Numerical Methods for Mean Field Games

Lecture 3
Classical Numerical Methods: Part II
FBPDE and FBSDE systems

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Outline

Reminder: FB systems

- Here we will focus on the continuous time and space setting
- We have seen two types of forward-backward systems:
  - PDE systems: Kolmogorov-Fokker-Planck (KFP) and Hamilton-Jacobi-Bellman (HJB)
  - SDE systems of McKean-Vlasov (MKV) type
- We will describe methods based on both approaches
- There are two questions to design a numerical method:
  - Discretization $\rightarrow$ numerical scheme
  - Computation $\rightarrow$ algorithm

MFG PDE System

Goal: (1) introduce and **(2)** solve a discrete version of the MFG PDE system:

$$
\begin{cases}
0 = -\dfrac{\partial u}{\partial t}(t,x) - \nu \Delta u(t,x) + H(x, m(t,\cdot), \nabla u(t,x)), \\
0 = \dfrac{\partial m}{\partial t}(t,x) - \nu \Delta m(t,x) - \operatorname{div}\left(m(t,\cdot)\partial_p H(\cdot, m(t), \nabla u(t,\cdot))\right)(x), \\
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Desirable properties for (1):

- **Mass** and **positivity** of distribution: $\int_{\mathcal{S}} m(t,x)dx = 1$, $m \geq 0$
- **Convergence** of discrete solution to continuous solution as mesh step $\to 0$

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For (2): Once we have a discrete system, how can we compute its solution?

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Discretization

Semi-implicit finite difference scheme from [Achdou and Capuzzo-Dolcetta, 2010]
Discretization:

- For simplicity we consider the domain $\mathbb{T} =$ one-dimensional (unit) torus.
- Let $\nu = \sigma^2/2$.
- We consider N_h and N_T steps respectively in space and time.
- Let $h = 1/N_h$ and $\Delta t = T/N_T$. Let $\mathbb{T}_h =$ discretized torus.
- We approximate $m_0(x_i)$ by ρ_i^0 such that $h \sum_i \rho_i^0 = 1$.

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Then we introduce the following **discrete operators** : for $\varphi \in \mathbb{R}^{N_T+1}$ and $\psi \in \mathbb{R}^{N_h}$

- **time derivative** : $$(D_t\varphi)^n := \frac{\varphi^{n+1} - \varphi^n}{\Delta t}, \qquad 0 \le n \le N_T - 1$$
- **Laplacian** : $$(\Delta_h\psi)_i := -\frac{1}{h^2}\left(2\psi_i - \psi_{i+1} - \psi_{i-1}\right), \qquad 0 \le i \le N_h$$
- **partial derivative** : $$(D_h\psi)_i := \frac{\psi_{i+1} - \psi_i}{h}, \qquad 0 \le i \le N_h$$
- **gradient** : $$[\nabla_h\psi]_i := \left((D_h\psi)_i, (D_h\psi)_{i-1}\right), \qquad 0 \le i \le N_h$$

Discrete Hamiltonian

For simplicity, we assume that the drift b and the costs f and g are of the form

$$b(x, m, \alpha) = \alpha, \qquad f(x, m, \alpha) = L(x, \alpha) + \mathtt{f}_0(x, m), \qquad g(x, m) = \mathtt{g}_0(x, m).$$

where $x \in \mathbb{R}^d$, $\alpha \in \mathbb{R}^d$, $m \in \mathbb{R}_+$. Then

$$H(x, m, p) = \max_{\alpha} \{-L(x, \alpha) - \langle \alpha, p \rangle\} - \mathtt{f}_0(x, m) = H_0(x, p) - \mathtt{f}_0(x, m)$$

where H_0 is the convex conjugate (also denoted L^*) of L with respect to α:

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Discrete Hamiltonian: $(x, p_1, p_2) \mapsto \tilde{H}_0(x, p_1, p_2)$ satisfying:

- Monotonicity: decreasing w.r.t. p_1 and increasing w.r.t. p_2
- Consistency with H_0: for every x, p, $\tilde{H}_0(x, p, p) = H_0(x, p)$
- Differentiability: for every x, $(p_1, p_2) \mapsto \tilde{H}_0(x, p_1, p_2)$ is $\mathcal{C}^1$
- Convexity: for every x, $(p_1, p_2) \mapsto \tilde{H}_0(x, p_1, p_2)$ is convex

Example: if $H_0(x, p) = |p|^2$, a possible choice is $\tilde{H}_0(x, p_1, p_2) = ({p_1}^-)^2 + ({p_2}^+)^2$

Discrete HJB

Discrete solution: We replace $u, m : [0,T] \times \mathbb{T} \to \mathbb{R}$ by vectors

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The **HJB equation**

$$\begin{cases} \partial_t u(t,x) + \nu \Delta u(t,x) + H_0(x, \nabla u(t,x)) = \mathtt{f}_0(x, m(t,x)) \\ u(T,x) = \mathtt{g}_0(x, m(T,x)) \end{cases}$$

is discretized as:

$$\begin{cases} -(D_t U_i)^n - \nu(\Delta_h U^n)_i + \tilde{H}_0(x_i, [D_h U^n]_i) = \mathtt{f}_0(x_i, M_i^{n+1}) \\ U_i^{N_T} = \mathtt{g}_0(x_i, M_i^{N_T}) \end{cases}$$

Discrete KFP

The **KFP equation**

$$\partial_t m(t,x) - \nu \Delta m(t,x) + \mathrm{div}\Big(m(t,x)\partial_q H(x, m(t), \nabla u(t,x)) \Big) = 0, \qquad m(0,x) = m_0(x)$$

is discretized as

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Here we use the **discrete transport operator** $\approx -\operatorname{div}(\dots)$

$$\mathcal{T}_i(U,M) := \frac{1}{h}\begin{pmatrix} M_i \partial_{p_1} \tilde{H}_0(x_i, [\nabla_h U]_i) - M_{i-1}\partial_{p_1}\tilde{H}_0(x_{i-1}, [\nabla_h U]_{i-1}) \\ + M_{i+1}\partial_{p_2}\tilde{H}_0(x_{i+1}, [\nabla_h U]_{i+1}) - M_i \partial_{p_2}\tilde{H}_0(x_i, [\nabla_h U]_i) \end{pmatrix}$$

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Intuition: weak formulation & integration by parts

$$\int_{\mathbb{T}} \mathrm{div}\,(m \partial_p H_0(x, \nabla u)) w = - \int_{\mathbb{T}} m \partial_p H_0(x, \nabla u) \cdot \nabla w$$

is discretized as

$$-h \sum_i \mathcal{T}_i(U,M) W_i = h \sum_i M_i \nabla_q \tilde{H}_0(x_i, [\nabla_h U]_i) \cdot [\nabla_h W]_i$$

Discrete System – Properties

Discrete forward-backward system:

$$
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-(D_t U_i)^n - \nu(\Delta_h U^n)_i + \tilde{H}_0(x_i, [D_h U^n]_i) = \mathtt{f}_0(x_i, M_i^{n+1}), & \forall n \le N_T - 1 \\
(D_t M_i)^n - \nu(\Delta_h M^{n+1})_i - \mathcal{T}_i(U^n, M^{n+1}) = 0, & \forall n \le N_T - 1 \\
M_i^0 = \rho_i^0, \qquad U_i^{N_T} = \mathtt{g}_0(x_i, M_i^{N_T}), \qquad i = 0, \dots, N_h &
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This scheme enjoys many nice properties, among which:

- It yields a monotone scheme for the KFP equation: **mass** and **positivity** are preserved
- **Convergence** to classical solution if monotonicity [Achdou and Capuzzo-Dolcetta, 2010, Achdou et al., 2012]
- Can sometimes be used to show existence of a **weak** solution [Achdou and Porretta, 2016]
- The discrete KFP operator is the **adjoint** of the linearized Bellman operator
- **Existence** and **uniqueness** result for the discrete system
- It corresponds to the **optimality condition** of a discrete optimization problem (details later)

Outline

Algo 1: Fixed Point Iterations

Input: Initial guess $(\tilde{M}, \tilde{U})$; damping $\delta(\cdot)$; number of iterations $\mathtt{K}$

Output: Approximation of $(\hat{M}, \hat{U})$ solving the finite difference system

Initialize $M^{(0)} = \tilde{M}^{(0)} = \tilde{M}, U^{(0)} = \tilde{U}$

for $\mathtt{k} = 0, 1, 2, \dots, \mathtt{K} - 1$ **do**

Let $U^{(\mathtt{k}+1)}$ be the solution to:

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Let $\tilde{M}^{(\mathtt{k}+1)} = \delta(\mathtt{k}) \tilde{M}^{(\mathtt{k})} + (1 - \delta(\mathtt{k})) M^{(\mathtt{k}+1)}$

return $(M^{(\mathtt{K})}, U^{(\mathtt{K})})$

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Remark: the HJB equation is **non-linear**

- **Idea 1:** replace $\tilde{H}_0(x_i, [D_h U^n]_i)$ by $\tilde{H}_0(x_i, [D_h U^{(\mathtt{k}), n}]_i)$

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- **Idea 2:** use non linear solver to find a zero of $\mathbb{R}^{N_h \times (N_T+1)} \ni U \mapsto \varphi(U) \in \mathbb{R}^{N_h \times N_T}$,

$$\varphi(U) = \Big(-(D_t U_i)^n - \nu(\Delta_h U^n)_i + \tilde{H}_0(x_i, [D_h U^n]_i) - \mathtt{f}_0(x_i, \tilde{M}_i^{(\mathtt{k}),n+1}) \Big)_{i=0,\ldots,N_h-1}^{n=0,\ldots,N_T-1}$$

Sample code

Code

Sample code to illustrate: IPython notebook

`https://colab.research.google.com/drive/1shJWSD2MA5Fo7_rB625dAvNTdZS1a7bG?usp=sharing`

- Finite difference scheme
- Solved by damped fixed point approach

Algo 2: Newton's Method for FD System

Idea: Directly look for a zero of $\varphi = (\varphi_{\mathcal{U}}, \varphi_{\mathcal{M}})^\top$ with $\varphi_{\mathcal{U}}$ and $\varphi_{\mathcal{M}}$ s.t.

$$\begin{cases} \varphi_{\mathcal{U}}(U,M) = 0 & \Leftrightarrow \ (U,M) \text{ solves discrete HJB equation} \\ \varphi_{\mathcal{M}}(U,M) = 0 & \Leftrightarrow \ (U,M) \text{ solves discrete KFP equation} \end{cases}$$

- Let $X^{(k)} = (U^{(k)}, M^{(k)})^\top$
- Iterate: $X^{(k+1)} = X^{(k)} - J_\varphi(X^{(k)})^{-1}\varphi(X^{(k)})$

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- Or rather: $J_\varphi(X^{(k)})Y = -\varphi(X^{(k)})$, then $X^{(k+1)} = Y + X^{(k)}$

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Key step: Solve a linear system of the form

$$\begin{pmatrix} A_{\mathcal{U},\mathcal{U}} & A_{\mathcal{U},\mathcal{M}} \\ A_{\mathcal{M},\mathcal{U}} & A_{\mathcal{M},\mathcal{M}} \end{pmatrix} \begin{pmatrix} U \\ M \end{pmatrix} = \begin{pmatrix} G_{\mathcal{U}} \\ G_{\mathcal{M}} \end{pmatrix}$$

where $A_{\mathcal{U},\mathcal{M}}(U, M) = \nabla_U \varphi_{\mathcal{M}}(U, M), \quad A_{\mathcal{U},\mathcal{U}}(U, M) = \nabla_U \varphi_{\mathcal{U}}(U, M), \quad \ldots$

Newton Method – Implementation

Linear system to be solved: $\begin{pmatrix} A_{\mathcal{U},\mathcal{U}} & A_{\mathcal{U},\mathcal{M}} \\ A_{\mathcal{M},\mathcal{U}} & A_{\mathcal{M},\mathcal{M}} \end{pmatrix} \begin{pmatrix} U \\ M \end{pmatrix} = \begin{pmatrix} G_{\mathcal{U}} \\ G_{\mathcal{M}} \end{pmatrix}$

Structure: $A_{\mathcal{U},\mathcal{M}}, A_{\mathcal{M},\mathcal{U}}$ are block-diagonal, $A_{\mathcal{U},\mathcal{U}} = A_{\mathcal{M},\mathcal{M}}^{\top}$, and

$$A_{\mathcal{U},\mathcal{U}} = \begin{pmatrix} D_1 & 0 & \dots & \dots & 0 \\ -\frac{1}{\Delta t}\mathrm{Id}_{N_h} & D_2 & \ddots & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 0 & -\frac{1}{\Delta t}\mathrm{Id}_{N_h} & D_{N_T} \end{pmatrix}$$

where D_n corresponds to the discrete operator

$$Z = (Z_{i,j})_{i,j} \mapsto \left(\frac{1}{\Delta t} Z_{i,j} - \nu(\Delta_h Z)_{i,j} + [\nabla_h Z]_{i,j} \cdot \nabla_p \tilde{H}_0(x_{i,j}, [\nabla_h U^{(k),n}]_{i,j}) \right)_{i,j}$$

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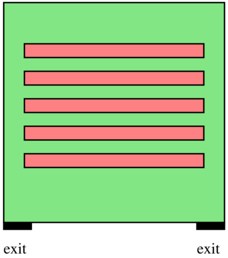
Rem. Initial guess $(U^{(0)}, M^{(0)})$ is important for Newton's method

- Idea 1: initialize with the ergodic solution
- Idea 2: continuation method w.r.t. ν (converges more easily with a large viscosity)

See [Achdou, 2013] for more details.

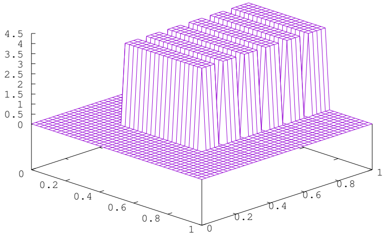
Example: Exit of a Room – Distribution

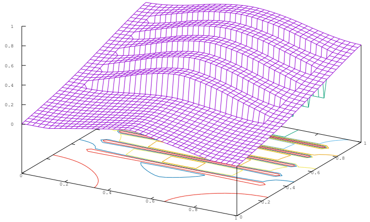
Evacuation of a room with obstacles & congestion [Achdou and Laurière, 2020]



Geometry of the room

Evacuation of a room with obstacles & congestion [Achdou and Laurière, 2020]





Initial density (left) and final cost (right)

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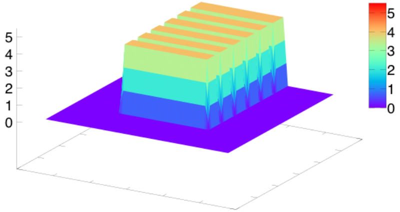

Density in MFGame (left) and MFControl (right)

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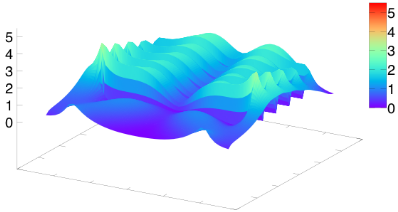

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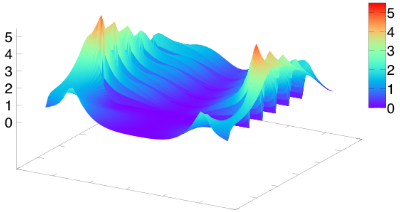

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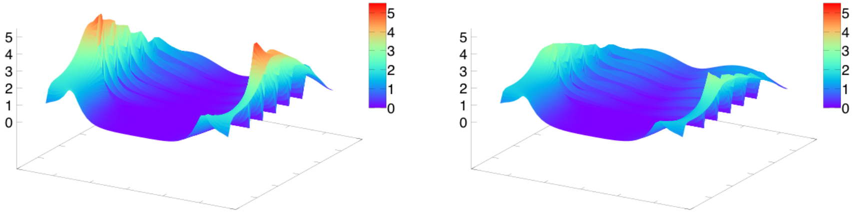

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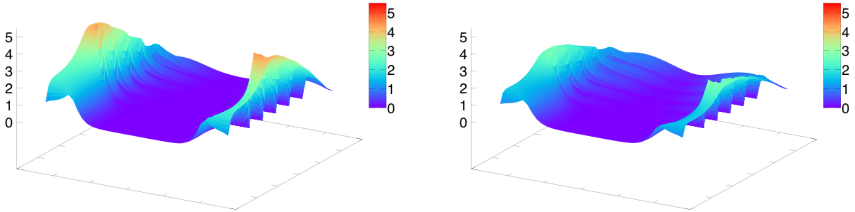

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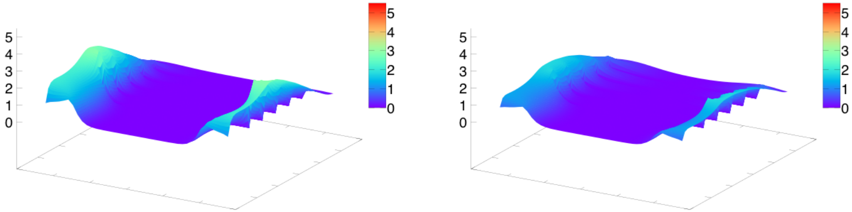

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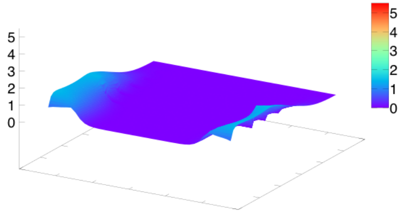

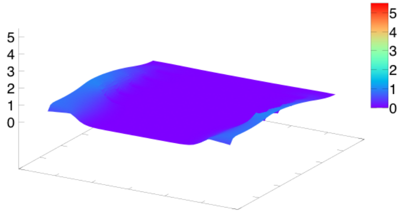

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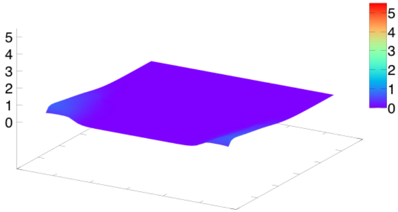

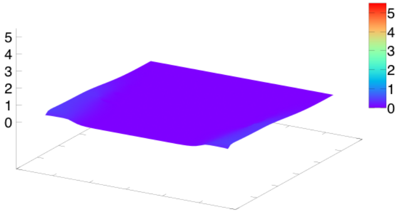

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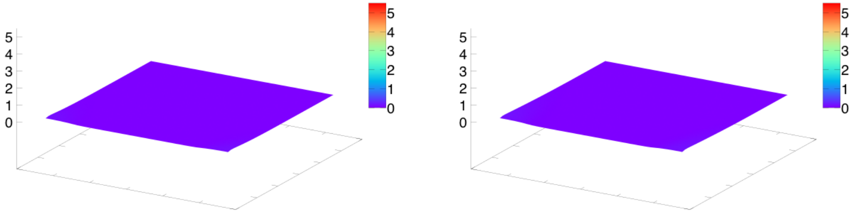

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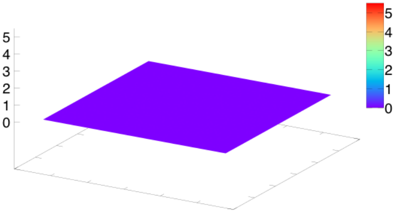

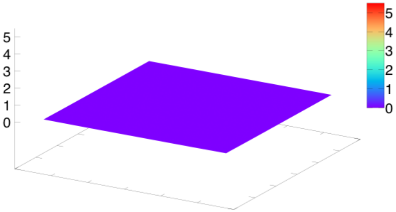

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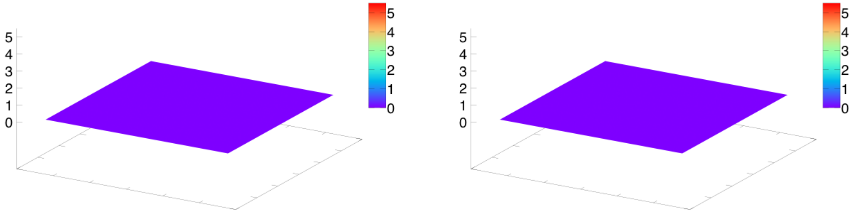

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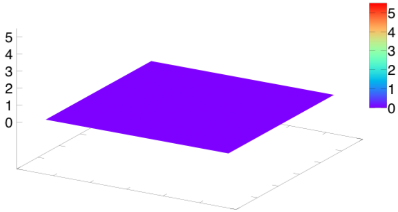

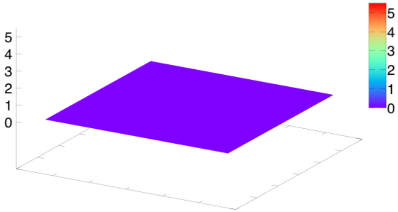

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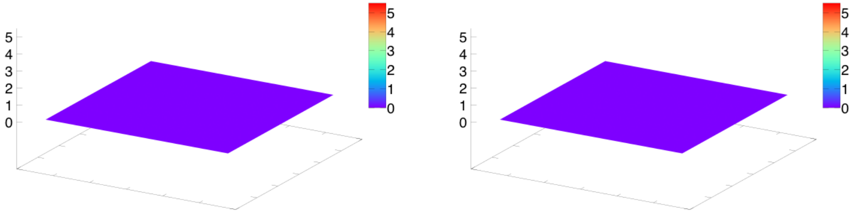
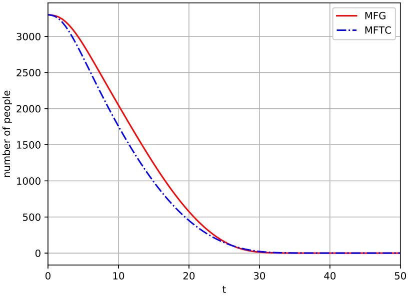


Density in MFGame (left) and MFControl (right)

Example: Exit of a Room – Remaining Mass

Evacuation of a room with obstacles & congestion [Achdou and Laurière, 2020]



Remaining mass inside the room

Outline

MFG Setup

- Scheme introduced by [Carlini and Silva, 2014]
- For simplicity: $d = 1$, domain $\mathcal{S} = \mathbb{R}$, $\mathcal{A} = \mathbb{R}$
- $\nu = 0$, degenerate second order case also possible; see [Carlini and Silva, 2015]
- Model:

$$\begin{aligned} b(x, m, \alpha) &= \alpha \\ f(x, m, \alpha) &= \frac{1}{2}|\alpha|^2 + f_0(x, m), \qquad g(x, m) \end{aligned}$$

where f_0 and g depend on $m \in \mathcal{P}_1(\mathbb{R})$ in a potentially non-local way

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- MFG PDE system:

$$\begin{cases} -\dfrac{\partial u}{\partial t}(t,x) + \dfrac{1}{2}|\nabla u(t,x)|^2 = f_0(x, m(t,\cdot)), & \text{in } [0,T) \times \mathbb{R}, \\ \dfrac{\partial m}{\partial t}(t,x) - \operatorname{div}\left(m(t,\cdot)\nabla u(t,\cdot)\right)(x) = 0, & \text{in } (0,T] \times \mathbb{R}, \\ u(T,x) = g(x, m(T,\cdot)), \qquad m(0,x) = m_0(x), & \text{in } \mathbb{R}. \end{cases}$$

Representation of the Value Function

- Dynamics:

$$X_t^{\alpha} = X_0^{\alpha} + \int_0^t \alpha(s)ds, \qquad t \geq 0.$$

- **Representation formula** for the value function given $m = (m_t)_{t\in[0,T]}$:

$$u[m](t,x) = \inf_{\alpha\in L^2([t,T];\mathbb{R})} \Big\{ \int_t^T \Big[\frac{1}{2}|\alpha(s)|^2 + f_0(X_s^{\alpha,t,x}, m(s,\cdot))\Big] ds + g(X_T^{\alpha,t,x}, m(T,\cdot)) \Big\},$$

where $X^{\alpha,t,x}$ starts from x at time t and is controlled by α

Discrete HJB equation

Discrete HJB: Given a flow of densities m,

$$\begin{cases} U_i^n = S_{\Delta t,h}[m](U^{n+1}, i, n), & (n,i) \in [\![N_T - 1]\!] \times \mathbb{Z}, \\ U_i^{N_T} = g(x_i, m(T,\cdot)), & i \in \mathbb{Z}, \end{cases}$$

where

- $S_{\Delta t,h}$ is defined as

$$S_{\Delta t,h}[m](W,n,i) = \inf_{\alpha \in \mathbb{R}} \left\{ \left(\frac{1}{2}|\alpha|^2 + f_0(x_i, m(t_n,\cdot)) \right) \Delta t + I[W](x_i + \alpha\,\Delta t) \right\},$$

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- with $I : \mathcal{B}(\mathbb{Z}) \to \mathcal{C}_b(\mathbb{R})$ is the **interpolation operator** defined as

$$I[W](\cdot) = \sum_{i \in \mathbb{Z}} W_i \beta_i(\cdot),$$

- where $\mathcal{B}(\mathbb{Z})$ is the set of bounded functions from $\mathbb{Z}$ to $\mathbb{R}$
- and $\beta_i = \left[1 - \frac{|x - x_i|}{h}\right]_+$: triangular function with support $[x_{i-1}, x_{i+1}]$ and s.t. $\beta_i(x_i) = 1$.

Discrete HJB equation – cont.

Before moving to the KFP equation:

- **Interpolation:** from $U = (U_i^n)_{n,i}$, construct the function $u_{\Delta t,h}[m](x,t) : [0,T] \times \mathbb{R} \to \mathbb{R}$,

$$u_{\Delta t,h}[m](t,x) = I[U^{[\frac{t}{\Delta t}]}](x), \qquad (t,x) \in [0,T] \times \mathbb{R}.$$

Discrete HJB equation – cont.

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- **Regularization of HJB solution** with a mollifier ρ_ϵ:

$$u^\epsilon_{\Delta t,h}[m](t,\cdot) = \rho_\epsilon * u_{\Delta t,h}[m](t,\cdot), \qquad t \in [0,T].$$

Discrete KFP equation: intuition

- **Eulerian** viewpoint:
  - focus on a location
  - look at the flow passing through it
  - evolution characterized by the velocity at (t, x)
- **Lagrangian** viewpoint:
  - focus on a fluid parcel
  - look at how it flows
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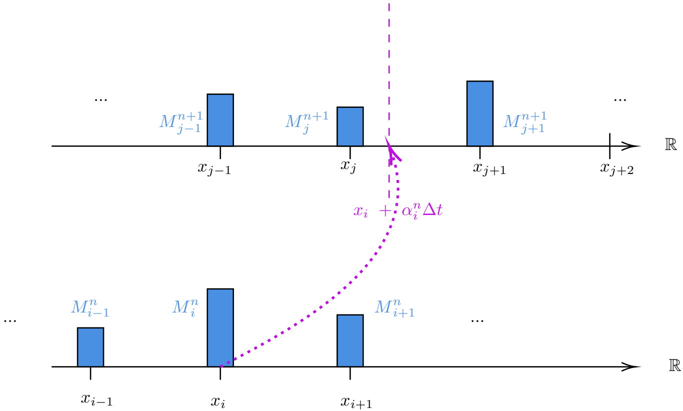
- Here, in our model:

$$X_t^{\alpha} = X_0^{\alpha} + \int_0^t \alpha(s)ds, \qquad t \geq 0.$$

- Time and space discretization?

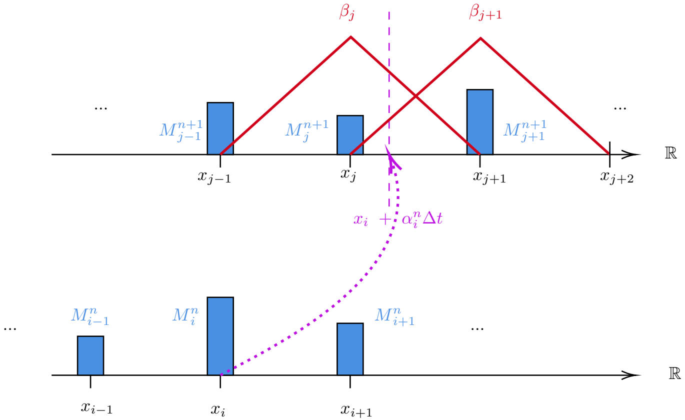
Discrete KFP equation: intuition – diagram



Movement of the mass when using control $v(t_n, x_i) = \alpha_i^n$.

Bottom: time t_n; top: time t_{n+1}.

Discrete KFP equation: intuition – diagram



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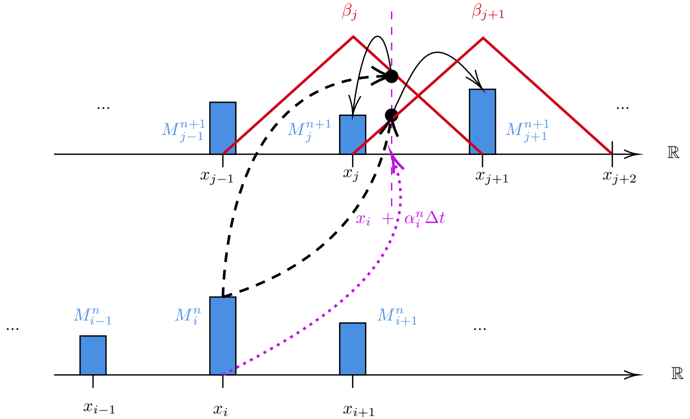

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Discrete KFP equation

- **Control** induced by value function:

$$\hat{\alpha}^{\epsilon}_{\Delta t,h}[m](t,x) = -\nabla u^{\epsilon}_{\Delta t,h}[m](t,x),$$

and its discrete counter part: $\hat{\alpha}^{\epsilon}_{n,i} = \hat{\alpha}^{\epsilon}_{\Delta t,h}[m](t_n, x_i)$.

- **Discrete flow:**

$$\Phi^{\epsilon}_{n,n+1,i}[m] = x_i + \hat{\alpha}^{\epsilon}_{\Delta t,h}[m](t_n, x_i)\Delta t\,.$$

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- **Discrete KFP equation:** for $M^{\epsilon}[m] = (M^{\epsilon,n}_i[m])_{n,i}$:

$$\begin{cases} M^{\epsilon,n+1}_i[m] = \sum_j \beta_i\left(\Phi^{\epsilon}_{n,n+1,j}[m]\right) M^{\epsilon,n}_j[m], & (n,i) \in [\![N_T - 1]\!] \times \mathbb{Z}, \\ M^{\epsilon,0}_i[m] = \displaystyle\int_{[x_i - h/2, x_i + h/2]} m_0(x)dx, & i \in \mathbb{Z}. \end{cases}$$

Fixed Point Formulation

- **Function** $m^{\epsilon}_{\Delta t,h}[m] : [0,T] \times \mathbb{R} \to \mathbb{R}$ defined as: for $n \in [\![N_T - 1]\!]$, for $t \in [t_n, t_{n+1})$,

$$m^{\epsilon}_{\Delta t,h}[m](t,x) = \frac{1}{h}\left[\frac{t_{n+1}-t}{\Delta t}\sum_{i\in\mathbb{Z}} M_i^{\epsilon,n}[m]\mathbf{1}_{[x_i-h/2,x_i+h/2]}(x) + \frac{t-t_n}{\Delta t}\sum_{i\in\mathbb{Z}} M_i^{\epsilon,n+1}[m]\mathbf{1}_{[x_i-h/2,x_i+h/2]}(x)\right].$$

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- **Goal: Fixed-point problem:** Find $\hat{M} = (\hat{M}_i^n)_{i,n}$ such that:

$$\hat{M}_i^n = M_i^n\left[m^{\epsilon}_{\Delta t,h}[\hat{M}]\right].$$

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- **Solution strategy:** Fixed point iterations for example
- See [Carlini and Silva, 2014] for more details

Numerical Illustration

Costs:

$$g \equiv 0, \qquad f(x, m, \alpha) = \frac{1}{2}|\alpha|^2 + (x - c^*)^2 + \kappa_{MF} V(x, m),$$

with

$$V(x, m) = \rho_{\sigma_V} * \big(\rho_{\sigma_V} * m\big)(x),$$

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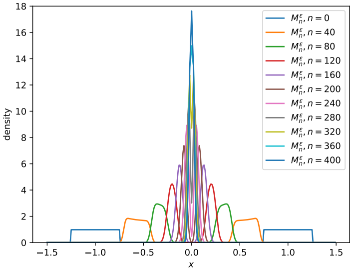
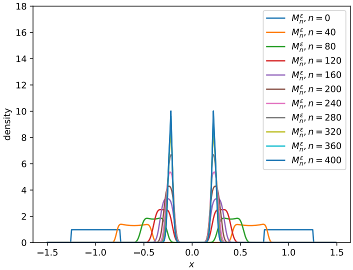
$$V(x, m) = \rho_{\sigma_V} * \big(\rho_{\sigma_V} * m\big)(x),$$

Experiments: target $c^* = 0$, m_0 = unif. on $[-1.25, -0.75]$ and on $[0.75, 1.25]$



$\kappa_{MF} = 0.5$



$\kappa_{MF} = 0.9$

See [Laurière, 2021] for more details on these experiments

Sample code

Code

Sample code to illustrate: IPython notebook

https://drive.google.com/file/d/1_S9680R_CAt20M83NENcyeHKsLLcxit9/view?usp=sharing

- Semi-Lagrangian scheme
- Solved by damped fixed point approach

Outline

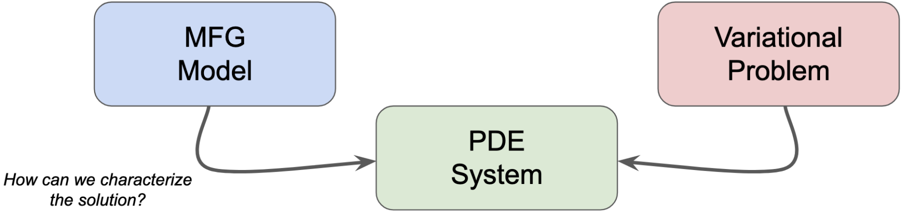
Outline

Variational MFGs

Key ideas:

- Variational MFG
- Duality
- Optimization techniques

Variational MFGs



In some cases, the MFG PDE system can be interpreted as the optimality conditions for a variational problem

MFG PDE system $\Leftrightarrow$ *optimality condition of two optimization problems in duality*

See [Lasry and Lions, 2007], [Cardaliaguet, 2015], [Cardaliaguet and Graber, 2015], [Cardaliaguet et al., 2015], [Benamou et al., 2017], . . .

A Variational MFG

- $d = 1$, domain = $\mathbb{T}$
- drift and costs:

$$b(x, m, \alpha) = \alpha, \qquad f(x, m, \alpha) = L(x, \alpha) + \mathtt{f}_0(x, m), \qquad g(x, m) = \mathtt{g}_0(x).$$

  where $x \in \mathbb{R}^d$, $\alpha \in \mathbb{R}^d$, $m \in \mathbb{R}_+$.
- Then

$$H(x, m, p) = \sup_{\alpha} \{-L(x, \alpha) - \alpha p\} - \mathtt{f}_0(x, m) = H_0(x, p) - \mathtt{f}_0(x, m)$$

- where H_0 is the convex conjugate (also denoted L^*) of L with respect to α:

$$H_0(x, p) = L^*(x, p) = \sup_{\alpha}\{\ \alpha p - L(x, \alpha)\}$$

- Further assume (for simplicity)

$$L(x, \alpha) = \frac{1}{2}|\alpha|^2, \qquad H_0(x, p) = \frac{1}{2}|p|^2$$

A Variational Problem

- At equilibrium, $\mathcal{L}(X_t) = \hat{\mu}_t$ and

$$J(\hat{\alpha};\hat{m}) = \mathbb{E}\left[\int_0^T f(X_t, \hat{m}(t, X_t), \hat{\alpha}(t, X_t))dt + g(X_T)\right]$$

$$= \int_0^T \int_{\mathbb{T}} \underbrace{f(x, \hat{m}(t,x), \hat{\alpha}(t,x))}_{=L(x,\hat{\alpha}(t,x))+f_0(x,\hat{m}(t,x))} \hat{m}(t,x)dxdt + \int_{\mathbb{T}} g(x)\hat{m}(T,x)dx$$

subject to:

$$0 = \frac{\partial \hat{m}}{\partial t}(t,x) - \nu\Delta\hat{m}(t,x) + \operatorname{div}\left(\hat{m}(t,\cdot)\underbrace{b(\cdot,\hat{m}(t),\hat{\alpha}(t,\cdot))}_{=\hat{\alpha}(t,\cdot)}\right)(x), \qquad \hat{m}_0 = m_0$$

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- At equilibrium, $\mathcal{L}(X_t) = \hat{\mu}_t$ and

$$J(\hat{\alpha};\hat{m}) = \mathbb{E}\left[\int_0^T f(X_t, \hat{m}(t,X_t), \hat{\alpha}(t,X_t))dt + g(X_T)\right]$$
$$= \int_0^T \int_{\mathbb{T}} \underbrace{f(x,\hat{m}(t,x),\hat{\alpha}(t,x))}_{=L(x,\hat{\alpha}(t,x))+\mathtt{f}_0(x,\hat{m}(t,x))} \hat{m}(t,x)dxdt + \int_{\mathbb{T}} g(x)\hat{m}(T,x)dx$$

subject to:

$$0 = \frac{\partial \hat{m}}{\partial t}(t,x) - \nu\Delta\hat{m}(t,x) + \mathrm{div}\left(\hat{m}(t,\cdot)\underbrace{b(\cdot,\hat{m}(t),\hat{\alpha}(t,\cdot))}_{=\hat{\alpha}(t,\cdot)}\right)(x), \qquad \hat{m}_0 = m_0$$

- Change of variable:

$$\hat{w}(t,x) = \hat{m}(t,x)\hat{\alpha}(t,x)$$

$$\mathcal{B}(\hat{m},\hat{w}) = \int_0^T \int_{\mathbb{T}} \left[L\left(x, \frac{\hat{w}(t,x)}{\hat{m}(t,x)}\right) + \mathtt{f}_0(x,\hat{m}(t,x))\right]\hat{m}(t,x)dxdt + \int_{\mathbb{T}} g(x)\hat{m}(T,x)dx$$

subject to:

$$0 = \frac{\partial \hat{m}}{\partial t}(t,x) - \nu\Delta\hat{m}(t,x) + \mathrm{div}\left(\hat{w}(t,\cdot)\right)(x), \qquad \hat{m}_0 = m_0$$

- Reformulation:

$$\mathcal{B}(\hat{m},\hat{w}) = \int_0^T \int_{\mathbb{T}} \Big[\underbrace{L\Big(x, \frac{\hat{w}(t,x)}{\hat{m}(t,x)}\Big)\hat{m}(t,x)}_{\widetilde{L}(x,\hat{m}(t,x),\hat{w}(t,x))} + \underbrace{\mathtt{f}_0(x,\hat{m}(t,x))\hat{m}(t,x)}_{\widetilde{F}(x,\hat{m}(t,x))} \Big] dxdt$$

$$+ \int_{\mathbb{T}} \underbrace{g(x)\hat{m}(T,x)}_{\widetilde{G}(x,\hat{m}(t,x))} dx$$

$$= \int_0^T \int_{\mathbb{T}} \Big[\widetilde{L}(x,\hat{m}(t,x),\hat{w}(t,x)) + \widetilde{F}(x,\hat{m}(t,x)) \Big] dxdt + \int_{\mathbb{T}} \widetilde{G}(x,\hat{m}(t,x))dx$$

subject to:

$$0 = \frac{\partial \hat{m}}{\partial t}(t,x) - \nu\Delta\hat{m}(t,x) + \operatorname{div}\Big(\hat{w}(t,\cdot)\Big)(x), \qquad \hat{m}_0 = m_0$$

Reformulation

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$$\mathcal{B}(\hat{m}, \hat{w}) = \int_0^T \int_{\mathbb{T}} \Big[\underbrace{L\Big(x, \frac{\hat{w}(t,x)}{\hat{m}(t,x)}\Big)\hat{m}(t,x)}_{\widetilde{L}(x,\hat{m}(t,x),\hat{w}(t,x))} + \underbrace{\mathtt{f}_0(x, \hat{m}(t,x))\hat{m}(t,x)}_{\widetilde{F}(x,\hat{m}(t,x))} \Big] dxdt$$

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subject to:

$$0 = \frac{\partial \hat{m}}{\partial t}(t,x) - \nu \Delta \hat{m}(t,x) + \operatorname{div}\Big(\hat{w}(t,\cdot)\Big)(x), \qquad \hat{m}_0 = m_0$$

- **Convex** problem under a **linear** constraint, provided $\widetilde{L}, \widetilde{F}, \widetilde{G}$ are convex

Primal Optimization Problem

Primal problem: Minimize over $(m, w) = (m, m\alpha)$:

$$\mathcal{B}(m, w) = \int_0^T \int_{\mathbb{T}} \left(\widetilde{L}(x, m(t,x), w(t,x)) + \widetilde{F}(x, m(t,x)) \right) dx dt + \int_{\mathbb{T}} \widetilde{G}(x, m(T,x)) dx$$

subject to the constraint:

$$\partial_t m - \nu \Delta m + \mathrm{div}(w) = 0, \qquad m(0,x) = m_0(x)$$

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where

$$\widetilde{F}(x, m) = \begin{cases} \int_0^m \tilde{f}(x,s)ds, & \text{if } m \geq 0, \\ +\infty, & \text{otherwise,} \end{cases} \qquad \widetilde{G}(x, m) = \begin{cases} m\, \mathsf{g}_0(x), & \text{if } m \geq 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$\widetilde{L}(x, m, w) = \begin{cases} mL\left(x, \frac{w}{m}\right), & \text{if } m > 0, \\ 0, & \text{if } m = 0 \text{ and } w = 0, \\ +\infty, & \text{otherwise} \end{cases}$$

where $\mathbb{R} \ni m \mapsto \tilde{f}(x, m) = \partial_m (m\, \mathtt{f}_0(x, m))$
is non-decreasing (hence $\widetilde{F}$ convex and l.s.c.) provided $m \mapsto m\, \mathtt{f}_0(x, m)$ is convex.

Duality

Dual problem: Maximize over ϕ such that $\phi(T,x) = g_0(x)$

$$\mathcal{A}(\phi) = \inf_m \mathcal{A}(\phi, m)$$

with $\mathcal{A}(\phi, m) = \int_0^T \int_{\mathbb{T}} m(t,x)\Big(\partial_t \phi(t,x) + \nu \Delta \phi(t,x) - H(x, m(t,x), \nabla \phi(t,x))\Big) dx dt$

$$+ \int_{\mathbb{T}} m_0(x) \phi(0,x) dx.$$

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Duality relation: $\mathcal{A}$ and $\mathcal{B}$ satisfy: **(A)** $= \sup_{\phi} \mathcal{A}(\phi) = \inf_{(m,w)} \mathcal{B}(m,w) =$ **(B)**

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Duality relation: $\mathcal{A}$ and $\mathcal{B}$ satisfy: **(A)** $= \sup_\phi \mathcal{A}(\phi) = \inf_{(m,w)} \mathcal{B}(m,w) =$ **(B)**

Proof idea: Fenchel-Rockafellar duality theorem and observe:

$$\textbf{(A)} = -\inf_{\phi}\left\{\mathcal{F}(\phi) + \mathcal{G}(\Lambda(\phi))\right\}, \qquad \textbf{(B)} = \inf_{(m,w)}\left\{\mathcal{F}^*(\Lambda^*(m,w)) + \mathcal{G}^*(-m,-w)\right\}$$

where $\mathcal{F}^*, \mathcal{G}^*$ are the convex conjugates of $\mathcal{F}, \mathcal{G}$, and Λ^* is the adjoint operator of Λ, and $\Lambda(\phi) = \left(\frac{\partial\phi}{\partial t} + \nu\Delta\phi, \nabla\,\phi\right)$,

$$\mathcal{F}(\phi) = \chi_T(\phi) - \int_{\mathbb{T}^d} m_0(x)\phi(0,x)dx, \qquad \chi_T(\phi) = \begin{cases} 0 & \text{if } \phi|_{t=T} = g_0 \\ +\infty & \text{otherwise,} \end{cases}$$

$$\mathcal{G}(\varphi_1,\varphi_2) = -\inf_{0\le m\in L^1((0,T)\times\mathbb{T}^d)} \int_0^T\int_{\mathbb{T}^d} m(t,x)\left(\varphi_1(t,x) - H(x, m(t,x), \varphi_2(t,x))\right)dxdt.$$

Outline

Augmented Lagrangian

Reformulation of the primal problem:

$$\textbf{(A)} = -\inf_{\phi}\left\{\mathcal{F}(\phi) + \mathcal{G}(\Lambda(\phi))\right\} = -\inf_{\phi}\ \inf_{q}\left\{\mathcal{F}(\phi) + \mathcal{G}(q),\ \text{subj. to } q = \Lambda(\phi)\right\}.$$

- The corresponding **Lagrangian** is

$$\mathcal{L}(\phi, q, \tilde{q}) = \mathcal{F}(\phi) + \mathcal{G}(q) - \langle \tilde{q}, \Lambda(\phi) - q \rangle.$$

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- We consider the **augmented Lagrangian** (with parameter $r > 0$)

$$\mathcal{L}^r(\phi, q, \tilde{q}) = \mathcal{L}(\phi, q, \tilde{q}) + \frac{r}{2}\|\Lambda(\phi) - q\|^2$$

- Goal: find a **saddle-point** of $\mathcal{L}^r$.

Alternating Direction Method of Multipliers (ADMM)

Reminder: $\mathcal{L}^r(\phi, q, \tilde{q}) = \mathcal{F}(\phi) + \mathcal{G}(q) - \langle \tilde{q}, \Lambda(\phi) - q \rangle + \frac{r}{2} \|\Lambda(\phi) - q\|^2$

Input: Initial guess $(\phi^{(0)}, q^{(0)}, \tilde{q}^{(0)})$; number of iterations $\mathtt{K}$
Output: Approximation of a saddle point $(\phi, q, \tilde{q})$ solving the finite difference system

Initialize $(\phi^{(0)}, q^{(0)}, \tilde{q}^{(0)})$
for $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**
(a) Compute

$$\phi^{(\mathtt{k}+1)} \in \operatorname*{argmin}_{\phi} \left\{ \mathcal{F}(\phi) - \langle \tilde{q}^{(\mathtt{k})}, \Lambda(\phi) \rangle + \frac{r}{2} \|\Lambda(\phi) - q^{(\mathtt{k})}\|^2 \right\}$$

References: ALG2 in the book of [Fortin and Glowinski, 1983]
→ in MFG: [Benamou and Carlier, 2015a], [Andreev, 2017]
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(b) Compute

$$q^{(\mathtt{k}+1)} \in \operatorname*{argmin}_{q} \Big\{ \mathcal{G}(q) + \langle \tilde{q}^{(\mathtt{k})}, q \rangle + \frac{r}{2} \|\Lambda(\phi^{(\mathtt{k}+1)}) - q\|^2 \Big\}$$

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(c) Compute

$$\tilde{q}^{(\mathtt{k}+1)} = \tilde{q}^{(\mathtt{k})} - r\left(\Lambda(\phi^{(\mathtt{k}+1)}) - q^{(\mathtt{k}+1)}\right)$$

return $(\phi^{(\mathtt{K})}, q^{(\mathtt{K})}, \tilde{q}^{(\mathtt{K})})$

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ADMM: Discrete Primal Problem

Notation: N_h, N_T steps resp. in space and time, $N = (N_T + 1)N_h$, $N' = N_T N_h$.

Recall: $H_0(x,p) = \frac{1}{2}|p|^2$. We take $\tilde{H}_0(x, p_1, p_2) = \frac{1}{2}|(p_1^-, p_2^+)|^2$.

Discrete version of the **dual** convex problem:

$$(\mathbf{A_h}) = -\inf_{\phi \in \mathbb{R}^N} \left\{ \mathcal{F}_h(\phi) + \mathcal{G}_h(\Lambda_h(\phi)) \right\},$$

where $\Lambda_h : \mathbb{R}^N \to \mathbb{R}^{3N'}$ is defined by : $\forall\, n \in \{1, \ldots, N_T\}, \forall\, i \in \{0, \ldots, N_h - 1\}$,

$$(\Lambda_h(\phi))_i^n = \left((D_t \phi_i)^n + \nu \left(\Delta_h \phi^{n-1} \right)_i, [\nabla_h \phi^{n-1}]_i \right),$$

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Discrete version of the **dual** convex problem:

$$(\mathbf{A_h}) = - \inf_{\phi \in \mathbb{R}^N} \left\{ \mathcal{F}_h(\phi) + \mathcal{G}_h(\Lambda_h(\phi)) \right\},$$

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where $\mathcal{F}_h, \mathcal{G}_h$ are the l.s.c. proper functions defined by:

$$\mathcal{F}_h : \mathbb{R}^N \ni \phi \mapsto \chi_T(\phi) - h \sum_{i=0}^{N_h - 1} \rho_i^0 \phi_i^0 \in \mathbb{R} \cup \{+\infty\},$$

$$\mathcal{G}_h : \mathbb{R}^{3N'} \ni (a, b, c) \mapsto -\, h \Delta t \sum_{n=1}^{N_T} \sum_{i=0}^{N_h - 1} \mathcal{K}_h(x_i, a_i^n, b_i^n, c_i^n) \in \mathbb{R} \cup \{+\infty\},$$

with

$$\mathcal{K}_h(x, a_0, p_1, p_2) = \min_{m \in \mathbb{R}_+} \left\{ m[a_0 + \tilde{H}_0(x, m, p_1, p_2)] \right\}, \quad \chi_T(\phi) = \begin{cases} 0 & \text{if } \phi_i^{N_T} \equiv \mathrm{g}_0(x_i) \\ +\infty & \text{otherwise.} \end{cases}$$

ADMM with Discretization

Discrete Aug. Lag.: $\mathcal{L}_h^r(\phi, q, \tilde{q}) = \mathcal{F}_h(\phi) + \mathcal{G}_h(q) - \langle \tilde{q}, \Lambda_h(\phi) - q \rangle + \frac{r}{2}\|\Lambda(\phi) - q\|^2$

Input: Initial guess $(\phi^{(0)}, q^{(0)}, \tilde{q}^{(0)})$; number of iterations $\mathtt{K}$
Output: Approximation of a saddle point $(\phi, q, \tilde{q})$

Initialize $(\phi^{(0)}, q^{(0)}, \tilde{q}^{(0)})$
for $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**
- (a) Compute $\phi^{(\mathtt{k}+1)} \in \operatorname{argmin}_\phi \left\{ \mathcal{F}_h(\phi) - \langle \tilde{q}^{(\mathtt{k})}, \Lambda_h(\phi) \rangle + \frac{r}{2}\|\Lambda_h(\phi) - q^{(\mathtt{k})}\|^2 \right\}$
- (b) Compute $q^{(\mathtt{k}+1)} \in \operatorname{argmin}_q \left\{ \mathcal{G}_h(q) + \langle \tilde{q}^{(\mathtt{k})}, q \rangle + \frac{r}{2}\|\Lambda_h(\phi^{(\mathtt{k}+1)}) - q\|^2 \right\}$
- (c) Compute $\tilde{q}^{(\mathtt{k}+1)} = \tilde{q}^{(\mathtt{k})} - r \left(\Lambda_h(\phi^{(\mathtt{k}+1)}) - q^{(\mathtt{k}+1)} \right)$

return $(\phi^{(\mathtt{K})}, q^{(\mathtt{K})}, \tilde{q}^{(\mathtt{K})})$

ADMM with Discretization

Discrete Aug. Lag.: $\mathcal{L}_h^r(\phi, q, \tilde{q}) = \mathcal{F}_h(\phi) + \mathcal{G}_h(q) - \langle \tilde{q}, \Lambda_h(\phi) - q\rangle + \frac{r}{2}\|\Lambda(\phi) - q\|^2$

Input: Initial guess $(\phi^{(0)}, q^{(0)}, \tilde{q}^{(0)})$; number of iterations $\mathtt{K}$
Output: Approximation of a saddle point $(\phi, q, \tilde{q})$

```
Initialize (φ^(0), q^(0), q̃^(0))
for k = 0, 1, 2, ..., K − 1 do
    (a) Compute φ^(k+1) ∈ argmin_φ { F_h(φ) − ⟨q̃^(k), Λ_h(φ)⟩ + r/2 ‖Λ_h(φ) − q^(k)‖² }
    (b) Compute q^(k+1) ∈ argmin_q { G_h(q) + ⟨q̃^(k), q⟩ + r/2 ‖Λ_h(φ^(k+1)) − q‖² }
    (c) Compute q̃^(k+1) = q̃^(k) − r (Λ_h(φ^(k+1)) − q^(k+1))
return (φ^(K), q^(K), q̃^(K))
```

$$\phi^{(\mathtt{k}+1)} \in \operatorname{argmin}_\phi\left\{\mathcal{F}_h(\phi) - \langle \tilde{q}^{(\mathtt{k})}, \Lambda_h(\phi)\rangle + \frac{r}{2}\|\Lambda_h(\phi) - q^{(\mathtt{k})}\|^2\right\}$$

$$q^{(\mathtt{k}+1)} \in \operatorname{argmin}_q\left\{\mathcal{G}_h(q) + \langle \tilde{q}^{(\mathtt{k})}, q\rangle + \frac{r}{2}\|\Lambda_h(\phi^{(\mathtt{k}+1)}) - q\|^2\right\}$$

$$\tilde{q}^{(\mathtt{k}+1)} = \tilde{q}^{(\mathtt{k})} - r\left(\Lambda_h(\phi^{(\mathtt{k}+1)}) - q^{(\mathtt{k}+1)}\right)$$

First-order Optimality Conditions:

Step (a): finite-difference equation

Step (b): minimization problem at each point of the grid

ADMM with Discretization

Discrete Aug. Lag.: $\mathcal{L}_h^r(\phi, q, \tilde{q}) = \mathcal{F}_h(\phi) + \mathcal{G}_h(q) - \langle \tilde{q}, \Lambda_h(\phi) - q \rangle + \frac{r}{2} \|\Lambda(\phi) - q\|^2$

Input: Initial guess $(\phi^{(0)}, q^{(0)}, \tilde{q}^{(0)})$; number of iterations $\mathtt{K}$
Output: Approximation of a saddle point $(\phi, q, \tilde{q})$

1 Initialize $(\phi^{(0)}, q^{(0)}, \tilde{q}^{(0)})$
2 **for** $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**
3 $\quad (a)$ Compute $\phi^{(\mathtt{k}+1)} \in \mathrm{argmin}_\phi \left\{ \mathcal{F}_h(\phi) - \langle \tilde{q}^{(\mathtt{k})}, \Lambda_h(\phi) \rangle + \frac{r}{2} \|\Lambda_h(\phi) - q^{(\mathtt{k})}\|^2 \right\}$
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5 $\quad (c)$ Compute $\tilde{q}^{(\mathtt{k}+1)} = \tilde{q}^{(\mathtt{k})} - r \left(\Lambda_h(\phi^{(\mathtt{k}+1)}) - q^{(\mathtt{k}+1)} \right)$
6 **return** $(\phi^{(\mathtt{K})}, q^{(\mathtt{K})}, \tilde{q}^{(\mathtt{K})})$

First-order Optimality Conditions:

Step (a): finite-difference equation

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Rem.: For (a): discrete PDE

- if $\nu = 0$, a direct solver can be used
- if $\nu > 0$, PDE with 4^{th} order linear elliptic operator $\Rightarrow$ needs preconditioner

See e.g. [Achdou and Perez, 2012], [Andreev, 2017], [Briceño Arias et al., 2018]

Numerical Example: Congestion Without Viscosity

- Domain $\Omega = [0,1]^2 \backslash [0.4, 0.6]^2$ (obstacle at the center)
- Define the Hamiltonian by duality (on $\partial\Omega$ the vector speed is towards the interior)

$$H(x, m, p) = \begin{cases} \sup_{\xi \in \mathbb{R}^2} \Big\{ -\xi \cdot p - L(x, m, \xi) \Big\} = m^{-\alpha} |p|^{\beta} - \ell(x, m), & \text{if } x \in \Omega, \\ \sup_{\xi \in \mathbb{R}^2 \,:\, \xi \cdot n \leq 0} \Big\{ -\xi \cdot p - L(x, m, \xi) \Big\}, & \text{if } x \in \partial\Omega. \end{cases}$$

- The associated Lagrangian (corresponding to the running cost) is:

$$L(x, m, \xi) = (\beta - 1)\beta^{-\beta^*} m^{\frac{\alpha}{\beta - 1}} |\xi|^{\beta^*} + \ell(x, m), \qquad 1 < \beta \leq 2, 0 \leq \alpha < 1$$

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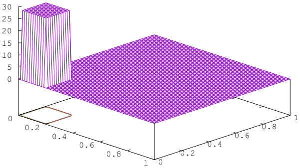
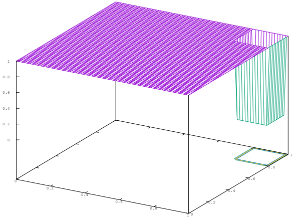
- Ex.: m_0 : & u_T : opposite corners; $\alpha = 0.01, \beta = 2, \ell(x, m) = 0.01m$.

Results for the mean field control (MFC) problem, with $\nu = 0$





Initial distribution (left) and final cost (right)

For more details, see [Achdou and Laurière, 2016b]

Numerical Example: Congestion Without Viscosity

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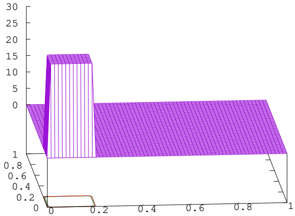
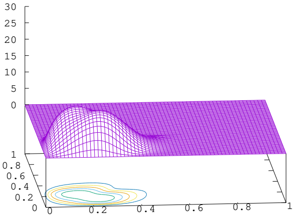


Density at time $t = 0$

For more details, see [Achdou and Laurière, 2016b]

Numerical Example: Congestion Without Viscosity

Results for the mean field control (MFC) problem, with $\nu = 0$



Density at time $t = T/8$

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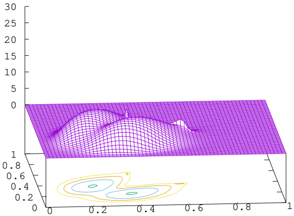

Density at time $t = T/4$

For more details, see [Achdou and Laurière, 2016b]

Numerical Example: Congestion Without Viscosity

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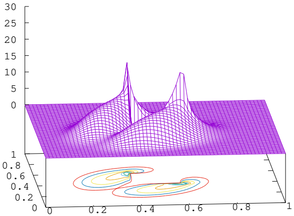
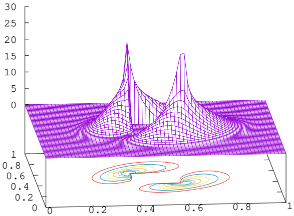


Density at time $t = 3T/8$

For more details, see [Achdou and Laurière, 2016b]

Numerical Example: Congestion Without Viscosity

Results for the mean field control (MFC) problem, with $\nu = 0$

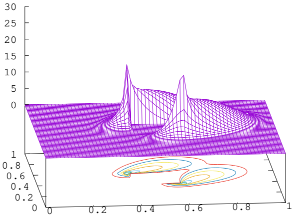


Density at time $t = T/2$

For more details, see [Achdou and Laurière, 2016b]

Numerical Example: Congestion Without Viscosity

Results for the mean field control (MFC) problem, with $\nu = 0$

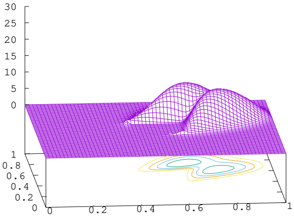


Density at time $t = 5T/8$

For more details, see [Achdou and Laurière, 2016b]

Numerical Example: Congestion Without Viscosity

Results for the mean field control (MFC) problem, with $\nu = 0$



Density at time $t = 3T/4$

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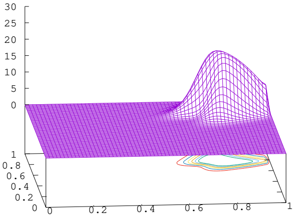
Results for the mean field control (MFC) problem, with $\nu = 0$



Density at time $t = 7T/8$

For more details, see [Achdou and Laurière, 2016b]

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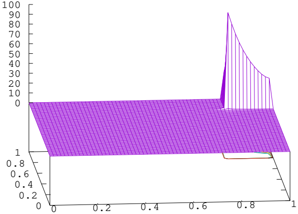

Density at time $t = T$

For more details, see [Achdou and Laurière, 2016b]

Outline

Optimality Conditions and Proximal Operator

- Let $\varphi, \psi\colon \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be convex l.s.c. proper functions.
- Consider the optimization problem

$$\min_{y \in \mathbb{R}^N} \varphi(y) + \psi(y),$$

and its dual

$$\min_{\sigma \in \mathbb{R}^N} \varphi^*(-\sigma) + \psi^*(\sigma).$$

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- The 1^{st}-order opt. cond. satisfied by a solution $(\hat{y}, \hat{\sigma})$ are

$$\begin{cases} -\hat{\sigma} \in \partial\varphi(\hat{y}) \\ \hat{y} \in \partial\psi^*(\hat{\sigma}) \end{cases} \Leftrightarrow \begin{cases} \hat{y} - \tau\hat{\sigma} \in \tau\partial\varphi(\hat{y}) + \hat{y} \\ \hat{\sigma} + \gamma\hat{y} \in \gamma\partial\psi^*(\hat{\sigma}) + \hat{\sigma} \end{cases} \Leftrightarrow \begin{cases} \mathrm{prox}_{\tau\varphi}(\hat{y} - \tau\hat{\sigma}) = \hat{y} \\ \mathrm{prox}_{\gamma\psi^*}(\hat{\sigma} + \gamma\hat{y}) = \hat{\sigma}, \end{cases}$$

where $\gamma > 0$ and $\tau > 0$ are arbitrary and

- The **proximal operator** of a l.s.c. convex proper $\phi \colon \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is:

$$\mathrm{prox}_{\gamma\phi}(x) := \operatorname*{argmin}_{y \in \mathbb{R}^N} \left\{ \phi(y) + \frac{|y-x|^2}{2\gamma} \right\} = (I + \partial(\gamma\phi))^{-1}(x), \qquad \forall\, x \in \mathbb{R}^N.$$

Chambolle-Pock's Primal-Dual Algorithm

The following algorithm has been proposed by [Chambolle and Pock, 2011]
It has been proved to converge when $\tau\gamma < 1$.

Input: Initial guess $(\sigma^{(0)}, y^{(0)}, \bar{y}^{(0)})$; $\theta \in [0,1]$; $\gamma > 0, \tau > 0$; number of iterations $\mathtt{K}$
Output: Approximation of $(\hat{\sigma}, \hat{y})$ solving the optimality conditions

Initialize $(\sigma^{(0)}, y^{(0)}, \bar{y}^{(0)})$
for $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**
    (a) Compute

$$\sigma^{(\mathtt{k}+1)} = \mathrm{prox}_{\gamma\psi^*}(\sigma^{(\mathtt{k})} + \gamma\bar{y}^{(\mathtt{k})}),$$

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(c) Compute

$$\bar{y}^{(\mathtt{k}+1)} = y^{(\mathtt{k}+1)} + \theta(y^{(\mathtt{k}+1)} - y^{(\mathtt{k})}).$$

return $(\sigma^{(\mathtt{K})}, y^{(\mathtt{K})}, \bar{y}^{(\mathtt{K})})$

Dual of Discrete Problem $(\mathbf{A_h})$

By Fenchel-Rockafellar theorem, the dual problem of $(\mathbf{A_h})$ is:

$$(\mathbf{B_h}) = \min_{(m,w_1,w_2)=\sigma\in\mathbb{R}^{3N'}} \Big\{ \mathcal{F}_h^*(\Lambda_h^*(\sigma)) + \mathcal{G}_h^*(-\sigma) \Big\},$$

where $\mathcal{G}_h^*$ and $\mathcal{F}_h^*$ are respectively the Legendre-Fenchel conjugates of $\mathcal{G}_h$ and $\mathcal{F}_h$, defined by:

- $\mathcal{F}_h^*(\mu) = \sup_{\phi\in\mathbb{R}^N} \Big\{ \langle \mu, \phi \rangle_{\ell^2(\mathbb{R}^N)} - \mathcal{F}_h(\phi) \Big\}, \quad \forall\, \mu \in \mathbb{R}^N$
- $\mathcal{G}_h^*(-\sigma) = \max\limits_{q\in\mathbb{R}^{3N'}} \Big\{ -\langle \sigma, q \rangle_{\ell^2(\mathbb{R}^{3N'})} - \mathcal{G}_h(q) \Big\} = h\Delta t \sum\limits_{n=1}^{N_T} \sum\limits_{i=0}^{N_h-1} \tilde{L}_h(x_i, \sigma_i^n), \quad \forall\, \sigma \in \mathbb{R}^{3N'}$
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Rem.: The max can be costly to compute but in some cases $\tilde{L}_h$ has a **closed-form** expression.

Finally $\Lambda_h^* : \mathbb{R}^{3N'} \to \mathbb{R}^N$ denotes the adjoint of Λ_h: for all $(m, y, z) \in \mathbb{R}^{3N'}, \phi \in \mathbb{R}^N$:

$$\langle \Lambda_h^*(m, y, z), \phi\rangle_{\ell^2(\mathbb{R}^N)} = \langle (m, y, z), \Lambda_h(\phi)\rangle_{\ell^2(\mathbb{R}^{3N'})}$$

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Rem.: We have $\mathcal{F}_h^*(\Lambda_h^*(m,y,z)) = \begin{cases} h\sum_{i=0}^{N_h-1} m_i^{N_T}\,\mathrm{g}_0(x_i), & \text{if } (m,y,z) \text{ satisfies } (\star) \text{ below,}\\ +\infty, & \text{otherwise,}\end{cases}$

with $\forall\, i\in\{0,\dots,N_h-1\}$, $m_i^0 = \rho_i^0$, and $\forall\, n \in \{0,\dots,N_T-1\}$:

$$(D_t m_i)^n - \nu\left(\Delta_h m^{n+1}\right)_i + \frac{y_i^{n+1} - y_{i-1}^{n+1}}{h} + \frac{z_{i+1}^{n+1} - z_i^{n+1}}{h} = 0. \qquad (\star)$$

Reformulation

The discrete dual problem can be recast as:

$$\inf_{(m,w)} \underbrace{\mathbb{B}_h(m,w) + \mathbb{F}_h(m)}_{\varphi(m,w)} + \underbrace{\iota_{\mathbb{G}^{-1}(\rho^0,0)}(m,w)}_{\psi(m,w)} \qquad (P_h)$$

with the costs

$$\mathbb{F}_h(m) := \sum_{i,n} \widetilde{F}(x_i, m_i^n) + \frac{1}{\Delta t}\sum_i \widetilde{G}(x_i, m_i^{N_T}), \qquad \mathbb{B}_h(m,w) := \sum_{i,n} \hat{b}(m_i^n, w_i^{n-1}),$$

$$\hat{b}(m,w) := \begin{cases} mL\left(x, -\frac{w}{m}\right), & \text{if } m > 0, w \in K = \mathbb{R}_- \times \mathbb{R}_+, \\ 0, & \text{if } (m,w) = (0,0), \\ +\infty, & \text{otherwise,} \end{cases}$$

and $\mathbb{G}(m,w) := (m_0, (Am^{n+1} + Bw^n)_{0\le n\le N_T-1})$ with

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Rem.: The optimality conditions of this problem correspond to the **finite-difference system**

So we can apply **Chambolle-Pock**'s method for (P_h) with

$$y = (m,w), \qquad \varphi(m,w) = \mathbb{B}_h(m,w) + \mathbb{F}_h(m), \qquad \psi(m,w) = \iota_{\mathbb{G}^{-1}(\rho^0,0)}(m,w)$$

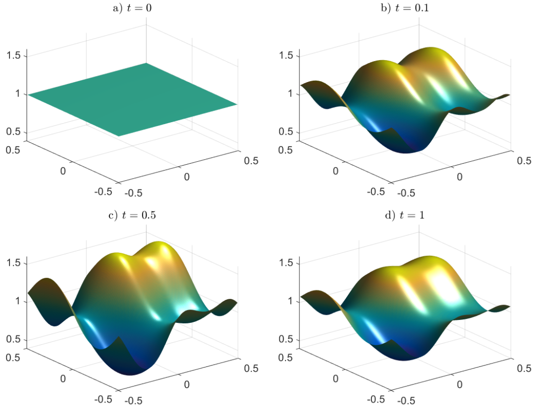
See [Briceño Arias et al., 2018] and [Briceño Arias et al., 2019] in stationary and dynamic cases.

Numerical Example

Setting: $g \equiv 0$ and $\mathbb{R}^2 \times \mathbb{R} \ni (x, m) \mapsto f(x, m) := m^2 - \overline{H}(x)$, with

$$\overline{H}(x) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(2\pi x_1)$$

We solve the corresponding MFG and obtain the following evolution of the density:



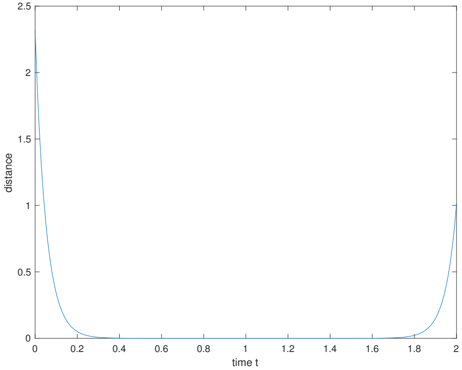
Evolution of the density

More details in [Briceño Arias et al., 2019]

Turnpike phenomenon

This example also illustrates the **turnpike phenomenon**, see e.g. [Porretta and Zuazua, 2013]

- the mass starts from an initial density;
- it **converges to a steady state**, influenced only by the running cost;
- as $t \to T$, the mass is influenced by the final cost and **converges to a final state**.



L^2 **distance between dynamic and stationary solutions**

More details in [Briceño Arias et al., 2019]

Summary so far

Outline

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1. Introduction

2. Methods for the PDE system

3. Optimization Methods for MFC and Variational MFG

4. Methods for MKV FBSDE

- A Picard Scheme for MKV FBSDE
- Stochastic Methods for some Finite-Dimensional MFC Problems

5. Conclusion

MKV FBSDE System

- Recall: generic form:

$$\begin{cases} dX_t = B(X_t, \mathcal{L}(X_t), Y_t, Z_t)dt + \sigma dW_t, & 0 \leq t \leq T \\ dY_t = -F(X_t, \mathcal{L}(X_t), Y_t, Z_t)dt + Z_t dW_t, & 0 \leq t \leq T \\ X_0 \sim m_0, \qquad Y_T = G(X_T, \mathcal{L}(X_T)) \end{cases}$$

- Decouple:
  - Given $(\mathcal{L}(X), Y, Z)$, solve for X
  - Given $(X, \mathcal{L}(X))$ solve for (Y, Z)
- Iterate
- Algorithm proposed by [Chassagneux et al., 2019, Angiuli et al., 2019]

Picard Scheme for MKV FBSDE System

Input: Initial guess (ξ, ζ); initial condition ξ; terminal condition ζ; time horizon T; number of iterations $\mathtt{K}$

Output: Approximation of (X, Y, Z) solving the MKV FBSDE system

Initialize $X_t^{(0)} = \xi, Y_t^{(0)} = 0, Z_t^{(0)} = 0, 0 \leq t \leq T$

for $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**

  Let $X^{(\mathtt{k}+1)}$ be the solution to:

$$\begin{cases} dX_t = B(X_t^{(\mathtt{k})}, \mathcal{L}(X_t^{(\mathtt{k})}), Y_t^{(\mathtt{k})}, Z_t^{(\mathtt{k})})dt + \sigma dW_t, & 0 \leq t \leq T \\ X_0 = \xi \end{cases}$$

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Let $(Y^{(\mathtt{k}+1)}, Z^{(\mathtt{k}+1)})$ be the solution to:

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return $\mathtt{Picard}[T](\xi, \zeta) = (X^{(\mathtt{K})}, Y^{(\mathtt{K})}, Z^{(\mathtt{K})})$

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Contraction? Small T or small Lipschitz constants for B, F, G

Continuation Method

- If T is big: Solve FBSDE on small intervals & “patch” the solutions together

Continuation Method

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- Grid: $0 = T_0 < T_1 < \cdots < T_{M-1} < T_M = T$
- Subproblem: Given $(\xi_{T_m}, \mathcal{L}(\xi_{T_m}))$ and $\zeta_{T_{m+1}}$, solve:

$$\begin{cases} dX_t = B(X_t, \mathcal{L}(X_t), Y_t, Z_t)dt + \sigma dW_t, \qquad T_m \le t \le T_{m+1} \\ dY_t = -F(X_t, \mathcal{L}(X_t), Y_t, Z_t)dt + Z_t dW_t, \qquad T_m \le t \le T_{m+1} \\ X_{T_m} = \xi_{T_m}, \qquad Y_{T_{m+1}} = \zeta_{T_{m+1}} \end{cases}$$

Continuation Method

- If T is big: Solve FBSDE on small intervals & "patch" the solutions together
- Grid: $0 = T_0 < T_1 < \cdots < T_{M-1} < T_M = T$
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- How to find ξ_{T_m} and $\zeta_{T_{m+1}}$?
  - $\rightarrow$ ξ_{T_m} from previous problem's solution (or initial condition)
  - $\rightarrow$ $\zeta_{T_{m+1}}$ from next problem's solution (or terminal condition)

Global Solver for MKV FBSDE System

Following [Chassagneux et al., 2019], define a global solver recursively, and then call:

$$\texttt{Solver}[m](\xi_0, \mu_0)$$

with ξ_0 a random variable with distribution μ_0

Input: Initial guess $(\xi, \mathcal{L}(\xi))$; time step index m; number of iterations $\mathtt{K}$
Output: Approximation of Y_{T_m} where (X, Y, Z) solves the MKV FBSDE system on $[T_m, T]$ starting with $(\xi, \mathcal{L}(\xi))$ at time T_m

Initialize $X_t^{(0)} = \xi, \mathcal{L}(X_t^{(0)}) = \mathcal{L}(\xi)$ for all $T_m \leq t \leq T_{m+1}$
for $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**
  If $T_{m+1} = T$, $Y_{T_{m+1}}^{(\mathtt{k}+1)} = G(X_{T_{m+1}}^{(\mathtt{k})}, \mathcal{L}(X_{T_{m+1}}^{(\mathtt{k})}))$

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3. If $T_{m+1} = T$, $Y_{T_{m+1}}^{(\texttt{k}+1)} = G(X_{T_{m+1}}^{(\texttt{k})}, \mathcal{L}(X_{T_{m+1}}^{(\texttt{k})}))$
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$$Y_{T_{m+1}}^{(\texttt{k}+1)} = \texttt{Solver}[m+1](X_{T_{m+1}}^{(\texttt{k})}, \mathcal{L}(X_{T_{m+1}}^{(\texttt{k})}))$$
5. Compute:
$$(X_t^{(\texttt{k}+1)}, \mathcal{L}(X_t^{(\texttt{k}+1)}), Y_t^{(\texttt{k}+1)}, Z_t^{(\texttt{k}+1)})_{T_m \leq t \leq T_{m+1}} = \texttt{Picard}[T_{m+1} - T_m](X_{T_m}^{(\texttt{k})}, Y_{T_{m+1}}^{(\texttt{k}+1)})$$
6. **return** $\texttt{Solver}[m](\xi, \mathcal{L}(\xi)) := Y_{T_m}^{(\texttt{K})}$

Implementation: Discretizations

In the sequel, we present two algorithms, following [Angiuli et al., 2019]

- Tree algorithm:
  - Time discretization
  - Space discretization: binomial tree structure
  - Look at trajectories
- Grid algorithm:
  - Time and space discretization on a grid
  - Look at time marginals

Tree-Based Algorithm: Time Discretization

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- Euler Scheme: $0 \le i \le N_t - 1$

$$
\begin{cases}
X_{t_{i+1}}^{(\mathtt{k+1})} = X_{t_i}^{(\mathtt{k+1})} + B(X_{t_i}^{(\mathtt{k+1})}, \mathcal{L}(X_{t_i}^{(\mathtt{k+1})}), Y_{t_i}^{(\mathtt{k})}, Z_{t_i}^{(\mathtt{k})})\Delta t + \sigma \Delta W_{t_{i+1}} \\
X_0^{(\mathtt{k+1})} = \xi \\
Y_{t_i}^{(\mathtt{k+1})} = \mathbb{E}_{t_i}[Y_{t_{i+1}}^{(\mathtt{k+1})}] + F(X_{t_i}^{(\mathtt{k+1})}, \mathcal{L}(X_{t_i}^{(\mathtt{k+1})}), Y_{t_i}^{(\mathtt{k})}, Z_{t_i}^{(\mathtt{k})})\Delta t \\
\qquad\quad \approx Y_{t_{i+1}}^{(\mathtt{k+1})} + F(X_{t_i}^{(\mathtt{k+1})}, \mathcal{L}(X_{t_i}^{(\mathtt{k+1})}), Y_{t_i}^{(\mathtt{k})}, Z_{t_i}^{(\mathtt{k})})\Delta t - Z_{t_i}^{(\mathtt{k+1})} \Delta W_{t_{i+1}} \\
Y_T^{(\mathtt{k+1})} = G(X_T^{(\mathtt{k+1})}, \mathcal{L}(X_T^{(\mathtt{k+1})})) \\
Z_{t_i}^{(\mathtt{k+1})} = \frac{1}{\Delta t} \mathbb{E}_{t_i}[Y_{t_{i+1}}^{(\mathtt{k+1})} \Delta W_{t_{i+1}}] \\
Z_T^{(\mathtt{k+1})} = 0
\end{cases}
$$

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$$\begin{cases} X_{t_{i+1}}^{(\mathtt{k}+1)} = X_{t_i}^{(\mathtt{k}+1)} + B(X_{t_i}^{(\mathtt{k}+1)}, \mathcal{L}(X_{t_i}^{(\mathtt{k}+1)}), Y_{t_i}^{(\mathtt{k})}, Z_{t_i}^{(\mathtt{k})})\Delta t + \sigma \Delta W_{t_{i+1}} \\ X_0^{(\mathtt{k}+1)} = \xi \\ Y_{t_i}^{(\mathtt{k}+1)} = \mathbb{E}_{t_i}[Y_{t_{i+1}}^{(\mathtt{k}+1)}] + F(X_{t_i}^{(\mathtt{k}+1)}, \mathcal{L}(X_{t_i}^{(\mathtt{k}+1)}), Y_{t_i}^{(\mathtt{k})}, Z_{t_i}^{(\mathtt{k})})\Delta t \\ \qquad \approx Y_{t_{i+1}}^{(\mathtt{k}+1)} + F(X_{t_i}^{(\mathtt{k}+1)}, \mathcal{L}(X_{t_i}^{(\mathtt{k}+1)}), Y_{t_i}^{(\mathtt{k})}, Z_{t_i}^{(\mathtt{k})})\Delta t - Z_{t_i}^{(\mathtt{k}+1)} \Delta W_{t_{i+1}} \\ Y_T^{(\mathtt{k}+1)} = G(X_T^{(\mathtt{k}+1)}, \mathcal{L}(X_T^{(\mathtt{k}+1)})) \\ Z_{t_i}^{(\mathtt{k}+1)} = \frac{1}{\Delta t}\mathbb{E}_{t_i}[Y_{t_{i+1}}^{(\mathtt{k}+1)} \Delta W_{t_{i+1}}] \\ Z_T^{(\mathtt{k}+1)} = 0 \end{cases}$$

- Questions:
  - How to represent $\mathcal{L}(X_{t_i}^{(\mathtt{k}+1)})$?
  - How to compute the conditional expectation $\mathbb{E}_{t_i}[Y_{t_{i+1}}^{(\mathtt{k}+1)}]$?

Tree-Based Algorithm: Remarks

- At each t_i, replace $\Delta W_{t_{i+1}}$ by a branch with 2 values: $\pm\sqrt{\Delta t}$ w.p. $1/2$
- Answers:
  - $\mathcal{L}(X_{t_i}^{(\mathtt{k}+1)}) \approx$ weighted empirical distribution:

$$\mathcal{L}(X_{t_0}^{(\mathtt{k}+1)}) \approx \sum_{n=1}^{N_{x_0}} p_0^k \delta_{x_0^k},$$

    and at time $t_i, i \geq 1$: look at values on the nodes at depth i
  - $\mathbb{E}_{t_i}[Y_{t_{i+1}}^{(\mathtt{k}+1)}] \approx$ weighted average of values on the two next branches

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  - $\mathbb{E}_{t_i}[Y_{t_{i+1}}^{(\mathtt{k}+1)}] \approx$ weighted average of values on the two next branches
- Starting from some x_0, doing N_t steps: 2^{N_t} paths
- N_{x_0} starting points i.i.d. $\sim \mu_0$: $N_{x_0} \times 2^{N_t}$ paths !

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- Save space thanks to recombinations? *Not really but . . .*

Grid-Based Algorithm: Time & Space Discretization

- Decoupling functions (see e.g., Section 6.4 in [Carmona and Delarue, 2018]):

$$Y_t = u(t, X_t, \mathcal{L}(X_t)), \qquad Z_t = v(t, X_t, \mathcal{L}(X_t))$$

$\rightarrow$ Approximate $u(\cdot,\cdot,\cdot), v(\cdot,\cdot,\cdot)$ instead of $(Y_t, Z_t)_{t\in[0,T]}$

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$\rightarrow$ Freeze it during each Picard iteration:

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- Picard iterations for distribution & decoupling functions:
  - **Step 1:** Given $(\mu^{(\mathtt{k})}, u^{(\mathtt{k})}, v^{(\mathtt{k})})$, compute $\mu_t^{(\mathtt{k}+1)} = \mathcal{L}(X_t^{(\mathtt{k}+1)}), 0 \le t \le T$, where

$$dX_t^{(\mathtt{k}+1)} = B\Big(X_t^{(\mathtt{k}+1)}, \mu_t^{(\mathtt{k})}, u^{(\mathtt{k})}(t, X_t^{(\mathtt{k}+1)}), v^{(\mathtt{k})}(t, X_t^{(\mathtt{k}+1)})\Big)dt + \sigma dW_t$$

  - **Step 2:** Given $(X^{(\mathtt{k})}, \mu^{(\mathtt{k}+1)})$, compute $(u^{(\mathtt{k}+1)}, v^{(\mathtt{k}+1)})$ such that ($\star$) holds, where

$$dY_t^{(\mathtt{k}+1)} = -F\Big(X_t^{(\mathtt{k}+1)}, \mu_t^{(\mathtt{k}+1)}, Y_t^{(\mathtt{k}+1)}, Z_t^{(\mathtt{k}+1)}\Big)dt + Z_t^{(\mathtt{k}+1)} dW_t$$

  - Return $(\mu^{(\mathtt{k}+1)}, u^{(\mathtt{k}+1)}, v^{(\mathtt{k}+1)})$

Grid-Based Algorithm: Forward Equation

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$$+ \sigma \Delta W_{t_{i+1}}\Big]$$

  - In fact $\mu_{t_{i+1}}^{(\mathtt{k}+1)}$ can be expressed in terms of $\mu_{t_i}^{(\mathtt{k}+1)}$ and a transition kernel
  - Ex: binomial approx. of W $\rightarrow$ efficient computation using quantization

Grid-Based Algorithm: Backward Equation

- Picard iterations for distribution & decoupling functions (continued):
  - **Step 2:** Update u, v: for all $0 \leq i \leq N_t$, $x \in \Gamma$,

$$
\begin{cases}
u_{t_i}^{(\mathtt{k}+1)}(x) = \mathbb{E}\Big[u_{t_{i+1}}^{(\mathtt{k}+1)}(X_{t_i}^{(\mathtt{k}+1)}) \\
\qquad\qquad + F\Big(X_{t_i}^{(\mathtt{k}+1)}, \mu_{t_i}^{(\mathtt{k}+1)}, u_{t_i}^{(\mathtt{k})}(X_{t_i}^{(\mathtt{k}+1)}), v_{t_i}^{(\mathtt{k})}(X_{t_i}^{(\mathtt{k}+1)})\Big)\Delta t \,\Big|\, X_{t_i}^{(\mathtt{k}+1)} = x\Big] \\
u_T^{(\mathtt{k}+1)}(x) = G(x, \mu_{t_i}^{(\mathtt{k}+1)}) \\
v_{t_i}^{(\mathtt{k}+1)}(x) = \mathbb{E}\Big[\frac{1}{\Delta t} u_{t_{i+1}}^{(\mathtt{k}+1)}(X_{t_i}^{(\mathtt{k}+1)}) \,\Big|\, X_{t_i}^{(\mathtt{k}+1)} = x\Big] \\
v_T^{(\mathtt{k}+1)}(x) = 0
\end{cases}
$$

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- Summary:
  - Forward: $(\mu^{(\mathtt{k})}, u^{(\mathtt{k})}, v^{(\mathtt{k})}) \mapsto \mu^{(\mathtt{k}+1)} = \mathcal{L}(X^{(\mathtt{k}+1)})$
  - Backward: $(\mu^{(\mathtt{k}+1)}, u^{(\mathtt{k})}, v^{(\mathtt{k})}) \mapsto (u^{(\mathtt{k}+1)}, v^{(\mathtt{k}+1)})$

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More details and numerical examples in
[Chassagneux et al., 2019, Angiuli et al., 2019]

Outline

Dependence on the Moments

- In general: b, f, g involve the whole distribution $\mu_t = \mathcal{L}(X_t)$ (infinite dim.)

- What if they involve only the first moment $\overline{\mu}_t = \mathbb{E}[X_t]$?

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- Ex. 1: LQ (see lecture 2)
  - optimal control is a function of X_t and $\overline{\mu}_t = \mathbb{E}[X_t]$
  - ODE for $\overline{\mu}_t$ of the form $\frac{d}{dt}\overline{\mu}_t = \varphi(t, \overline{\mu}_t)$

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- Ex. 2:

$$\begin{cases} b(x, \mu, \alpha) = b(x, \overline{\mu}, \alpha) = (\cos(x) + \cos(\overline{\mu}))\alpha \\ f(x, \mu, \alpha) = |\alpha|^2, \qquad g(x, \mu) = 0 \end{cases}$$

  - Can the optimal control be expressed as a function of $X_t, \mathbb{E}[X_t]$ only?
  - ODE for $\overline{\mu}_t$?

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\begin{cases} b(x, \mu, \alpha) = b(x, \overline{\mu}, \alpha) = (\cos(x) + \cos(\overline{\mu}))\alpha \\ f(x, \mu, \alpha) = |\alpha|^2, \qquad g(x, \mu) = 0 \end{cases}
$$

  - Can the optimal control be expressed as a function of $X_t, \mathbb{E}[X_t]$ only?
  - ODE for $\overline{\mu}_t$?

$$
\frac{d}{dt}\overline{\mu}_t = \mathbb{E}\left[(\cos(X_t) + \cos(\overline{\mu}_t))\alpha(t, X_t)\right]
$$

  It involves not only $\mathbb{E}[X_t] = \overline{\mu}_t$ but also $\mathbb{E}[\cos(X_t)]$

Dependence on the Moments

- In general: b, f, g involve the whole distribution $\mu_t = \mathcal{L}(X_t)$ (infinite dim.)
- What if they involve only the first moment $\overline{\mu}_t = \mathbb{E}[X_t]$?
- Ex. 1: LQ (see lecture 2)
  - optimal control is a function of X_t and $\overline{\mu}_t = \mathbb{E}[X_t]$
  - ODE for $\overline{\mu}_t$ of the form $\frac{d}{dt}\overline{\mu}_t = \varphi(t, \overline{\mu}_t)$
- Ex. 2:

$$\begin{cases} b(x, \mu, \alpha) = b(x, \overline{\mu}, \alpha) = (\cos(x) + \cos(\overline{\mu}))\alpha \\ f(x, \mu, \alpha) = |\alpha|^2, \qquad g(x, \mu) = 0 \end{cases}$$

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- Class of MFC s.t. the problem can be solved with a finite number of moments?

Finite-Dimensional Reformulation

Following [Balata et al., 2019]

- In some cases, MFC problems can be written as:

$$J(\alpha) = \mathbb{E}\left[\int_0^T \mathcal{F}(\underline{X}_t, \alpha_t)dt + \mathcal{G}(\underline{X}_T)\right]$$

subject to:

$$d\underline{X}_t = \mathcal{B}(\underline{X}_t, \alpha_t)dt + \Sigma d\mathbb{W}_t$$

where the state is: $\underline{X}_t = (\mathbb{E}[X_t], \mathbb{E}[|X_t|^2], \dots, \mathbb{E}[|X_t|^p]) \in (\mathbb{R}^d)^p$

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- Time discretization: $0 = t_0 < t_1 < \dots < t_{N_t} = T$, $t_{i+1} - t_i = \Delta t$
- DPP for $V : [0,T] \times (\mathbb{R}^d)^p \to \mathbb{R}$ or rather $V_{\Delta t} : \{t_0, \dots, t_{N_t}\} \times (\mathbb{R}^d)^p \to \mathbb{R}$:

$$\begin{cases} V_{\Delta t}(T, \underline{x}) = \mathcal{G}(\underline{x}) \\ V_{\Delta t}(t_n, \underline{x}) = \sup_\alpha \left\{ \mathcal{F}(\underline{x}, \alpha)\Delta t + \mathbb{E}^{t_n, \underline{x}, \alpha}\left[V_{\Delta t}(t_{n+1}, \underline{X}_{t_{n+1}})\right]\right\}, n = N_t - 1, \dots, 1, 0 \end{cases}$$

  where $\mathbb{E}^{t_n, \underline{x}, \alpha}\left[V_{\Delta t}(t_{n+1}, \underline{X}_{t_{n+1}})\right] = \mathbb{E}\left[V_{\Delta t}(t_{n+1}, \underline{X}^\alpha_{t_{n+1}}) \,|\, \underline{X}^\alpha_{t_n} = \underline{x}\right]$

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→ **Key difficulty:** estimation of the conditional expectation

Estimation Method 1: Regression Monte Carlo

- Family of basis functions $\phi = (\phi^m)_{m=1,\ldots,M}$
- Projection:

$$\mathbb{E}\left[V_{\Delta t}(t_{n+1}, \underline{X}^{\alpha}_{t_{n+1}}) \,|\, \underline{X}^{\alpha}_{t_n}\right] \approx \sum_{m=1}^{M} \beta^m_{t_n} \phi^m(\underline{X}^{\alpha}_{t_n})$$

where

$$\beta^m_{t_n} = \operatorname*{argmin}_{\beta \in \mathbb{R}^M} \mathbb{E}\left[\left| V_{\Delta t}(t_{n+1}, \underline{X}^{\alpha}_{t_{n+1}}) - \sum_{m=1}^{M} \beta^m \phi^m(\underline{X}^{\alpha}_{t_n}) \right|^2\right]$$

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- Explicit expression:

$$\beta^m_{t_n} = \mathbb{E}[\phi(\underline{X}^{\alpha}_{t_n})\phi(\underline{X}^{\alpha}_{t_n})^\top]^{-1}\, \mathbb{E}[V_{\Delta t}(t_{n+1}, \underline{X}^{\alpha}_{t_{n+1}})\phi(\underline{X}^{\alpha}_{t_n})]$$

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- Estimation with N_{MC} Monte Carlo samples:

$$\mathbb{E}[\phi(\underline{X}^{\ell,\alpha}_{t_n})\phi(\underline{X}^{\ell,\alpha}_{t_n})^{\top}] \approx \frac{1}{N_{MC}} \sum_{\ell=1}^{N_{MC}} \phi(\underline{X}^{\ell,\alpha}_{t_n})\phi(\underline{X}^{\ell,\alpha}_{t_n})^{\top}$$

and

$$\mathbb{E}[V_{\Delta t}(t_{n+1}, \underline{X}^{\ell,\alpha}_{t_{n+1}})\phi(\underline{X}^{\ell,\alpha}_{t_n})] \approx \frac{1}{N_{MC}} \sum_{\ell=1}^{N_{MC}} V_{\Delta t}(t_{n+1}, \underline{X}^{\ell,\alpha}_{t_{n+1}})\phi(\underline{X}^{\ell,\alpha}_{t_n})$$

with training set $\{\left(\underline{X}^{\ell,\alpha}_{t_n}, \underline{X}^{\ell,\alpha}_{t_{n+1}}\right); \ell = 1, \ldots, N_{MC}\}$

Estimation Method 1: Regression Monte Carlo

- Family of basis functions $\phi = (\phi^m)_{m=1,\dots,M}$ *Not always easy to choose !*
- Projection:

$$\mathbb{E}\left[V_{\Delta t}(t_{n+1}, \underline{X}^{\alpha}_{t_{n+1}}) \,|\, \underline{X}^{\alpha}_{t_n}\right] \approx \sum_{m=1}^{M} \beta^m_{t_n} \phi^m(\underline{X}^{\alpha}_{t_n})$$

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Estimation Method 2: Quantization

- Two space discretizations:
  - Set of points Γ on which we want to approximate $V_{\Delta t}$; projection Π_Γ
  - Quantization of noise (see e.g. [Pagès, 2018]):
    - ★ Set of cells $\mathcal{C}_Q = \{C_j; j = 1, \dots, J_Q\}$
    - ★ Associated grid points $\mathcal{G}_Q = \{\zeta_j; j = 1, \dots, J_Q\}$
    - ★ Weights for Gaussian r.v. $\Delta\mathbb{W} \sim \mathcal{N}(0, \Delta t)$: $p_j = \mathbb{P}(\Delta\mathbb{W} \in C_j)$
    - ★ Discrete version: $\Delta\hat{\mathbb{W}} \in \mathcal{G}_Q$: $\mathbb{P}(\Delta\hat{\mathbb{W}} = \zeta_j) = p_j$
    - ★ Can be optimized[1]; particularly helpful when $d > 1$

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  for all $\underline{x} \in \Gamma$
- Other interpolations are possible

For more details and numerical examples, see [Balata et al., 2019]

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Outline

1. Introduction

2. Methods for the PDE system

3. Optimization Methods for MFC and Variational MFG

4. Methods for MKV FBSDE

5. Conclusion

Summary

Other numerical methods

The previous presentation is not exhaustive!

Some other references:

- Gradient descent based methods [Laurière and Pironneau, 2016], [Pfeiffer, 2016], [Lavigne and Pfeiffer, 2022]
- Monotone operators [Almulla et al., 2017], [Gomes and Saúde, 2018], [Gomes and Yang, 2020]
- Policy iteration [Cacace et al., 2021], [Cui and Koeppl, 2021], [Camilli and Tang, 2022], [Tang and Song, 2022], [Laurière et al., 2023]
- Finite elements [Benamou and Carlier, 2015b], [Andreev, 2017]
- Cubature [de Raynal and Trillos, 2015]
- ...

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- Cubature [de Raynal and Trillos, 2015]
- ...

However efficient, these methods are usually limited to problems with:

- (relatively) small dimension
- (relatively) simple structure

$\Rightarrow$ motivations to develop machine learning methods (see next lectures)

References I

[Achdou, 2013] Achdou, Y. (2013).
Finite difference methods for mean field games.
In *Hamilton-Jacobi equations: approximations, numerical analysis and applications*, volume 2074 of *Lecture Notes in Math.*, pages 1–47. Springer, Heidelberg.

[Achdou et al., 2012] Achdou, Y., Camilli, F., and Capuzzo-Dolcetta, I. (2012).
Mean field games: numerical methods for the planning problem.
SIAM J. Control Optim., 50(1):77–109.

[Achdou and Capuzzo-Dolcetta, 2010] Achdou, Y. and Capuzzo-Dolcetta, I. (2010).
Mean field games: numerical methods.
SIAM J. Numer. Anal., 48(3):1136–1162.

[Achdou and Laurière, 2016a] Achdou, Y. and Laurière, M. (2016a).
Mean field type control with congestion (ii): An augmented lagrangian method.
Applied Mathematics & Optimization, 74(3):535–578.

[Achdou and Laurière, 2016b] Achdou, Y. and Laurière, M. (2016b).
Mean Field Type Control with Congestion (II): An augmented Lagrangian method.
Appl. Math. Optim., 74(3):535–578.

[Achdou and Laurière, 2020] Achdou, Y. and Laurière, M. (2020).
Mean field games and applications: Numerical aspects.
Mean Field Games: Cetraro, Italy 2019, 2281:249–307.

References II

[Achdou and Perez, 2012] Achdou, Y. and Perez, V. (2012).
Iterative strategies for solving linearized discrete mean field games systems.
Netw. Heterog. Media, 7(2):197–217.

[Achdou and Porretta, 2016] Achdou, Y. and Porretta, A. (2016).
Convergence of a finite difference scheme to weak solutions of the system of partial differential equations arising in mean field games.
SIAM J. Numer. Anal., 54(1):161–186.

[Almulla et al., 2017] Almulla, N., Ferreira, R., and Gomes, D. (2017).
Two numerical approaches to stationary mean-field games.
Dyn. Games Appl., 7(4):657–682.

[Andreev, 2017] Andreev, R. (2017).
Preconditioning the augmented lagrangian method for instationary mean field games with diffusion.
SIAM Journal on Scientific Computing, 39(6):A2763–A2783.

[Angiuli et al., 2019] Angiuli, A., Graves, C. V., Li, H., Chassagneux, J.-F., Delarue, F., and Carmona, R. (2019).
Cemracs 2017: numerical probabilistic approach to MFG.
ESAIM: ProcS, 65:84–113.

References III

[Balata et al., 2019] Balata, A., Huré, C., Laurière, M., Pham, H., and Pimentel, I. (2019).
A class of finite-dimensional numerically solvable mckean-vlasov control problems.
ESAIM: Proceedings and Surveys, 65:114–144.

[Baudelet et al., 2023] Baudelet, S., Frénais, B., Laurière, M., Machtalay, A., and Zhu, Y. (2023).
Deep learning for mean field optimal transport.
arXiv preprint arXiv:2302.14739.

[Benamou and Carlier, 2015a] Benamou, J.-D. and Carlier, G. (2015a).
Augmented Lagrangian methods for transport optimization, mean field games and degenerate elliptic equations.
J. Optim. Theory Appl., 167(1):1–26.

[Benamou and Carlier, 2015b] Benamou, J.-D. and Carlier, G. (2015b).
Augmented lagrangian methods for transport optimization, mean field games and degenerate elliptic equations.
Journal of Optimization Theory and Applications, 167(1):1–26.

[Benamou et al., 2017] Benamou, J.-D., Carlier, G., and Santambrogio, F. (2017).
Variational mean field games.
In *Active Particles, Volume 1*, pages 141–171. Springer.

References IV

[Briceño Arias et al., 2019] Briceño Arias, L. M., Kalise, D., Kobeissi, Z., Laurière, M., Mateos González, A., and Silva, F. J. (2019).
On the implementation of a primal-dual algorithm for second order time-dependent mean field games with local couplings.
ESAIM: ProcS, 65:330–348.

[Briceño Arias et al., 2018] Briceño Arias, L. M., Kalise, D., and Silva, F. J. (2018).
Proximal methods for stationary mean field games with local couplings.
SIAM J. Control Optim., 56(2):801–836.

[Cacace et al., 2021] Cacace, S., Camilli, F., and Goffi, A. (2021).
A policy iteration method for mean field games.
ESAIM: Control, Optimisation and Calculus of Variations, 27:85.

[Camilli and Tang, 2022] Camilli, F. and Tang, Q. (2022).
Rates of convergence for the policy iteration method for mean field games systems.
Journal of Mathematical Analysis and Applications, 512(1):126138.

[Cardaliaguet, 2015] Cardaliaguet, P. (2015).
Weak solutions for first order mean field games with local coupling.
In *Analysis and geometry in control theory and its applications*, pages 111–158. Springer.

[Cardaliaguet and Graber, 2015] Cardaliaguet, P. and Graber, P. J. (2015).
Mean field games systems of first order.
ESAIM Control Optim. Calc. Var., 21(3):690–722.

References V

[Cardaliaguet et al., 2015] Cardaliaguet, P., Graber, P. J., Porretta, A., and Tonon, D. (2015).
Second order mean field games with degenerate diffusion and local coupling.
NoDEA Nonlinear Differential Equations Appl., 22(5):1287–1317.

[Carlini and Silva, 2014] Carlini, E. and Silva, F. J. (2014).
A fully discrete semi-Lagrangian scheme for a first order mean field game problem.
SIAM J. Numer. Anal., 52(1):45–67.

[Carlini and Silva, 2015] Carlini, E. and Silva, F. J. (2015).
A semi-Lagrangian scheme for a degenerate second order mean field game system.
Discrete Contin. Dyn. Syst., 35(9):4269–4292.

[Carmona and Delarue, 2018] Carmona, R. and Delarue, F. (2018).
Probabilistic theory of mean field games with applications. I, volume 83 of *Probability Theory and Stochastic Modelling*.
Springer, Cham.
Mean field FBSDEs, control, and games.

[Chambolle and Pock, 2011] Chambolle, A. and Pock, T. (2011).
A first-order primal-dual algorithm for convex problems with applications to imaging.
J. Math. Imaging Vision, 40(1):120–145.

[Chassagneux et al., 2019] Chassagneux, J.-F., Crisan, D., and Delarue, F. (2019).
Numerical method for FBSDEs of McKean-Vlasov type.
Ann. Appl. Probab., 29(3):1640–1684.

References VI

[Cui and Koeppl, 2021] Cui, K. and Koeppl, H. (2021).
Approximately solving mean field games via entropy-regularized deep reinforcement learning.
In *International Conference on Artificial Intelligence and Statistics*, pages 1909–1917. PMLR.

[de Raynal and Trillos, 2015] de Raynal, P. C. and Trillos, C. G. (2015).
A cubature based algorithm to solve decoupled mckean–vlasov forward–backward stochastic differential equations.
Stochastic Processes and their Applications, 125(6):2206–2255.

[Fortin and Glowinski, 1983] Fortin, M. and Glowinski, R. (1983).
Augmented Lagrangian methods: applications to the numerical solution of boundary-value problems.
North-Holland.

[Gomes and Saúde, 2018] Gomes, D. A. and Saúde, J. (2018).
Numerical methods for finite-state mean-field games satisfying a monotonicity condition.
Applied Mathematics & Optimization.

[Gomes and Yang, 2020] Gomes, D. A. and Yang, X. (2020).
The hessian riemannian flow and newton's method for effective hamiltonians and mather measures.
ESAIM: Mathematical Modelling and Numerical Analysis, 54(6):1883–1915.

References VII

[Lasry and Lions, 2007] Lasry, J.-M. and Lions, P.-L. (2007).
Mean field games.
Jpn. J. Math., 2(1):229–260.

[Laurière, 2021] Laurière, M. (2021).
Numerical methods for mean field games and mean field type control.
arXiv preprint arXiv:2106.06231.

[Laurière and Pironneau, 2016] Laurière, M. and Pironneau, O. (2016).
Dynamic programming for mean-field type control.
J. Optim. Theory Appl., 169(3):902–924.

[Laurière et al., 2023] Laurière, M., Song, J., and Tang, Q. (2023).
Policy iteration method for time-dependent mean field games systems with non-separable hamiltonians.
Applied Mathematics & Optimization, 87(2):17.

[Lavigne and Pfeiffer, 2022] Lavigne, P. and Pfeiffer, L. (2022).
Generalized conditional gradient and learning in potential mean field games.
arXiv preprint arXiv:2209.12772.

[Pagès, 2018] Pagès, G. (2018).
Numerical probability.
In *Universitext*. Springer.

References VIII

[Pfeiffer, 2016] Pfeiffer, L. (2016).
Numerical methods for mean-field type optimal control problems.
Pure Appl. Funct. Anal., 1(4):629–655.

[Porretta and Zuazua, 2013] Porretta, A. and Zuazua, E. (2013).
Long time versus steady state optimal control.
SIAM J. Control Optim., 51(6):4242–4273.

[Tang and Song, 2022] Tang, Q. and Song, J. (2022).
Learning optimal policies in potential mean field games: Smoothed policy iteration algorithms.
arXiv preprint arXiv:2212.04791.