

Numerical Methods for Mean Field Games

Lecture 4 *Deep Learning Methods: Part I* *MFC and MKV FBSDE*

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Outline

1. Introduction

2. Deep Learning for MFC

3. Deep Learning for MKV FBSDE

4. Two Examples of Extensions

5. Conclusion

Numerical methods discussed so far:

- ODE system for LQ setting
- FBPDE system
- FBSDE system

“Classical” Numerical Methods for MFG: Some references

Some methods based on the deterministic approach to MFG/MFC:

- Finite difference & Newton method: [Achdou and Capuzzo-Dolcetta, 2010], [Achdou et al., 2012], ...
- (Semi-)Lagrangian approach: [Carlini and Silva, 2014, Carlini and Silva, 2015], [Carlini and Silva, 2018], [Calzola et al., 2022], ...
- Augmented Lagrangian & ADMM: [Benamou and Carlier, 2015], [Andreev, 2017a], [Achdou and Laurière, 2016], ...
- Primal-dual algo.: [Briceño Arias et al., 2018], [Briceño Arias et al., 2019], ...
- Gradient descent based methods [Laurière and Pironneau, 2016], [Pfeiffer, 2016], [Lavigne and Pfeiffer, 2022], ...
- Monotone operators [Almulla et al., 2017], [Gomes and Saúde, 2018], [Gomes and Yang, 2020], ...
- Policy iteration [Cacace et al., 2021], [Cui and Koeppl, 2021], [Camilli and Tang, 2022], [Tang and Song, 2022], [Laurière et al., 2023], ...
- Finite elements [Benamou and Carlier, 2015], [Andreev, 2017b], ...

Some methods based on the probabilistic approach to MFG/MFC:

- Cubature [[de Raynal and Trillos, 2015](#)], ...
- Markov chain approximation: [[Bayraktar et al., 2018](#)], ...
- Probabilistic approach and Picard: [[Chassagneux et al., 2019](#)], [[Angiuli et al., 2019](#)], ...
- Probabilistic approach and regression: [[Balata et al., 2019](#)], ...
- ...

Many of these methods are very **efficient** and have been **analyzed** in detail

However, they are usually limited to problems with:

- (relatively) **small dimension**
- (relatively) **simple structure**

⇒ motivations to develop **machine learning** methods (see lectures 4, 5, 6)

- In this lecture and the following one, we will use deep learning to solve MFGs
- At a high level, there are two main ingredients:
 - ▶ Approximation using [deep neural networks](#)
 - ▶ Minimization of a loss function using [stochastic gradient descent](#)
- Many variants and refinements, ...
- See e.g. [\[LeCun et al., 2015, Goodfellow et al., 2016\]](#), ...

- **Goal:** Minimize over $\varphi(\cdot)$, $\mathbb{J}(\varphi) := \mathbb{E}_{\xi}[\mathbb{L}(\varphi, \xi)]$
- Example: Regression: $\xi = (x, f(x))$ for some f , $\mathbb{L}(\varphi, \xi) = \|\varphi(x) - f(x)\|^2$

Ingredient 1: Neural Networks

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- Example: Regression: $\xi = (x, f(x))$ for some f , $\mathbb{L}(\varphi, \xi) = \|\varphi(x) - f(x)\|^2$
- **Idea:** Instead of min. over all $\varphi(\cdot)$, min. over parameters θ of $\varphi_{\theta}(\cdot)$
- Example: **Feedforward fully-connected neural network:**
 - ▶ $\varphi_{\theta}(\cdot)$
 - ▶ with **weights & biases** $\theta = (\beta^{(k)}, w^{(k)})_{k=1, \dots, \ell}$
 - ▶ activation functions $\psi^{(i)}$: sigmoid, tanh, ReLU, ...; applied coordinate-wise

$$\underbrace{\varphi_{\theta}(x)}_{\varphi(\theta, x)} = \psi^{(\ell)} \left(\beta^{(\ell)} + w^{(\ell)} \dots \psi^{(2)} \left(\beta^{(2)} + w^{(2)} \underbrace{\psi^{(1)}(\beta^{(1)} + w^{(1)}x)}_{\text{one hidden layer}} \right) \dots \right)$$

- ▶ Depth = number of layers; width of a layer = dimension of bias vector

- Many other architectures (convolutional neural networks, recurrent neural networks, ...), see e.g. [\[Leijnen and Veen, 2020\]](#)
- Successes of deep learning in many fields: natural language processing, computer vision, drug design, ... and even games!
- Combination with reinforcement learning (see lecture 6)
- Universal approximation theorems [\[Cybenko, 1989\]](#), [\[Hornik, 1991\]](#), ...
- Connections with numerical analysis, see e.g. [\[Després, 2022\]](#)

Differentiation: can compute partial derivatives by automatic differentiation (AD) at every (θ, x) :

- With respect to parameters: $\nabla_{\theta} \varphi(\theta, x)$

$$\nabla_{\beta^{(\ell)}} \varphi(\theta, x) = \dots, \quad \nabla_{w^{(2)}} \varphi(\theta, x) = \dots$$

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- With respect to state variable: $\nabla_x \varphi(\theta, x)$ can be computed by AD too

$$\partial_{x_1} \varphi(\theta, x) = \dots$$

\Rightarrow can be used in PDEs (see lecture 5)

Ingredient 2: Stochastic Gradient Descent

- **Goal:** Minimize over $\varphi(\cdot)$, $\mathbb{J}(\varphi) := \mathbb{E}_{\xi}[\mathbb{L}(\varphi, \xi)]$
- **Parameterization:** $\tilde{\mathbb{J}}(\theta) := \mathbb{E}_{\xi}[\tilde{\mathbb{L}}(\theta, \xi)]$, where $\tilde{\mathbb{L}}(\theta, \xi) := \mathbb{L}(\varphi_{\theta}, \xi)$

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 - ▶ we have some samples (i.e. random realizations) of ξ
 - ▶ we know \mathbb{L}

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Algorithm: Stochastic Gradient Descent

Input: Initial param. θ_0 ; data $S = (\xi_s)_{s=1, \dots, |S|}$; nb of steps K ; learning rates $(\eta^{(k)})_k$

Output: Parameter θ^* s.t. φ_{θ^*} (approximately) minimizes $\tilde{\mathbb{J}}$

- 1 Initialize $\theta^{(0)} = \theta_0$
 - 2 **for** $k = 0, 1, 2, \dots, K - 1$ **do**
 - 3 Pick $s \in S$ randomly
 - 4 Compute the gradient $\nabla_{\theta} \tilde{\mathbb{L}}(\theta^{(k-1)}, \xi_s) = \frac{d}{d\theta} \mathbb{L}(\varphi_{\theta^{(k-1)}}, \xi_s)$
 - 5 Set $\theta^{(k)} = \theta^{(k-1)} - \eta^{(k)} \nabla_{\theta} \tilde{\mathbb{L}}(\theta^{(k-1)}, \xi_s)$
 - 6 **return** $\theta^{(K)}$
-

Ingredient 2: Stochastic Gradient Descent – Comments

- Many variants:
 - ▶ Learning rate: `ADAM` (Adaptive Moment Estimation) [\[Kingma and Ba, 2014\]](#), ...
 - ▶ Samples: Mini-batches, ...
- Proofs of convergence e.g. using stochastic approximation [\[Robbins and Monro, 1951\]](#), [\[Borkar, 2009\]](#)
- In practice: many details to be discussed, see e.g. [\[Bottou, 2012\]](#); choice of hyperparameters
 - ▶ architecture
 - ▶ initialization
 - ▶ learning rate
 - ▶ loss function
 - ▶ ...

- Consider the task: minimize over φ the **population risk**:

$$\mathcal{R}(\varphi) = \mathbb{E}_{x,y}[L(\varphi(x), y)]$$

with $x \sim \mu$ and $y = f(x) + \epsilon$ for some noise ϵ where f is unknown

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- In practice:
 - ▶ minimize over a **hypothesis class** \mathcal{F} of φ
 - ▶ finite number of samples, $S = (x_m, y_m)_{m=1,\dots,M}$: **empirical risk**:

$$\hat{\mathcal{R}}_S(\varphi) = \frac{1}{M} \sum_{m=1}^M L(\varphi(x_m), y_m) \quad (+ \text{ regu})$$

- ▶ finite number of **optimization steps**, say k

We are interested in:

- **Approximation error:** Letting $\varphi^* = \operatorname{argmin}_{\varphi \in \mathcal{F}} \operatorname{dist}(\varphi, f)$,

$$\epsilon_{\text{approx}} = \operatorname{dist}(\varphi^*, f)$$

- **Estimation error:** Letting $\hat{\varphi}_S = \operatorname{argmin}_{\varphi \in \mathcal{F}} \hat{\mathcal{R}}_S(\varphi)$

$$\epsilon_{\text{estim}} = \operatorname{dist}(\hat{\varphi}_S, \varphi^*)$$

- **Optimization error:** After k steps, we get $\varphi_S^{(k)}$;

$$\epsilon_{\text{optim}} = \operatorname{dist}(\varphi_S^{(k)}, \hat{\varphi}_S)$$

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- **Optimization error:** After k steps, we get $\varphi_S^{(k)}$;

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- **Generalization error** of the learnt $\varphi_S^{(k)}$:

$$\epsilon_{\text{gene}} = \epsilon_{\text{approx}} + \epsilon_{\text{estim}} + \epsilon_{\text{optim}}$$

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- An optimal control is a “temporally extended” optimization problem
- Numerically, we cannot minimize over all possible controls
- We can parameterize the control function
- and then optimize over the parameters
- See e.g. [\[Gobet and Munos, 2005\]](#), [\[Han and E, 2016\]](#), ...

Stochastic optimal control problem:

Minimize over $\alpha(\cdot, \cdot)$

$$J(\alpha(\cdot, \cdot)) = \mathbb{E} \left[\int_0^T f(X_t, \alpha(t, X_t)) dt + g(X_T) \right],$$

with

$$X_0 \sim m_0, \quad dX_t = b(X_t, \alpha(t, X_t)) dt + \sigma dW_t$$

Stochastic optimal control problem: (1) neural network φ_θ ,

Minimize over **neural network** parameters θ

$$J(\theta) = \mathbb{E} \left[\int_0^T f(X_t, \varphi_\theta(t, X_t)) dt + g(X_T) \right],$$

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Stochastic optimal control problem: (1) neural network φ_θ , (2) time discretization

Minimize over **neural network** parameters θ and N_T time steps

$$J^{N_T}(\theta) = \mathbb{E} \left[\sum_{n=0}^{N_T-1} f(X_n, \varphi_\theta(t_n, X_n)) \Delta t + g(X_{N_T}) \right],$$

with

$$X_0 \sim m_0, \quad X_{n+1} - X_n = b(X_n, \varphi_\theta(t_n, X_n)) \Delta t + \sigma \Delta W_n$$

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To implement SGD, at each iteration we pick a sample $\xi = (X_0, \Delta W_0, \dots, \Delta W_{N_T-1})$

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where $\mu_t = \mathcal{L}(X_t)$ with

$$X_0 \sim m_0, \quad dX_t = b(X_t, \mu_t, \alpha(t, X_t)) dt + \sigma dW_t$$

MFC problem: (1) Finite pop.,

Minimize over **decentralized** controls $\alpha(\cdot, \cdot)$ with N agents

$$J^N(\alpha(\cdot, \cdot)) = \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N \int_0^T f(X_t^i, \mu_t^N, \alpha(t, X_t^i)) dt + g(X_T^i, \mu_T^N) \right],$$

where $\mu_t^N = \frac{1}{N} \sum_{j=1}^N \delta_{X_t^j}$, with

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MFC: Approximate Problem

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MFC: Approximate Problem

MFC problem: (1) Finite pop., (2) neural network φ_θ , (3) time discretization

Minimize over **neural network** parameters θ with N agents and N_T time steps

$$J^{N, N_T}(\theta) = \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N \sum_{n=0}^{N_T-1} f(X_n^i, \mu_n^N, \varphi_\theta(t_n, X_n^i)) \Delta t + g(X_{N_T}^i, \mu_{N_T}^N) \right],$$

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Note: we aim for a decentralized control, whereas for a general N -agent control problem, the optimal control is not always of this type

- The following kind of convergence result (bound on the **approximation error**) can be proved, see [\[Carmona and Laurière, 2022\]](#):

Approximation theorem

Under suitable assumptions (in particular regularity of the value function),

$$\left| \inf_{\alpha(\cdot, \cdot)} J(\alpha(\cdot, \cdot)) - \inf_{\theta \in \Theta} J^{N, N_T}(\theta) \right| \leq \epsilon_1(N) + \epsilon_2(\dim(\theta)) + \epsilon_3(N_T)$$

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- The **optimization error** remains to be studied
- Many extensions and refinements to be investigated

Approximation Error Analysis: Main Ingredients of the Proof

Proposition 1 (N agents & decentralized controls):

Under suitable assumptions, there exists a decentralized control α^* s.t. ($d = \text{dimension of } X_t$)

$$\left| \inf_{\alpha(\cdot)} J(\alpha(\cdot)) - J^N(\alpha^*(\cdot)) \right| \leq \epsilon_1(N) \in \tilde{O}(N^{-1/d}).$$

Proof: propagation of chaos type argument [Carmona and Delarue, 2018]

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Proposition 2 (approximation by neural networks): Under suitable assumptions

There exists a set of parameters $\theta \in \Theta$ for a one-hidden layer $\hat{\varphi}_\theta$ s.t.

$$\left| J^N(\alpha^*(\cdot)) - J^N(\hat{\varphi}_\theta(\cdot)) \right| \leq \epsilon_2(\dim(\theta)) \in O\left(\dim(\theta)^{-\frac{1}{3(d+1)}}\right).$$

Proof: Key difficulty: approximate $v^*(\cdot)$ by $\hat{\varphi}_\theta(\cdot)$ while controlling $\|\nabla \hat{\varphi}_\theta(\cdot)\|$ by $\|\nabla v^*(\cdot)\|$

→ universal approximation without rate of convergence is not enough

→ approximation rate for the derivative too, e.g. from [Mhaskar and Micchelli, 1995]

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Proposition 3 (Euler-Maruyama scheme):

For a specific neural network $\hat{\varphi}_\theta(\cdot)$,

$$\left| J^N(\hat{\varphi}_\theta(\cdot)) - J^{N,N_T}(\hat{\varphi}_\theta(\cdot)) \right| \leq \epsilon_3(N_T) \in O\left(N_T^{-1/2}\right).$$

Key point: $O(\cdot)$ independent of N and $\text{dim}(\theta)$

Proof: analysis of **strong error rate** for Euler scheme (reminiscent of [Bossy and Talay, 1997])

- Key idea: replace optimal control problem by (finite dim.) optimization problem:

- ▶ Loss function = cost: $J^{N, N_T}(\theta) = \mathbb{E}[\mathbb{L}(\varphi_\theta, \xi)]$
- ▶ One sample: $\xi = (X_0^j, (\Delta W_n^j)_{n=0, \dots, N_T-1})_{j=1, \dots, N}$

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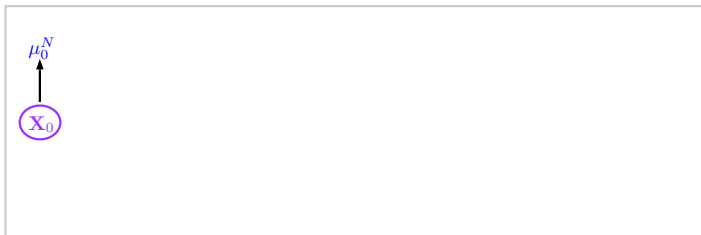
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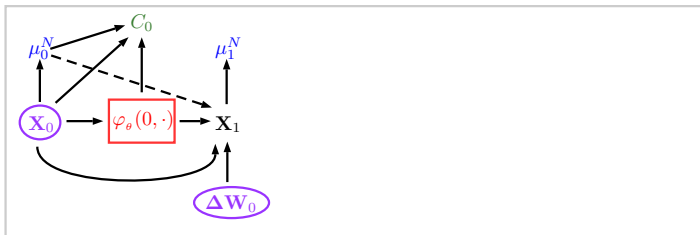


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- ▶ One sample: $\xi = (X_0^j, (\Delta W_n^j)_{n=0, \dots, N_T-1})_{j=1, \dots, N}$

→ can use **Stochastic Gradient Descent**

- Structure:

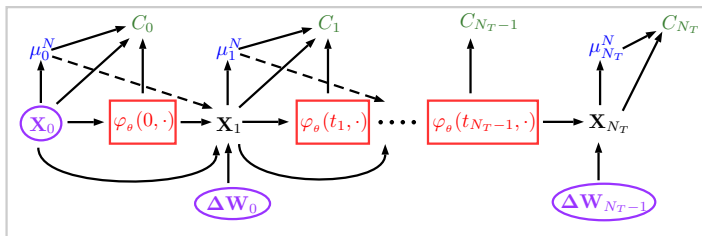


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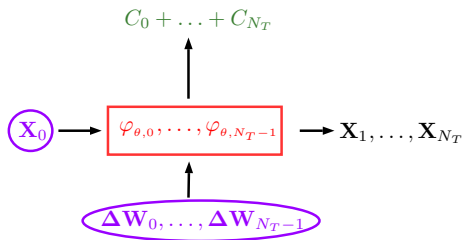


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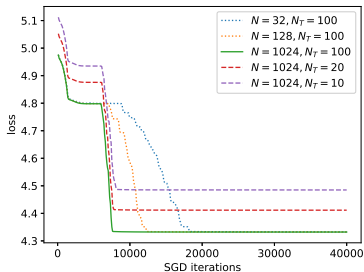
Numerical Illustration 1: LQ MFC

Benchmark to assess **empirical convergence of SGD**: LQ problem with explicit sol.

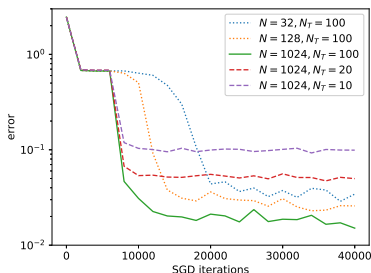
Example: Linear dynamics, quadratic costs of the type

$$f(x, \mu, v) = \underbrace{(\bar{\mu} - x)^2}_{\text{distance to mean position}} + \underbrace{v^2}_{\text{cost of moving}}, \quad \bar{\mu} = \underbrace{\int \mu(\xi) d\xi}_{\text{mean position}}, \quad g(x) = x^2$$

Numerical example with $d = 10$ (see [\[Carmona and Laurière, 2022\]](#)):



total cost (= loss function)



L^2 -error on the control

Numerical Illustration 2: min-LQ MFC with common noise

The following model is inspired by [Salhab et al., 2015] and [Achdou and Lasry, 2019].

MFC with simple CN:

Dynamics: $dX_t = \phi_t(X_t, \epsilon_t^0)dt + \sigma dW_t$, $\epsilon_t^0 = 0$ until $t = T/2$, and then ξ_1 or ξ_2 w.p. $1/2$

Running cost $|\phi_t(X_t, \epsilon_t^0)|^2$, final cost $(X_T - \epsilon_T^0)^2 + \bar{Q}_T(\bar{m}_T - X_T)^2$

Parameter values: $\sigma = 0.1$, $T = 1$, $\xi_1 = -1.5$, $\xi_2 = +1.5$

Numerical results:

- **neural network** $\varphi_\theta(t, X_t, \epsilon_t^0)$, taking as an input the **common noise**
- benchmark solution computed by solving a **system of 6 PDEs** (see [Achdou and Lasry, 2019])

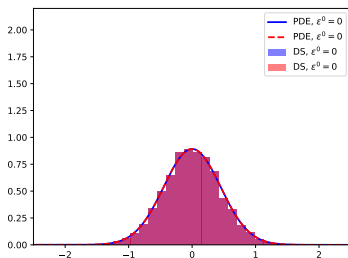
Numerical Illustration 2: min-LQ MFC with common noise

Here the common noise takes one among two values, at time $T/2$.

More details in [\[Carmona and Laurière, 2022\]](#)

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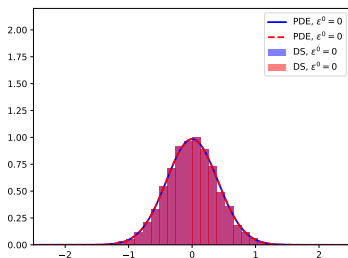
$t = 0$

Until $T/2$: concentrate around mid-point = 0

More details in [\[Carmona and Laurière, 2022\]](#)

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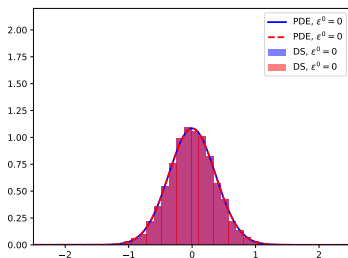
$t = 0.1$

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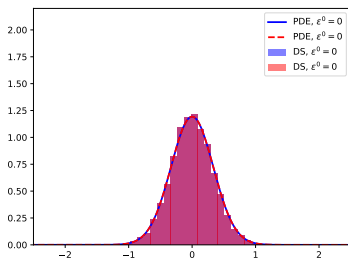
$t = 0.2$

Until $T/2$: concentrate around mid-point = 0

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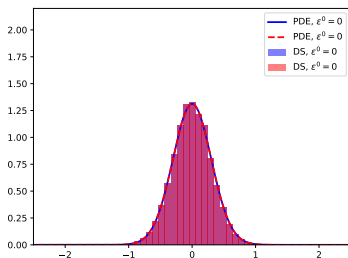
$t = 0.3$

Until $T/2$: concentrate around mid-point = 0

More details in [\[Carmona and Laurière, 2022\]](#)

Numerical Illustration 2: min-LQ MFC with common noise

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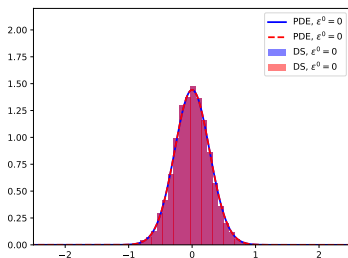
$t = 0.4$

Until $T/2$: concentrate around mid-point = 0

More details in [\[Carmona and Laurière, 2022\]](#)

Numerical Illustration 2: min-LQ MFC with common noise

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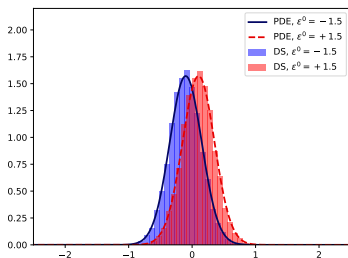
$t = 0.5$

Until $T/2$: concentrate around mid-point = 0

More details in [\[Carmona and Laurière, 2022\]](#)

Numerical Illustration 2: min-LQ MFC with common noise

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$t = 0.6$

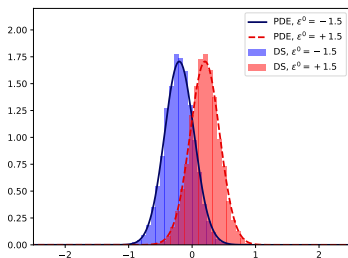
Until $T/2$: concentrate around mid-point = 0

After $T/2$: move towards the target selected by common noise

More details in [\[Carmona and Laurière, 2022\]](#)

Numerical Illustration 2: min-LQ MFC with common noise

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$t = 0.7$

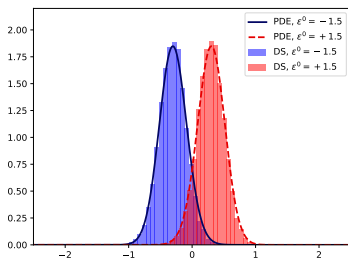
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$t = 0.8$

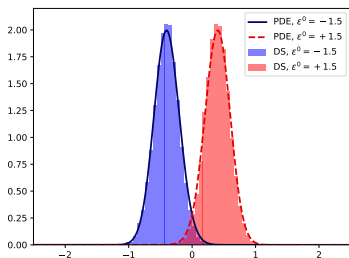
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More details in [\[Carmona and Laurière, 2022\]](#)

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$t = 0.9$

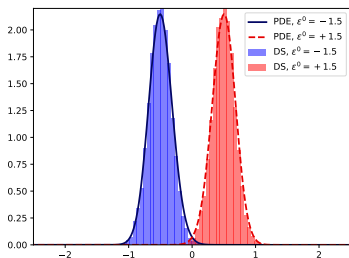
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More details in [\[Carmona and Laurière, 2022\]](#)

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$t = 1$

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After $T/2$: move towards the target selected by common noise

More details in [\[Carmona and Laurière, 2022\]](#)

Numerical Illustration 3: MFC with Interactions Through the Controls

Price Impact Model [Carmona and Lacker, 2015, Carmona and Delarue, 2018]:

- Price process: with $\nu^\alpha =$ population's distribution over actions,

$$dS_t^\alpha = \gamma \int_{\mathbb{R}} a d\nu_t^\alpha(a) dt + \sigma_0 dW_t^0$$

- Typical agent's inventory: $dX_t^\alpha = \alpha_t dt + \sigma dW_t$
- Typical agent's wealth: $dK_t^\alpha = -(\alpha_t S_t^\alpha + c_\alpha(\alpha_t)) dt$
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Equivalent problem:

$$J(\alpha) = \mathbb{E} \left[\int_0^T \left(c_\alpha(\alpha_t) + c_X(X_t^\alpha) - \gamma X_t^\alpha \int_{\mathbb{R}} a d\nu_t^\alpha(a) \right) dt + g(X_T^\alpha) \right]$$

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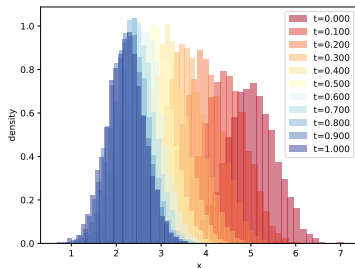
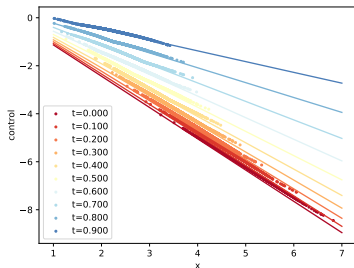
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We take: $c_\alpha(v) = \frac{1}{2} c_\alpha v^2$, $c_X(x) = \frac{1}{2} c_X x^2$ and $g(x) = \frac{1}{2} c_g x^2$

Numerical Illustration 3: MFC with Interactions Through the Controls

Control learnt (left) and associated state distribution (right)

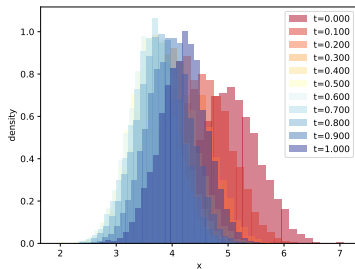
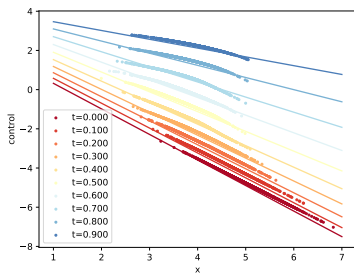


$$T = 1, c_X = 2, c_\alpha = 1, c_g = 0.3, \sigma = 0.5, \gamma = 0.2$$

See Section 2 in [\[Carmona and Laurière, 2023\]](#) for more details.

Numerical Illustration 3: MFC with Interactions Through the Controls

Control learnt (left) and associated state distribution (right)



$$T = 1, c_X = 2, c_\alpha = 1, c_g = 0.3, \sigma = 0.5, \gamma = 1$$

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Code

Sample code to illustrate: [IPython notebook](#)

<https://colab.research.google.com/drive/1QYWz4Sclw9goRZsbd0uB6wR6a0Uu0a3k?usp=sharing>

- Deep learning for MFC using a direct approach where the control is parameterized as a neural network
- Applied to the price impact model discussed above

- DL for stochastic control [[Gobet and Munos, 2005](#)], [[Han and E, 2016](#)], ...
- Various possible implementations; example: 1 NN per time step instead of a single 1 NN as a function of time
- Extensions to finite-player games [[Hu, 2021](#)]
- Extension to MFC presented here [[Carmona and Laurière, 2022](#)]; see also [[Carmona and Laurière, 2023](#)]
- Related works with mean field: [[Fouque and Zhang, 2020](#)] (MFC with delay), [[Germain et al., 2019](#)], [[Agram et al., 2020](#)], [[Dayanikli et al., 2023](#)] (with population-dependent controls), ...

Outline

1. Introduction

2. Deep Learning for MFC

3. Deep Learning for MKV FBSDE

4. Two Examples of Extensions

5. Conclusion

- Goal: solve an FBSDE system
- The backward process has a value Y_0 at time 0, but it is not known
- Try to guess the correct initial condition so that the terminal condition is satisfied
- This yields a new optimal control problem
- See e.g. [\[Kohlmann and Zhou, 2000\]](#), [\[Sannikov, 2008\]](#), ...
- For the new optimal control problem, use deep learning [\[E et al., 2017\]](#)

Solutions of sto. control problems can be characterized by **FBSDEs** of the form

$$\begin{cases} dX_t = B(t, X_t, Y_t)dt + dW_t, & X_0 \sim m_0 & \rightarrow \text{state} \\ dY_t = -F(t, X_t, Y_t)dt + Z_t \cdot dW_t, & Y_T = G(X_T) & \rightarrow \text{control/cost} \end{cases}$$

(stemming from sto. Pontryagin's or Bellman's principle: $F = f$ or $F = \partial_x H$)

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Reformulation as a new optimal control problem

Minimize over $y_0(\cdot)$ and $\mathbf{z}(\cdot) = (z_t(\cdot))_{t \geq 0}$

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under the constraint that $(X^{y_0, \mathbf{z}}, Y^{y_0, \mathbf{z}})$ solve: $\forall t \in [0, T]$

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Note: This problem is *not* the original stochastic control problem !

Application to Solve PDEs

This method can be used to solve PDEs [\[E et al., 2017\]](#)

Feynman-Kac formula: correspondence $u(t, X_t) = Y_t$ where

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- u solves the PDE

$$\begin{cases} u(T, x) = G(x) \\ \frac{\partial u}{\partial t}(t, x) + B(t, x) \frac{\partial u}{\partial x}(t, x) + \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2}(t, x) + F(t, x) = 0 \end{cases}$$

- X solves the SDE:

$$dX_t = B(t, x)dt + \sigma dW_t$$

- (Y, Z) solves the BSDE:

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- Connection also works with $dX_t = dW_t$ and a different $Y_t \dots$

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- Connection also works with $dX_t = dW_t$ and a different $Y_t \dots$
- Application: solve a PDE by solving the corresponding (F)BSDE

This method can be used to solve PDEs [E et al., 2017]

Feynman-Kac formula: correspondence $u(t, X_t) = Y_t$ where

- u solves the PDE

$$\begin{cases} u(T, x) = G(x) \\ \frac{\partial u}{\partial t}(t, x) + B(t, x) \frac{\partial u}{\partial x}(t, x) + \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2}(t, x) + F(t, x) = 0 \end{cases}$$

- X solves the SDE:

$$dX_t = B(t, x)dt + \sigma dW_t$$

- (Y, Z) solves the BSDE:

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- Ex. HJB equation. Many variations/extensions

Solutions of MFG (and MFC) can be characterized by **MKV FBSDEs** of the form

$$\begin{cases} dX_t = B(t, X_t, \mathcal{L}(X_t), Y_t)dt + dW_t, & X_0 \sim m_0 & \rightarrow \text{state} \\ dY_t = -F(t, X_t, \mathcal{L}(X_t), Y_t)dt + Z_t \cdot dW_t, & Y_T = G(X_T, \mathcal{L}(X_T)) & \rightarrow \text{control/cost} \end{cases}$$

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Reformulation as a MFC problem [Carmona and Laurière, 2022]

Minimize over $y_0(\cdot)$ and $\mathbf{z}(\cdot) = (z_t(\cdot))_{t \geq 0}$

$$\mathfrak{J}(y_0(\cdot), \mathbf{z}(\cdot)) = \mathbb{E} \left[\|Y_T^{y_0, \mathbf{z}} - G(X_T^{y_0, \mathbf{z}}, \mathcal{L}(X_T^{y_0, \mathbf{z}}))\|^2 \right],$$

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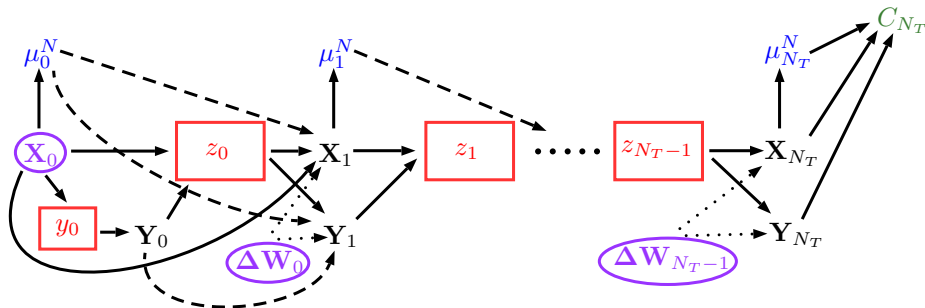
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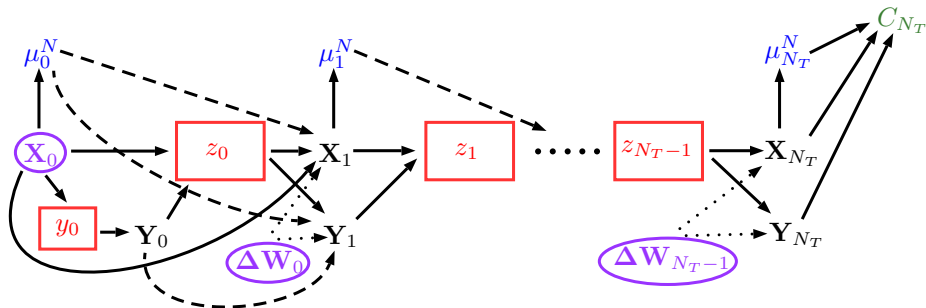
NB: This problem is *not* the original MFG or MFC

Implementation



- **Inputs:** initial positions $\mathbf{X}_0 = (X_0^i)_i$, BM increments: $\Delta \mathbf{W}_n = (\Delta W_n^i)_i$, for all n
- **Loss function:** total cost = C_{N_T} = terminal penalty; state = (X_n, Y_n)
- **SGD** to optimize over the **param.** θ_y, θ_z of 2 NN for
 $y_{\theta_y}(\cdot) \approx y_0(\cdot), z_{\theta_z}(\cdot, \cdot) \approx z(\cdot, \cdot)$

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 $y_{\theta_y}(\cdot) \approx y_0(\cdot), z_{\theta_z}(\cdot, \cdot) \approx z(\cdot, \cdot)$
- Alternative implementation: $1 + N_T$ NNs for $y_0(\cdot), z_0(\cdot), \dots, z_{N_T-1}(\cdot)$

Numerical Illustration 1: Comparison with Picard Solver

Example of MKV FBSDE from [Chassagneux et al., 2019] (ρ = coupling parameter)

$$\begin{aligned}dX_t &= -\rho Y_t dt + \sigma dW_t, & X_0 &= x_0 \\dY_t &= \operatorname{atan}(\mathbb{E}[X_t])dt + Z_t dW_t, & Y_T &= G'(X_T) := \operatorname{atan}(X_T)\end{aligned}$$

Comes from the **MFG** defined by $dX_t^\alpha = \alpha_t dt + dW_t$ and

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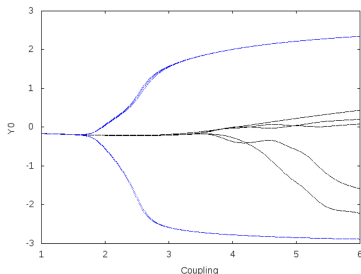
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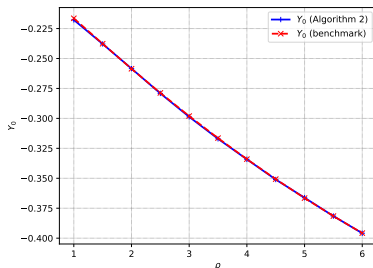
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[Chassagneux et al., 2019]



NN (FBSDE system)

More details in [Carmona and Laurière, 2022]

Example: MFG for inter-bank borrowing/lending

[Carmona et al., 2015]

X = log-monetary reserve, α = rate of borrowing/lending to central bank, cost:

$$J(\alpha; \bar{m}) = \mathbb{E} \left[\int_0^T \left[\frac{1}{2} \alpha_t^2 - q \alpha_t (\bar{m}_t - X_t) + \frac{\epsilon}{2} (\bar{m}_t - X_t)^2 \right] dt + \frac{c}{2} (\bar{m}_T - X_T)^2 \right]$$

where $\bar{m} = (\bar{m}_t)_{t \geq 0}$ = conditional mean of the population states given W^0 , and

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Numerical Illustration 2: LQ MFG with Common Noise

Example: MFG for inter-bank borrowing/lending

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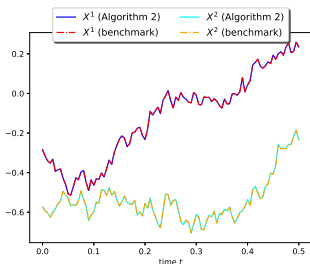
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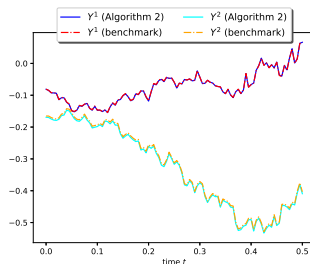
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NN for FBSDE system VS (semi) analytical solution (LQ structure)



Samples of X



Samples of Y

More details in [Carmona and Laurière, 2022]

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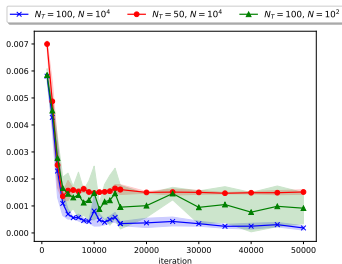
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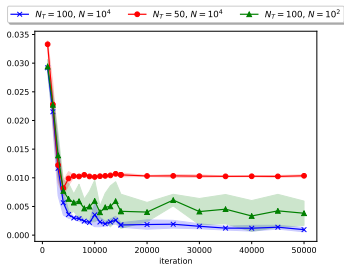
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L^2 error on X



L^2 error on Y

More details in [Carmona and Laurière, 2022]

Code

Sample code to illustrate: [IPython notebook](#)

<https://colab.research.google.com/drive/1RXmRaf2OqtNensLMt0ASn25PfSC58saY?usp=sharing>

- Deep learning for MKV FBSDEs
- Applied to the systemic risk model discussed above

- Convergence of the DeepBSDE method [[Han and Long, 2020](#)]
- Extension to finite-player games [[Han et al., 2022](#)]
- Analysis of the different types of errors to be done for MKV case
- The new MFC problem is not standard
- Deep learning of MKV FBSDEs as presented here [[Carmona and Laurière, 2022](#)]; see also [[Carmona and Laurière, 2023](#)]
- Related works on deep learning for MKV FBSDEs: [[Fouque and Zhang, 2020](#)] (MFC with delay), [[Germain et al., 2019](#)], [[Aurell et al., 2022b](#)], ...
- Similar “shooting” strategy can be applied to (infinite-dimensional) ODE systems obtained in graphon games [[Aurell et al., 2022a](#)]. Code (Gökçe Dayanıklı):

`https://github.com/gokce-d/GraphonEpidemics`

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2. Deep Learning for MFC

3. Deep Learning for MKV FBSDE

4. Two Examples of Extensions

- Solving Stackelberg MFG with Deep MKV FBSDE
- Computing MFC Value Function with DBDP

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MFG with a **Stackelberg** (leader-follower) structure:

- A Principal chooses a policy λ
- A population of agents react and form a Nash equilibrium:

$$J^\lambda(\alpha, \mu) := \mathbb{E} \left[\int_0^T f(t, X_t, \alpha_t, \mu_t; \lambda(t)) dt + g(X_T, \mu_T; \lambda(T)) \right],$$

- This is an MFG parameterized by λ
- The resulting mean field flow $\hat{\mu}^\lambda$ incurs a cost to the principal

$$J^0(\lambda) := \int_0^T f_0(t, \hat{\mu}_t^\lambda, \lambda(t)) dt + g_0(\hat{\mu}_T^\lambda, \lambda(T))$$

Related works: Holmström-Milgrom (1987), Sannikov (2008, 2013), Djehiche-Helgesson (2014), Cvitanović *et al* (2018), Carmona-Wang (2018), Elie *et al* (2019)

Reminder:

- MFG solution can be characterized using a MKV FBSDE system
- This MKV FBSDE can be rewritten as a control problem
 - ▶ 2 forward equations
 - ▶ **terminal cost**

Stackelberg MFG:

- The above **terminal cost** can be combined with the **principal's cost**
- We obtain an **MFC problem** [Elie et al., 2019]
- From here we can apply the methods discussed previously

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For more details, see:

- [Aurell et al., 2022b] with application to epidemics management (finite state MFG): principal gives guidelines (social distancing, etc.) and population reacts
- **Code available** ((Gökçe Dayanıklı)):

<https://github.com/gokce-d/StackelbergMFG>

- Extension to other Stackelberg MFGs: [Dayanikli and Lauriere, 2023]
- Similarities with DA for **mean field optimal transport** [Baudalet et al., 2023]

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Social optimum: Mean Field Control

Reminder from lecture 2 about mean field (type) control or control of McKean-Vlasov (MKV) dynamics

Definition (Mean field control (MFC) problem)

α^* is a solution to the MFC problem if it minimizes

$$J^{MFC}(\alpha) = \mathbb{E} \left[\int_0^T f(X_t^\alpha, \alpha_t, m_t^\alpha) dt + g(X_T^\alpha, m_T^\alpha) \right].$$

Main difference with MFG: here not only X but m too is controlled by α .

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Main difference with MFG: here not only X but m too is controlled by α .

Optimality conditions? Several approaches:

- Dynamic programming value function depending on m ; [value function \$V\$](#)
- Calculus of variations taking m as a state; adjoint state u
- Pontryagin's maximum principle for the (MKV process) X ; adjoint state Y

Dynamic programming for MFC [\[Laurière and Pironneau, 2014\]](#),
[\[Bensoussan et al., 2015\]](#), [\[Pham and Wei, 2017\]](#), [\[Djete et al., 2022\]](#), ...

→ [Algorithm?](#)

For standard (non-mean field) stochastic optimal control problems, [Huré et al., 2019] have introduced the **Deep Backward Dynamic Programming (DBDP)**:

Idea: learn Y_n and Z_n at each n as functions of X_n , backward in time:

- Initialize $\hat{Y}_{N_T} = g$ and then, for $n = N_T - 1, \dots, 0$, either:
- Version 1: Let $(\hat{Y}_n, \hat{Z}_n) = \text{minimizer over } (Y_n, Z_n) \text{ of:}$

$$\mathbb{E} \left[|\hat{Y}_{n+1}(X_{n+1}) - Y_n(X_n) - f(t_n, X_n, Y_n(X_n), \textcolor{red}{Z}_n(X_n))\Delta t - \textcolor{red}{Z}_n(X_n) \cdot \Delta W_{n+1}| \right]$$

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- or Version 2: Let $(\hat{Y}_n, \hat{Z}_n) = \text{minimizer over } (Y_n, Z_n) \text{ of:}$

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For more details on deep learning methods for (non-mean field) optimal control problems, see e.g. [\[Germain et al., 2021b\]](#)

- Can we apply the same idea to MFC, replacing V by a neural network?
- Main challenge: the value function V takes $m \in \mathcal{P}(\mathbb{R}^d)$ as an input
- We need to approximate m

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- Main challenge: the value function V takes $m \in \mathcal{P}(\mathbb{R}^d)$ as an input
- We need to approximate m
- One possibility:

$$V(t, m_t) \approx \tilde{V}(t, m_t^N) \approx \tilde{V}_\theta(t, X_t^1, \dots, X_t^N)$$

where \tilde{V}_θ is a neural network which is **symmetric** with respect to the inputs

- See the **lecture 5** for more details
- See [\[Germain et al., 2021a\]](#) for more details about the implementation and [\[Germain et al., 2022\]](#) for the analysis
- See also e.g. [\[Dayanikli et al., 2023\]](#) for different approximations of the population (combined with direct approach instead of DBDP)

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Summary

- Two algorithms based on the stochastic approach
- Direct approach without any optimality condition
- DeepBSDE: recasting (MKV) FBSDEs as control problems
- Many possible extensions and variations
- Many **open questions** for mathematicians (proofs of approximation, rates of convergence, ...)
- Some surveys on DL for control/games:
[Germain et al., 2021b, Carmona and Laurière, 2023, Hu and Laurière, 2023]

Next lecture: deep learning methods for the PDE approach

Thank you for your attention

Questions?

Feel free to reach out: `mathieu.lauriere@nyu.edu`

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