q-Hodge filtrations, Habiro cohomology & ku

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4 Refined $\mathrm{THH}/\mathrm{TC}^-$ & analytic Habiro cohomology

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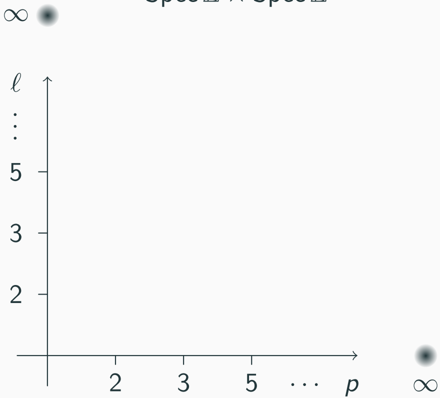

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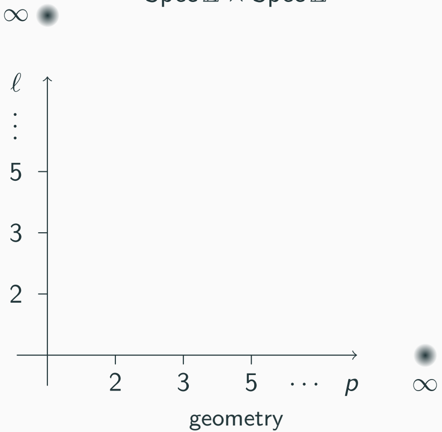

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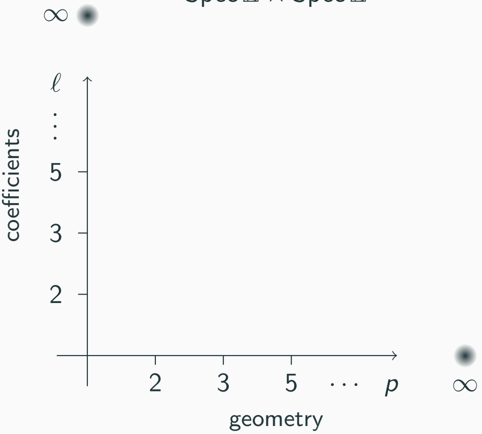

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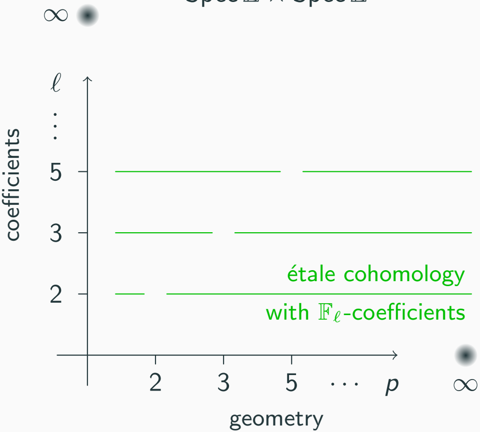

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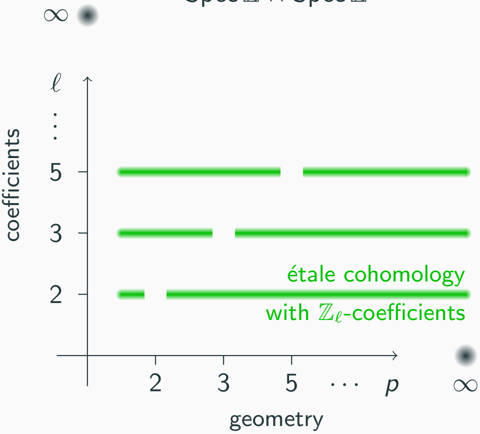

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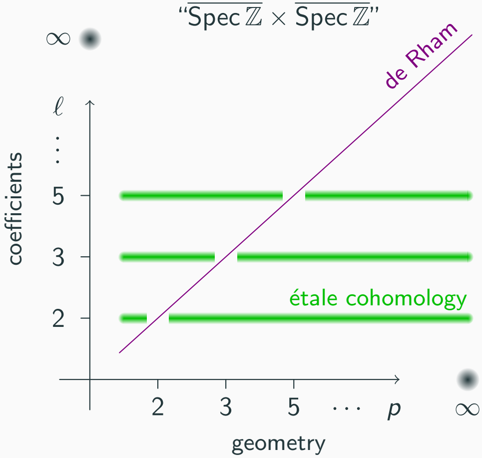

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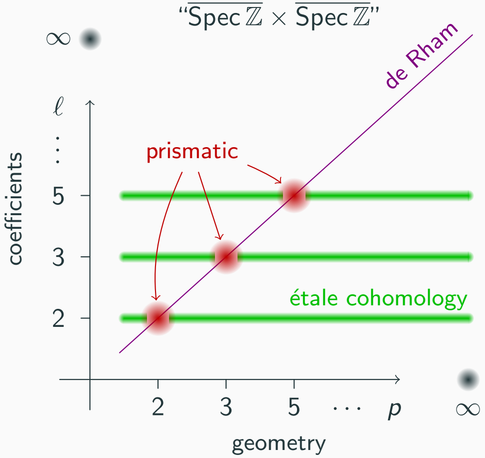

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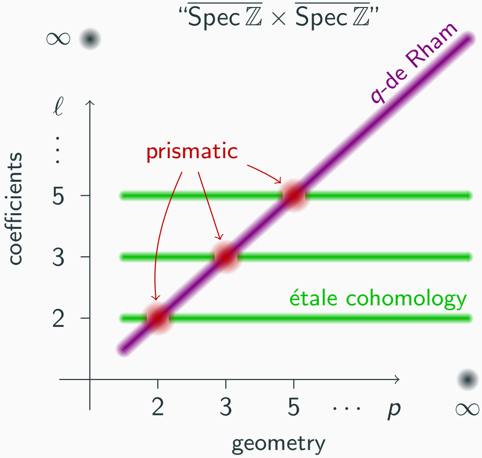

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***q*-de Rham complex** (Bhatt–Scholze)**.** There is a functor

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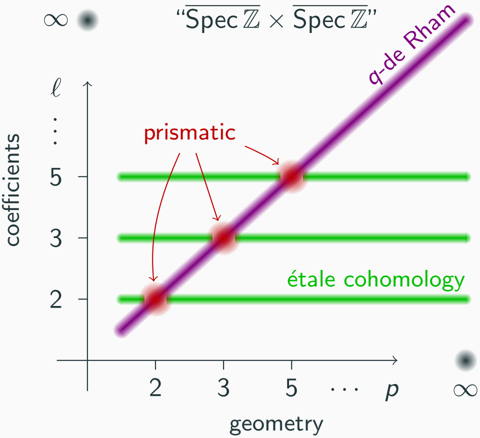

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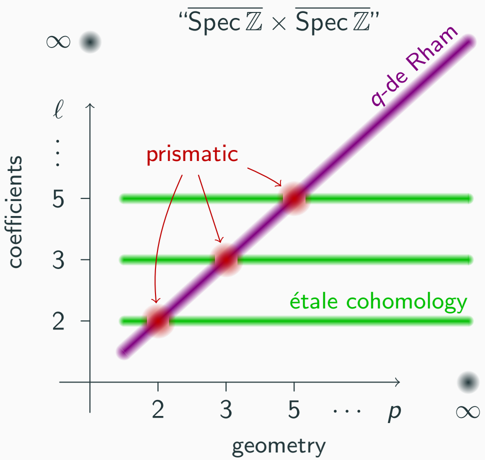

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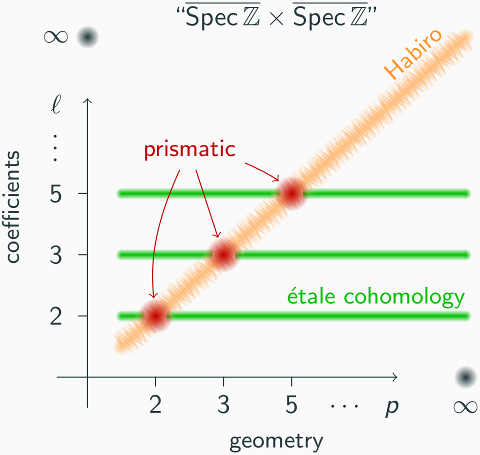

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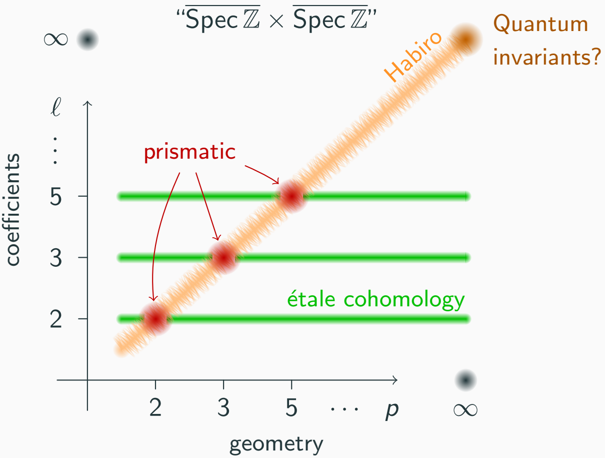

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$\Rightarrow$ **We should look for q-deformations of the Hodge filtration!**

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Theorem (W. 2025).

The forgetful functor $\mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}}\to\mathrm{Sm}_{\mathbb{Z}}$ has *no* section

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Theorem (W. 2025).

The forgetful functor $\mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}} \to \mathrm{Sm}_{\mathbb{Z}}$ has *no* section, but over the full subcategory

$$\left\{R \;\middle|\; p \in R^\times \text{ for all primes } p \leqslant \dim(R/\mathbb{Z})\right\} \subseteq \mathrm{Sm}_{\mathbb{Z}}$$

there exists a section.

The q-Hodge complex. Given $(R, \mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\Omega_R)$, define the *$q$-Hodge complex*

$$q\text{-}\mathrm{Hdg}_R :=$$

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$$q\text{-}\mathcal{H}\mathrm{dg}_{\mathcal{O}_F[1/\Delta]} \simeq \mathcal{H}_{\mathcal{O}_F[1/\Delta]}$$

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3. q-Hodge filtrations from homotopy theory

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If R is quasisyntomic, there is a complete exhaustive "motivic" filtration $\mathrm{fil}^{\star}_{\mathrm{mot}}\,\mathrm{HC}^-(R)$ with associated graded

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$\mathrm{TC}^-(-/\mathrm{ku})$ & q-Hodge filtrations

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4. Refined THH/TC^- & analytic Habiro cohomology

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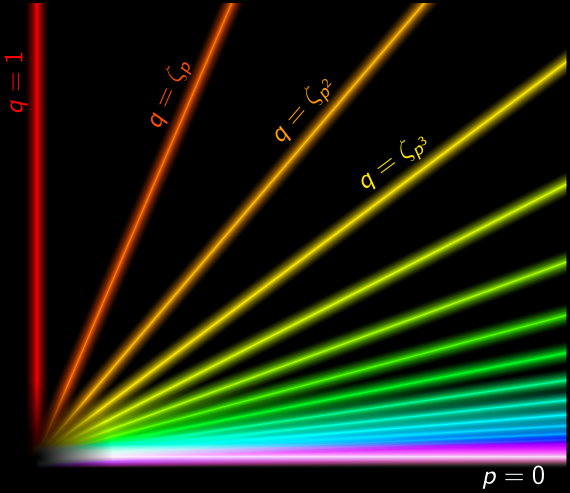

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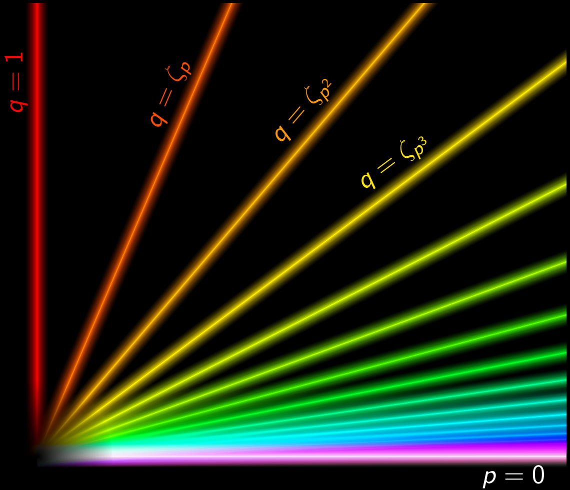

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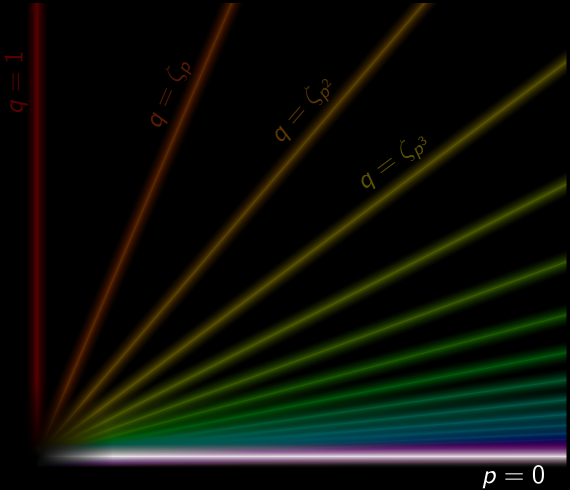

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Outlook

Analytic Habiro cohomology (Scholze)**.** Coefficients in the *analytic Habiro ring* $\mathcal{H}^{\mathrm{an}}$.

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$$\left(\qquad \text{Recall: } \mathrm{TCn}(-) := \lim_{m \in \mathbb{N}} \left(\mathrm{THH}(-)^{C_m}\right)^{h(S^1/C_m)}. \qquad\right)$$

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Question.

Can we compute this for bases other than KU (e.g. elliptic cohomology, MU, $\mathbb{S}$)?

Thank you!