NORMALIZED FREQUENCY

For continuous-time (analog) signals, frequency is usually expressed as the number of cycles per *second*. (1 Hz = 1 cycle/second.) However, for discrete-time signals, frequency may be expressed as either the number of cycles per second or as the number of cycles per *sample*. When it is expressed as the number of cycles per sample, it is referred to as *normalized* frequency.

Consider an analog sinusoid signal with a frequency of $f_1 = 5$ cycles/second (i.e., 5 Hz). Suppose this signal is sampled at a rate of 20 samples/second. What is its frequency f_1 expressed in units of cycles/sample? Since the sampling rate is 20 samples/second, we can write '20 samples = 1 second' and use this for unit conversion, i.e.,

$$1 = \frac{20 \text{ samples}}{\text{second}} = \frac{\text{second}}{20 \text{ samples}}$$

We can always multiply by 1 to convert units.

$$\begin{aligned} f &= 5 \ \frac{\text{cycle}}{\text{second}} \\ &= 5 \ \frac{\text{cycle}}{\text{second}} \times \frac{\text{second}}{20 \text{ samples}} \\ &= 0.25 \ \frac{\text{cycle}}{\text{sample}} \end{aligned}$$

So, at a sampling rate of 20 samples/sec, 5 Hz is the same as 0.25 cycles/sample. Similarly, at a sampling rate of 20 samples/sec, we have: 10 Hz = 0.5 cycles/sample.

In general, at a sampling rate of F_s samples/sec, we have $\dfrac{F_s}{2}$ cycles/second = 0.5 cycles/sample. So, the frequency response of a digital filter is usually shown from zero to 0.5 cycles/sample in normalized frequency.

The frequency of analog and discrete-time signals can also be expressed in units of *radians* per second. The normalized frequency of a discrete-time signal can likewise be expressed in units of radians per sample. To convert to rad/sec or to rad/sample, use the conversion factor. Since 2π radians = 1 cycle, we write

$$1 = \frac{2\pi \text{ radians}}{\text{cycle}} = \frac{\text{cycle}}{2\pi \text{ radians}}$$

EL 6113 lecture notes, Ivan Selesnick.

So, 5 Hz is the same as 20 π rad/second because

$$\begin{aligned} 5 \text{ Hz} &= 5 \ \frac{\text{cycle}}{\text{second}} \\ &= 5 \ \frac{\text{cycle}}{\text{second}} \times \frac{2\pi \text{ radians}}{\text{cycle}} \\ &= 10\pi \ \frac{\text{radians}}{\text{second}} \end{aligned}$$

Likewise, at a sampling rate of 20 samples/sec, we have 5 Hz $= 0.5\pi$ rad/sample because

$$\begin{aligned} 5 \text{ Hz} &= 5 \ \frac{\text{cycle}}{\text{second}} \\ &= 5 \ \frac{\text{cycle}}{\text{second}} \times \frac{2\pi \text{ radians}}{\text{cycle}} \times \frac{\text{second}}{20 \text{ samples}} \\ &= 0.5\pi \ \frac{\text{radians}}{\text{sample}} \end{aligned}$$

In general, at a sampling rate of F_s samples/sec, we have $\frac{F_s}{2}$ cycles/sec $= \pi$ rad/sample. So, the frequency response of a digital filter is usually shown from zero to π rad/sample in normalized frequency.