# Mean Field Games: Numerical Methods and Applications in Machine Learning

# Part 8: Learning in MFGs

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# RECAP

⚠ Terminology **"learning"**:

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1

: *"The theory of learning
in games [. . . ] examines how, which, and what kind of equilibrium might arise as a consequence of a long-run nonequilibrium
process of learning, adaptation, and/or imitation"*

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: *"A computer program is said
to learn from experience E with respect to some class of tasks T and performance measure P if its performance at tasks in T, as
measured by P, improves with experience E."*

⚠ Terminology **"learning"**:

● Game theory, economics, . . . :
[Fudenberg, Levine][1]

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⚠ Terminology **"learning"**:

● Game theory, economics, . . . :
[Fudenberg, Levine][1]

● Machine learning, RL, . . . :
[Mitchell][2]

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[2] Mitchell, T. M. (1997). *Machine Learning*. New York: McGraw-Hill. ISBN: 978-0-07-042807-2: *"A computer program is said to learn from experience E with respect to some class of tasks T and performance measure P if its performance at tasks in T, as measured by P, improves with experience E."*

**Learning/optimization** methods:

- Fixed point iteration
    - ▶ Banach-Picard iterations
    - ▶ idem + damping/mixing/smoothing
    - ▶ Fictitious Play (FP)
- Online Mirror Descent (OMD)
- . . .

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3

4

5

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in

- Games, particularly in economics, see e.g. [Fudenberg, Levine][3]
- Non-atomic games. see e.g. [Hadikhanloo *et al.*'21][4]
- Mean Field Games, see e.g. [Hadikhanloo'18][5]

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[3] Fudenberg, D., & Levine, D. (1998). The Theory of Learning in Games. *The MIT Press*.

[4] Hadikhanloo, S., Laraki, R., Mertikopoulos, P., & Sorin, S. (2021). Learning in nonatomic games, Part I: Finite action spaces and population games. arXiv preprint arXiv:2107.01595.

[5] Hadikhanloo, S. (2018). *Learning in Mean Field Games* (Doctoral dissertation, Université Paris sciences et lettres).

**Generic structure:** repeated game (iterations)

- Update the representative agent behavior
  - value function
  - policy (control)

- Update the population behavior

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*Where is there **learning**?*

→ First type of "Learning": meta-algorithm / outside loop

→ Second type of "Learning": agent's viewpoint / inner loop

**Generic Mean Field model:** for a typical infinitesimal agent

- **Dynamics:** discrete time

$$X_{n+1}^{\alpha,\mu} = F(X_n^{\alpha,\mu}, \alpha_n, \mu_n, \epsilon_{n+1}, \epsilon_{n+1}^0), \quad n \geq 0, \qquad X_0^{\alpha,\mu} \sim \mu_0$$

  - $X_n^{\alpha,\mu} \in \mathcal{X} \subseteq \mathbb{R}^d$ : state, $\alpha_n \in \mathcal{U} \subseteq \mathbb{R}^k$ : action
  - $\epsilon_n \sim \nu$ : idiosyncratic noise, $\epsilon_n^0 \sim \nu^0$ : common noise (random env.)
  - $p(x'|x, a, \mu)$: corresponding transition probability distribution
  - $\mu_n \in \mathcal{P}(\mathcal{X} \times \mathcal{A})$: a state-action distribution
  - $\pi_n$: a policy; randomized actions: $\alpha_n \sim \pi_n(\cdot|s_n, \mu_n)$

# MFG Setup

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- **Reward** ⚠ **:** $\mathbb{J}(\pi; \mu) = \mathbb{E}_{\epsilon,\epsilon^0}\left[\sum_{n=0}^{\infty} \gamma^n r\left(X_n^{\alpha,\mu}, \alpha_n, \mu_n\right)\right]$

- **Two scenarios:**
  - ▶ **Cooperative (MFC):** Find $\pi^*$ s.t.

    $$\pi^* \text{ maximizes } \pi \mapsto J^{MFC}(\pi) = \mathbb{J}(\pi; \mu^\pi) \text{ where } \mu_n^\pi = \mathbb{P}_{X_n^{\alpha,\mu^\pi}}^0$$

  - ▶ **Non-Cooperative (MFG):** Find $(\hat{\pi}, \hat{\mu})$ s.t.

    $$\begin{cases} \hat{\pi} \text{ maximizes } \pi \mapsto J^{MFG}(\pi; \hat{\mu}) = \mathbb{J}(\pi; \hat{\mu}) \\ \hat{\mu}_n = \mathbb{P}_{X_n^{\hat{\alpha},\hat{\mu}}}^0 \end{cases}$$

## Best Response and Population Behavior Maps

We focus on MFG and write $J = J^{MFG}$. For simplicity let's forget the common noise.

Two important functions:

- **Best Response map:**

$$\mathrm{BR} : \mu \mapsto \pi \in \mathrm{argmax}\, J^{MFG}(\cdot; \mu)$$

- **Population Behavior** induced when everyone using a policy:

$$\mathrm{PB} : \pi \mapsto \mu : \mu_{n+1} = \Phi(\mu_n, \pi_n)$$

where:

$$\Phi(\mu, \pi)(x) := \sum_{x \in \mathcal{S}} \sum_{a \in \mathcal{A}} p(x|x_0, a, \mu)\pi(a|x_0, \mu)\mu(x_0), \qquad x \in \mathcal{S}$$

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**Mean Field Nash equilibrium:** $(\hat{\mu}, \hat{\pi})$ such that

$$\begin{cases} \hat{\mu} = \text{PB}(\pi) \\ \hat{\pi} = \text{BR}(\hat{\mu}) \end{cases}$$

⚠ $\hat{\mu}$ can be unique without $\hat{\pi}$ being unique!

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- Update agent's policy: $\pi^{(k+1)} \in \text{BR}(\mu^{(k)})$
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See e.g., [Caines *et al.*; Guo *et al.*; Anahtarci *et al.*; ...]

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Consequence:

$$0 \ge \big[ J(\pi; \mu) - J(\pi; \mu') \big] - \big[ J(\pi'; \mu) - J(\pi'; \mu') \big] =: \mathcal{M}(\pi, \mu, \pi', \mu')$$

If $(\hat{\mu}, \hat{\pi})$ and $(\hat{\mu}', \hat{\pi}')$ are two Nash equilibria,

$$\mathcal{M}(\hat{\pi}, \hat{\mu}, \hat{\pi}', \hat{\mu}') = \big[ J(\hat{\pi}; \hat{\mu}) - J(\hat{\pi}'; \hat{\mu}) \big] + \big[ J(\hat{\pi}'; \hat{\mu}') - J(\hat{\pi}; \hat{\mu}') \big]$$

$$\ge \mathcal{E}(\hat{\pi}'; \hat{\mu}) + \mathcal{E}(\hat{\pi}; \hat{\mu}') \ge 0$$

where $\mathcal{E}$ denotes the **exploitability** of $\pi$ facing $\mu$: $\mathcal{E}(\pi; \mu) = \sup J(\cdot; \mu) - J(\pi; \mu) \ge 0$

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See e.g., [Cardaliaguet & Hadikhanloo; Elie *et al.*; Perrin *et al.*; Geist *et al.*; . . . ]

# Banach-Picard Fixed Point Iterations – with Q-function

Reminder:
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Or, using a Q-function defined as:

$$
\begin{aligned}
&Q_{\pi,\mu}(x, a) \\
&= \mathbb{E}\Big[\sum_{n \geq 0} \gamma^n r(x_n, a_n, \mu)\Big], \quad x_{n+1} \sim p(\cdot|x_n, a_n, \mu), a_{n+1} \sim \pi(\cdot|x_{n+1}), x_0 = x, a_0 = a \\
&= r(x, a, \mu) + \gamma \mathbb{E}[Q_{\pi,\mu}(x', a')], \quad x' \sim p(\cdot|x, a, \mu), a' \sim \pi(\cdot|x')
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- Update agent's policy: $\pi^{(\mathtt{k}+1)}(x) = \mathrm{argmax}_{a \in \mathcal{A}} Q^{(\mathtt{k}+1)}(x, a), x \in \mathcal{S}$
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where

$$\Gamma(y) := \nabla h^*(y) = \underset{p \in \mathcal{P}(\mathcal{A})}{\operatorname{argmax}}[\langle y, p \rangle - h(\pi)].$$

with a regularizer $h : \mathcal{P}(\mathcal{A}) \to \mathbb{R}$ and $h^* : \mathbb{R}^{|\mathcal{A}|} \to \mathbb{R}$ its convex conjugate defined by
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- Update the population behavior

*Where is there **learning**?*

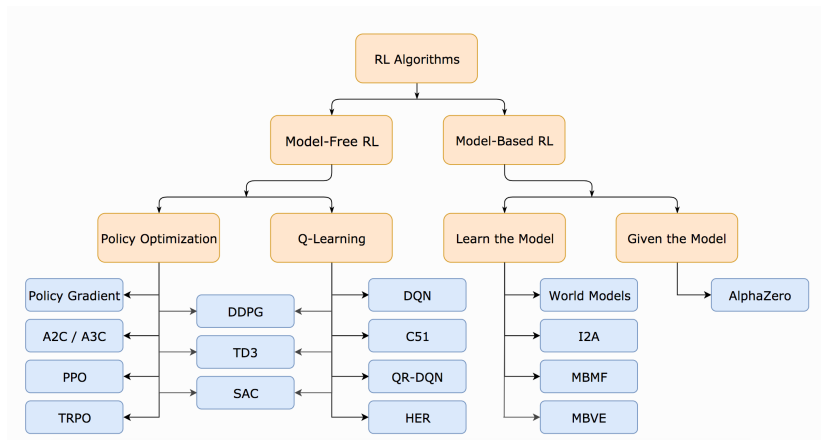→ First type of "Learning": meta-algorithm / outside loop

→ Second type of "Learning": agent's viewpoint / inner loop

# RL Taxonomy



Source: [OpenAI Spinning Up][6]

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**Algorithm 1** Deep Q-learning with Experience Replay

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Initialize replay memory $\mathcal{D}$ to capacity $N$
Initialize action-value function $Q$ with random weights
**for** episode $= 1, M$ **do**
    Initialise sequence $s_1 = \{x_1\}$ and preprocessed sequenced $\phi_1 = \phi(s_1)$
    **for** $t = 1, T$ **do**
        With probability $\epsilon$ select a random action $a_t$
        otherwise select $a_t = \max_a Q^*(\phi(s_t), a; \theta)$
        Execute action $a_t$ in emulator and observe reward $r_t$ and image $x_{t+1}$
        Set $s_{t+1} = s_t, a_t, x_{t+1}$ and preprocess $\phi_{t+1} = \phi(s_{t+1})$
        Store transition $(\phi_t, a_t, r_t, \phi_{t+1})$ in $\mathcal{D}$
        Sample random minibatch of transitions $(\phi_j, a_j, r_j, \phi_{j+1})$ from $\mathcal{D}$
        Set $y_j = \begin{cases} r_j & \text{for terminal } \phi_{j+1} \\ r_j + \gamma \max_{a'} Q(\phi_{j+1}, a'; \theta) & \text{for non-terminal } \phi_{j+1} \end{cases}$
        Perform a gradient descent step on $(y_j - Q(\phi_j, a_j; \theta))^2$ according to equation 3
    **end for**
**end for**

---

Source: [Mnih *et al.*'13][7]

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[7] Mnih, V., Kavukcuoglu, K., Silver, D., Graves, A., Antonoglou, I., Wierstra, D. & Riedmiller, M. (2013). Playing Atari with Deep Reinforcement Learning.

# DDPG

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**Algorithm 1** DDPG algorithm

Randomly initialize critic network $Q(s, a|\theta^Q)$ and actor $\mu(s|\theta^\mu)$ with weights $\theta^Q$ and $\theta^\mu$.
Initialize target network $Q'$ and $\mu'$ with weights $\theta^{Q'} \leftarrow \theta^Q$, $\theta^{\mu'} \leftarrow \theta^\mu$
Initialize replay buffer $R$
**for** episode = 1, M **do**
    Initialize a random process $\mathcal{N}$ for action exploration
    Receive initial observation state $s_1$
    **for** t = 1, T **do**
        Select action $a_t = \mu(s_t|\theta^\mu) + \mathcal{N}_t$ according to the current policy and exploration noise
        Execute action $a_t$ and observe reward $r_t$ and observe new state $s_{t+1}$
        Store transition $(s_t, a_t, r_t, s_{t+1})$ in $R$
        Sample a random minibatch of $N$ transitions $(s_i, a_i, r_i, s_{i+1})$ from $R$
        Set $y_i = r_i + \gamma Q'(s_{i+1}, \mu'(s_{i+1}|\theta^{\mu'})|\theta^{Q'})$
        Update critic by minimizing the loss: $L = \frac{1}{N} \sum_i (y_i - Q(s_i, a_i|\theta^Q))^2$
        Update the actor policy using the sampled policy gradient:

$$\nabla_{\theta^\mu} J \approx \frac{1}{N} \sum_i \nabla_a Q(s, a|\theta^Q)|_{s=s_i, a=\mu(s_i)} \nabla_{\theta^\mu} \mu(s|\theta^\mu)|_{s_i}$$

        Update the target networks:

$$\theta^{Q'} \leftarrow \tau\theta^Q + (1-\tau)\theta^{Q'}$$

$$\theta^{\mu'} \leftarrow \tau\theta^\mu + (1-\tau)\theta^{\mu'}$$

    **end for**
**end for**

---

Source: [Lillicrap *et al.*'16][8]

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[8] Lillicrap, T. P., Hunt, J. J., Pritzel, A., Heess, N., Erez, T., Tassa, Y., Silver, D. & Wierstra, D. (2016). Continuous control with deep reinforcement learning. ICLR 2016.

# SAC

---

**Algorithm 1** Soft Actor-Critic

Initialize parameter vectors $\psi, \bar{\psi}, \theta, \phi$.
**for** each iteration **do**
    **for** each environment step **do**
        $\mathbf{a}_t \sim \pi_\phi(\mathbf{a}_t|\mathbf{s}_t)$
        $\mathbf{s}_{t+1} \sim p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t)$
        $\mathcal{D} \leftarrow \mathcal{D} \cup \{(\mathbf{s}_t, \mathbf{a}_t, r(\mathbf{s}_t, \mathbf{a}_t), \mathbf{s}_{t+1})\}$
    **end for**
    **for** each gradient step **do**
        $\psi \leftarrow \psi - \lambda_V \hat{\nabla}_\psi J_V(\psi)$
        $\theta_i \leftarrow \theta_i - \lambda_Q \hat{\nabla}_{\theta_i} J_Q(\theta_i)$ for $i \in \{1, 2\}$
        $\phi \leftarrow \phi - \lambda_\pi \hat{\nabla}_\phi J_\pi(\phi)$
        $\bar{\psi} \leftarrow \tau \psi + (1 - \tau)\bar{\psi}$
    **end for**
**end for**

---

Source: [Haarnoja *et al.*'18][9]

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[9] Haarnoja, T., Zhou, A., Abbeel, P. & Levine, S. (2018). Soft Actor-Critic: Off-Policy Maximum Entropy Deep Reinforcement Learning with a Stochastic Actor. ICML 2018.

*What about the population behavior $\mu$?*

## Population Distribution Approximation

*What about the population behavior $\mu$?*

- Empirical distribution $\mu^N$

- Histogram (state space discretization)

- $\epsilon$-net in $\mathcal{P}(\mathcal{X})$

- Function approximation for the density:
    - Kernels
    - Neural nets: normalizing flows, . . .
    - . . .

- . . .

## Systemic Risk

Revisiting: Systemic risk model of [Carmona, Fouque, Sun]

$$J((a_n)_n; (m_n)_n) = -\mathbb{E}\Big[\sum_{n=0}^{N_T} \Big(a_n^2 \underbrace{-qa_n(m_n - X_n)}_{\substack{\text{borrow if } X_n < m_n \\ \text{lend if } X_n > m_n}} +\kappa(m_n - X_n)^2\Big) + c(m_{N_T} - X_{N_T})^2\Big]$$
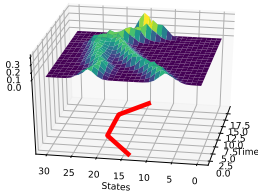
Subj. to: $\quad X_{n+1} = X_n + [K(m_n - X_n) + a_n] + \epsilon_{n+1} + \epsilon_{n+1}^0$

At equilibrium: $\quad m_n = \mathbb{E}[X_n | \epsilon^0], \, n \geq 0$
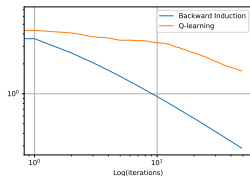
# Systemic Risk

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Subj. to: $\quad X_{n+1} = X_n + [K(m_n - X_n) + a_n] + \epsilon_{n+1} + \epsilon_{n+1}^0$

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[Perrin *et al.*, NeurIPS'20]: Fictitious Play with Backward Induction or tabular Q-learning
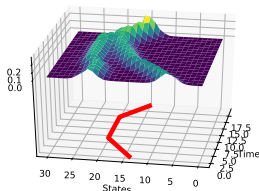


Exact solution



Fictitious Play & RL



Exploitability

## Crowd Aversion

Revisiting: Crowd aversion model of [Alumulla, Ferreira, Gomes]
MFG on $\mathbb{T}$,

$$f(x, m, v) = \frac{1}{2}|v|^2 + \tilde{f}(x) + \ln(m(x)),$$

with $\tilde{f}(x) = 2\pi^2 \left[ -\sum_{i=1}^d c\sin(2\pi x_i) + \sum_{i=1}^d |c\cos(2\pi x_i)|^2 \right] - 2\sum_{i=1}^d c\sin(2\pi x_i)$,
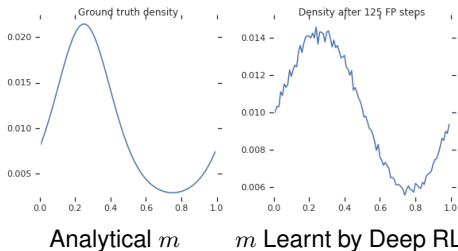
then the solution is given by $u(x) = c\sum_{i=1}^d \sin(2\pi x_i)$ and $m(x) = e^{2u(x)}/\int e^{2u}$

## Crowd Aversion

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[Elie *et al.*, AAAI'20]: Fictitious Play & DDPG (continuous spaces)



Analytical $m$     $m$ Learnt by Deep RL

## Flocking

Revisiting: Flocking aversion model of [Nourian, Caines, Malhamé]
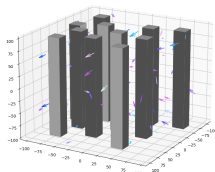[Perrin *et al.*, IJCAI'21]: For continuous space problems: **Deep RL**
- Deep RL (SAC) for the policy ($\approx$ control)
- Deep NN (normalizing flow) for the population distribution

state = (position, velocity) = $(x, v) \in \mathbb{R}^{2d}$, $\qquad \left\{ \begin{array}{l} x_{n+1} = x_n + v_n \Delta t, \\ v_{n+1} = v_n + a_n \Delta t + \epsilon_{n+1}, \end{array} \right.$

with running cost: $\qquad f_\beta^{\text{flock}}(x, v, \mu) = \left\| \int_{\mathbb{R}^{2d}} \frac{(v - v')}{(1 + \|x - x'\|^2)^\beta} \, d\mu(x', v') \right\|^2$,

where $\beta \geq 0$, and $\mu$ is the position-velocity distribution.

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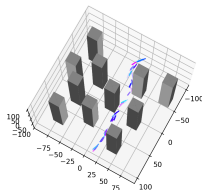- Deep NN (normalizing flow) for the population distribution

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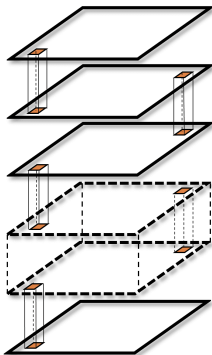


Initial distribution      At convergence

Video: https://www.youtube.com/watch?v=TdXysW_FA3k

## Building Evacuation

A model for crowd motion during building evacuation:

$$r(x, a, \mu) = -\eta \log(\mu(x)) + 10 \times \mathbb{1}_{floor=0}$$
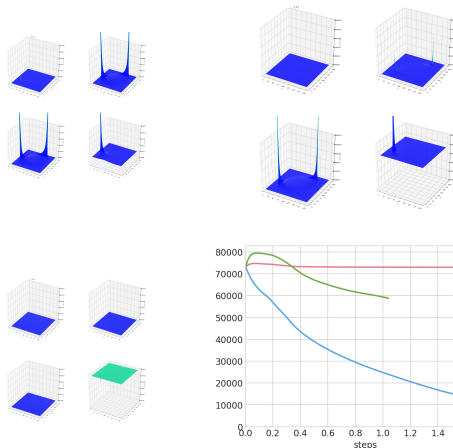
[Pérolat *et al.*'21]: OMD (no RL)



Initial distribution

# Building Evacuation

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[Pérolat *et al.*'21]: OMD (no RL)



FP (red, $\alpha = 10^{-5}$), FP damped (green, $\alpha = 10^{-3}$) and OMD (blue, $\alpha = 10^{-4}$)

# Summary