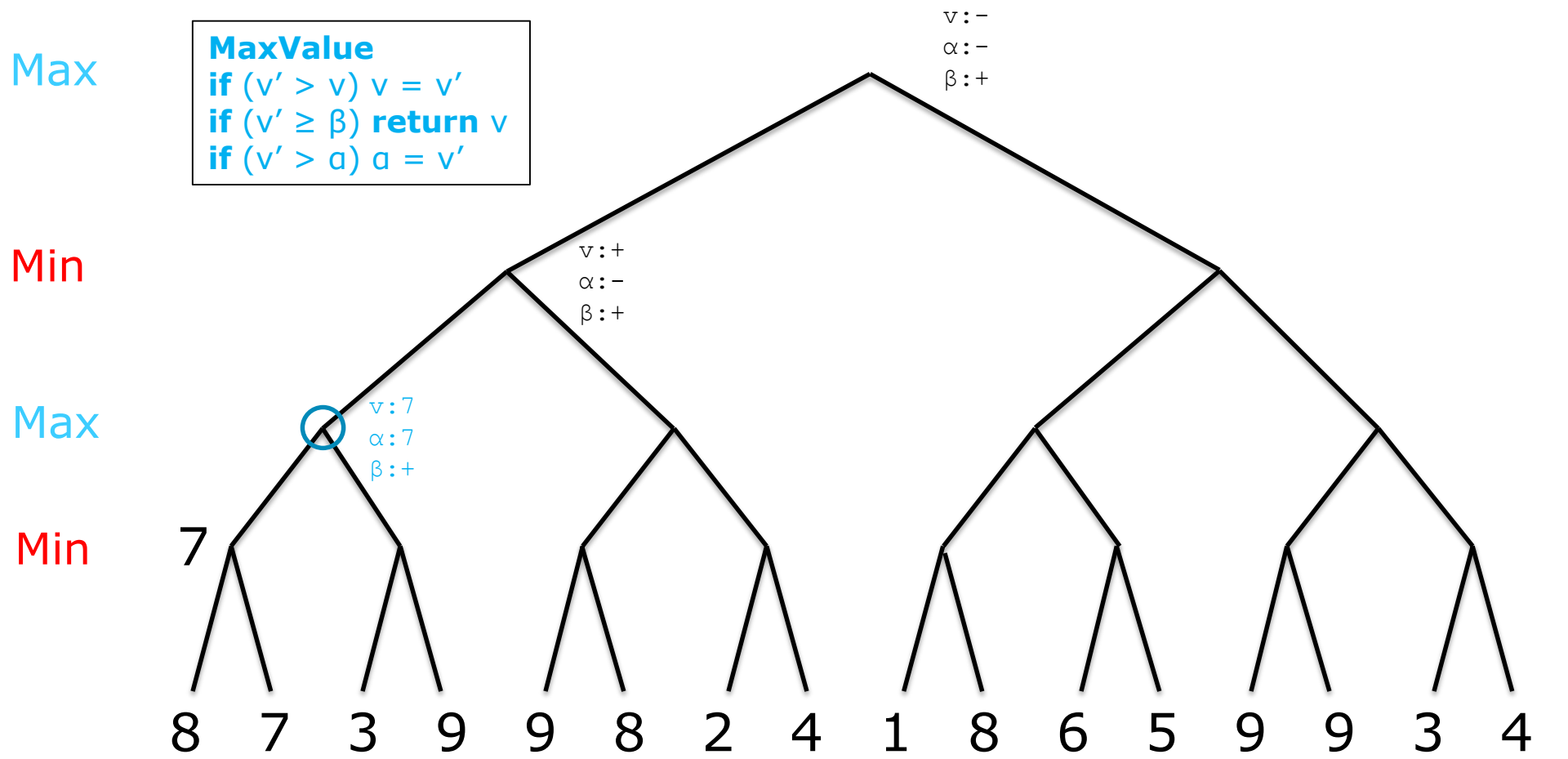
v : −
α : −
β : +
v : +
α : −
β : +
v : −
α : −
β : +
7
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4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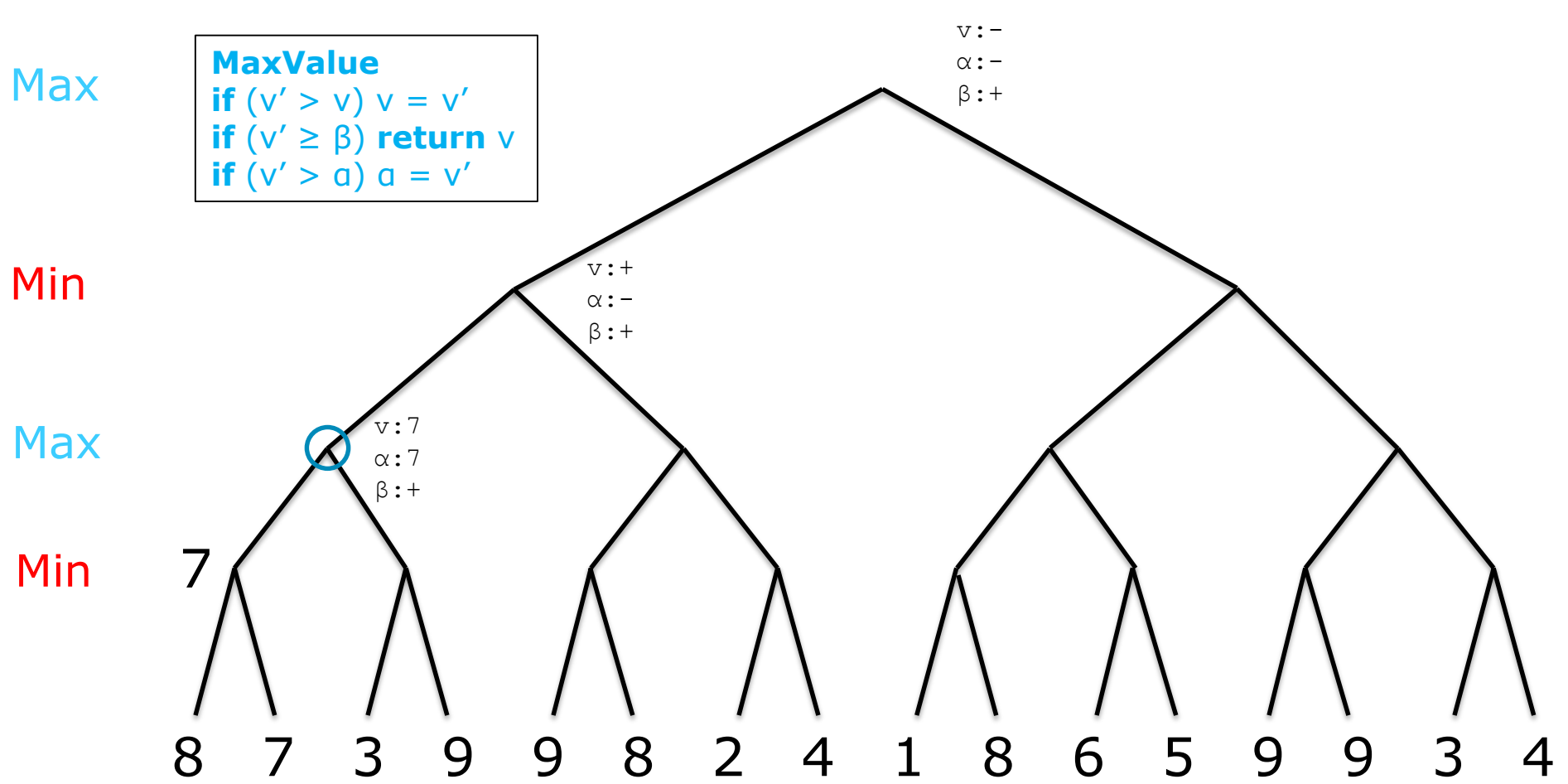
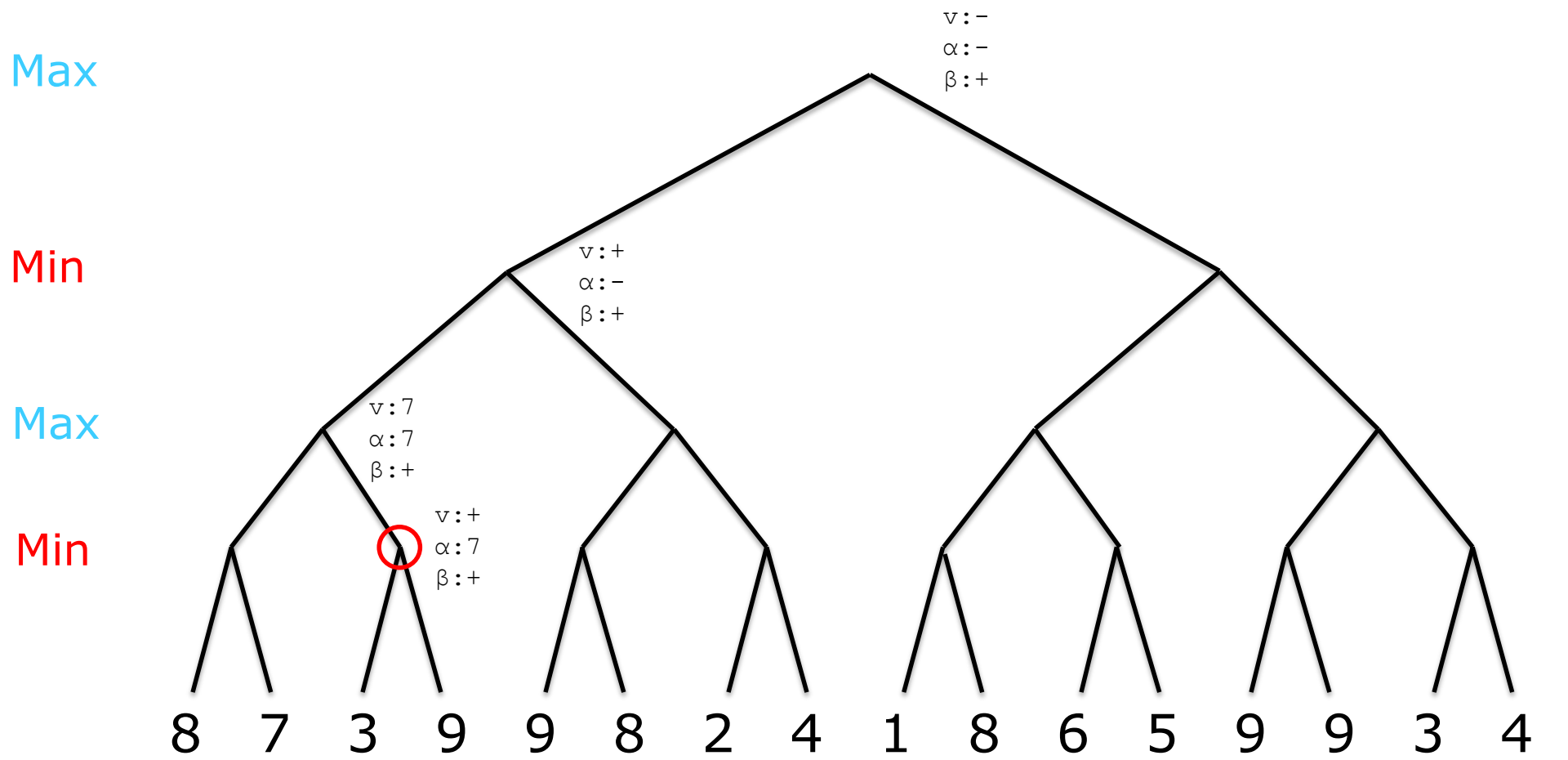
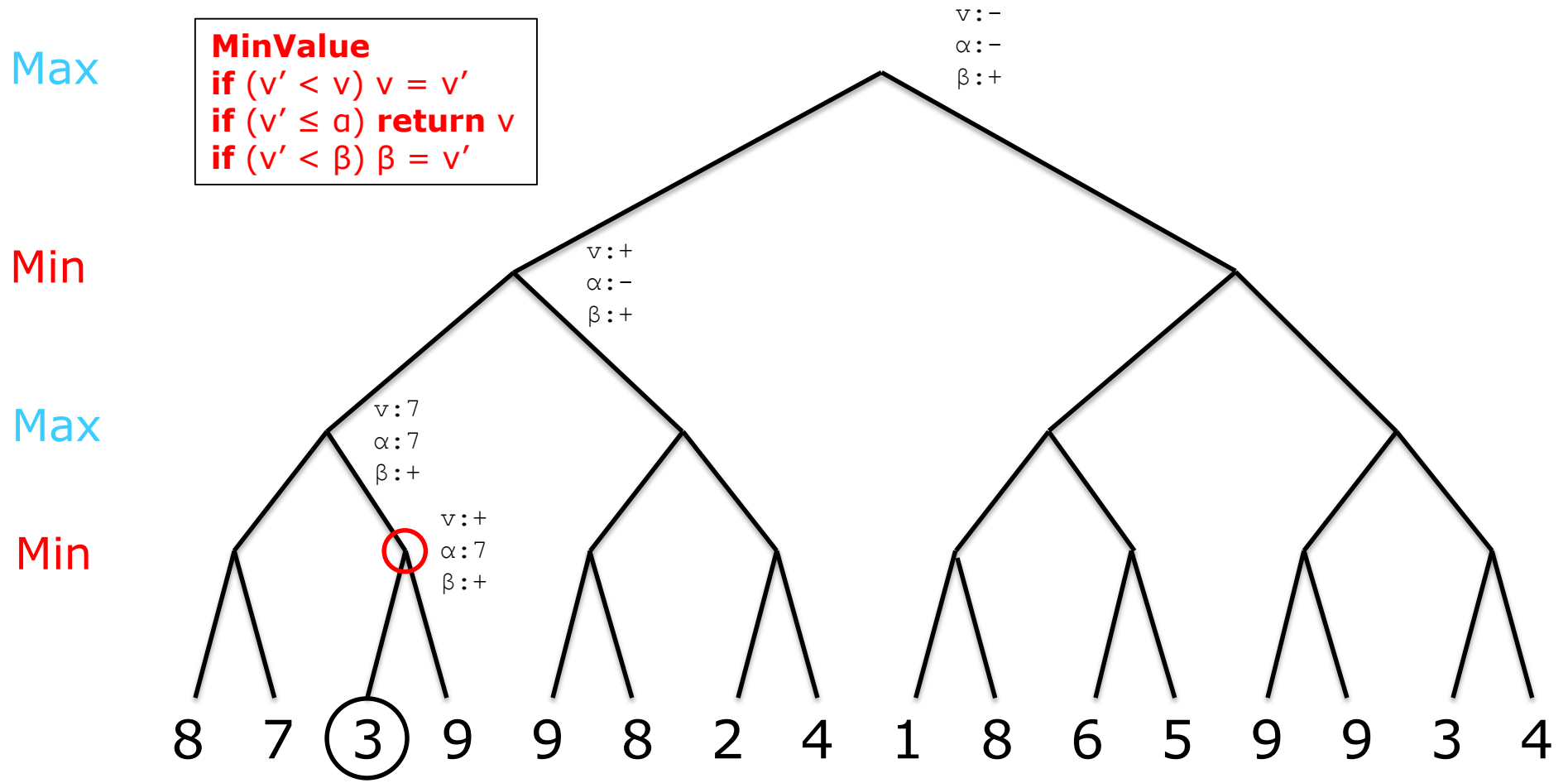
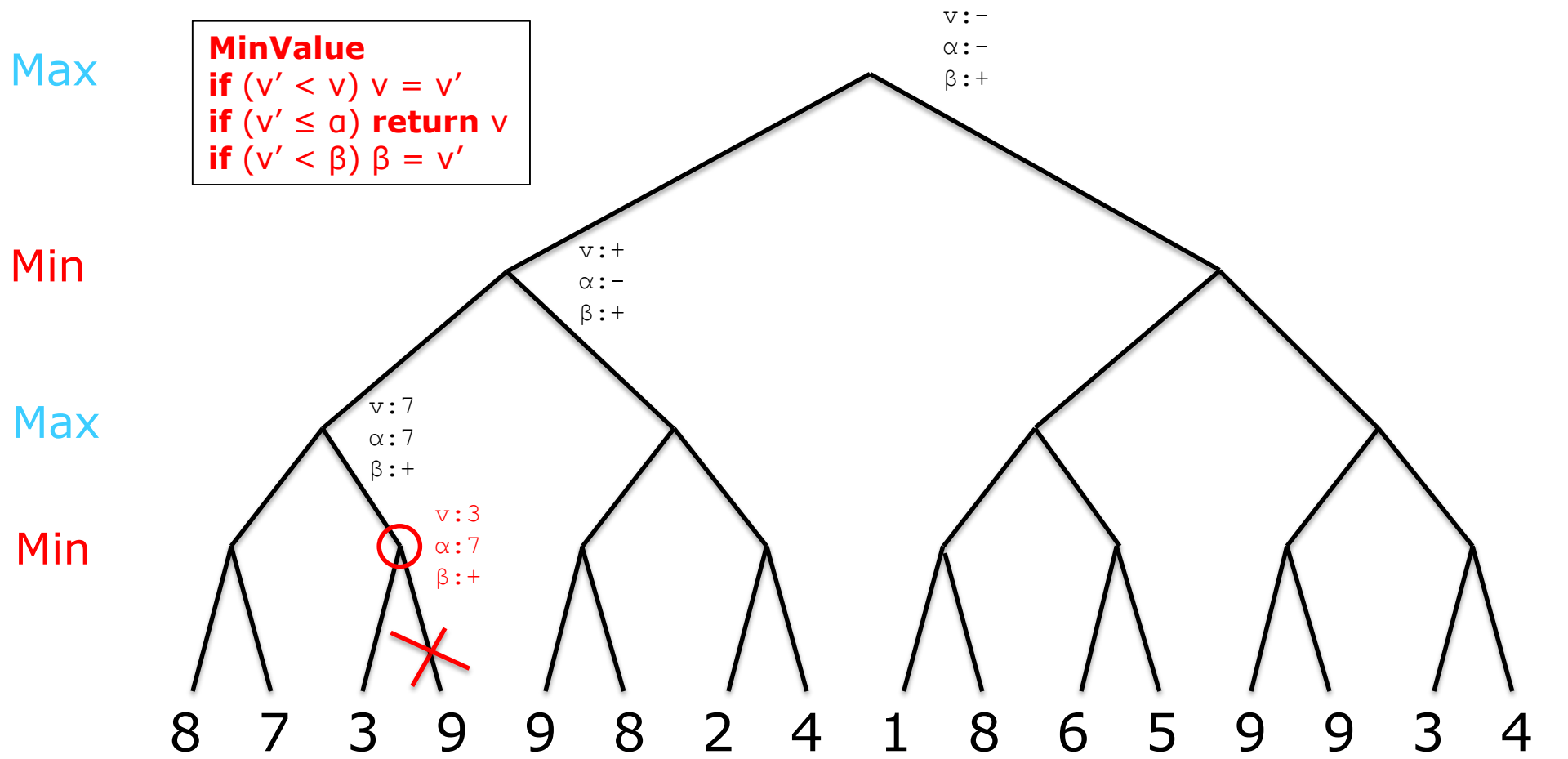
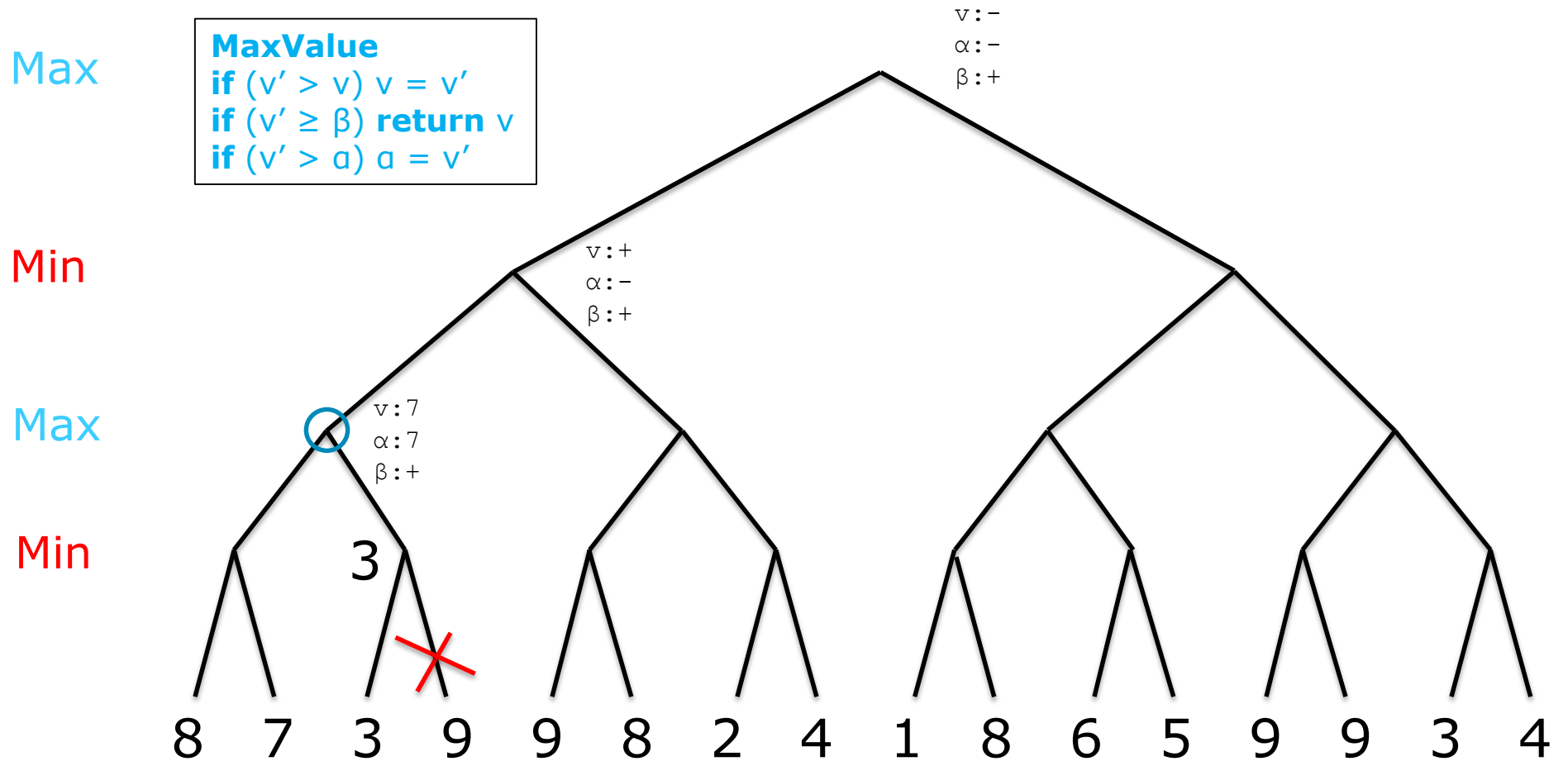
Min
Max
Min
v : −
α : −
β : +
v : +
α : −
β : +
v : 7
α : 7
β : +
7
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

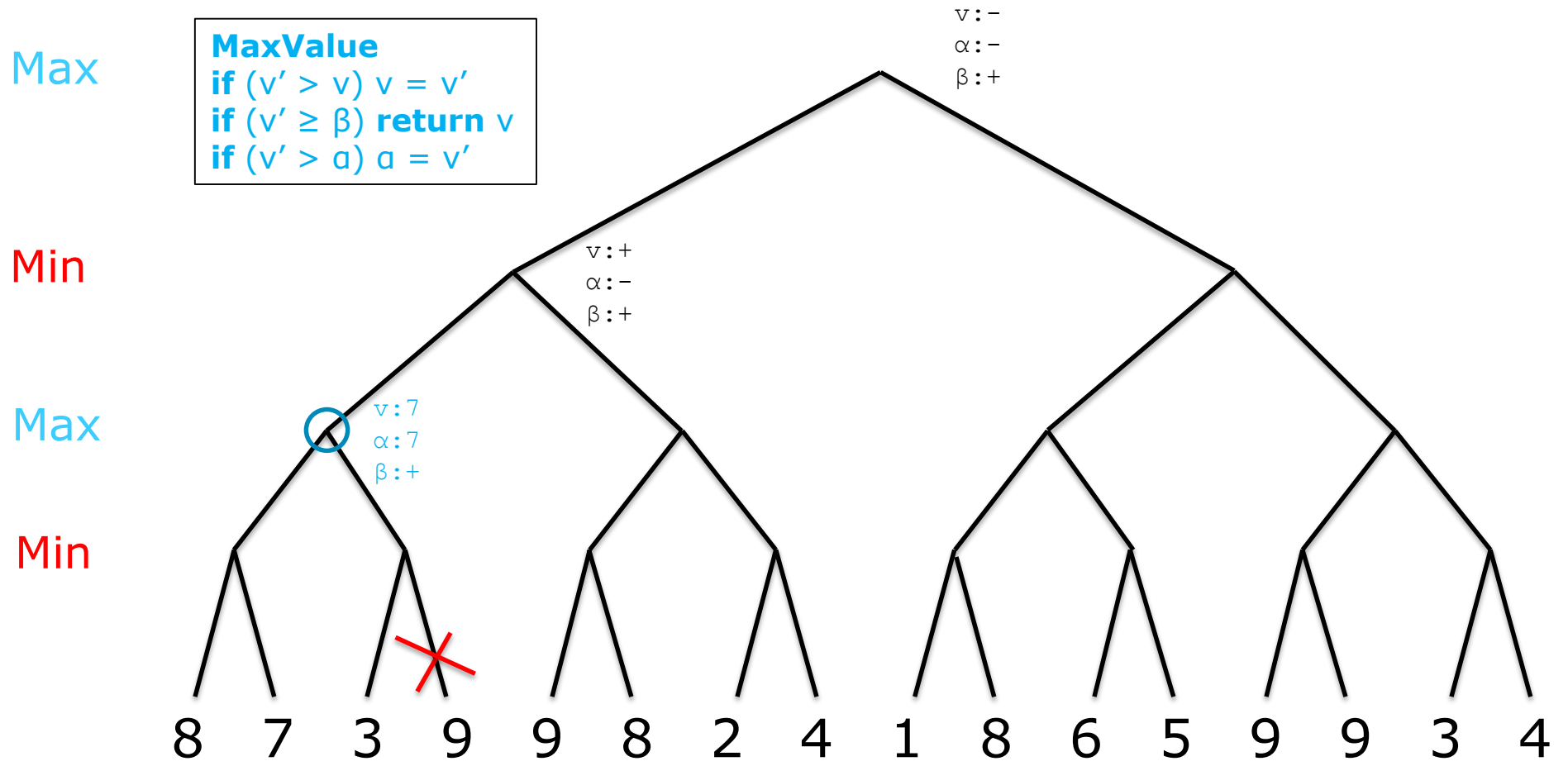

MaxValue
if (v′ > v) v = v′
if (v′ ≥ β) return v
if (v′ > α) α = v′
Max
Min
Max
Min
v : −
α : −
β : +
v : +
α : −
β : +
v : 7
α : 7
β : +
7
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

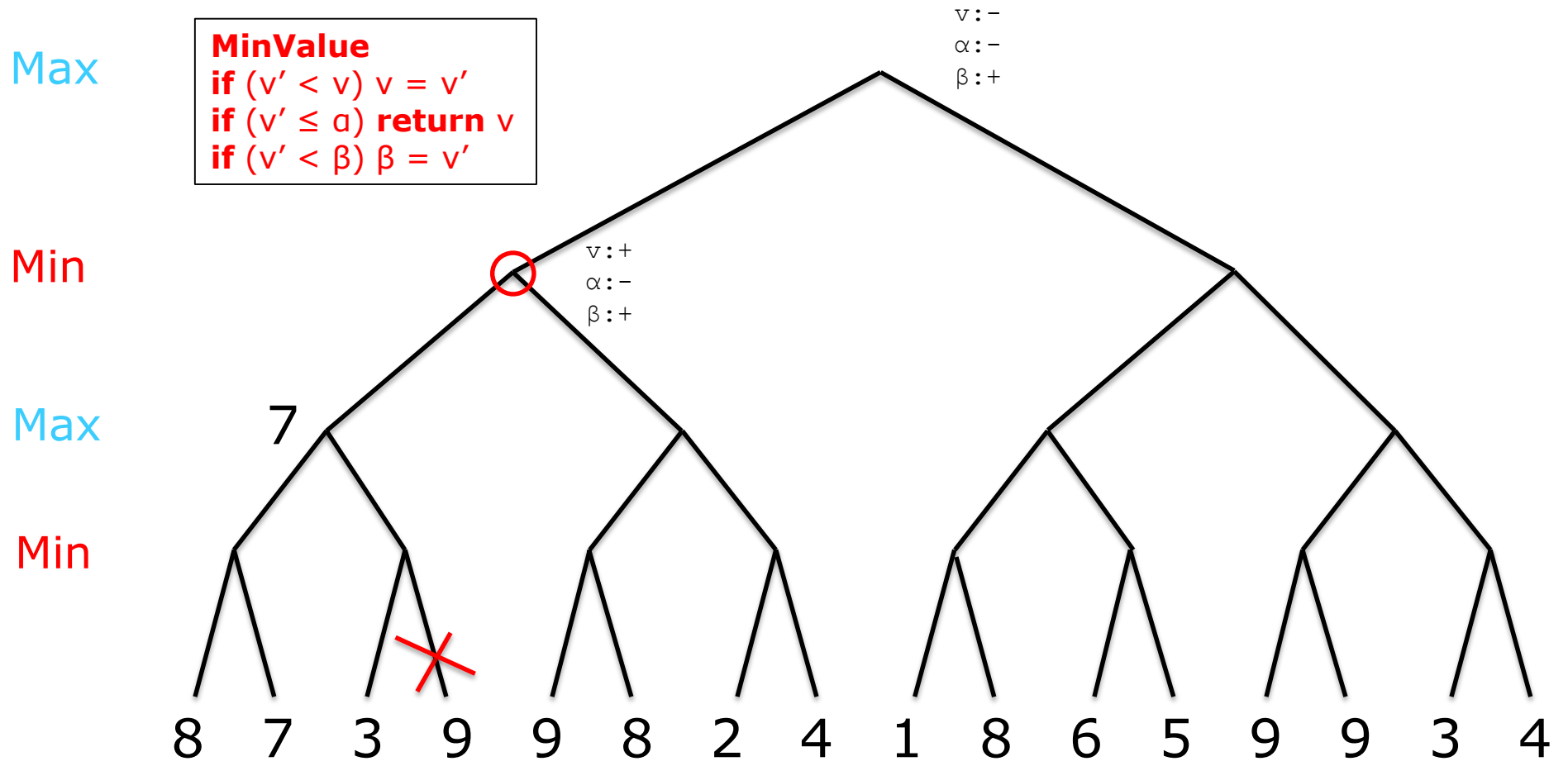

Max
Min
Max
Min
v : −
α : −
β : +
v : +
α : −
β : +
v : 7
α : 7
β : +
v : +
α : 7
β : +
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

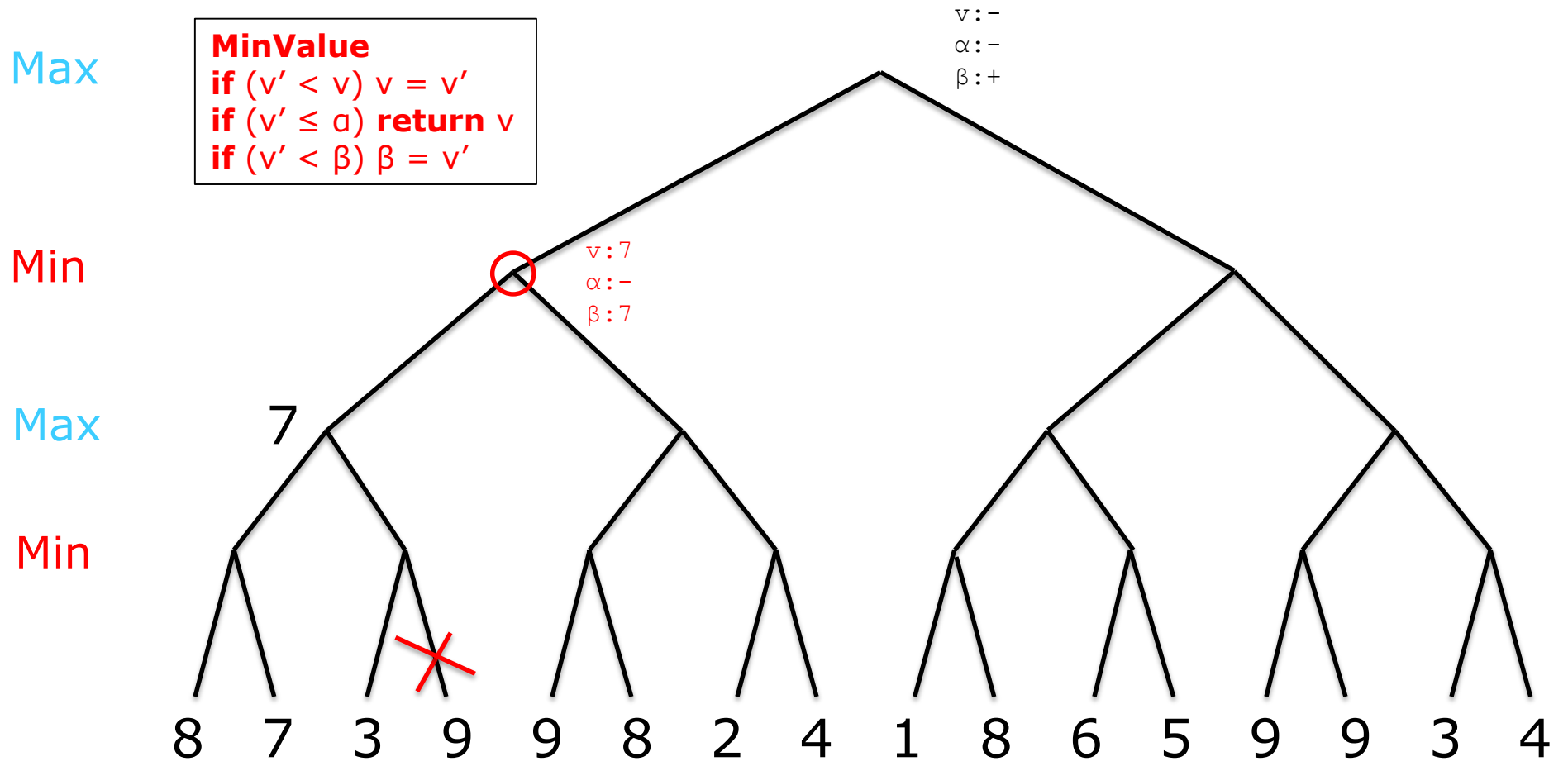

Max
Min
Max
Min
MinValue
if (v′ < v) v = v′
if (v′ ≤ α) return v
if (v′ < β) β = v′
v : −
α : −
β : +
v : +
α : −
β : +
v : 7
α : 7
β : +
v : +
α : 7
β : +
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

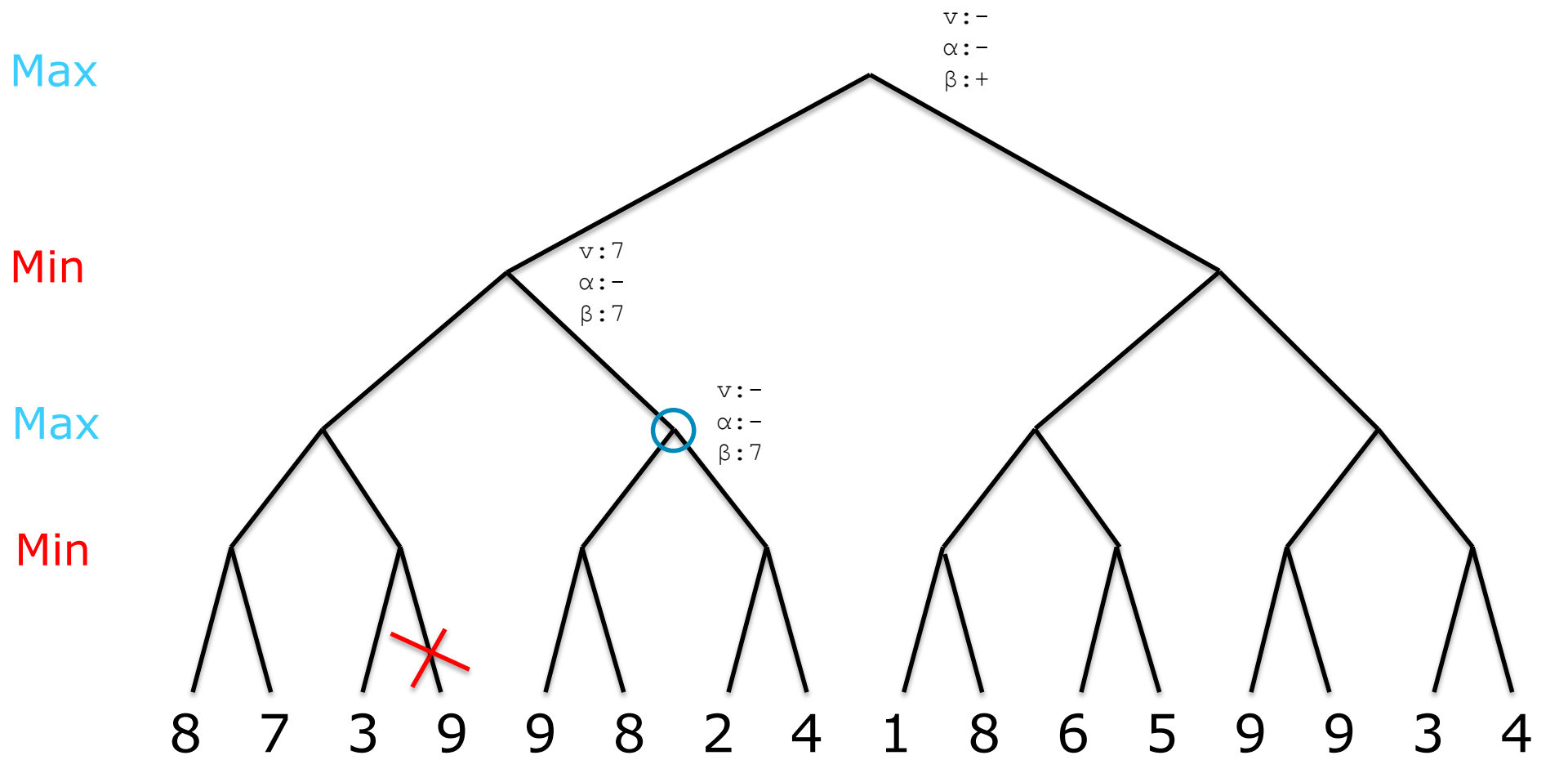

MinValue
if (v′ < v) v = v′
if (v′ ≤ α) return v
if (v′ < β) β = v′
Max
Min
Max
Min
v : −
α : −
β : +
v : +
α : −
β : +
v : 7
α : 7
β : +
v : 3
α : 7
β : +
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

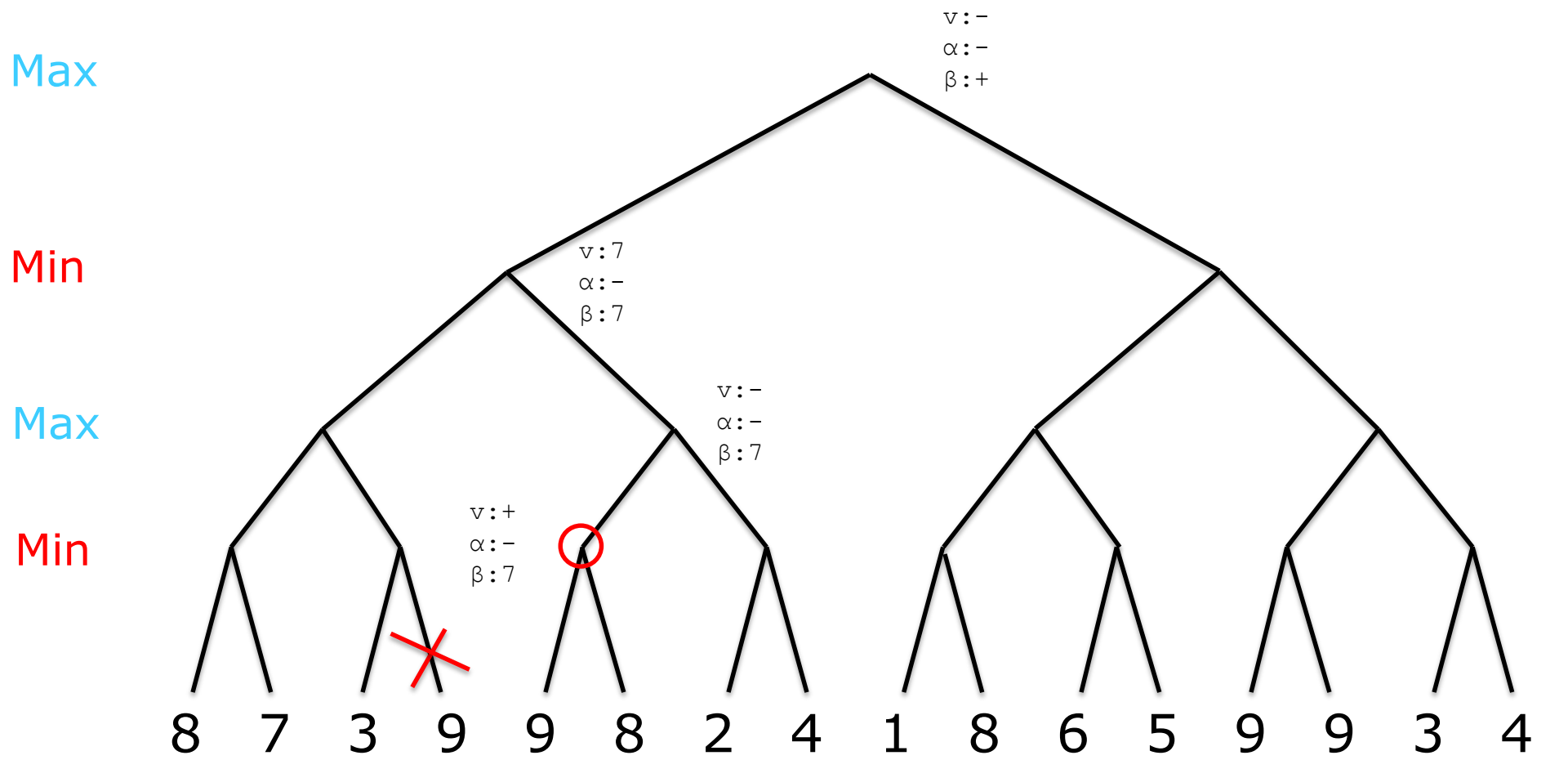

Max
MaxValue
if (v′ > v) v = v′
if (v′ ≥ β) return v
if (v′ > α) α = v′
Min
Max
Min
v : −
α : −
β : +
v : +
α : −
β : +
v : 7
α : 7
β : +
3
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

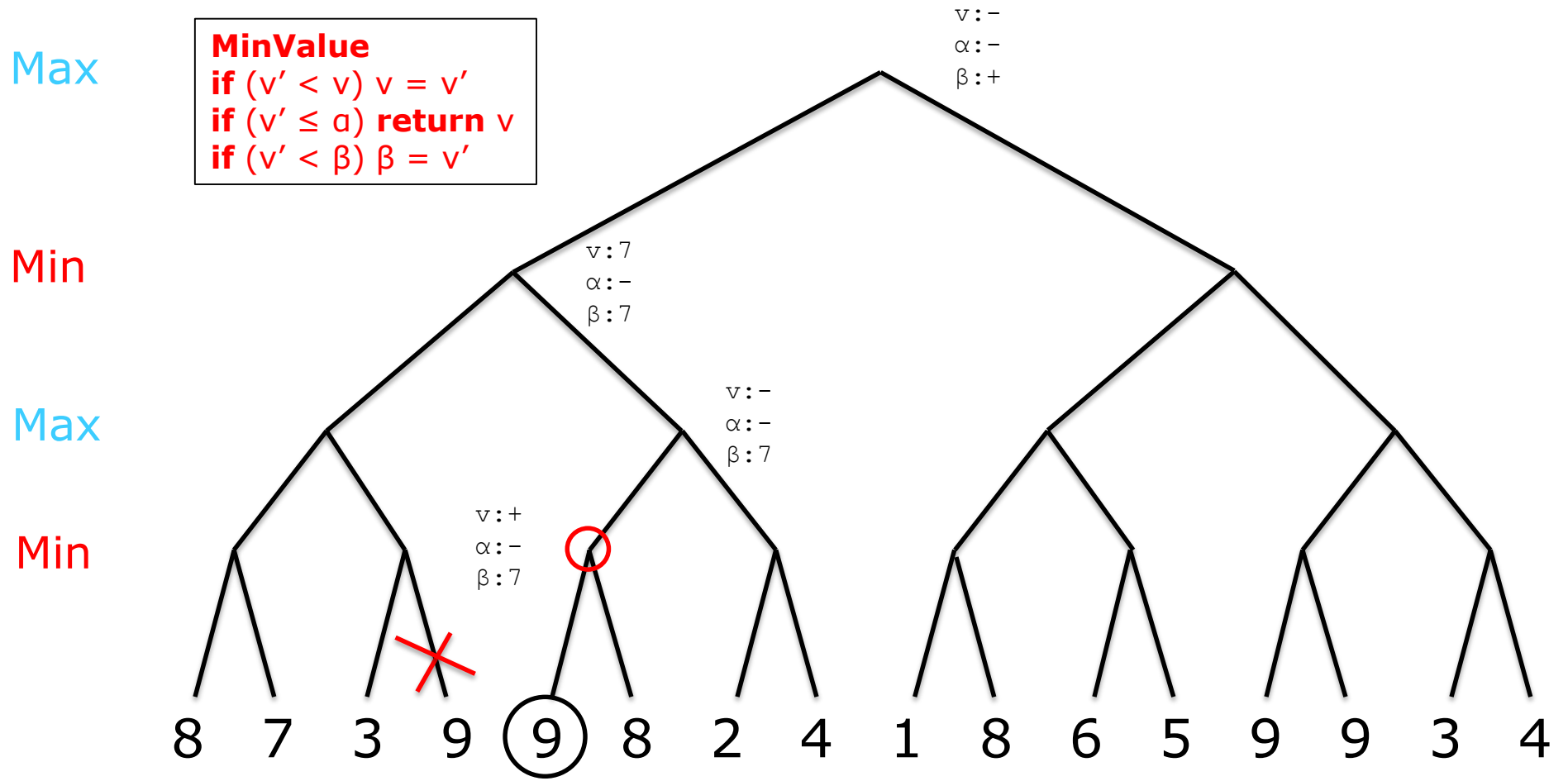

MaxValue
if (v′ > v) v = v′
if (v′ ≥ β) return v
if (v′ > α) α = v′
Max
Min
Max
Min
v : −
α : −
β : +
v : +
α : −
β : +
v : 7
α : 7
β : +
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4


MinValue
if (v′ < v) v = v′
if (v′ ≤ α) return v
if (v′ < β) β = v′
Max
Min
Max
Min
v : −
α : −
β : +
v : +
α : −
β : +
7
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4


MinValue
if (v′ < v) v = v′
if (v′ ≤ α) return v
if (v′ < β) β = v′
Max
Min
Max
Min
v : −
α : −
β : +
v : 7
α : −
β : 7
7
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4


Max
Min
Max
Min
v : –
α : –
β : +
v : 7
α : –
β : 7
v : –
α : –
β : 7
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4


Max
Min
Max
Min
v : −
α : −
β : +
v : 7
α : −
β : 7
v : −
α : −
β : 7
v : +
α : −
β : 7
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4


Max
Min
Max
Min
MinValue
if (v′ < v) v = v′
if (v′ ≤ α) return v
if (v′ < β) β = v′
v : −
α : −
β : +
v : 7
α : −
β : 7
v : −
α : −
β : 7
v : +
α : −
β : 7
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

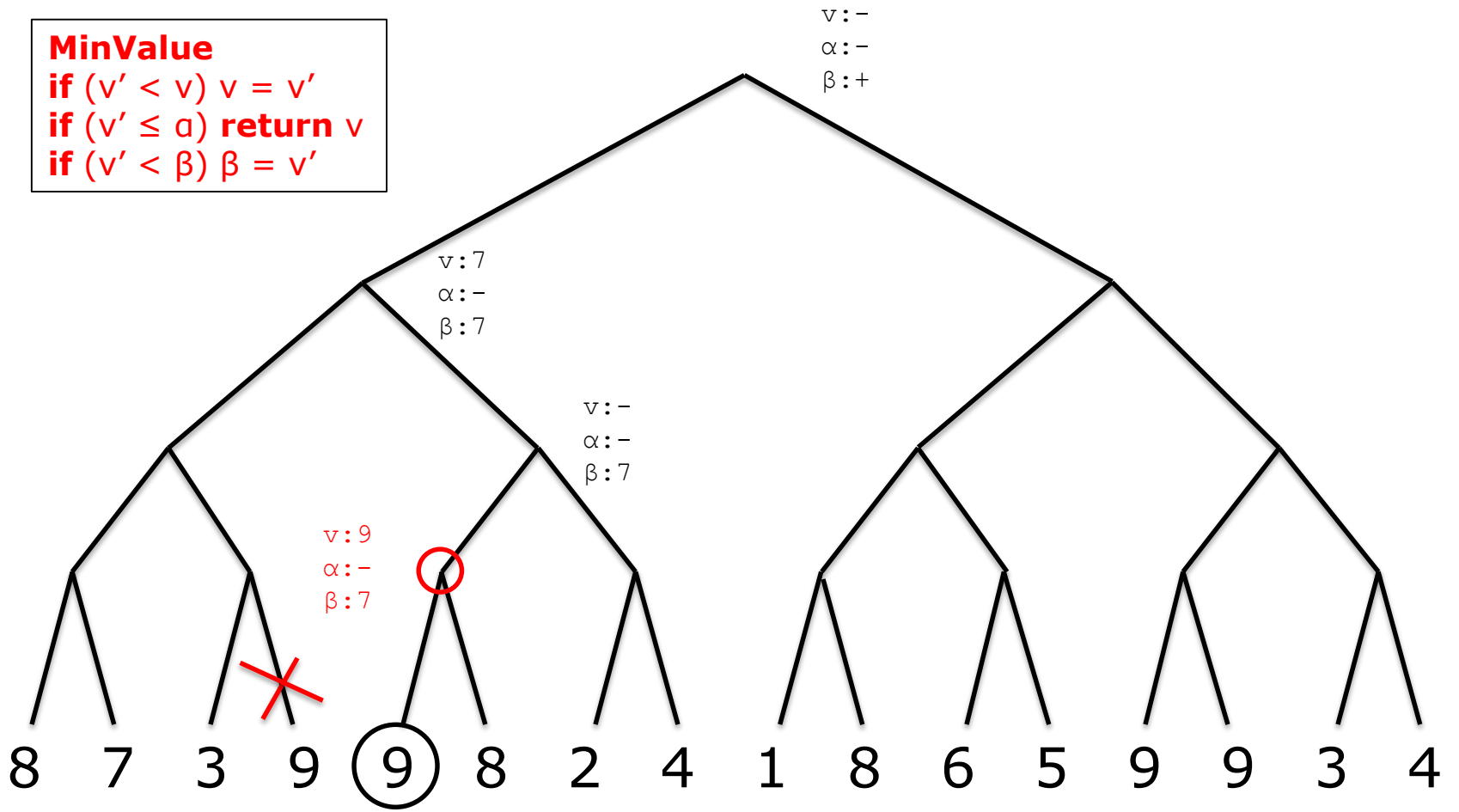
Max

Min

Max

Min

MinValue
if (v′ < v) v = v′
if (v′ ≤ α) **return** v
if (v′ < β) β = v′



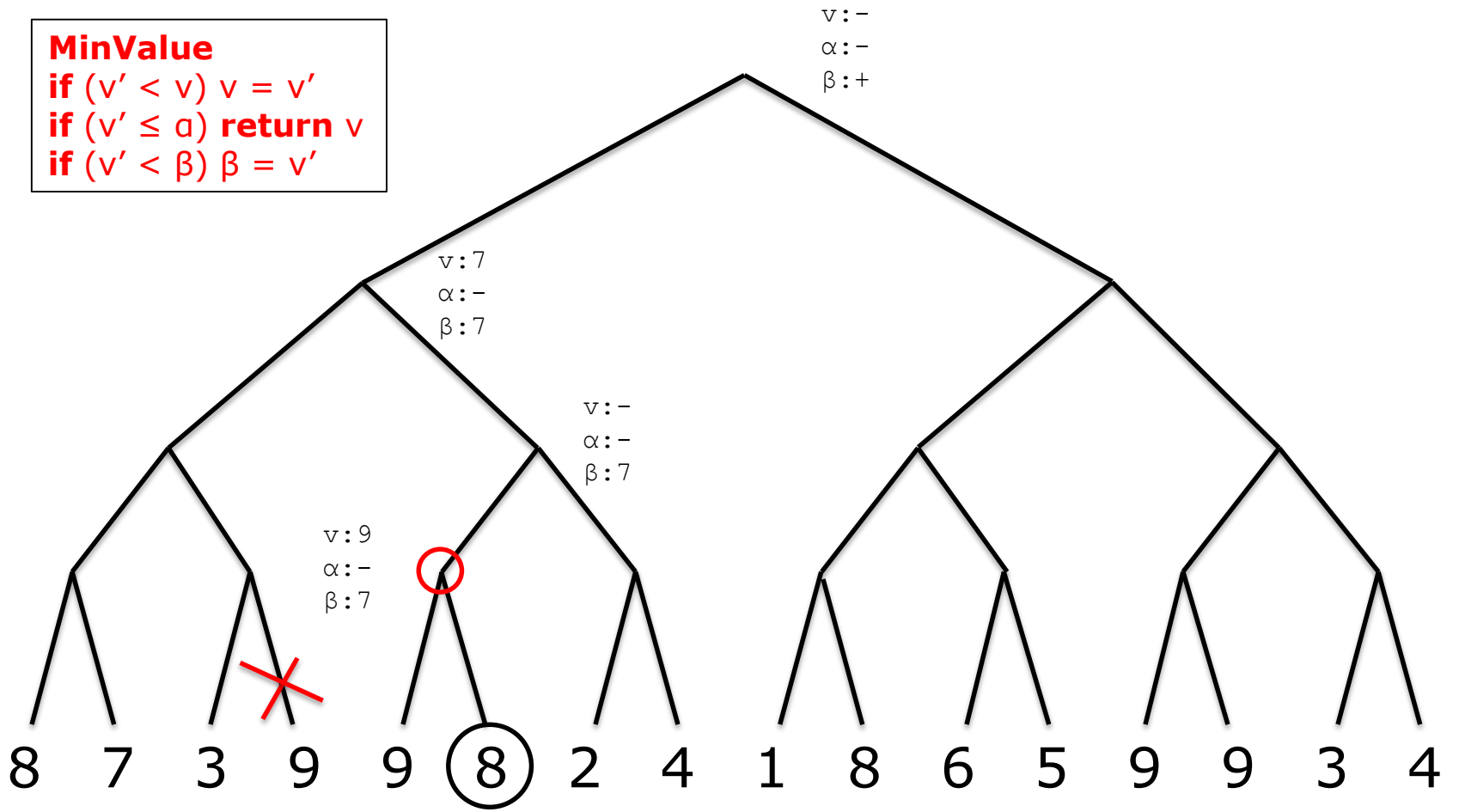
Max

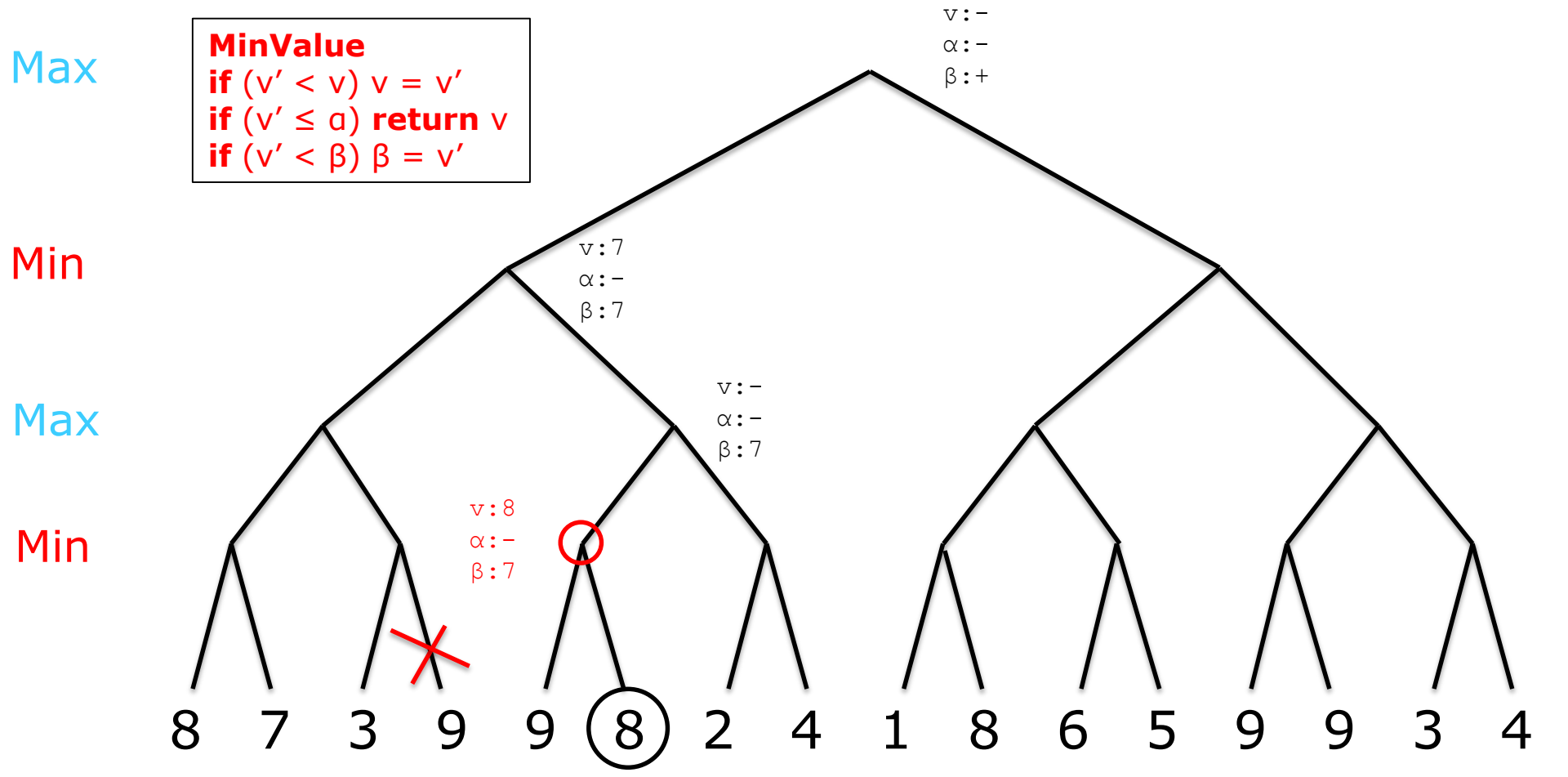
Min

Max

Min

MinValue
if (v′ < v) v = v′
if (v′ ≤ α) **return** v
if (v′ < β) β = v′




Max
MinValue
if (v′ < v) v = v′
if (v′ ≤ α) return v
if (v′ < β) β = v′
Min
Max
Min
v : −
α : −
β : +
v : 7
α : −
β : 7
v : −
α : −
β : 7
v : 8
α : −
β : 7
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

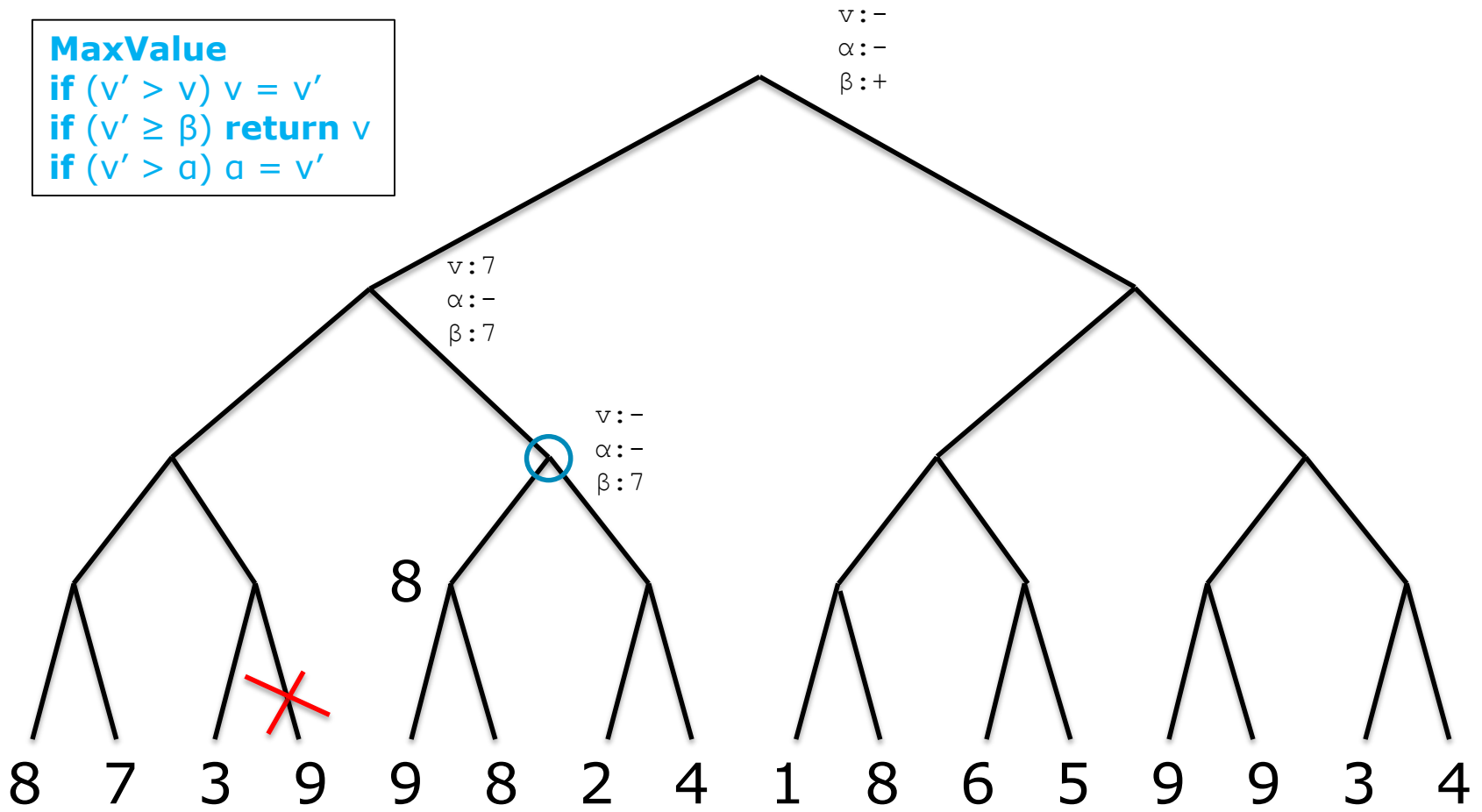
Max

MaxValue
if (v′ > v) v = v′
if (v′ ≥ β) **return** v
if (v′ > α) α = v′

Min

Max

Min



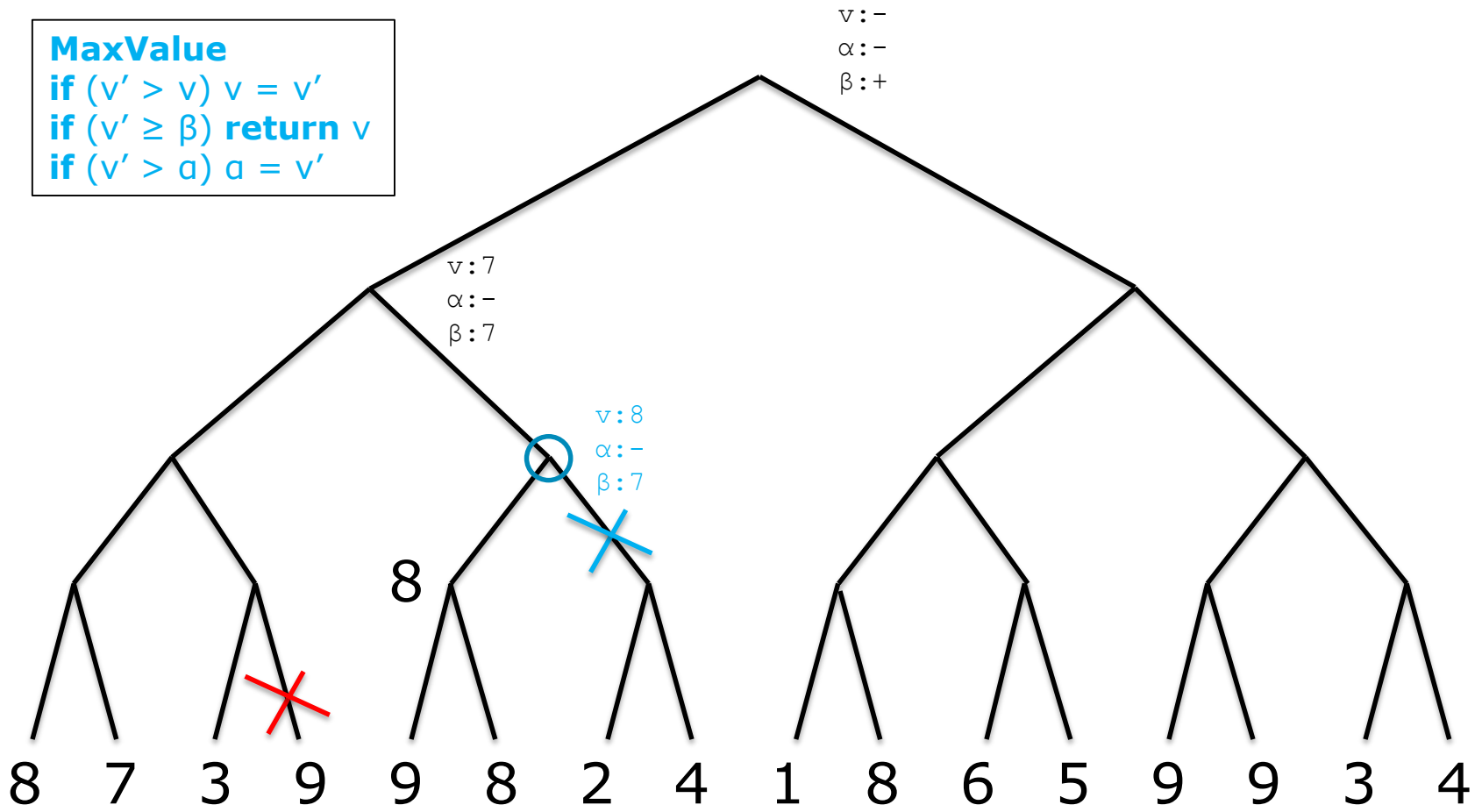
Max

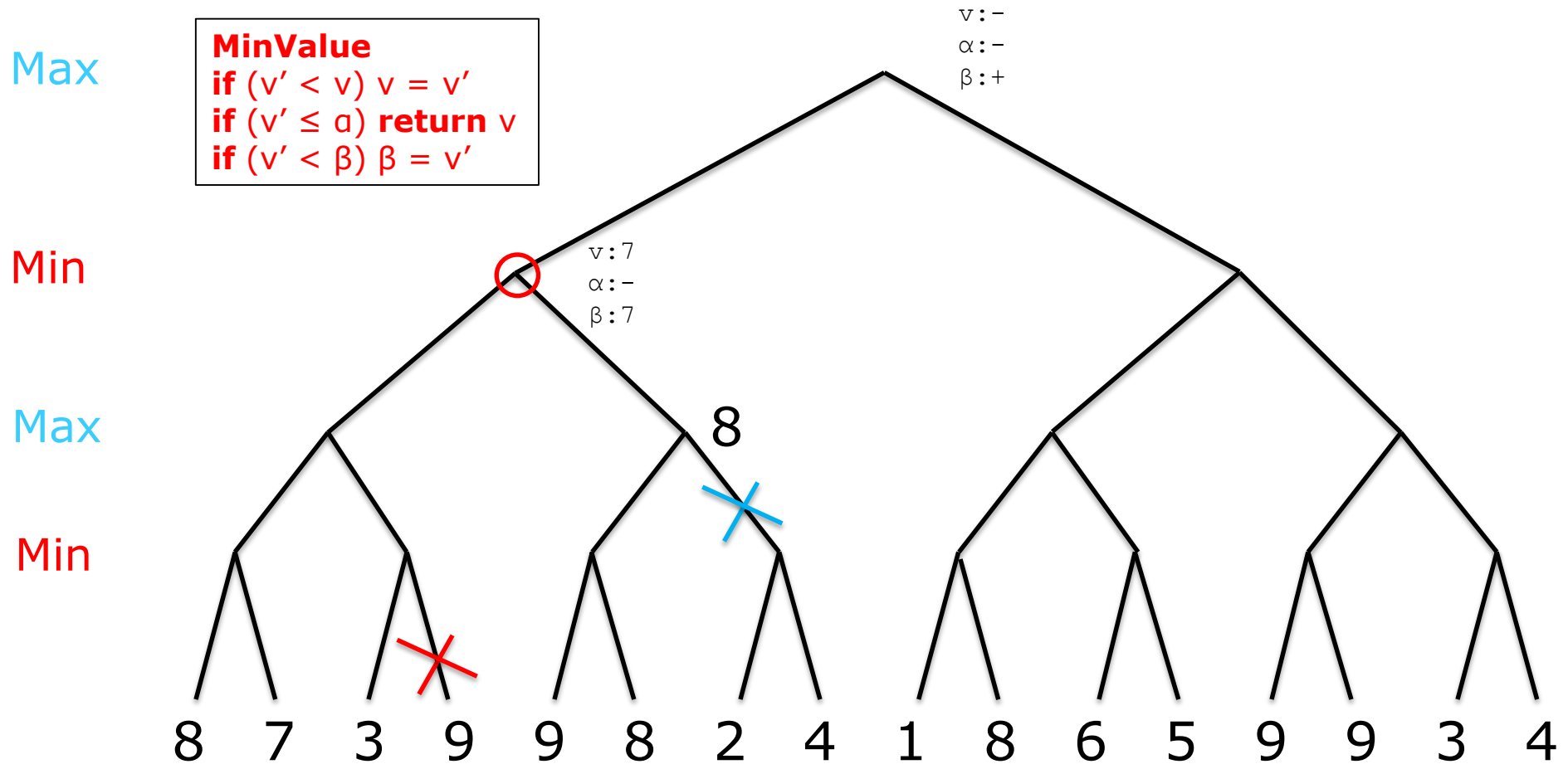
Min

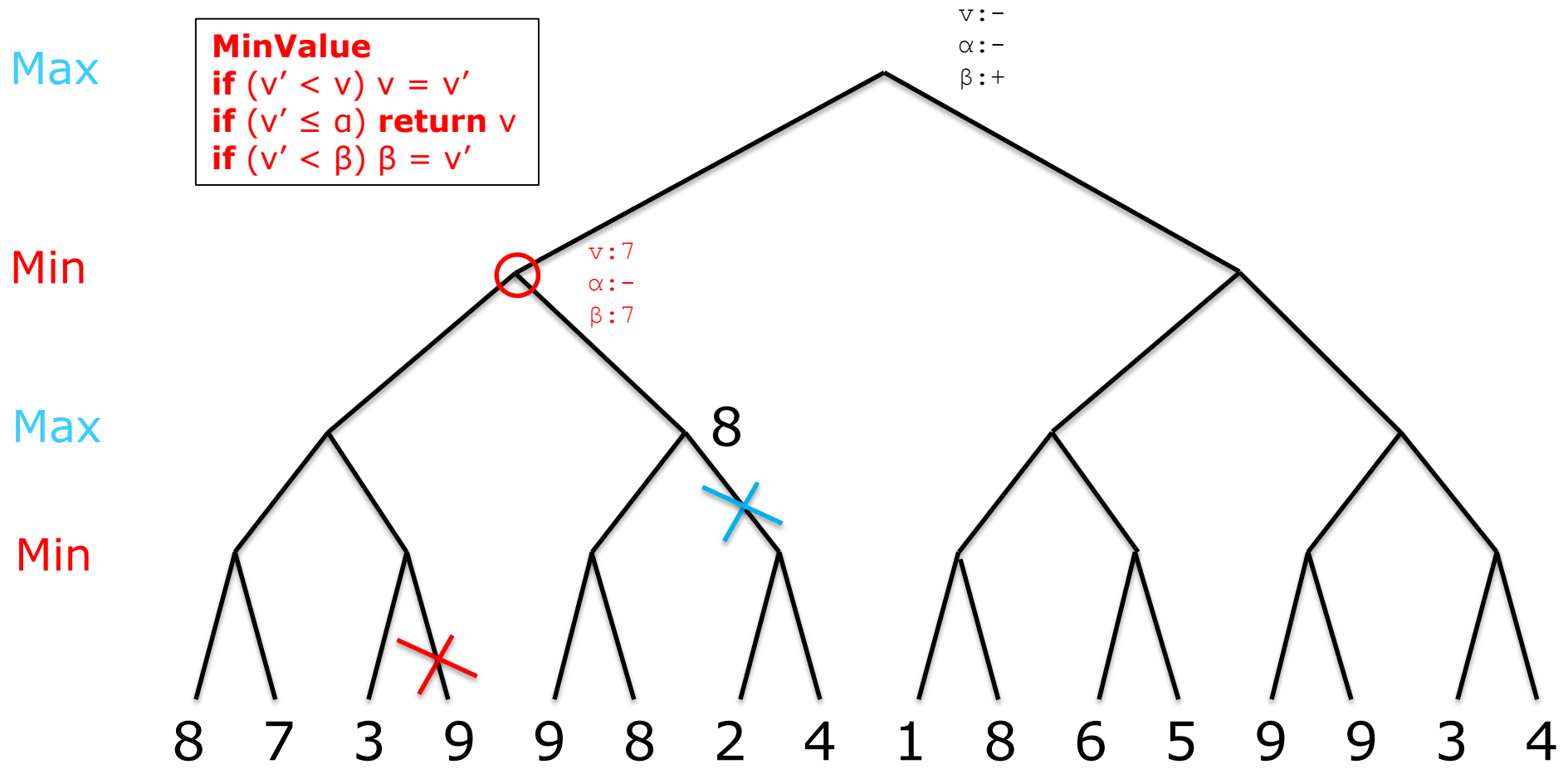
Max

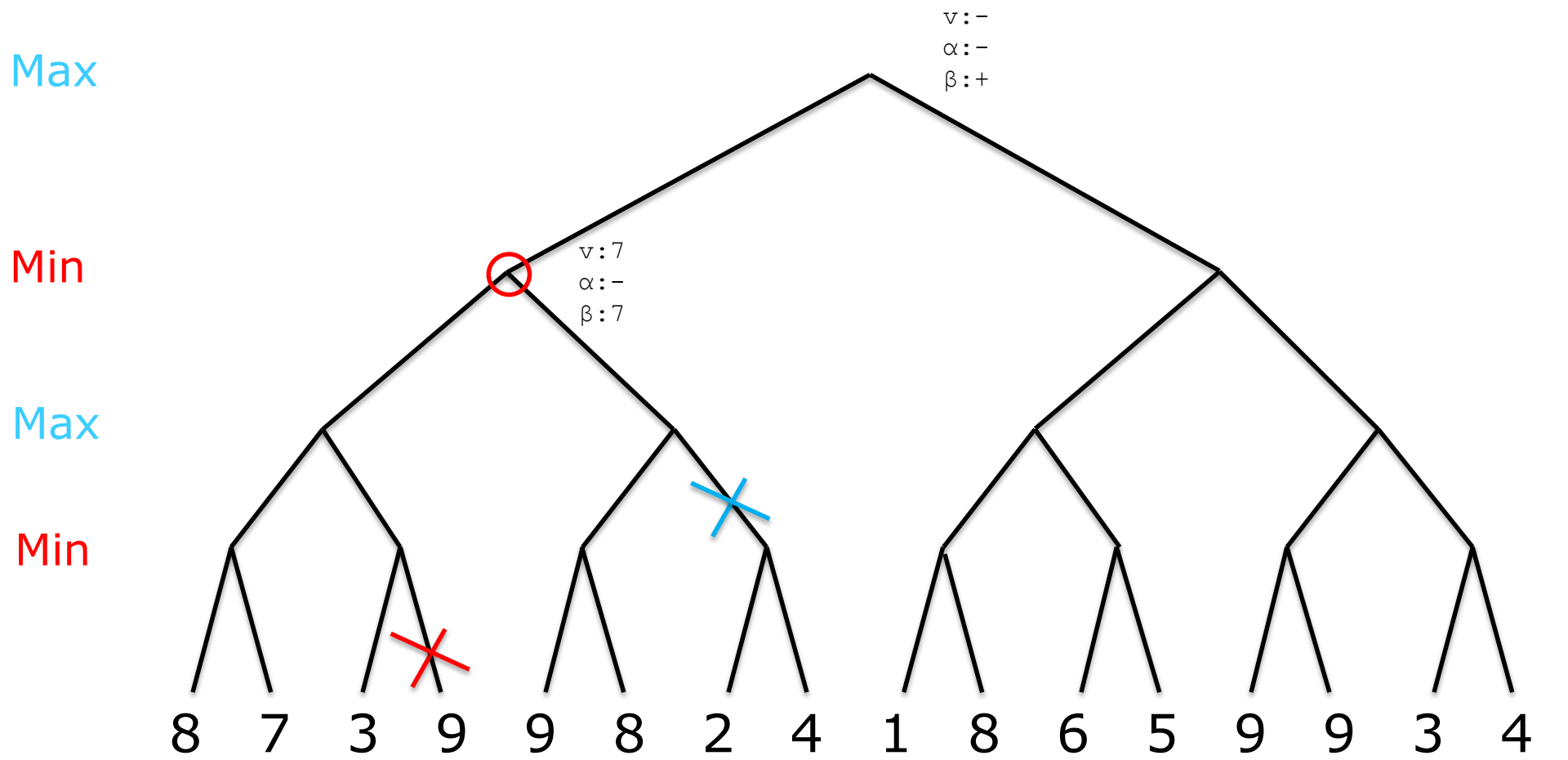
Min

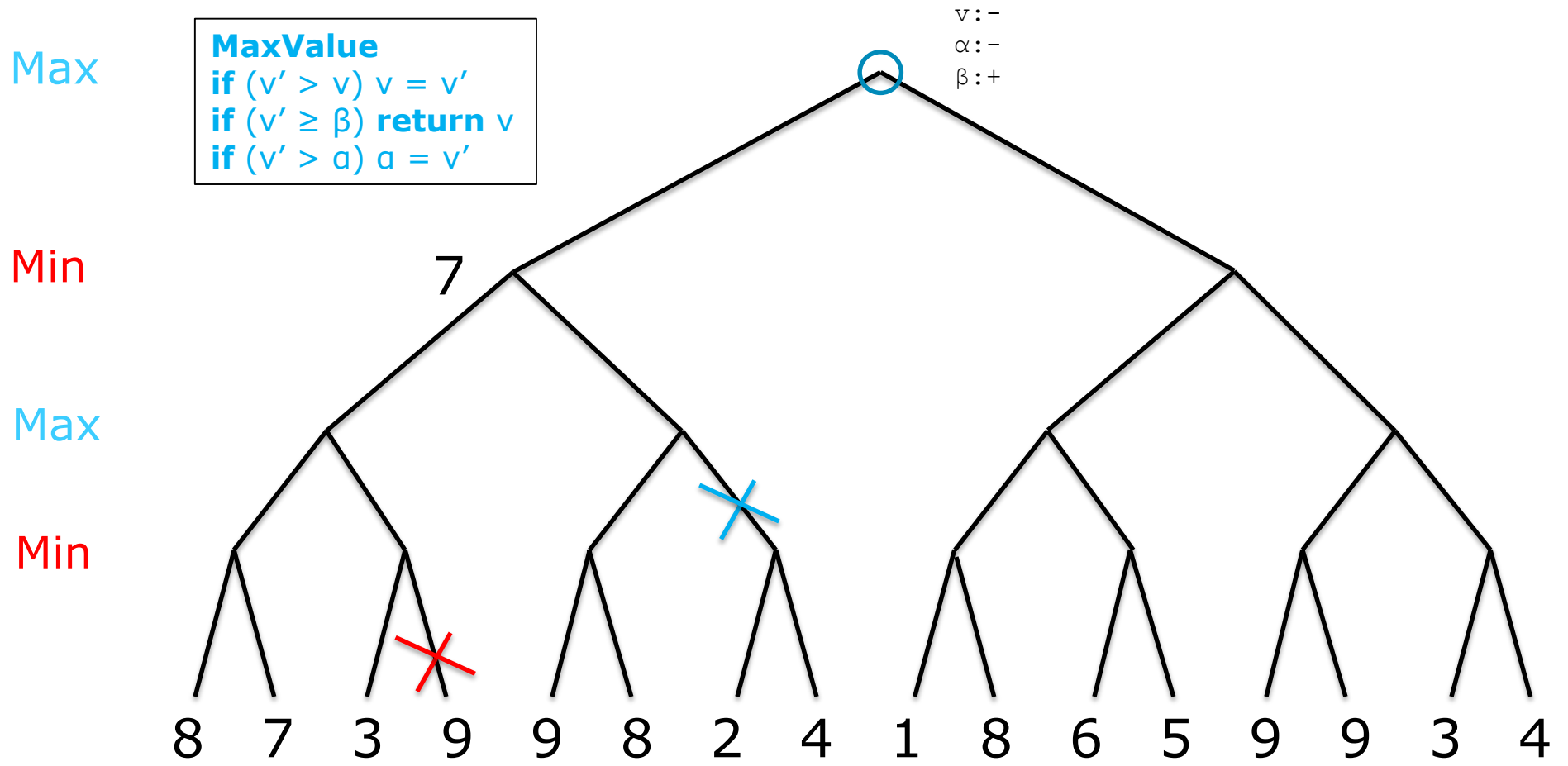
MaxValue
if (v′ > v) v = v′
if (v′ ≥ β) **return** v
if (v′ > α) α = v′




MinValue
if (v′ < v) v = v′
if (v′ ≤ α) return v
if (v′ < β) β = v′
Max
Min
Max
Min
v : −
α : −
β : +
v : 7
α : −
β : 7
8
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4


MinValue
if (v′ < v) v = v′
if (v′ ≤ α) return v
if (v′ < β) β = v′
Max
Min
Max
Min
v : −
α : −
β : +
v : 7
α : −
β : 7
8
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4


Max
Min
Max
Min
v : –
α : –
β : +
v : 7
α : –
β : 7
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4


MaxValue
if (v′ > v) v = v′
if (v′ ≥ β) return v
if (v′ > α) α = v′
v : −
α : −
β : +
Max
Min
Max
Min
7
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

Max

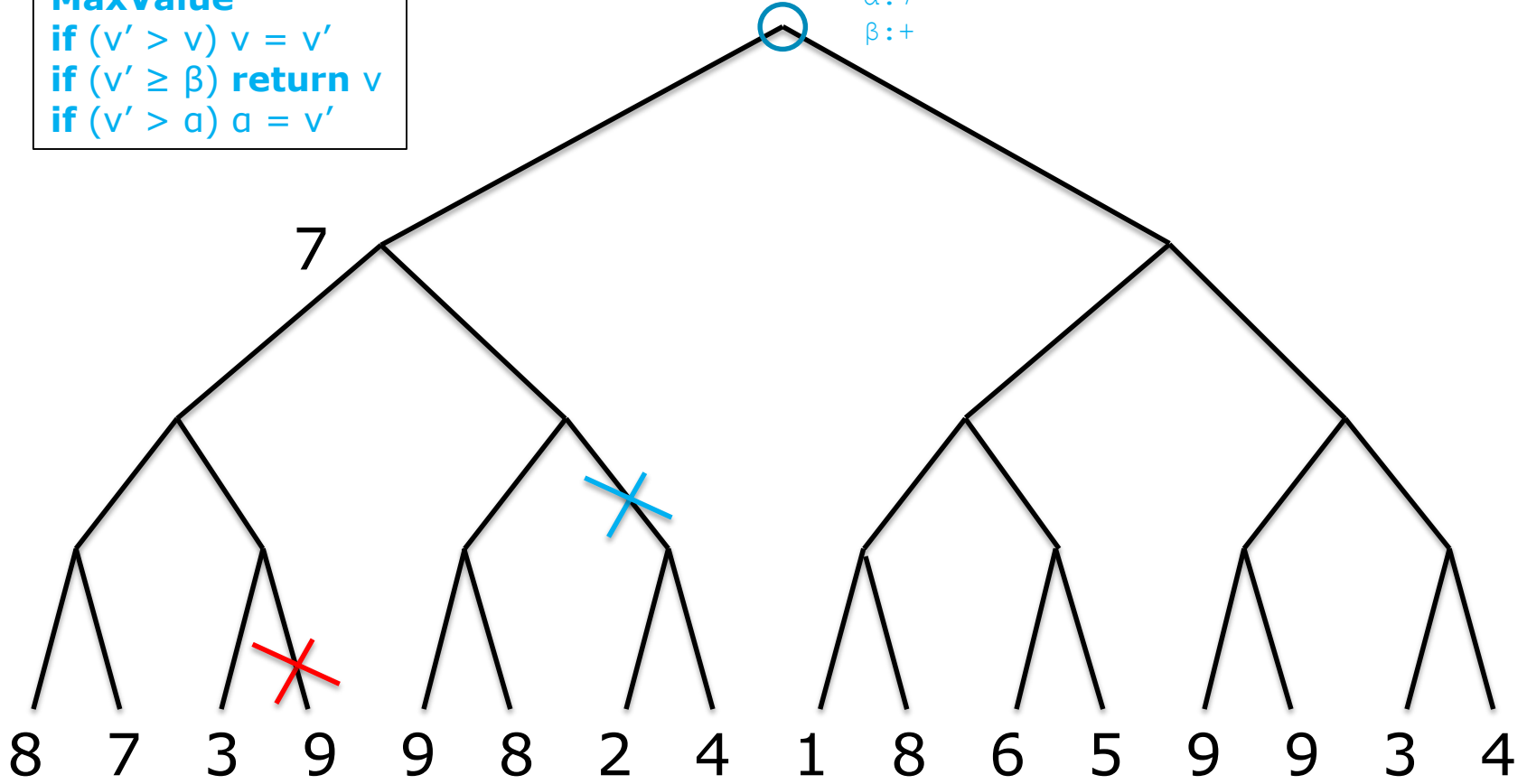
Min

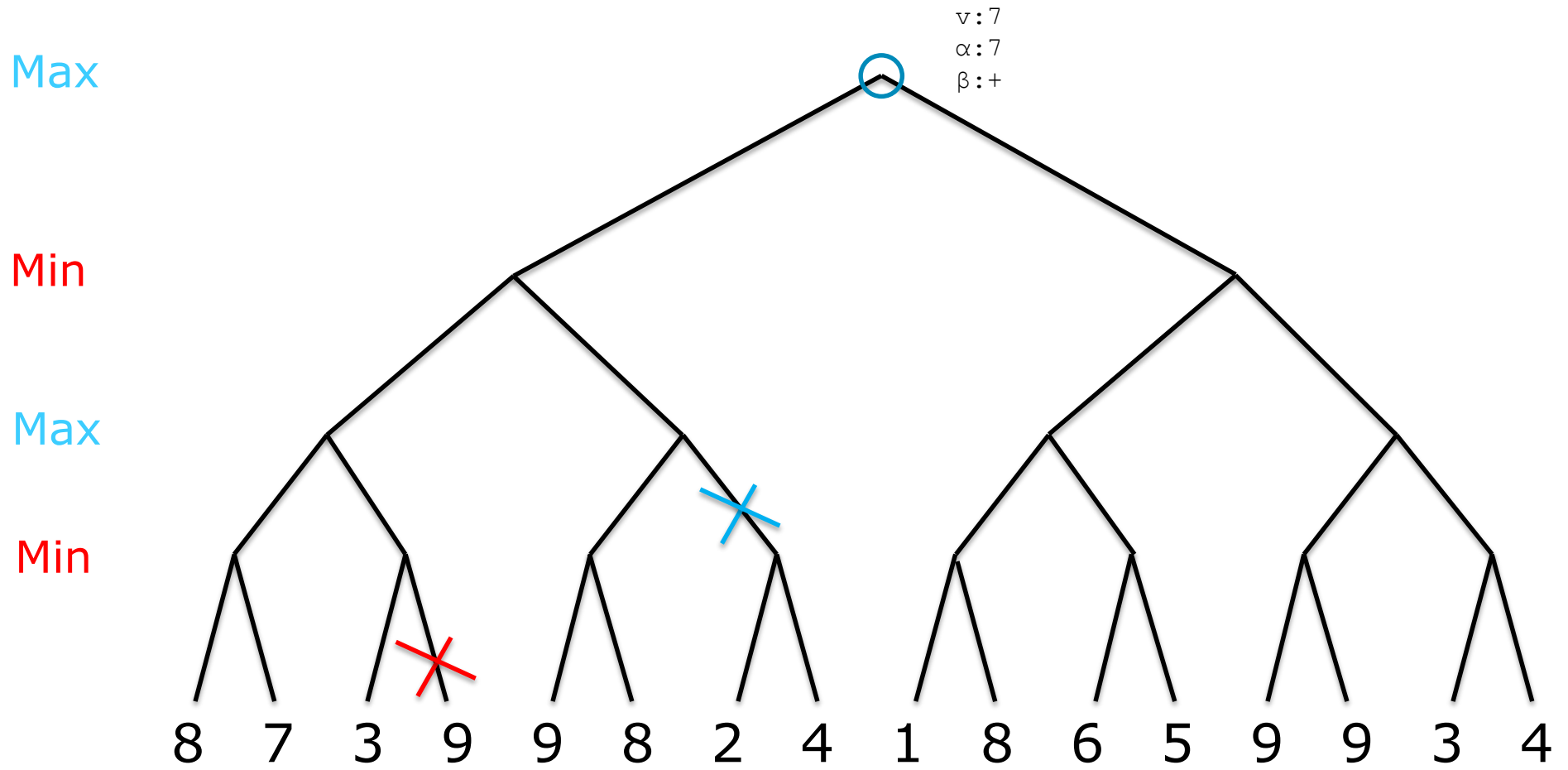
Max

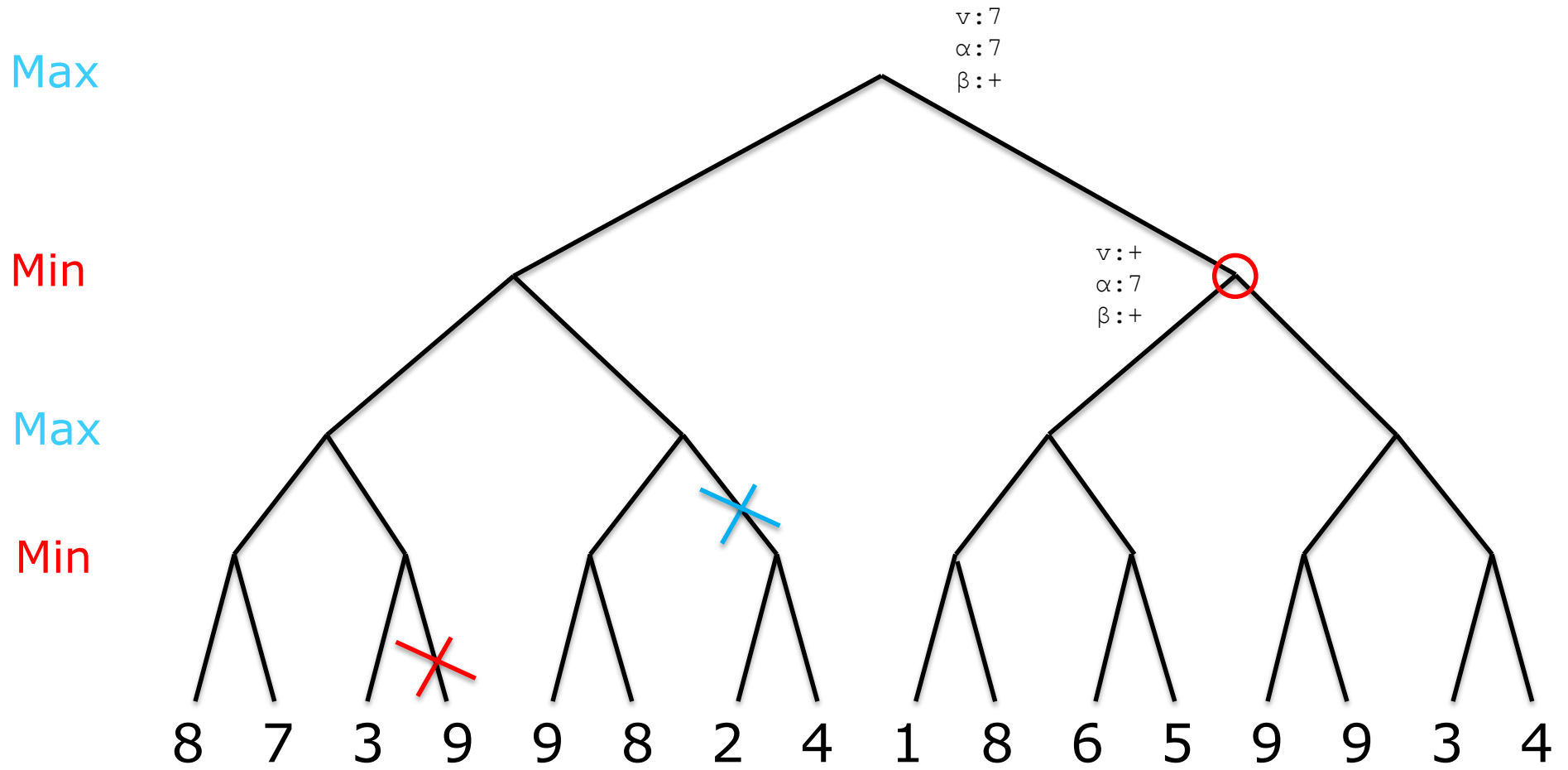
Min

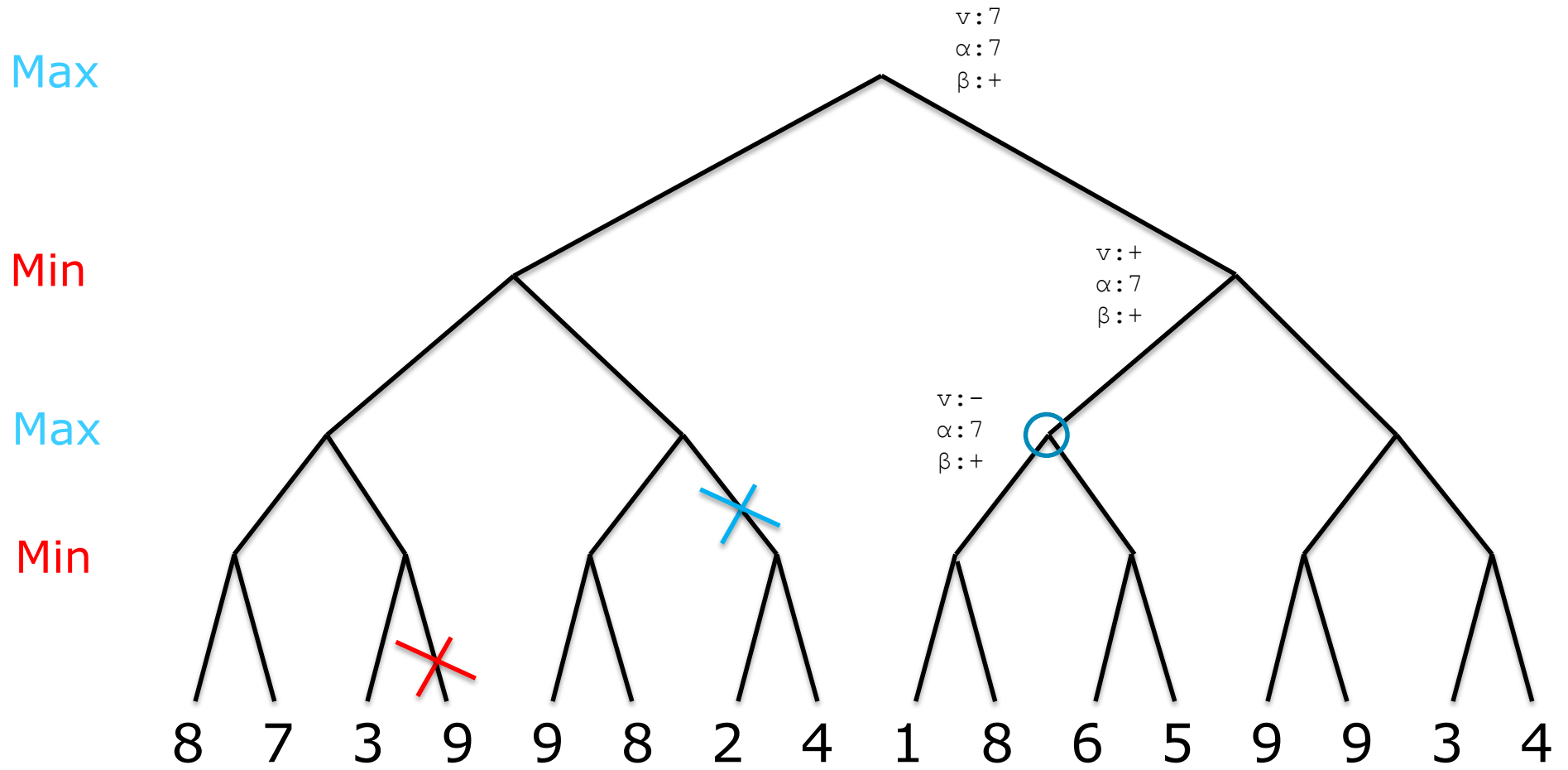
MaxValue
if (v′ > v) v = v′
if (v′ ≥ β) **return** v
if (v′ > α) α = v′

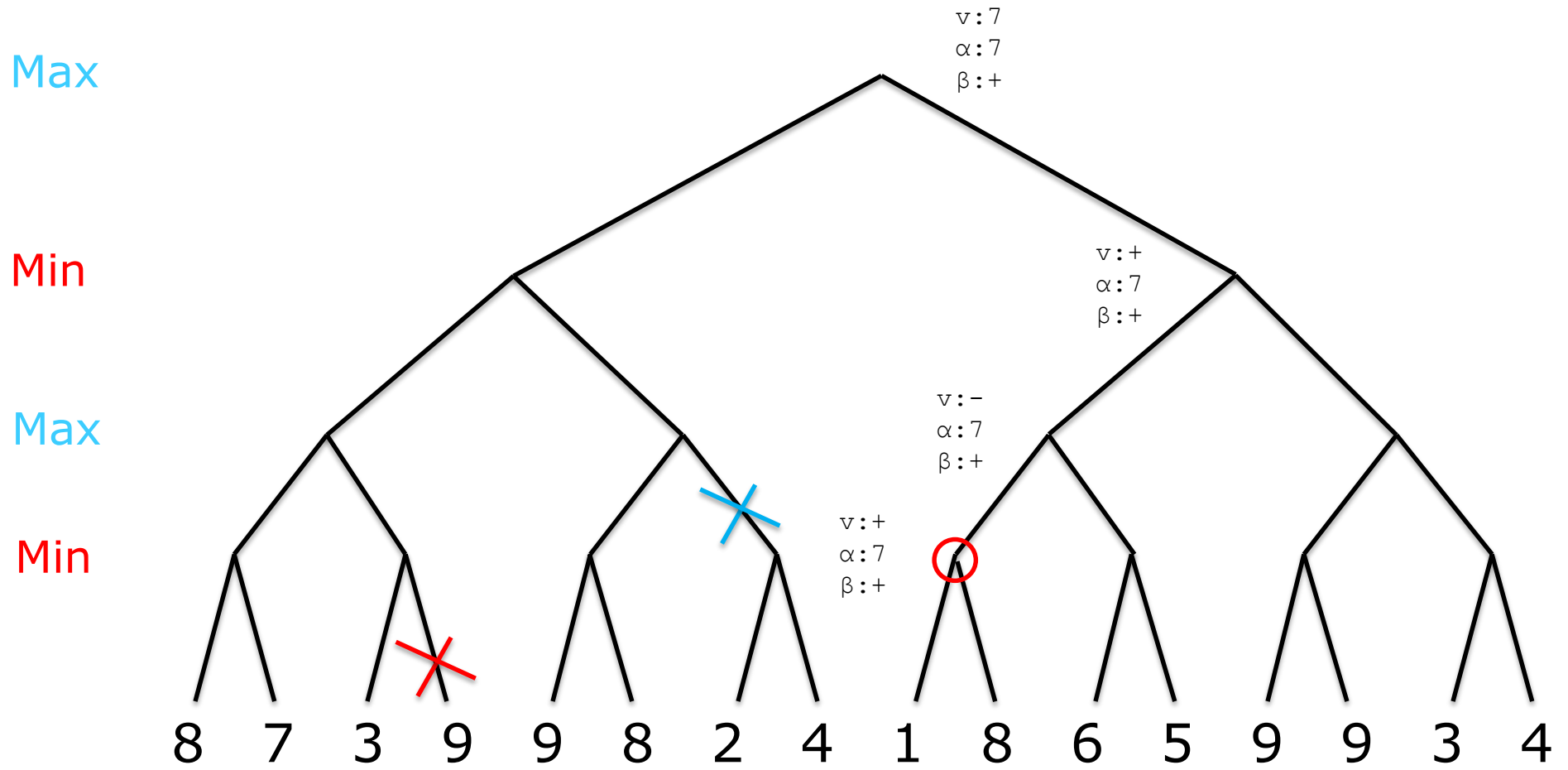
v : 7
α : 7
β : +

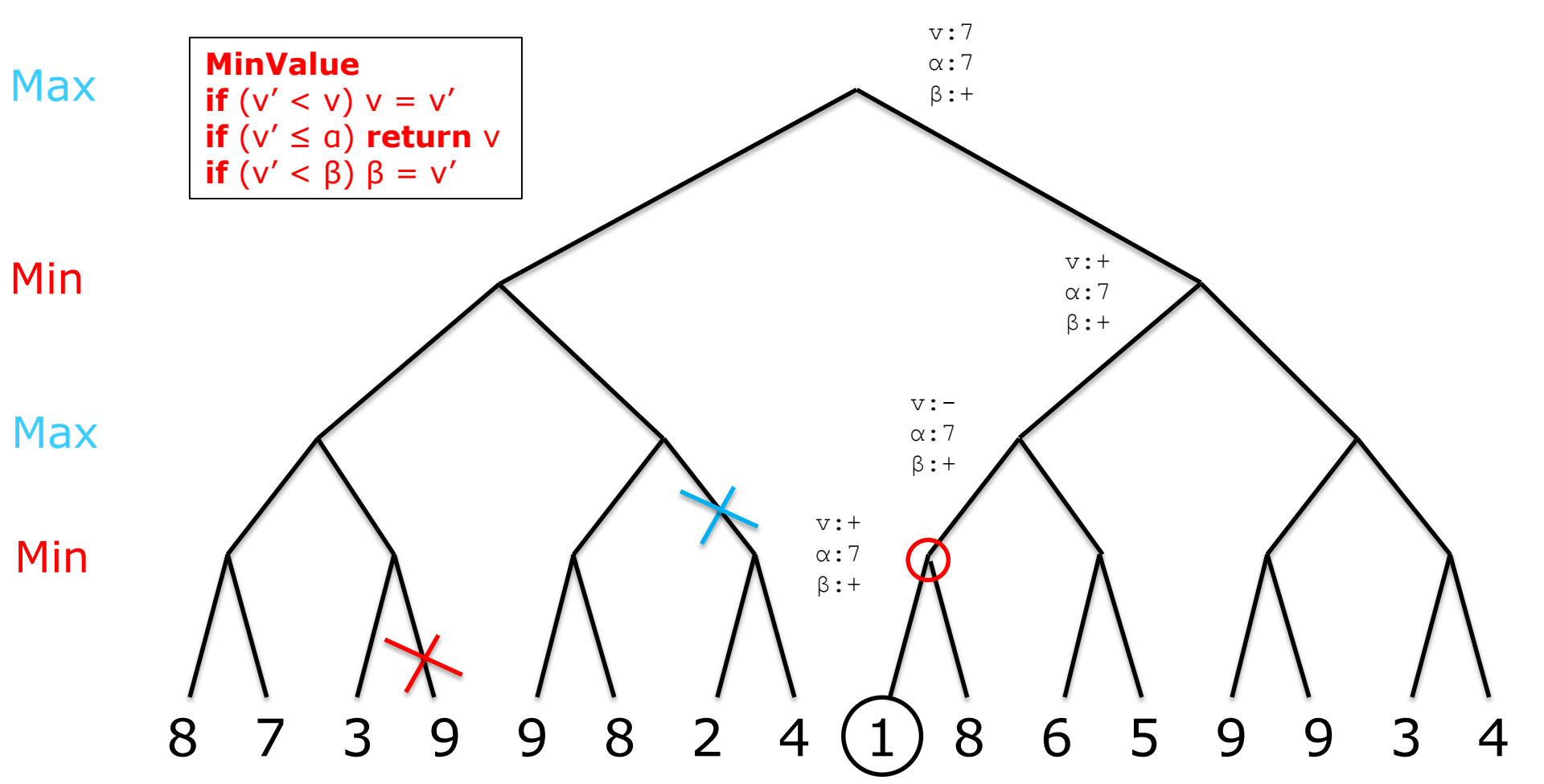


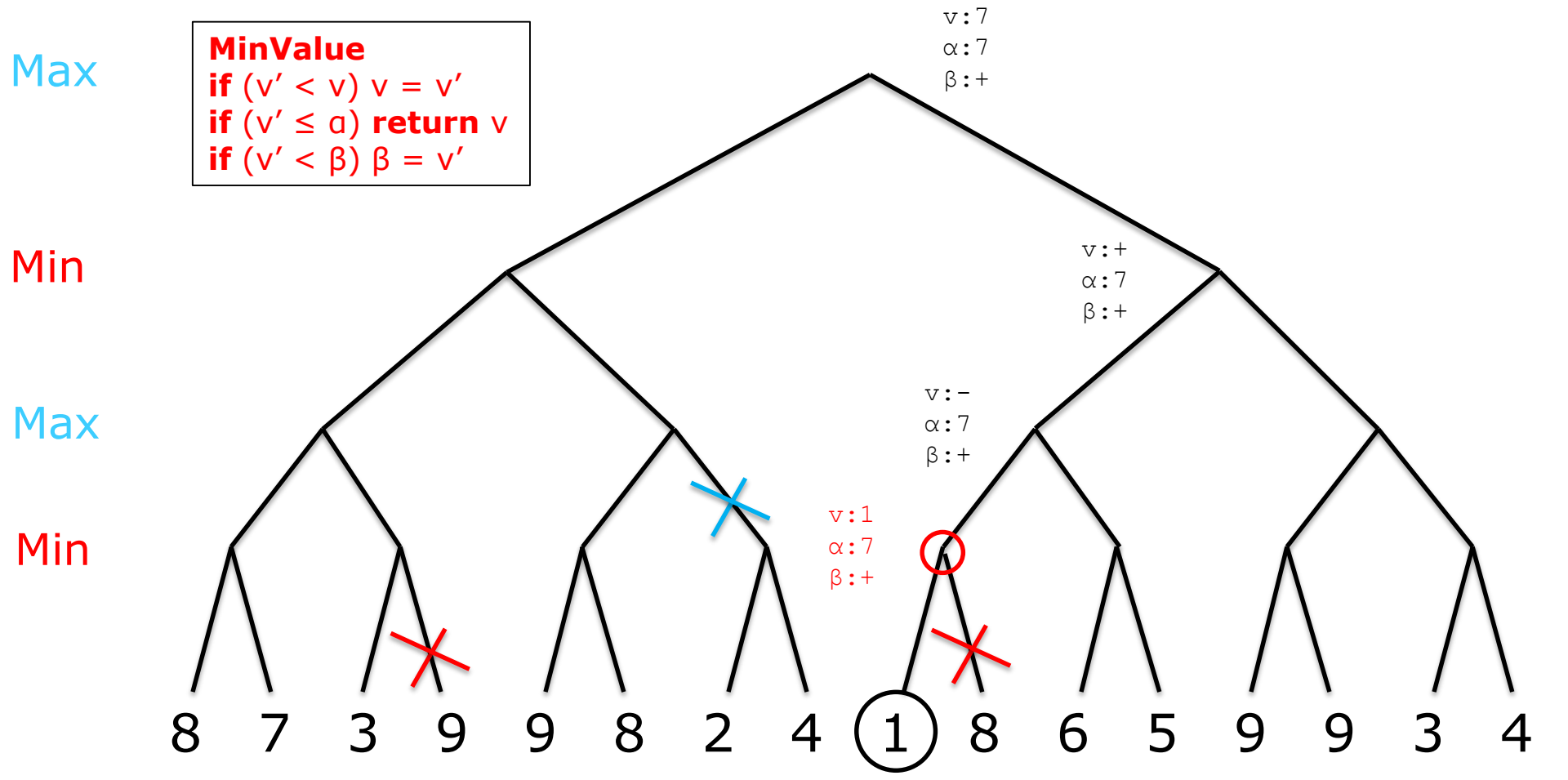

v : 7
α : 7
β : +
Max
Min
Max
Min
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

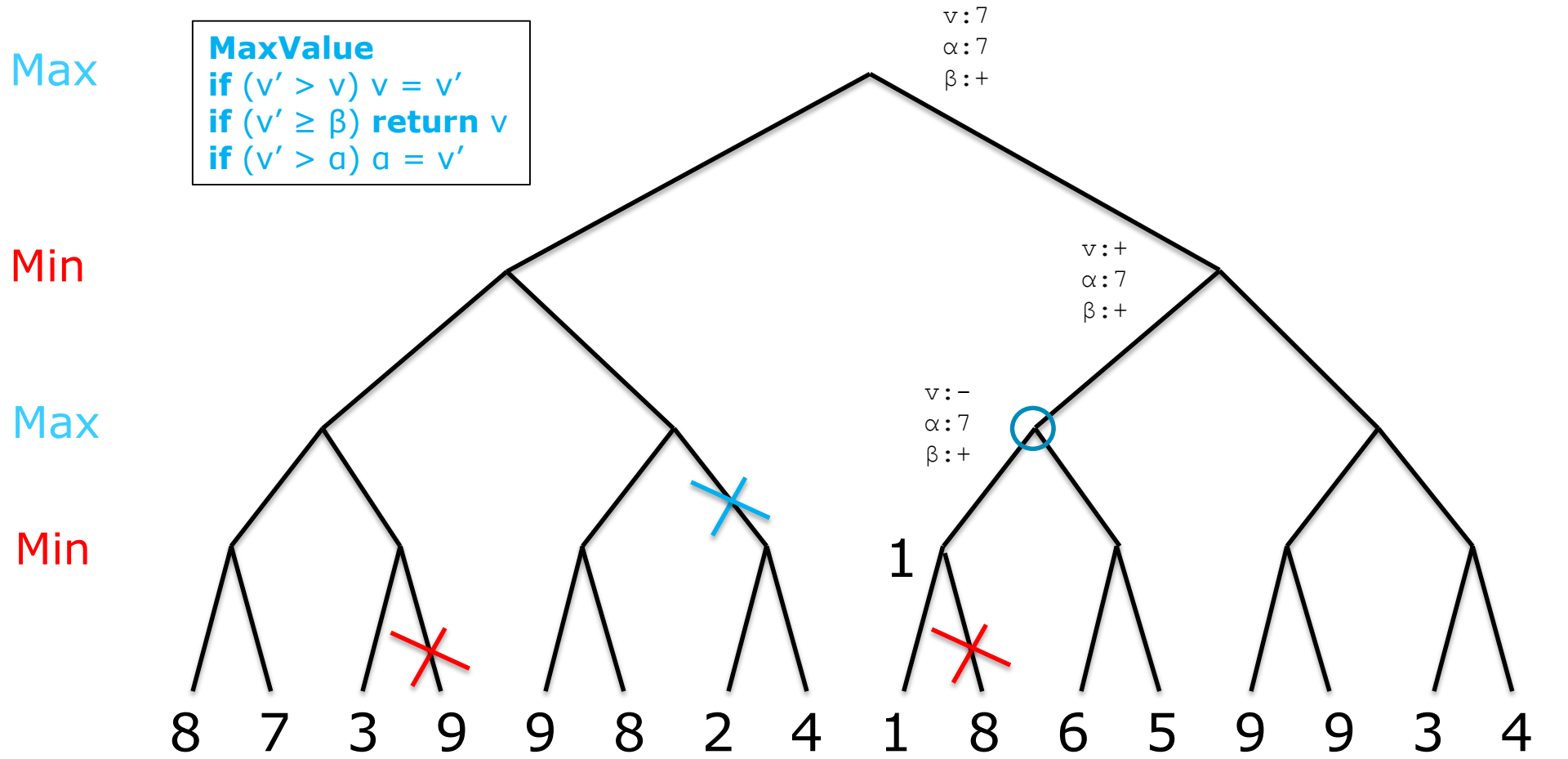

Max
Min
Max
Min
v : 7
α : 7
β : +
v : +
α : 7
β : +
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

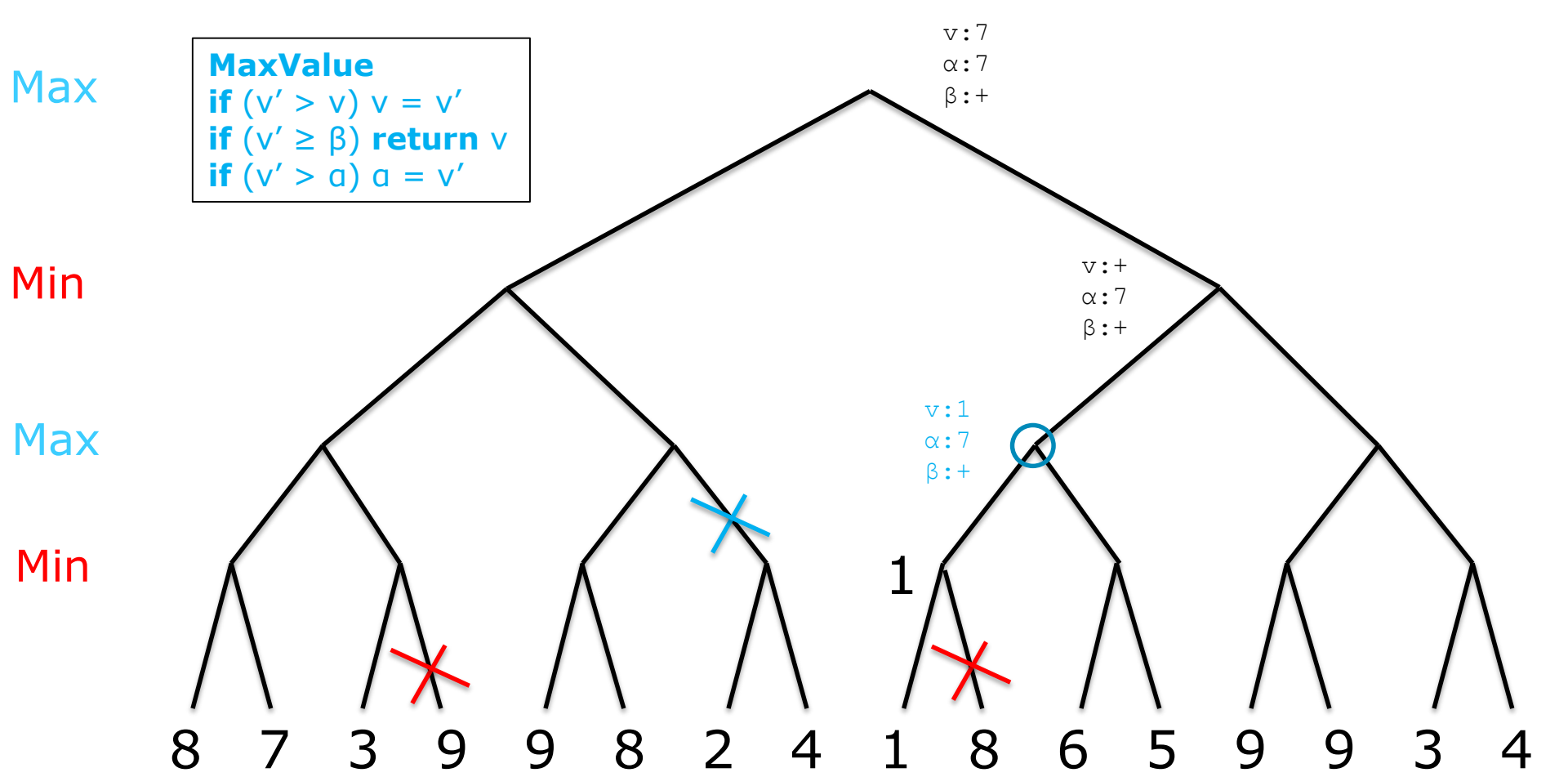

Max
Min
Max
Min
v : 7
α : 7
β : +
v : +
α : 7
β : +
v : −
α : 7
β : +
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

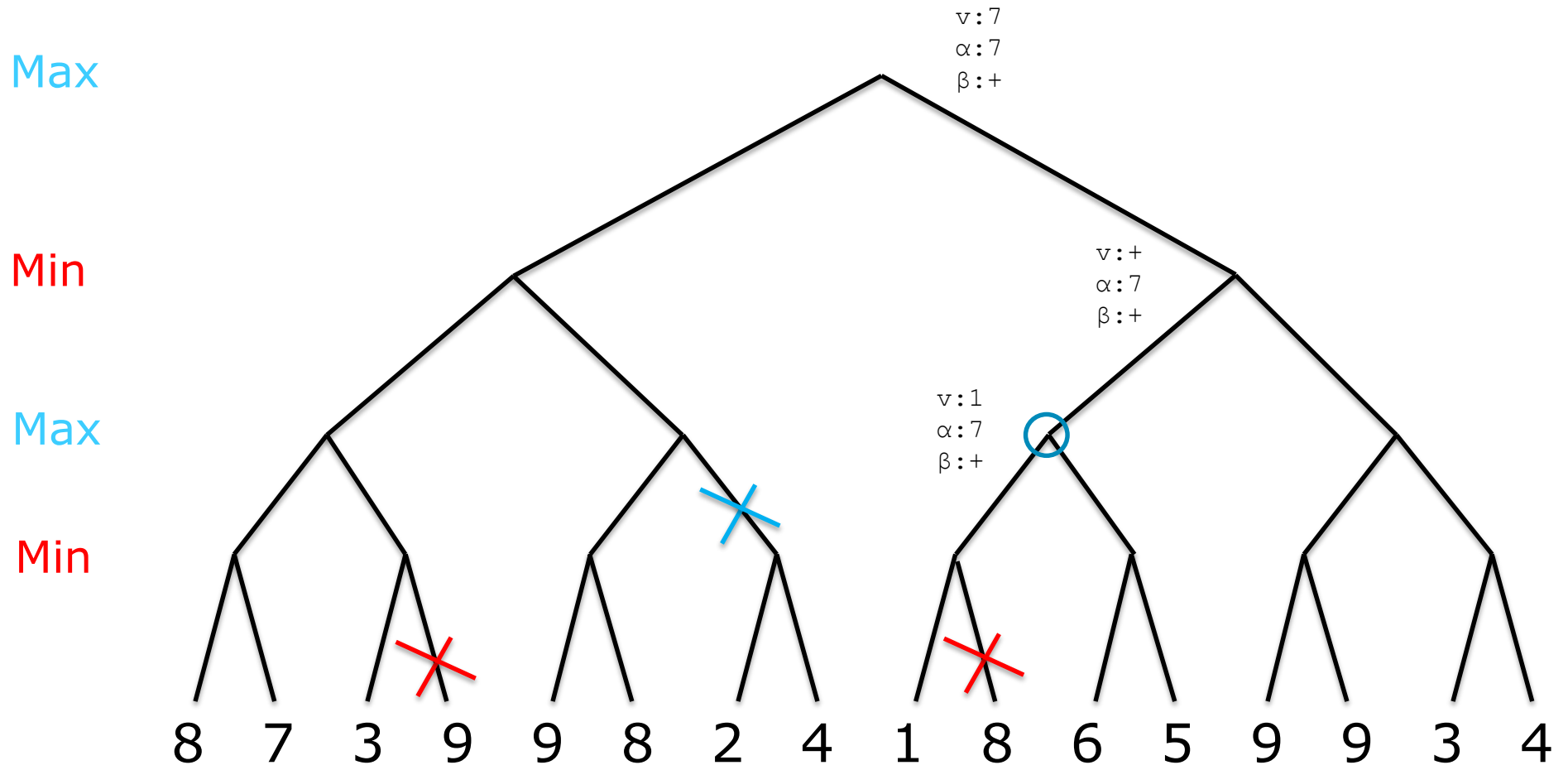

Max
Min
Max
Min
v : 7
α : 7
β : +
v : +
α : 7
β : +
v : −
α : 7
β : +
v : +
α : 7
β : +
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

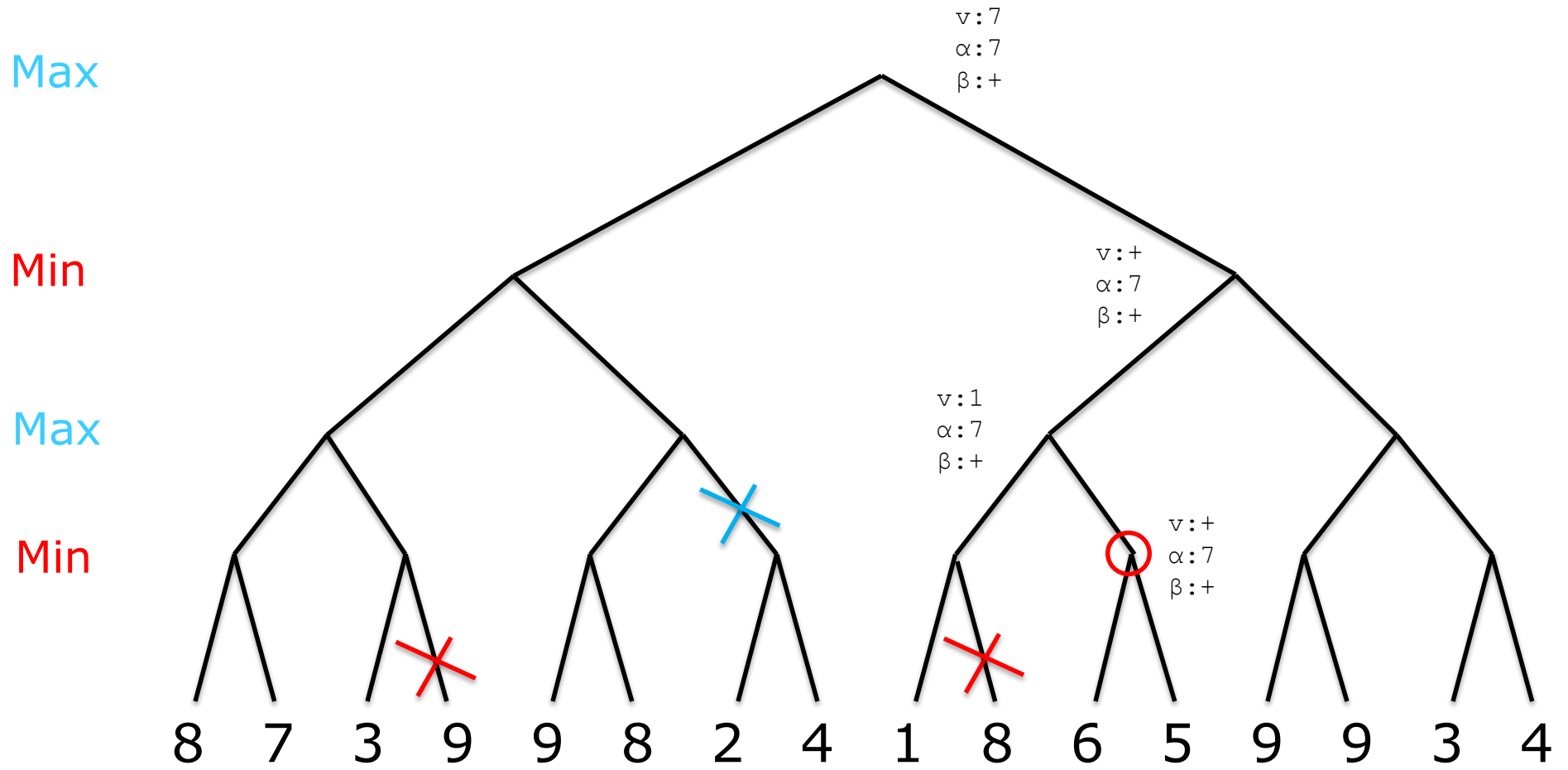

MinValue
if (v′ < v) v = v′
if (v′ ≤ α) return v
if (v′ < β) β = v′
Max
Min
Max
Min
v : 7
α : 7
β : +
v : +
α : 7
β : +
v : −
α : 7
β : +
v : +
α : 7
β : +
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

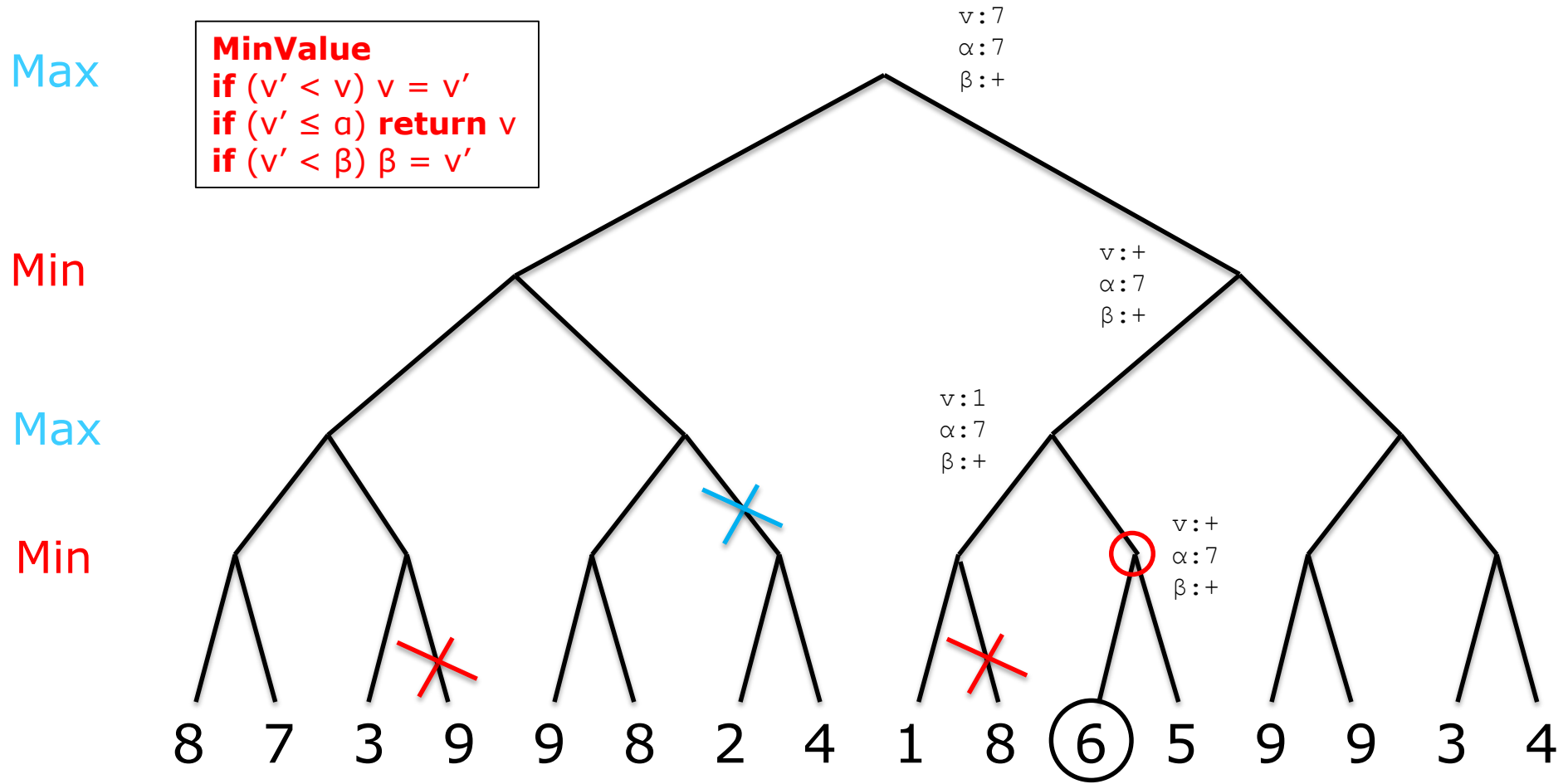

MinValue
if (v′ < v) v = v′
if (v′ ≤ α) return v
if (v′ < β) β = v′
Max
Min
Max
Min
v : 7
α : 7
β : +
v : +
α : 7
β : +
v : −
α : 7
β : +
v : 1
α : 7
β : +
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

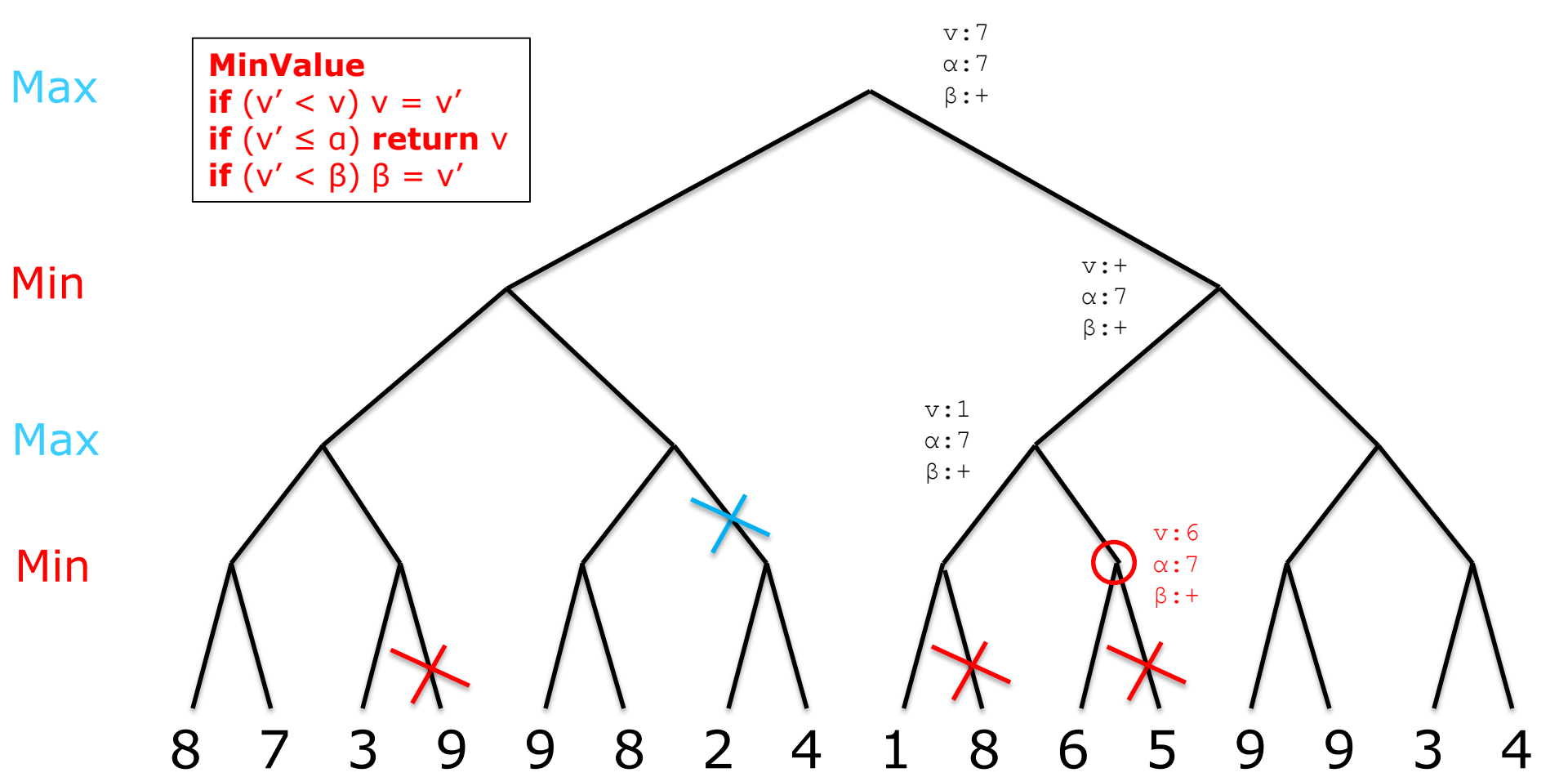

MaxValue
if (v′ > v) v = v′
if (v′ ≥ β) return v
if (v′ > α) α = v′
Max
Min
Max
Min
v : 7
α : 7
β : +
v : +
α : 7
β : +
v : −
α : 7
β : +
1
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

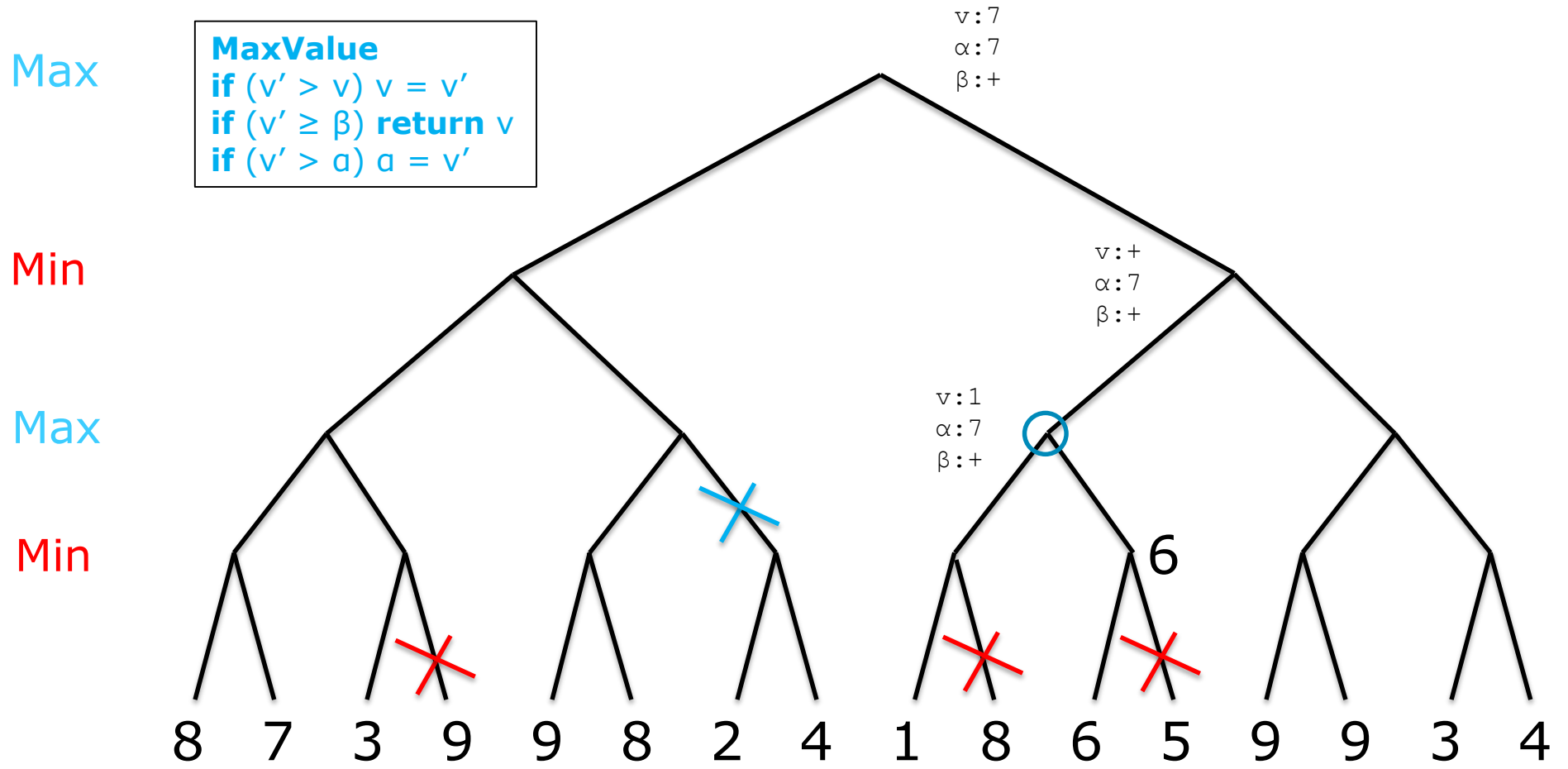

MaxValue
if (v′ > v) v = v′
if (v′ ≥ β) return v
if (v′ > α) α = v′
Max
Min
Max
Min
v : 7
α : 7
β : +
v : +
α : 7
β : +
v : 1
α : 7
β : +
1
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

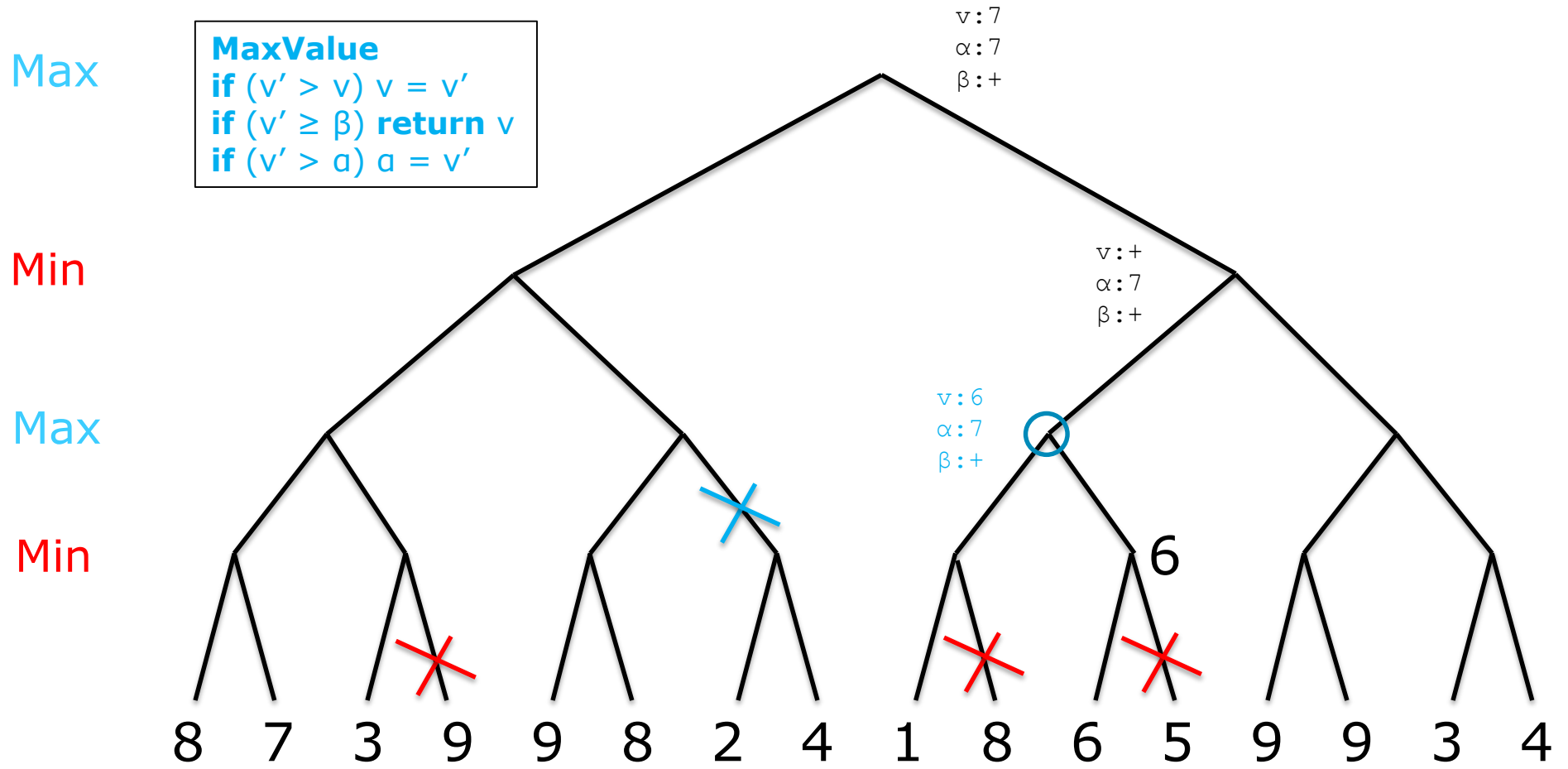

Max
Min
Max
Min
v : 7
α : 7
β : +
v : +
α : 7
β : +
v : 1
α : 7
β : +
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

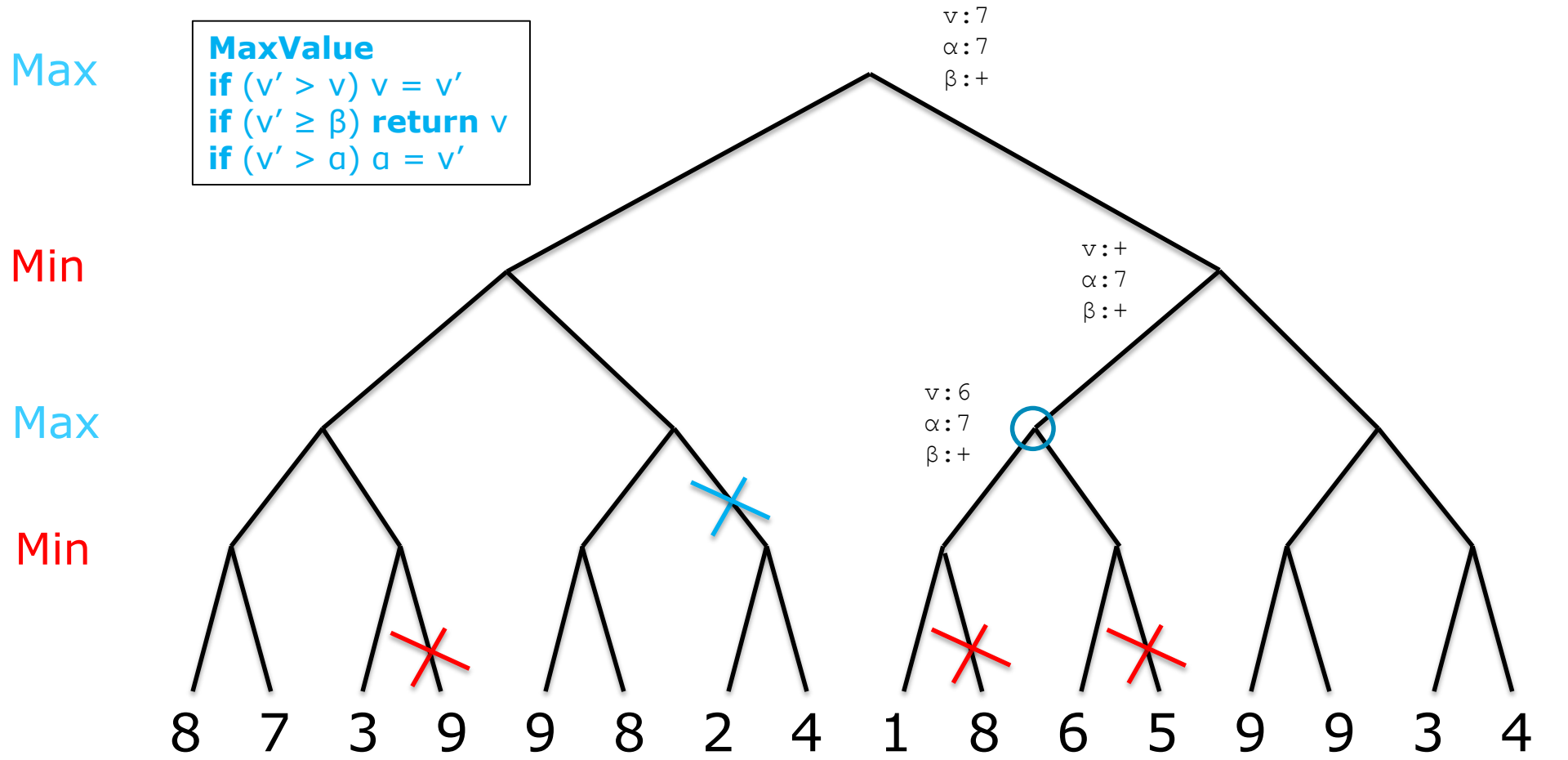

Max
Min
Max
Min
v : 7
α : 7
β : +
v : +
α : 7
β : +
v : 1
α : 7
β : +
v : +
α : 7
β : +
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

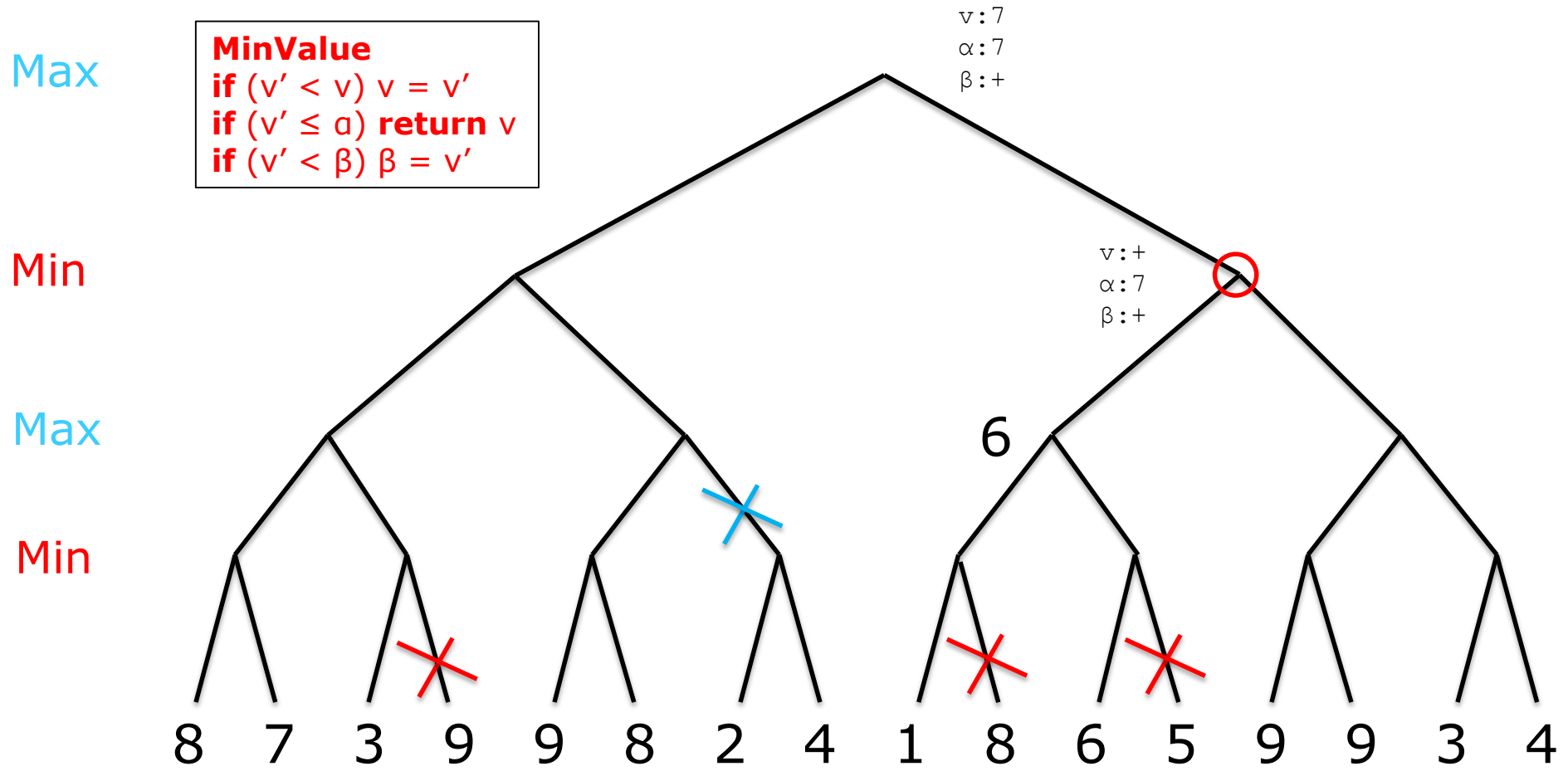

MinValue
if (v′ < v) v = v′
if (v′ ≤ α) return v
if (v′ < β) β = v′
Max
Min
Max
Min
v : 7
α : 7
β : +
v : +
α : 7
β : +
v : 1
α : 7
β : +
v : +
α : 7
β : +
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

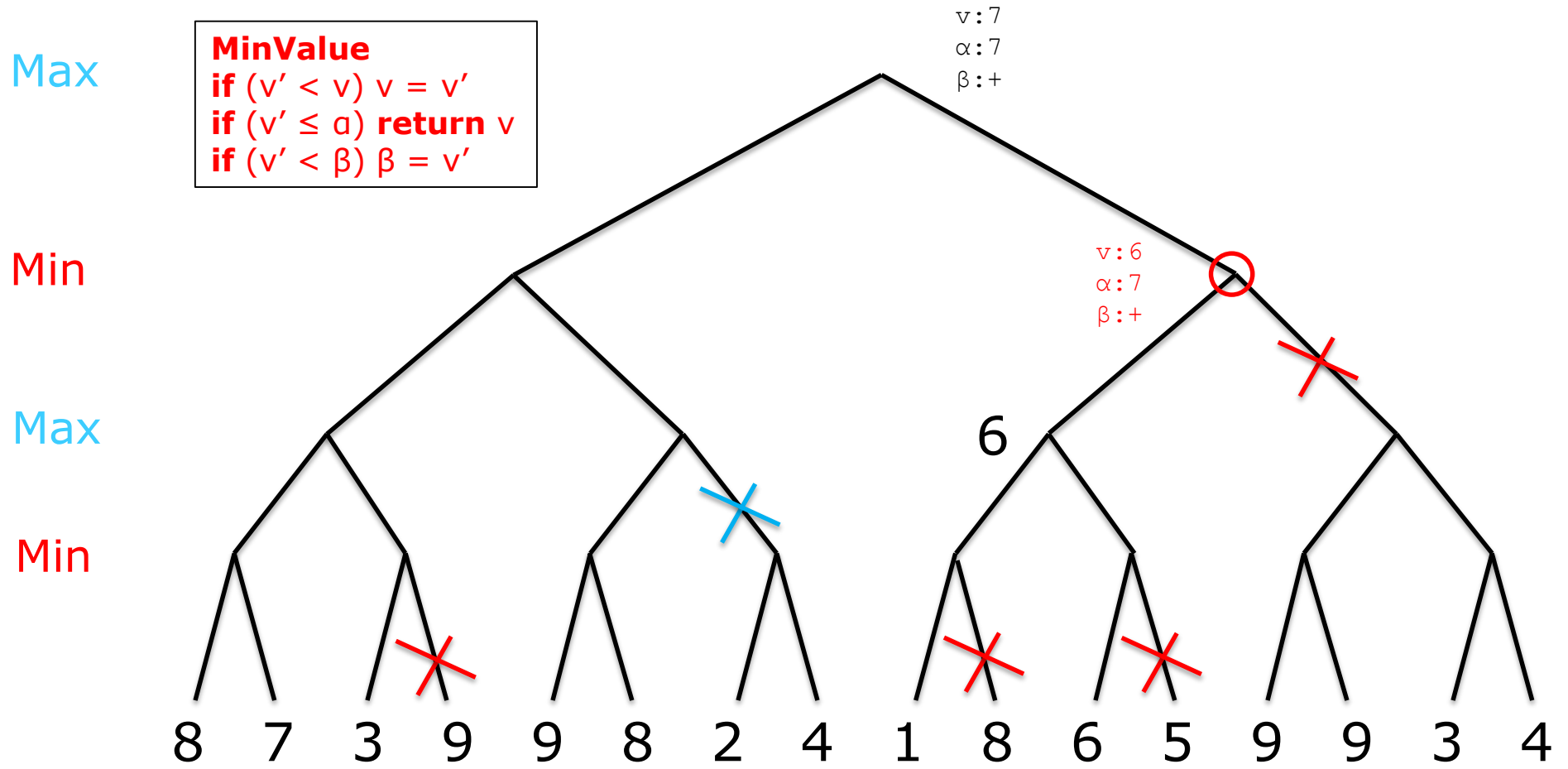

MinValue
if (v′ < v) v = v′
if (v′ ≤ α) return v
if (v′ < β) β = v′
Max
Min
Max
Min
v : 7
α : 7
β : +
v : +
α : 7
β : +
v : 1
α : 7
β : +
v : 6
α : 7
β : +
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

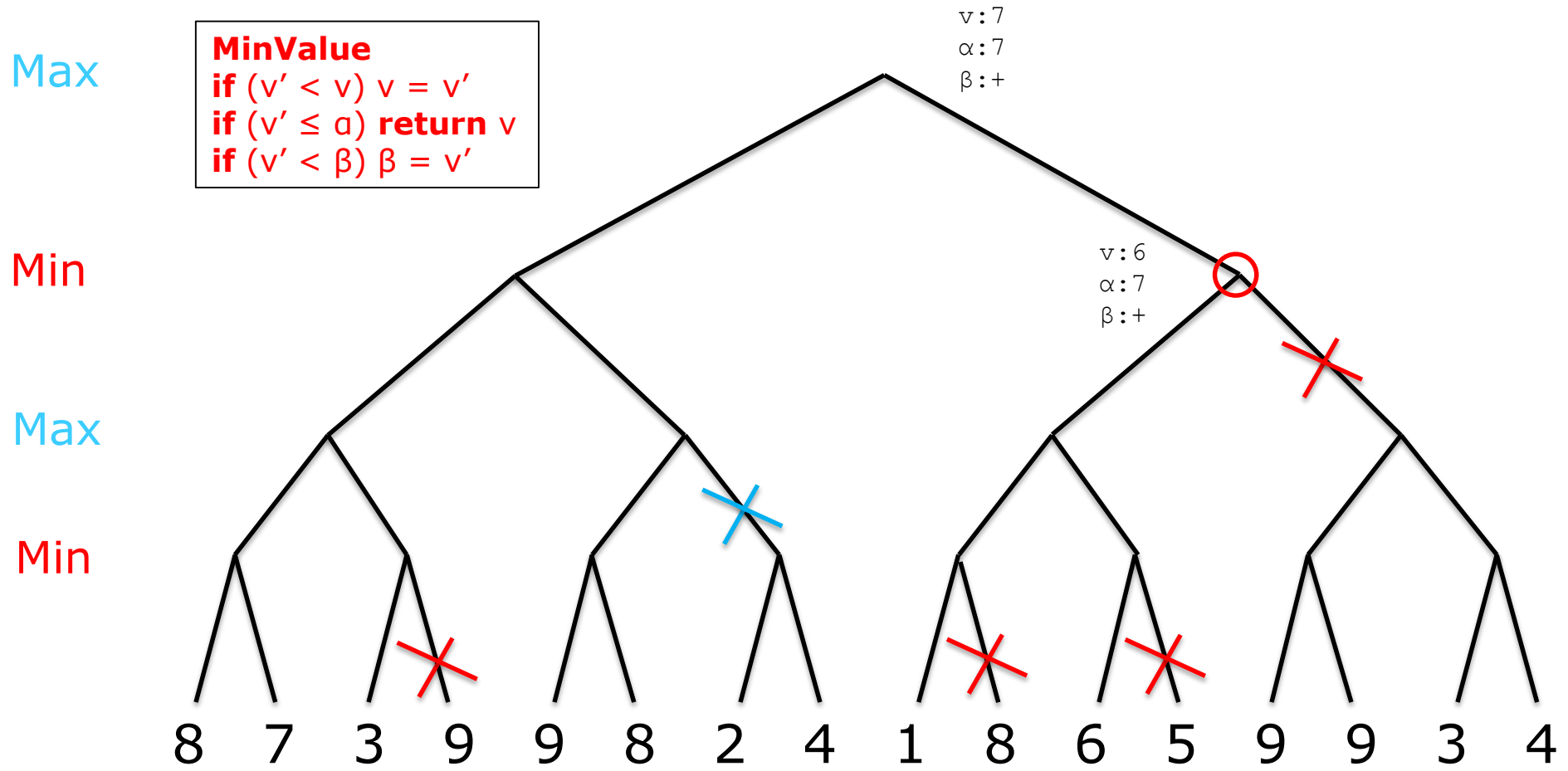

Max
Min
Max
Min
MaxValue
if (v′ > v) v = v′
if (v′ ≥ β) return v
if (v′ > α) α = v′
v : 7
α : 7
β : +
v : +
α : 7
β : +
v : 1
α : 7
β : +
6
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

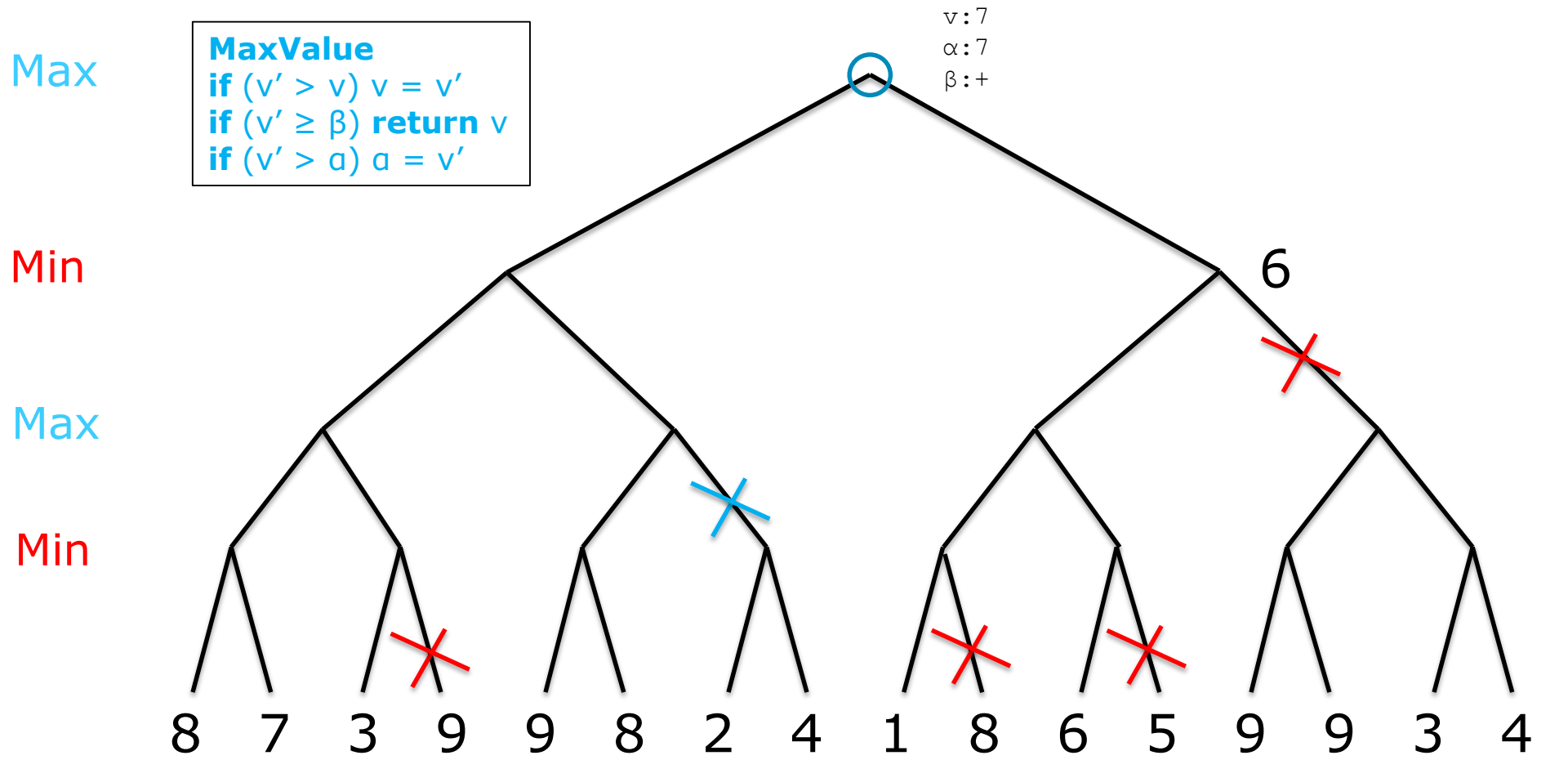

Max
Min
Max
Min
MaxValue
if (v′ > v) v = v′
if (v′ ≥ β) return v
if (v′ > α) α = v′
v : 7
α : 7
β : +
v : +
α : 7
β : +
v : 6
α : 7
β : +
6
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

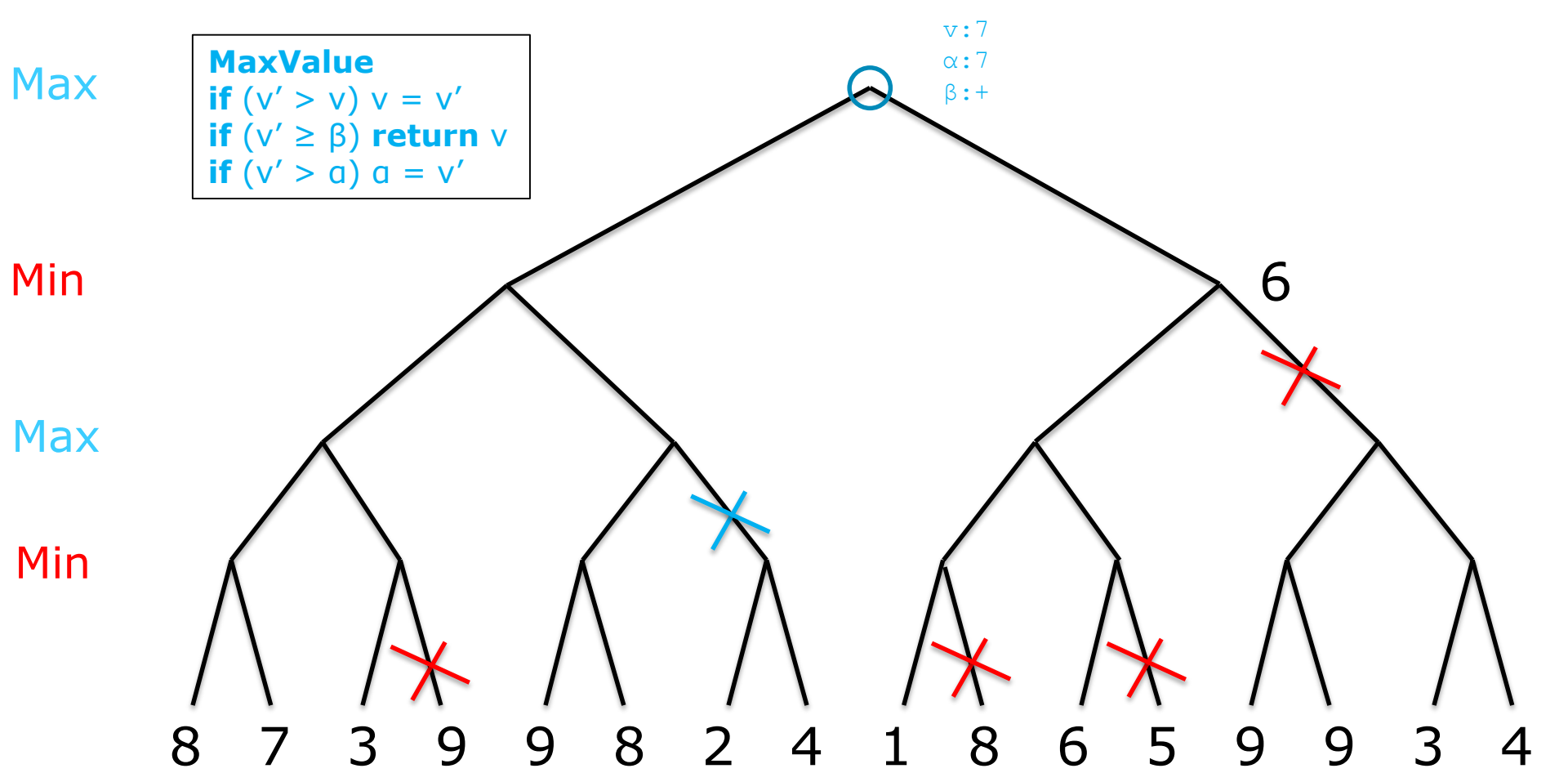

MaxValue
if (v′ > v) v = v′
if (v′ ≥ β) return v
if (v′ > α) α = v′
Max
Min
Max
Min
v : 7
α : 7
β : +
v : +
α : 7
β : +
v : 6
α : 7
β : +
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

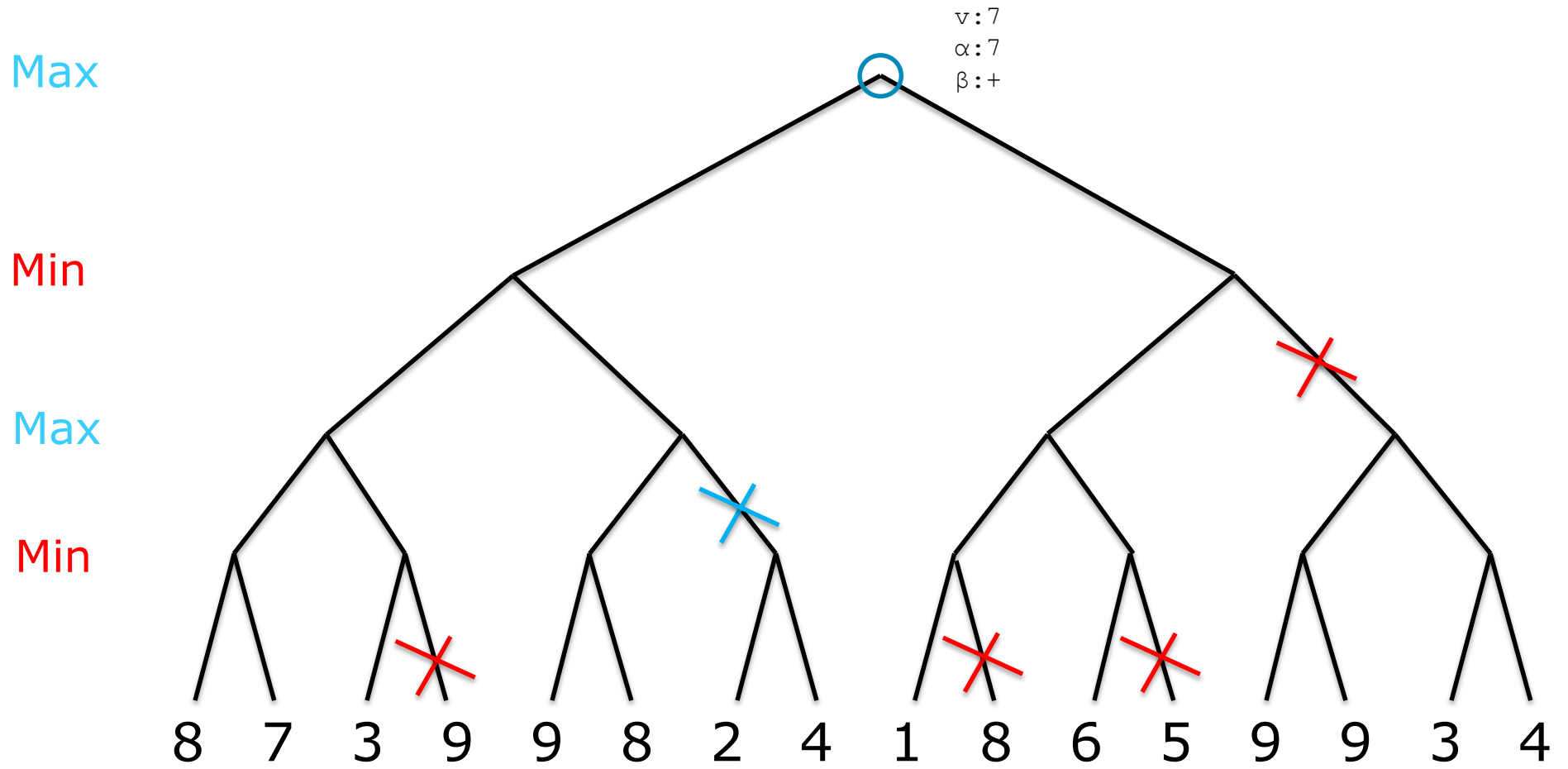

MinValue
if (v′ < v) v = v′
if (v′ ≤ α) return v
if (v′ < β) β = v′
Max
Min
Max
Min
v : 7
α : 7
β : +
v : +
α : 7
β : +
6
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

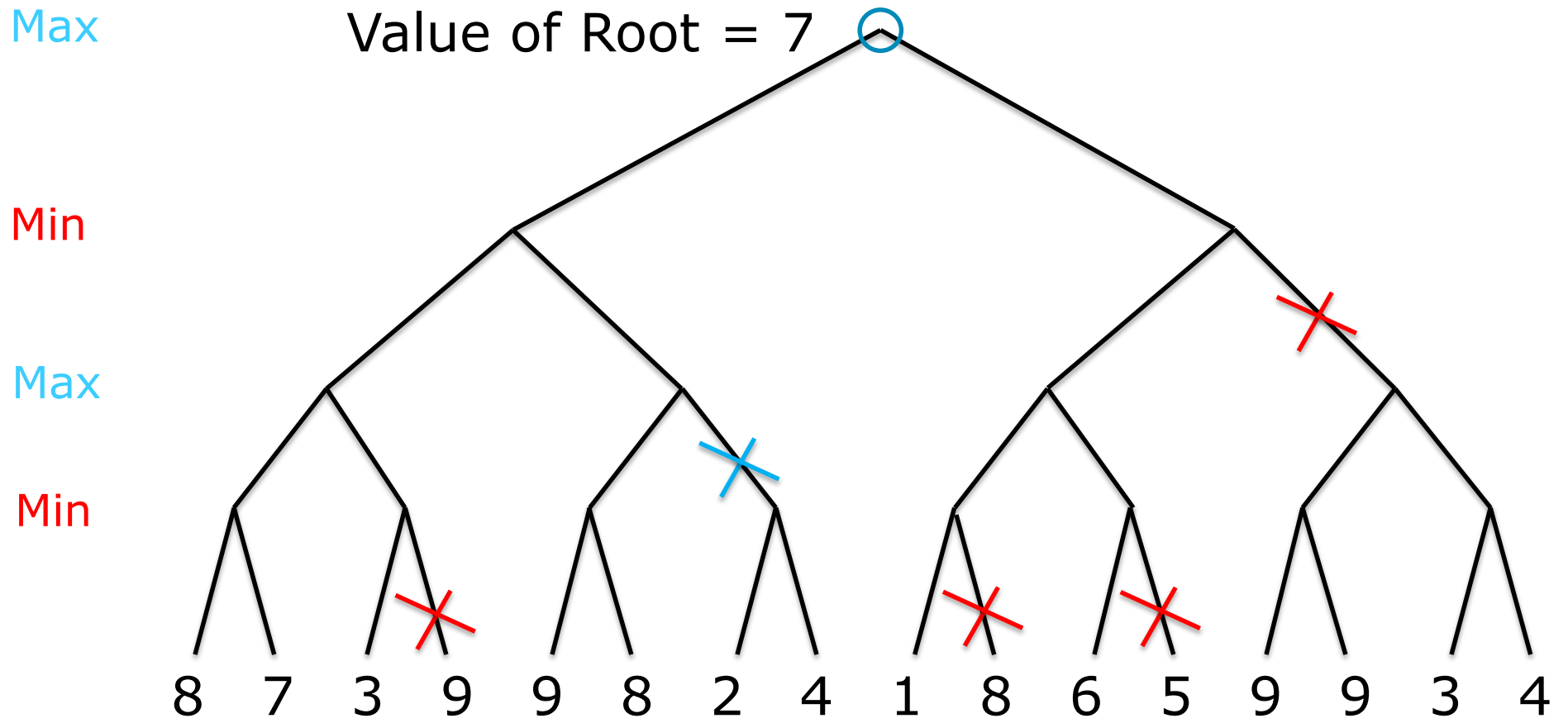

MinValue
if (v′ < v) v = v′
if (v′ ≤ α) return v
if (v′ < β) β = v′
Max
Min
Max
Min
v : 7
α : 7
β : +
v : 6
α : 7
β : +
6
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4


MinValue
if (v′ < v) v = v′
if (v′ ≤ α) return v
if (v′ < β) β = v′
Max
Min
Max
Min
v : 7
α : 7
β : +
v : 6
α : 7
β : +
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4


MaxValue
if (v′ > v) v = v′
if (v′ ≥ β) return v
if (v′ > α) α = v′
Max
Min
Max
Min
v : 7
α : 7
β : +
6
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4


MaxValue
if (v′ > v) v = v′
if (v′ ≥ β) return v
if (v′ > α) α = v′
v : 7
α : 7
β : +
Max
Min
Max
Min
6
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4


v : 7
α : 7
β : +
Max
Min
Max
Min
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4


Max
Value of Root = 7
Min
Max
Min
8 7 3 9 9 8 2 4 1 8 6 5 9 9 3 4

Computational Savings

- MiniMax Tree Search
  - Nodes searched = b^d
- Alpha-Beta Search
  - Nodes searched ~ $2b^{d/2}$ (optimal savings)
- Depth 7 search becomes depth 14
  - Bad program beats world champion

Recall MaxValue and MinValue

```
1.  Function MaxValue(s, α, β, d)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      v = -infinity
5.      for (c in children(s))
6.          v'= MinValue(c, α, β, d+1)
7.          if (v' > v) v = v'
8.          if (v' ≥ β) return v
9.          if (v' > α) α = v'
10.     return v
```

```
1.  Function MinValue(s, α, β, d)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      v = +infinity
5.      for (c in children(s))
6.          v'= MaxValue(c, α, β, d+1)
7.          if (v' < v) v = v'
8.          if (v' ≤ α) return v
9.          if (v' < β) β = v'
10.     return v
```

eval(s) returns score w.r.t. the maximizing player

Alpha-Beta Pruning (minimax)

```
1.  Function AlphaBeta(s, α, β, d, max)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      if (max) // maximizing player
5.          v = -infinity
6.          for (c in children(s))
7.              v'= MinValue(c, α, β, d+1)
8.              if (v' > v) v = v'
9.              if (v' ≥ β) return v
10.             if (v' > α) α = v'
11.         return v
12.     else  // minimizing player
13.         v = +infinity
14.         for (c in children(s))
15.             v'= MaxValue(c, α, β, d+1)
16.             if (v' < v) v = v'
17.             if (v' ≤ α) return v
18.             if (v' < β) β = v'
19.         return v
```

Can we shorten it?

MaxValue

```
1.  Function MaxValue(s, α, β, d)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      v = -infinity
5.      for (c in children(s))
6.          v'= MinValue(c, α, β, d+1)
7.          if (v' > v) v = v'
8.          if (v' ≥ β) return v
9.          if (v' > α) α = v'
10.     return v
```

MaxValue

```
1.  Function MaxValue(s, α, β, d)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      v = -infinity
5.      for (c in children(s))
6.          v'= MinValue(c, α, β, d+1)
7.          if (v' > v) v = v'
8.          if (v' > α) α = v'
9.          if (v' ≥ β) return v
10.     return v
```

Re-order the return statement, has same net effect

MaxValue

```
1.  Function MaxValue(s, α, β, d)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      v = -infinity
5.      for (c in children(s))
6.          v'= MinValue(c, α, β, d+1)
7.          if (v' > v) v = v'
8.          if (v' > α) α = v'
9.          if (α ≥ β) return v
10.     return v
```

Re-ordering allows us to compare α which will now store max(v')

MaxValue

```
1.  Function MaxValue(s, α, β, d)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      v = -infinity
5.      for (c in children(s))
6.          v'= MinValue(c, α, β, d+1)
7.          if (v' > v) v = v'
8.          if (v' > α) α = v'
9.          if (α ≥ β) return α
10.     return v
```

We can also return α since it is storing max(v)

MaxValue

```
1.  Function MaxValue(s, α, β, d)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      v = -infinity
5.      for (c in children(s))
6.          v'= MinValue(c, α, β, d+1)
7.          if (v' > v) v = v'
8.          if (v' > α) α = v'
9.          if (α ≥ β) return α
10.     return v
```

v is now only used as a max calculation placeholder

MaxValue

```
1.  Function MaxValue(s, α, β, d)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.
5.      for (c in children(s))
6.          v'= MinValue(c, α, β, d+1)
7.
8.          if (v' > α) α = v'
9.          if (α ≥ β) return α
10.     return α
```

We can completely remove v and use α

MaxValue

```
1.  Function MaxValue(s, α, β, d)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.
5.      for (c in children(s))
6.          α = max(α, MinValue(c, α, β, d+1))
7.
8.
9.          if (α ≥ β) return α
10.     return α
```

we can now get rid of the placeholder variable v’ as well

MaxValue

```
1. Function MaxValue(s, α, β, d)
2.     if (terminal(s) or d > maxD)
3.         return eval(s)
4.     for (c in children(s))
5.         α = max(α, MinValue(c, α, β, d+1))
6.         if (α ≥ β) return α
7.     return α
```

This is the most compact version of MaxValue

MaxValue

```
1. Function MaxValue(s, α, β, d)
2.     if (terminal(s) or d > maxD)
3.         return eval(s)
4.     for (c in children(s))
5.         α = max(α, MinValue(c, α, β, d+1))
6.         if (α ≥ β) break
7.     return α
```

We can use break instead of return, since we return outside the loop

MaxValue and MinValue

```
1. Function MaxValue(s, α, β, d)
2.     if (terminal(s) or d > maxD)
3.         return eval(s)
4.     for (c in children(s))
5.         α = max(α, MinValue(c, α, β, d+1))
6.         if (α ≥ β) break
7.     return α
```

```
1. Function MinValue(s, α, β, d)
2.     if (terminal(s) or d > maxD)
3.         return eval(s)
4.     for (c in children(s))
5.         β = min(β, MaxValue(c, α, β, d+1))
6.         if (β ≤ α) break
7.     return β
```

Same shortening can be applied in MinValue algorithm

MaxValue and MinValue

```
1. Function MaxValue(s, α, β, d)
2.     if (terminal(s) or d > maxD)
3.         return eval(s)
4.     for (c in children(s))
5.         α = max(α, MinValue(c, α, β, d+1))
6.         if (α ≥ β) break
7.     return α
```

```
1. Function MinValue(s, α, β, d)
2.     if (terminal(s) or d > maxD)
3.         return eval(s)
4.     for (c in children(s))
5.         β = min(β, MaxValue(c, α, β, d+1))
6.         if (α ≥ β) break
7.     return β
```

Reverse comparison in MinValue to get the same one as MaxValue

Shortened Max/Min

```
1.  Function AlphaBeta(s, α, β, d, max)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      if (max) // maximizing player
5.          for (c in children(s))
6.              α = max(α, AlphaBeta(c, α, β, d+1, !max))
7.              if (α ≥ β) break
8.          return α
9.      else                  // minimizing player
10.         for (c in children(s))
11.             β = min(β, AlphaBeta(c, α, β, d+1, !max))
12.             if (α ≥ β) break
13.         return β
```

Notice Similarities

```
1.  Function AlphaBeta(s, α, β, d, max)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      if (max) // maximizing player
5.          for (c in children(s))
6.              α = max(α, AlphaBeta(c, α, β, d+1, !max))
7.              if (α ≥ β) break
8.          return α
9.      else                // minimizing player
10.         for (c in children(s))
11.             β = min(β, AlphaBeta(c, α, β, d+1, !max))
12.             if (α ≥ β) break
13.         return β
```

There is duplicated code required due to if(max) being on the outside of the child loop

Bring Max Condition Inside the Loop

```
1. Function AlphaBeta(s, α, β, d, max)
2.     if (terminal(s) or d > maxD)
3.         return eval(s)
4.     for (c in children(s))
5.         if (max)  α = max(α, AlphaBeta(c, α, β, d+1, !max))
6.         if (!max) β = min(β, AlphaBeta(c, α, β, d+1, !max))
7.         if (α ≥ β) break
8.     return max ? α : β
```

Final shortened version of Alpha-Beta

Alpha-Beta (short version)

```
1. Function AlphaBeta(s, α, β, d, max)
2.     if (terminal(s) or d > maxD)
3.         return eval(s)
4.     for (c in children(s))
5.         if (max)  α = max(α, AlphaBeta(c, α, β, d+1, !max))
6.         if (!max) β = min(β, AlphaBeta(c, α, β, d+1, !max))
7.         if (α ≥ β) break
8.     return max ? α : β
```

Final shortened version of AB, returns VALUE of state
Result is exactly the same as the long version
Implement whichever one you feel more comfortable with

Which action do we take?

```
1.  Function AlphaBeta(s, α, β, d, max)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      for (c in children(s))
5.          if (max)  α = max(α, AlphaBeta(c, α, β, d+1, !max))
6.          if (!max) β = min(β, AlphaBeta(c, α, β, d+1, !max))
7.          if (α ≥ β) break
8.      return max ? α : β
```

So far only calculating value, need to record the action

Record Best Action

```
1.  Function AlphaBeta(s, α, β, d, max)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      for (c in children(s))
5.          v' = AlphaBeta(c, α, β, d+1, !max)
6.          if (max)  α = max(α, v')
7.          if (!max) β = min(β, v')
8.          if (α ≥ β) break
9.      return max ? α : β
```

Re-introduce temporary value variable v' and is it for comparison

Record Best Action

```
1.  Function AlphaBeta(s, α, β, d, max)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      for (c in children(s))
5.          v' = AlphaBeta(c, α, β, d+1, !max)
6.          if (max and (v' > α))  α = v'
7.          if (!max and (v' < β)) β = v'
8.          if (α ≥ β) break
9.      return max ? α : β
```

Re-introduce temporary value variable v' and is it for comparison

Record Best Action

```
1.  Function AlphaBeta(s, α, β, d, max)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      for (c in children(s))
5.          v' = AlphaBeta(c, α, β, d+1, !max)
6.          if (max and (v' > α))
7.              a = v'
8.              if (d == 0) bestAction = c.action
9.          if (!max and (v' < β)) β = v'
10.         if (α ≥ β) break
11.     return max ? α : β
```

We can use a global/state variable to store best action at the root node (d=0)

Record Action AB Long Version

```
1.  Function AlphaBeta(s, α, β, d, max)
2.      if (terminal(s) or d > maxD)
3.          return eval(s)
4.      if (max) // maximizing player
5.          v = -infinity
6.          for (c in children(s))
7.              v'= AlphaBeta(c, α, β, d+1, !max)
8.              if (v' > v) v = v'
9.              if (v' ≥ β) return v
10.             if (v' > α)
11.                 α = v'
12.                 if (d == 0) bestAction = c.action
13.         return v
14.     else    // minimizing player
15.         v = +infinity
16.         for (c in children(s))
17.             v'= AlphaBeta(c, α, β, d+1, !max)
18.             if (v' < v) v = v'
19.             if (v' ≤ α) return v
20.             if (v' < β) β = v'
21.         return v
```

Time Limit

- What if we want a time limit?
- Search to a given depth time varies
- Incorporate iterative deepening
- Search d=1, 2, …, infinity
- When time runs out, return the best action from the last fully completed depth
- “Iterative Deepening Alpha Beta”

Iterative Deepening AlphaBeta

```
1. Function IDAlphaBeta(s, maxDepth)
2.     depth = 0
3.     bestAction = null
4.     while (depth < maxDepth)
5.         try { bestAction = AlphaBeta(s, -infinity, infinity, 0, true); }
6.         catch (TimeOutException) { break; }
7.         depth = depth + 1
```

AB With Time Limit

```
1.  Function AlphaBeta(s, α, β, d, max)
2.      if (terminal(s) or d > maxD) return eval(s)
3.      if (timeElapsed > timeLimit) throw TimeOutException;
4.      for (c in children(s))
5.          v' = AlphaBeta(c, α, β, d+1, !max)
6.          if (max and (v' > α))
7.              a = v'
8.              if (d == 0) bestAction = c.action
9.          if (!max and (v' < β)) β = v'
10.         if (α ≥ β) break
11.     return max ? α : β
```

Throw timeout Exception to immediately halt AB execution