Mean Field Games: Numerical Methods and Applications in Machine Learning

Part 7: Mean Field Reinforcement Learning

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https://mlauriere.github.io/teaching/MFG-PKU-7.pdf

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RECAP

Outline

From Optimal Control to MFRL

$$\begin{tikzcd}
\text{Mean-Field Control} \arrow[r, "\textit{unknown model}"] & \text{Mean-Field Reinforcement Learning} \\
\text{Optimal Control} \arrow[u, "\substack{N\to\infty \\ MKV}"'] \arrow[r, "\textit{unknown model}"'] & \text{Reinforcement Learning} \arrow[u, "\substack{N\to\infty \\ MKV}"]
\end{tikzcd}$$

Reinforcement Learning – Setup

- **Markov Decision Process (MDP):** $(\mathcal{S}, \mathcal{A}, p, r, \gamma)$, where:
  - $\mathcal{S}$: state space, $\mathcal{A}$: action space,
  - $p : \mathcal{S} \times \mathcal{A} \to \mathcal{P}(\mathcal{S})$: transition kernel, $p(\cdot|s, a)$ gives next state's distribution
  - $r : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$: reward function, $\gamma \in (0, 1)$: discount factor
- **Goal:** Find (stationary, mixed) policy $\pi^* : \mathcal{S} \to \mathcal{P}(\mathcal{A})$ maximizing:

$$R(\pi) = \mathbb{E}\left[\sum_{n\geq 0} \gamma^n r(s_n, a_n)\right], \qquad \text{with } a_n \sim \pi(\cdot|s_n), s_{n+1} \sim p(\cdot|s_n, a_n)$$

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- **Model:** p, r
- **Two settings:**

  (1) **Known model** : Optimal control theory & methods

  (2) **Sample transitions & rewards**: Reinforcement Learning (RL) framework

Reinforcement Learning – Paradigm

We want to **learn** the best control by performing **experiments** of the form:

Given the current state S_t,
(1) *Take an action A_t*
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[1] Sutton, R. S., & Barto, A. G. (2018). *Reinforcement learning: An introduction.* MIT press.

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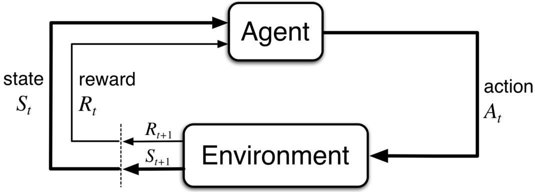

Source: [Sutton, Barto][1]

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$$Q^*(s, a) = r(s, a) + \gamma \max_{\pi} \mathbb{E}_{a' \sim \pi(\cdot|s), s' \sim p(\cdot|s, a')} \left[Q^*(s', a') \right]$$

Note: $V^*(s) = \max_{a \in \mathcal{A}} Q^*(s, a)$, $v^*(s) = \mathrm{argmax}_{a \in \mathcal{A}} Q^*(s, a)$

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- **Hybrid:**
  - Deep Deterministic Policy Gradient (DDPG)
  - Soft Actor Critic (SAC)
  - ...

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Problem Formulation

Generic Mean Field model: for a typical infinitesimal agent

- **Dynamics:** discrete time

$$X_{n+1}^{\alpha,\mu} = F(X_n^{\alpha,\mu}, \alpha_n, \mu_n, \epsilon_{n+1}, \epsilon_{n+1}^0), \quad n \geq 0, \qquad X_0^{\alpha,\mu} \sim \mu_0$$

◇ $X_n^{\alpha,\mu} \in \mathcal{X} \subseteq \mathbb{R}^d$: state, $\alpha_n \in \mathcal{U} \subseteq \mathbb{R}^k$: action

◇ $\epsilon_n \sim \nu$: idiosyncratic noise, $\epsilon_n^0 \sim \nu^0$: common noise (random env.)

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- **Cooperative (MFC):** Find π^* s.t.

$$\pi^* \text{ minimizes } \pi \mapsto J^{MFC}(\pi) = \mathbb{J}(\pi; \mu^\pi) \text{ where } \mu_n^\pi = \mathbb{P}^0_{X_n^{\alpha,\mu^\pi}}$$

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$$\begin{cases} \hat{\pi} \text{ minimizes } \pi \mapsto J^{MFG}(\pi; \hat{\mu}) = \mathbb{J}(\pi; \hat{\mu}) \\ \hat{\mu}_n = \mathbb{P}^0_{X_n^{\hat{\alpha},\hat{\mu}}} \end{cases}$$

Mean Field MDP

Key Remark:

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Theorem: DPP for MFMDP [Carmona, L., Tan'21]

Under suitable conditions,

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Here: discrete time, infinite horizon, common noise, feedback controls, . . .

$\rightarrow$ well-suited for **RL**

$\rightarrow$ Mean-field Q-learning algorithm

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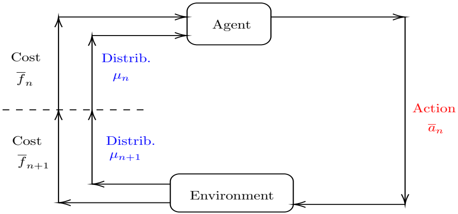

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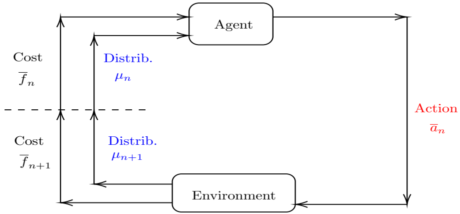
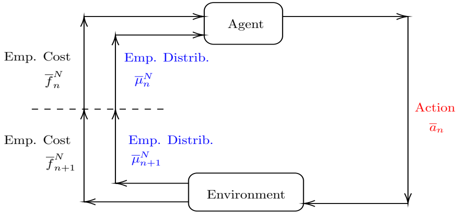
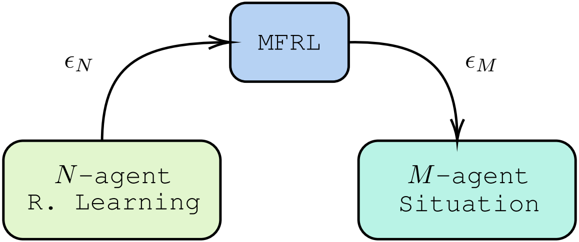


- **Setting 3:** unknown model but **samples from N-agent MDP**: approx. MF learning



Mean Field Control: Finite Population Approximation



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Policy Gradient (PG) to minimize $J(\theta)$

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Theorem: For **Linear-Quadratic** MFC [Carmona, L., Tan'19]

In each case, convergence holds at a linear rate:

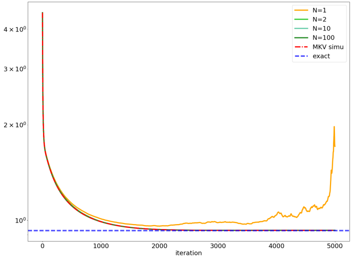
Taking $\mathtt{k} \approx \mathcal{O}\Big(\log(1/\epsilon)\Big)$ is sufficient to ensure $J(\theta^{(\mathtt{k})}) - J(\theta^*) < \epsilon$.

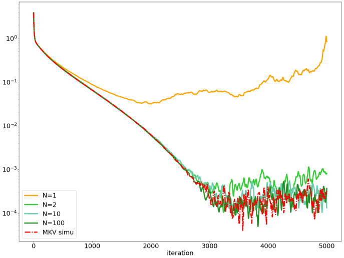
Proof: builds on [Fazel et al.'18], analysis of perturbation of Riccati equations

Numerical Illustration

Example: Linear dynamics, quadratic costs of the type:

$$f(x, \mu, v) = \underbrace{(\bar{\mu} - x)^2}_{\text{distance to mean position}} + \underbrace{v^2}_{\text{cost of moving}}, \qquad \bar{\mu} = \underbrace{\int \mu(\xi) d\xi}_{\text{mean position}},$$





Value of the MF cost

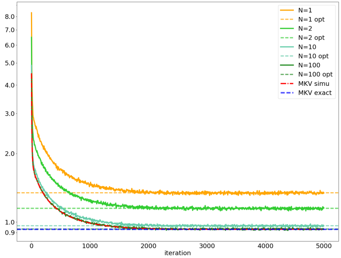
Rel. err. on MF cost

MF cost = cost in the mean field problem

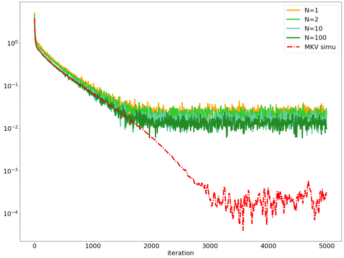
Numerical Illustration

Example: Linear dynamics, quadratic costs of the type:

$$f(x,\mu,v) = \underbrace{(\bar{\mu}-x)^2}_{\text{distance to mean position}} + \underbrace{v^2}_{\text{cost of moving}}, \qquad \bar{\mu} = \underbrace{\int \mu(\xi)d\xi}_{\text{mean position}},$$



Value of the social cost



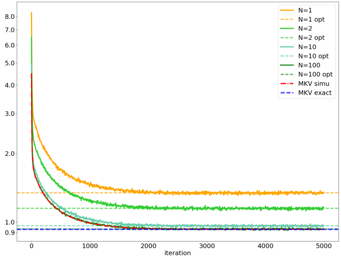
Rel. err. on social cost

Social cost = average over the N-agents

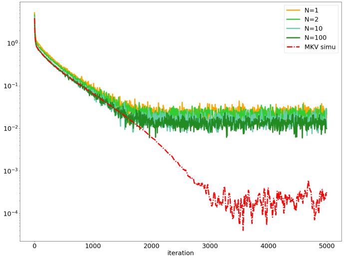
Numerical Illustration

Example: Linear dynamics, quadratic costs of the type:

$$f(x,\mu,v) = \underbrace{(\bar{\mu}-x)^2}_{\text{distance to mean position}} + \underbrace{v^2}_{\text{cost of moving}}, \qquad \bar{\mu} = \underbrace{\int \mu(\xi)d\xi}_{\text{mean position}},$$



Value of the social cost



Rel. err. on social cost

Social cost = average over the N-agents

Main take-away:

Trying to learn the mean-field regime solution can be efficient even for N small

Outline

Idea 2: *Generalize Q-learning to mean-field control*