Numerical Methods for Mean Field Games

Lecture 2
Classical Numerical Methods – Part I
Linear-Quadratic MFGs

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Outline

1. Introduction

2. Linear-Quadratic Setting

3. Algorithms

4. Preview of numerical schemes for the PDE system

5. Conclusion

Continuous time, continuous space MFG

- Time horizon $T < +\infty$, $t \in [0, T]$
- Player's control (deterministic) α_t, typically:
  - closed-loop Markovian: $\alpha_t = \alpha(t, X_t)$
  - open-loop: $\alpha_t = \alpha(t, \omega)$ progressively measurable
- Player's dynamics:

$$dX_t = b(t, X_t, \alpha_t, m_t)dt + \sigma dW_t, \qquad X_0 \sim m_0$$

- Population dynamics: Kolmogorov-Fokker-Planck equation

$$\partial_t m(t,x) - \frac{\sigma^2}{2}\Delta m(t,x) + \mathrm{div}(b(t,x,\alpha(t,x))m(t,x)) = 0, \qquad m_{|t=0} = m_0$$

- To stress the dependence on the control, we will sometimes write X^α and m^α.

Continuous time, continuous space MFG

Cost: dependence on the mean field

- non-local (typically “regularizing” operator)

$$f(t, X_t, \alpha_t, m_t)$$

- local (if the population distribution has a density, still denoted by m)

$$f(t, X_t, \alpha_t, m(t, X_t))$$

HJB equation

- Hamiltonian:

$$H(x, m, p) = \max_a -L(x, a, m, p), \quad L(x, a, m, p) = f(x, a, m) + b(x, a, m) \cdot p$$

- Hamilton-Jacobi-Bellman equation, given the mean field flow:

$$\begin{cases} -\partial_t u(t,x) - \frac{\sigma^2}{2}\Delta u(t,x) + H(x, m(t), \nabla u(t,x))) = 0, \\ u(T,x) = g(x, m(T)) \end{cases}$$

- Recovering the optimal control: optimizer of the Hamiltonian

- Unique action minimizes H under strict convexity assumptions

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- Recovering the optimal control: optimizer of the Hamiltonian
- Unique action minimizes H under strict convexity assumptions
- **Warning:** Another convention: $H(x,m,p) = \min_a L(x,a,m,p) \Rightarrow -H$ in HJB.

Forward-backward PDE system for MFG

The equilibrium control minimizes the Hamiltonian:

$$\hat{\alpha}(t,x) = \operatorname*{argmax}_a -L(t,x,a,\nabla u(t,x))$$

where (m,u) solve the forward-backward PDE system:

- Forward equation for the mean field:

$$\begin{cases} \partial_t m(t,x) - \frac{\sigma^2}{2}\Delta m(t,x) + \mathrm{div}(m(t,x)H_p(x,m(t),\nabla u(t,x))) = 0, \\ m(0,x) = m_0(x) \end{cases}$$

- Backward equation for the value function:

$$\begin{cases} -\partial_t u(t,x) - \frac{\sigma^2}{2}\Delta u(t,x) + H(x,m(t),\nabla u(t,x))) = 0, \\ u(T,x) = g(x,m(T)) \end{cases}$$

Challenge: We cannot (fully) solve one equation before the other!

Exercises

Exercise

For the following drift and running cost functions ($d = 1$ to simplicity), write the KFP equation, the Hamiltonian and the HJB equation:

- Linear-quadratic (LQ):

$$b(x,a,m) = Ax+Ba+\bar{A}\bar{m}^2, f(x,a,m) = Qx^2+Ra^2+\bar{Q}\bar{m}^2, g(x,m) = Q_T x^2+\bar{Q}_T\bar{m}^2$$

  with $\bar{m} = \int \xi m(\xi) d\xi$
- Congestion: $b(x,a,m) = a$, $f(x,a,m) = m(x)|a|^2$
- Aversion: $b(x,a,m) = a$, $f(x,a,m) = |a|^2 + m(x)$

Exercise

Derive optimality conditions for the social optimum problem.

Social optimum: Mean Field Control

The social optimum problem is referred to as

- mean field (type) control
- control of McKean-Vlasov (MKV) dynamics

Definition (Mean field control (MFC) problem)

α^* is a solution to the MFC problem if it minimizes

$$J^{MFC}(\alpha) = \mathbb{E}\left[\int_0^T f(X_t^\alpha, \alpha_t, m_t^\alpha)dt + g(X_T^\alpha, m_T^\alpha)\right].$$

Main difference with MFG: here not only X but m too is controlled by α.

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Main difference with MFG: here not only X but m too is controlled by α.

Optimality conditions? Several approaches:

- Dynamic programming value function depending on m; value function V
- Calculus of variations taking m as a state; adjoint state u
- Pontryagin's maximum principle for the (MKV process) X; adjoint state Y

Forward-backward PDE system for MFC

Approach by calculus of variations, assuming that X has a density in L^2. The optimal control minimizes the Hamiltonian:

$$\alpha^*(t,x) = \underset{a}{\operatorname{argmax}} -L(t,x,a,\nabla u(t,x))$$

where (m,u) solve the forward-backward PDE system:

- Forward equation for the mean field:

$$\begin{cases} \partial_t m(t,x) - \frac{\sigma^2}{2}\Delta m(t,x) + \operatorname{div}(m(t,x) H_p(x, m(t), \nabla u(t,x))) = 0, \\ m(0,x) = m_0(x) \end{cases}$$

- Backward equation for the ~~value function~~ adjoint state:

$$\begin{cases} -\partial_t u(t,x) - \frac{\sigma^2}{2}\Delta u(t,x) + H(x, m(t), \nabla u(t,x))) \\ \qquad\qquad + \displaystyle\int \partial_m H(\xi, m(t), \nabla u(t,\xi))(x) m(t,\xi) d\xi = 0, \\ u(T,x) = g(x, m(T)) + \int \partial_m g(\xi, m(T))(x) m(t,\xi) d\xi \end{cases}$$

where $\partial_m H$ denotes the derivative wrt m, so that for a differentiable $\varphi : L^2(\mathbb{R}^d) \to \mathbb{R}$,

$$\frac{d}{d\theta}\varphi(m + \theta\tilde{m})\Big|_{\theta=0} = \int \partial_m \varphi(m)(\xi)\tilde{m}(\xi) d\xi.$$

See e.g. [Bensoussan et al., 2013], Section 4.1.

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Linear-Quadratic (LQ) Setting

In this section, we are going to focus on the following example.

Example (Linear-Quadratic (LQ) Setting)

$$
\begin{aligned}
b(x,a,m) &= Ax + \bar{A}\bar{m} + Ba \\
f(x,a,m) &= \frac{1}{2}\left[x^\top Q x + (x - S\bar{m})^\top \bar{Q}(x - S\bar{m}) + a^\top C a\right] \\
g(x,m) &= \frac{1}{2}\left[x^\top Q_T x + (x - S_T\bar{m})^\top \bar{Q}_T(x - S_T\bar{m})\right] \\
\bar{m} &= \int \xi m(\xi) d\xi
\end{aligned}
$$

where $A, \bar{A}, \ldots$ are constant matrices of suitable dimensions.

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\bar{m} &= \int \xi m(\xi) d\xi
\end{aligned}
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where $A, \bar{A}, \ldots$ are constant matrices of suitable dimensions.

So:

- The interactions are only through the mean.
- The drift is linear in the state, the action and the mean.
- The costs are quadratic in these variables.

Key point: MFG equilibrium can be computed with ODEs. No need to solve PDEs.

HJB equation

For simplicity, consider the case $d = 1$.

Hamiltonian:

$$H(x, m, p) = \max_a -L(x, a, m, p), \quad L(x, a, m, p) = f(x, a, m) + b(x, a, m) \cdot p$$

Here

$$L(x, a, m, p) = \frac{1}{2}(Qx^2 + \bar{Q}(x - S\bar{m})^2 + Ca^2) + (Ax + \bar{A}\bar{m} + Ba)p$$

The optimal a satisfies (first order optimality condition):

$$Ca + Bp = 0 \Rightarrow a = -\frac{B}{C}p$$

So

$$\begin{aligned} H(x, m, p) &= -[\frac{1}{2}(Qx^2 + \bar{Q}(x - S\bar{m})^2 + \frac{B^2}{C}p^2) + (Ax + \bar{A}\bar{m} - \frac{B^2}{C}p)p] \\ &= -\frac{1}{2}[Qx^2 + \bar{Q}(x - S\bar{m})^2] - [Ax + \bar{A}\bar{m}]p + \frac{B^2}{2C}p^2 \end{aligned}$$

and $H_p(x, m, p) = -[Ax + \bar{A}\bar{m}] + \frac{B^2}{C}p$

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Hamilton-Jacobi-Bellman equation:

$$\begin{cases} -\partial_t u(t, x) - \frac{\sigma^2}{2}\Delta u(t, x) \\ \qquad -\frac{1}{2}[Qx^2 + \bar{Q}(x - S\bar{m}_t)^2] - [Ax + \bar{A}\bar{m}_t]\nabla u(t, x) + \frac{B^2}{2C}|\nabla u(t, x)|^2 = 0, \\ u(T, x) = Q_T x^2 + \bar{Q}_T(x - S\bar{m}(T))^2 \end{cases}$$

HJB equation: solution

Hamilton-Jacobi-Bellman equation:

$$\begin{cases} -\partial_t u(t,x) - \frac{\sigma^2}{2}\Delta u(t,x) \\ \qquad -\frac{1}{2}[Qx^2 + \bar{Q}(x - S\bar{m}_t)^2] - [Ax + \bar{A}\bar{m}_t]\nabla u(t,x) + \frac{B^2}{2C}|\nabla u(t,x)|^2 = 0, \\ u(T,x) = Q_T x^2 + \bar{Q}_T(x - S\bar{m}(T))^2 \end{cases}$$

First remark: The value function has a special form (ansatz):

$$u(t,x) = \frac{1}{2}p_t x^2 + r_t x + s_t,$$

with $p, r, s : [0,T] \to \mathbb{R}$ to be determined. We have:

- $\partial_t u(t,x) = \frac{1}{2}\dot{p}_t x^2 + \dot{r}_t x + \dot{s}_t$
- $\nabla u(t,x) = p_t x + r_t$, and $\Delta u(t,x) = p_t$

HJB equation: solution

Hamilton-Jacobi-Bellman equation:

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with $p, r, s : [0,T] \to \mathbb{R}$ to be determined. We have:

- $\partial_t u(t,x) = \frac{1}{2}\dot{p}_t x^2 + \dot{r}_t x + \dot{s}_t$
- $\nabla u(t,x) = p_t x + r_t$, and $\Delta u(t,x) = p_t$

Second remark: This equation depends on m only through $\bar{m}$. We do not need the full KFP equation

$$\partial_t m(t,x) - \frac{\sigma^2}{2}\Delta m(t,x) + \mathrm{div}(m(t,x)H_p(x, m(t), \nabla u(t,x))) = 0$$

but only the ODE for the mean, obtained by integrating the KFP:

$$\frac{d\bar{m}}{dt} - \int m(t,x)H_p(x, m(t), \nabla u(t,x))dx = 0,$$

Note: $\int m(t,x)H_p(x, m(t), \nabla u(t,x))dx = -[A\bar{m}_t + \bar{A}\bar{m}_t] + \frac{B^2}{C}[p_t\bar{m}_t + r_t]$

Forward-backward ODE system for MFG

Consequence: the MFG solution is given by:

$$
\begin{cases}
\text{Mean:} & \bar{m}^{\hat{\alpha}}_t = z_t, \\
\text{Control:} & \hat{\alpha}(t,x) = -\dfrac{B}{C}(p_t x + r_t), \\
\text{Value function:} & u(t,x) = \dfrac{1}{2} p_t x^2 + r_t x + s_t,
\end{cases}
$$

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Consequence: the MFG solution is given by:

$$\begin{cases} \text{Mean:} & \bar{m}_t^{\hat{\alpha}} = z_t, \\ \text{Control:} & \hat{\alpha}(t,x) = -\dfrac{B}{C}(p_t x + r_t), \\ \text{Value function:} & u(t,x) = \dfrac{1}{2}p_t x^2 + r_t x + s_t, \end{cases}$$

where (z, p, r, s) solve the following system of ordinary differential equations (ODEs):

$$\begin{cases} \dfrac{dz}{dt} = (A + \bar{A} - B^2 C^{-1} p_t) z_t - B^2 C^{-1} r_t, & z_0 = \bar{m}_0, \\ -\dfrac{dp}{dt} = 2A p_t - B^2 C^{-1} p_t^2 + Q + \bar{Q}, & p_T = Q_T + \bar{Q}_T, \\ -\dfrac{dr}{dt} = (A - B^2 C^{-1} p_t) r_t + (p_t \bar{A} - \bar{Q} S) z_t, & r_T = -\bar{Q}_T S_T z_T, \\ -\dfrac{ds}{dt} = \nu p_t - \dfrac{1}{2} B^2 C^{-1} r_t^2 + r_t \bar{A} z_t + \dfrac{1}{2} S^2 \bar{Q} z_t^2, & s_T = \dfrac{1}{2}\bar{Q}_T S_T^2 z_T^2. \end{cases}$$

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Key points:

- coupling between z and r
- **forward-backward** structure

LQ MFC

We can apply the same strategy to the MFC PDE system.
Recall:

$$H(x,m,p) = -\frac{1}{2}[Qx^2 + \bar{Q}(x - S\bar{m})^2] - [Ax + \bar{A}\bar{m}]p + \frac{B^2}{2C}p^2$$

So:

$$\begin{aligned}\frac{d}{d\theta}H(x, m + \theta\bar{\tilde{m}}, p)\Big|_{\theta=0} &= [\bar{Q}(x - S\bar{m})S\bar{\tilde{m}}] - [\bar{A}\bar{\tilde{m}}]p \\ &= \int \left[\bar{Q}(x - S\bar{m})S - \bar{A}p\right] \xi\tilde{m}(\xi)d\xi\end{aligned}$$

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Hence, by definition, $\partial_m H(x, m, p)(\xi) = [\bar{Q}(x - S\bar{m})S - \bar{A}p]\xi$, and thus (swap x and ξ)

$$\begin{aligned}\int \partial_m H(\xi, m, \nabla u(t, \xi))(x)m(\xi)d\xi &= \int [\bar{Q}(\xi - S\bar{m})S - \bar{A}\nabla u(t, \xi)]xm(\xi)d\xi \\ &= \left[\bar{Q}(S - S^2)\bar{m} - \bar{A}\int \nabla u(t, \xi)m(\xi)d\xi\right] x \\ &= \left[\bar{Q}(S - S^2)\bar{m} - \bar{A}(\check{p}_t\bar{m}_t + \check{r}_t)\right] x\end{aligned}$$

where we use an ansatz $u(t, x) = \frac{1}{2}\check{p}_t x^2 + \check{r}_t x + \check{s}_t$

Forward-backward ODE system for MFC

We obtain that the MFC optimum is given by:

$$
\begin{cases}
\text{Mean:} & \bar{m}_t^{\alpha^*} = \check{z}_t, \\
\text{Control:} & \alpha^*(t,x) = -\dfrac{B}{C}(\check{p}_t x + \check{r}_t), \\
\text{Value:} & J^{MFC}(\alpha^*) = \dfrac{1}{2}\check{p}_0(\sigma_0^2 + \bar{m}_0^2) + \check{r}_0\bar{m}_0 + \check{s}_0 + (1 - S_T)\bar{Q}_T S_T \check{z}_T^2 \\
& \qquad - \displaystyle\int_0^T \left[(\check{p}_t\check{z}_t + \check{r}_t)\bar{A}\check{z}_t - (1 - S_t)\bar{Q}S\check{z}_t^2\right] dt
\end{cases}
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where $(\check{z}, \check{p}, \check{r}, \check{s})$ solve the following system of ODEs:

$$\begin{cases} \frac{d\check{z}}{dt} = (A + \bar{A} - B^2C^{-1}\check{p}_t)\check{z}_t - B^2C^{-1}\check{r}_t, & \check{z}_0 = \bar{m}_0, \\ -\frac{d\check{p}}{dt} = 2A\check{p}_t - B^2C^{-1}\check{p}_t^2 + Q + \bar{Q}, & \check{p}_T = Q_T + \bar{Q}_T, \\ -\frac{d\check{r}}{dt} = (A + \bar{A} - B^2C^{-1}\check{p}_t)\check{r}_t + (2\check{p}_t\bar{A} - 2\bar{Q}S + \bar{Q}S^2)\check{z}_t, & \check{r}_T = (-2\bar{Q}_T S_T + \bar{Q}_T S_T^2)\check{z}_T, \\ -\frac{ds}{dt} = \nu\check{p}_t - \frac{1}{2}B^2C^{-1}\check{r}_t^2 + \check{r}_t\bar{A}\check{z}_t + \frac{1}{2}S^2\bar{Q}\check{z}_t^2, & \check{s}_T = \frac{1}{2}\bar{Q}_T S_T^2\check{z}_T^2. \end{cases}$$

Same system as for MFG, except for a few terms

Linear-Quadratic (LQ) Setting

Remarks:

- LQ models are useful because they have (almost) analytical solutions
- The above model is inspired by [Bensoussan et al., 2013], Chapter 6
- It is possible to have much more general LQ MFG models (see e.g., [Huang et al., 2006], [Barreiro-Gomez and Tembine, 2021], [Graber, 2016], . . .)
- Extension with common noise, see e.g. [Carmona et al., 2015, Graber, 2016]
- In some cases, using a different ansatz, the equations can be decoupled, see [Malhamé and Graves, 2020] (AMS'20 minicourse lecture notes)
- The equation for p can be solved by itself; sometimes it has an analytical solution, see e.g. [Carmona and Delarue, 2018], p. 110
- The equation for s can be solved by itself after computing p, z, r
- In the sequel, we focus on computing z and r

Outline

1. Introduction

2. Linear-Quadratic Setting

3. Algorithms
   - Pure Fixed Point Iterations (Banach-Picard)
   - Damped Fixed Point Iterations
   - Fictitious Play
   - Shooting Method
   - Newton Method
   - MFC & Price of Anarchy

4. Preview of numerical schemes for the PDE system

5. Conclusion

Time Discretization

The experiments that follow are borrowed from [Laurière, 2021], Section 2.

In practice, the following algorithms are implemented a discrete time system:

- We focus on the coupled system for (z, r)
- Uniform grid on $[0, T]$, step Δt, $t_n = n \times \Delta t, n = 0, \ldots, N_T$
- Approximate $z, r : [0, T] \to \mathbb{R}$ by vectors $Z, R \in \mathbb{R}^{N_T+1}$

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- Approximate $z, r : [0, T] \to \mathbb{R}$ by vectors $Z, R \in \mathbb{R}^{N_T+1}$
- Discrete ODE system:

$$\begin{cases} \dfrac{Z^{n+1} - Z^n}{\Delta t} = (A + \bar{A} - B^2 C^{-1} P^n) Z^{n+1} - B^2 C^{-1} R^n, \\ Z^0 = \bar{m}_0, \\ -\dfrac{R^{n+1} - R^n}{\Delta t} = (A - B^2 C^{-1} P^n) R^n + (P^n \bar{A} - \bar{Q} S) Z^{n+1}, \\ R^{N_T} = -\bar{Q}_T S_T Z^{N_T}. \end{cases}$$

- To alleviate the notation, most of the algorithms are described using the ODEs

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3. Algorithms
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Algorithm 1: Banach-Picard Iterations

Algorithm: Fixed-point iterations

Input: Initial guess $(\tilde{z}, \tilde{r})$; number of iterations $\mathtt{K}$

Output: Approximation of $(\hat{z}, \hat{r})$

Initialize $z^{(0)} = \tilde{z}, r^{(0)} = \tilde{r}$

for $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**

Let $r^{(\mathtt{k}+1)}$ be the solution to:

$$-\frac{dr}{dt} = (A - P_t B^2 C^{-1}) r_t + (P_t \bar{A} - \bar{Q} S) z_t^{(\mathtt{k})}, \qquad r_T = -\bar{Q}_T S_T z_T^{(\mathtt{k})}$$

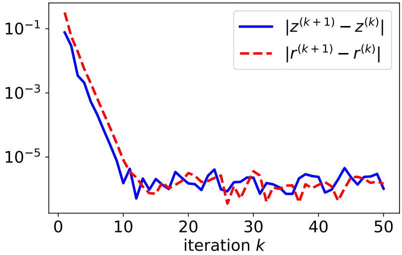
Let $z^{(\mathtt{k}+1)}$ be the solution to:

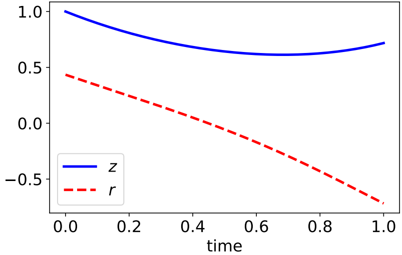
$$\frac{dz}{dt} = (A + \bar{A} - B^2 C^{-1}) z_t - B^2 C^{-1} r_t^{(\mathtt{k}+1)}, \qquad z_0 = \bar{m}_0$$

return $(z^{(\mathtt{K})}, r^{(\mathtt{K})})$

Algorithm 1: Banach-Picard Iterations – Illustration 1

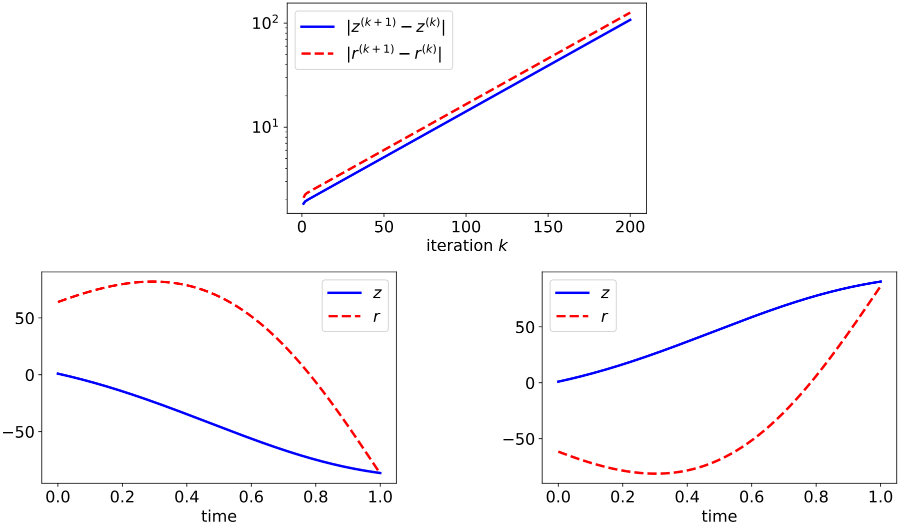
Test case 1 (for the values of $A, \bar{A}, \dots$, see [Laurière, 2021], Section 2)





Algorithm 1: Banach-Picard Iterations – Illustration 2

Test case 2 (for the values of $A, \bar{A}, \ldots$, see [Laurière, 2021], Section 2)



Algorithm 1: Banach-Picard Iterations – Remarks

- In fact this algorithm is related to a proof technique for the existence and uniqueness of a Nash equilibrium (see lecture 1)
- See e.g. [Huang et al., 2006]
- Here, the approach converges if $z^{(k)} \mapsto r^{(k)} \mapsto z^{(k+1)}$ is a strict contraction
- Typically true if T is small enough or the coefficients are small enough
- Otherwise, it is common to see non-convergence
- Can we “fix” this algorithm?

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Banach-Picard Iterations with Damping

Algorithm: Fixed-point iterations with damping

Input: Initial guess $(\tilde{z}, \tilde{r})$; damping $\delta \in [0, 1)$; number of iterations $\mathtt{K}$

Output: Approximation of $(\hat{z}, \hat{r})$

Initialize $z^{(0)} = \tilde{z}^{(0)} = \tilde{z}, r^{(0)} = \tilde{r}$

for $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**

Let $r^{(\mathtt{k}+1)}$ be the solution to:

$$-\frac{dr}{dt} = (A - P_t B^2 C^{-1}) r_t + (P_t \bar{A} - \bar{Q} S) \tilde{z}_t^{(\mathtt{k})}, \qquad r_T = -\bar{Q}_T S_T \tilde{z}_T^{(\mathtt{k})}$$

Let $z^{(\mathtt{k}+1)}$ be the solution to:

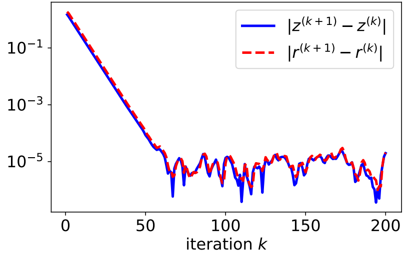
$$\frac{dz}{dt} = (A + \bar{A} - B^2 C^{-1}) z_t - B^2 C^{-1} r_t^{(\mathtt{k}+1)}, \qquad z_0 = \bar{m}_0$$

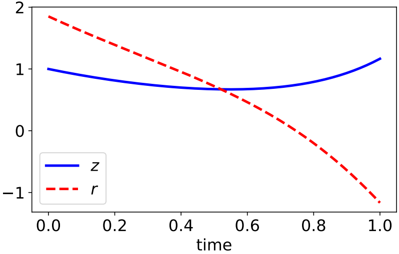
Let $\tilde{z}^{(\mathtt{k}+1)} = \delta \tilde{z}^{(\mathtt{k})} + (1 - \delta) z^{(\mathtt{k}+1)}$

return $(z^{(\mathtt{K})}, r^{(\mathtt{K})})$

Algorithm 1': Banach-Picard Iterations with Damping – Illustration 1

Test case 2
Damping = 0.1

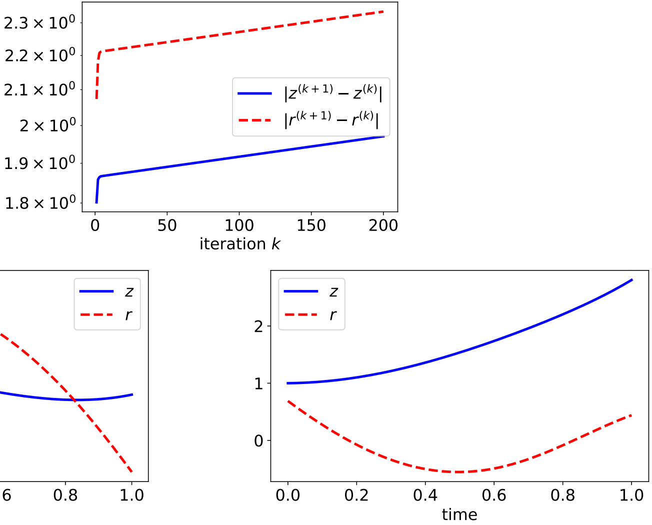




Algorithm 1': Banach-Picard Iterations with Damping – Illustration 2

Test case 2

Damping = 0.01



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Algorithm 2: Fictitious Play

- Introduced by [Brown, 1951], [Robinson, 1951]

- Converge proof for several classes of games

- In the MFG setting, FP has been introduced in [Cardaliaguet and Hadikhanloo, 2017], with a proof of convergence for potential MFGs; then extended to MFGs with monotonicity [Hadikhanloo, 2018], [Hadikhanloo and Silva, 2019]

- Related to learning in MFGs: [Perrin et al., 2020] for continuous-time FP under monotonicity condition, [Geist et al., 2022, Lavigne and Pfeiffer, 2022] for discrete-time FP in some potential MFGs; In linear-quadratic MFGs, a rate of convergence has been obtained by [Delarue and Vasileiadis, 2021]

- See Lecture 8 for more details on FP with RL for MFGs

Algorithm 2: Fictitious Play

Algorithm: Fictitious Play

Input: Initial guess $(\tilde{z}, \tilde{r})$; number of iterations $\mathtt{K}$

Output: Approximation of $(\hat{z}, \hat{r})$

Initialize $z^{(0)} = \tilde{z}^{(0)} = \tilde{z}, r^{(0)} = \tilde{r}$

for $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**

Let $r^{(\mathtt{k}+1)}$ be the solution to:

$$-\frac{dr}{dt} = (A - P_t B^2 C^{-1}) r_t + (P_t \bar{A} - \bar{Q} S) \tilde{z}_t^{(\mathtt{k})}, \qquad r_T = -\bar{Q}_T S_T \tilde{z}_T^{(\mathtt{k})}$$

Let $z^{(\mathtt{k}+1)}$ be the solution to:

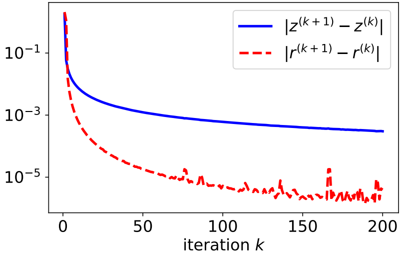
$$\frac{dz}{dt} = (A + \bar{A} - B^2 C^{-1}) z_t - B^2 C^{-1} r_t^{(\mathtt{k}+1)}, \qquad z_0 = \bar{m}_0$$

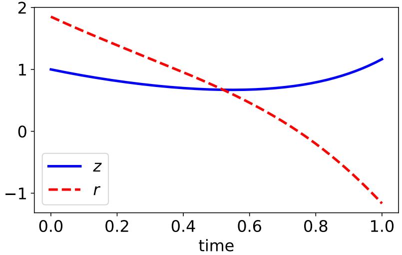
Let $\tilde{z}^{(\mathtt{k}+1)} = \frac{\mathtt{k}}{\mathtt{k}+1} \tilde{z}^{(\mathtt{k})} + \frac{1}{\mathtt{k}+1} z^{(\mathtt{k}+1)}$

return $(z^{(\mathtt{K})}, r^{(\mathtt{K})})$

Algorithm 2: Fictitious Play – Illustration

Test case 2





Algorithms 1, 1' & 2: Common Framework

Algorithm: General fixed-point iterations

Input: Initial guess $(\tilde{z}, \tilde{r})$; damping $\delta(\cdot)$; number of iterations $\mathtt{K}$

Output: Approximation of $(\hat{z}, \hat{r})$

Initialize $z^{(0)} = \tilde{z}^{(0)} = \tilde{z}, r^{(0)} = \tilde{r}$

for $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**

Let $r^{(\mathtt{k}+1)}$ be the solution to:

$$-\frac{dr}{dt} = (A - P_t B^2 C^{-1}) r_t + (P_t \bar{A} - \bar{Q} S) \tilde{z}_t^{(\mathtt{k})}, \qquad r_T = -\bar{Q}_T S_T \tilde{z}_T^{(\mathtt{k})}$$

Let $z^{(\mathtt{k}+1)}$ be the solution to:

$$\frac{dz}{dt} = (A + \bar{A} - B^2 C^{-1}) z_t - B^2 C^{-1} r_t^{(\mathtt{k}+1)}, \qquad z_0 = \bar{m}_0$$

Let $\tilde{z}^{(\mathtt{k}+1)} = \delta(\mathtt{k}) \tilde{z}^{(\mathtt{k})} + (1 - \delta(\mathtt{k})) z^{(\mathtt{k}+1)}$

return $(z^{(\mathtt{K})}, r^{(\mathtt{K})})$

Pure fixed point and Fictitious play are special cases

Remark: Could put the damping on r instead of z.

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Algorithm 3: Shooting Method

- Intuition: *instead of solving a backward equation, choose a starting point and try to shoot for the right terminal point*
- Concretely: replace the forward-backward system

$$\begin{cases} \dfrac{dz}{dt} = (A + \bar{A} - B^2 C^{-1} p_t) z_t - B^2 C^{-1} r_t, & z_0 = \bar{m}_0, \\ -\dfrac{dr}{dt} = (A - B^2 C^{-1} p_t) r_t + (p_t \bar{A} - \bar{Q} S) z_t, & r_T = -\bar{Q}_T S_T z_T \end{cases}$$

by the forward-forward system

$$\begin{cases} \dfrac{d\zeta}{dt} = (A + \bar{A} - B^2 C^{-1} p_t) \zeta_t - B^2 C^{-1} \rho_t, & z_0 = \bar{m}_0, \\ -\dfrac{d\rho}{dt} = (A - B^2 C^{-1} p_t) \rho_t + (p_t \bar{A} - \bar{Q} S) \zeta_t, & \rho_0 = \text{chosen} \end{cases}$$

and try to ensure: $\rho_T = -\bar{Q}_T S_T \zeta_T$

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Algorithm 4: Newton Method – Intuition

Newton method in dimension 1:

- Look for x^* such that: $\mathtt{f}(x^*) = 0$
- Start from initial guess x_0
- Repeat:

$$x_{k+1} = x_k - \frac{\mathtt{f}(x_k)}{\mathtt{f}'(x_k)}$$

Algorithm 4: Newton Method – Intuition

Newton method in dimension 1:

- Look for x^* such that: $\mathtt{f}(x^*) = 0$
- Start from initial guess x_0
- Repeat:

$$x_{k+1} = x_k - \frac{\mathtt{f}(x_k)}{\mathtt{f}'(x_k)}$$

- In high dimension, we avoid computing the inverse of $\mathtt{f}'(x_k)$
- $x_{k+1} = x_k + \tilde{x}_k$, where $\tilde{x}_k$ solves:

$$\mathtt{f}'(x_k)\,\tilde{x}_k = -\mathtt{f}(x_k)$$

which boils down to solving a linear system

Algorithm 4: Newton Method – Implementation

- Recast the problem:

$$(Z, R) \text{ solve forward-forward discrete system} \Leftrightarrow \mathcal{F}(Z, R) = 0$$

- $\mathcal{F}$ takes into account the initial and terminal conditions

- $D\mathcal{F} =$ differential of this operator

Exercise

Express $\mathcal{F}$ and $D\mathcal{F}$.

Algorithm 4: Newton Method – Implementation

Algorithm: Newton Iterations

Input: Initial guess $(\tilde{Z}, \tilde{R})$; number of iterations $\mathtt{K}$
Output: Approximation of $(\hat{z}, \hat{r})$

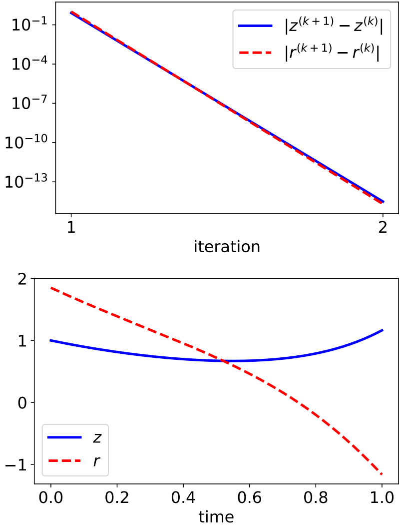
Initialize $(Z^{(0)}, R^{(0)}) = (\tilde{Z}, \tilde{R})$
for $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**
- Let $(\tilde{Z}^{(\mathtt{k}+1)}, \tilde{R}^{(\mathtt{k}+1)})$ solve

$$D\mathcal{F}(Z^{(\mathtt{k})}, R^{(\mathtt{k})})(\tilde{Z}^{(\mathtt{k}+1)}, \tilde{R}^{(\mathtt{k}+1)}) = -\mathcal{F}(Z^{(\mathtt{k})}, R^{(\mathtt{k})})$$

- Let $(Z^{(\mathtt{k}+1)}, R^{(\mathtt{k}+1)}) = (\tilde{Z}^{(\mathtt{k}+1)}, \tilde{R}^{(\mathtt{k}+1)}) + (Z^{(\mathtt{k})}, R^{(\mathtt{k})})$
- **return** $(Z^{(\mathtt{K})}, R^{(\mathtt{K})})$

Algorithm 4: Newton Method – Illustration

Test case 2



Algorithm 4: Newton Method – Explanation

- Reminder: Discrete ODE system:

$$\begin{cases} \dfrac{Z^{n+1} - Z^n}{\Delta t} = (A + \bar{A} - B^2 C^{-1} P^n) Z^{n+1} - B^2 C^{-1} R^n, \\ Z^0 = \bar{m}_0, \\ -\dfrac{R^{n+1} - R^n}{\Delta t} = (A - B^2 C^{-1} P^n) R^n + (P^n \bar{A} - \bar{Q} S) Z^{n+1}, \\ R^{N_T} = -\bar{Q}_T S_T Z^{N_T}. \end{cases}$$

Algorithm 4: Newton Method – Explanation

- Reminder: Discrete ODE system:

$$\begin{cases} \dfrac{Z^{n+1} - Z^n}{\Delta t} = (A + \bar{A} - B^2 C^{-1} P^n) Z^{n+1} - B^2 C^{-1} R^n, \\ Z^0 = \bar{m}_0, \\ -\dfrac{R^{n+1} - R^n}{\Delta t} = (A - B^2 C^{-1} P^n) R^n + (P^n \bar{A} - \bar{Q} S) Z^{n+1}, \\ R^{N_T} = -\bar{Q}_T S_T Z^{N_T}. \end{cases}$$

- Can be rewritten as a linear system:

$$\mathbf{M} \begin{pmatrix} Z \\ R \end{pmatrix} + \mathbf{B} = 0$$

- Newton's method solves a linear system in a single iteration.

- In hindsight: we did not need any of the previous methods! We could have simply used a solver for linear systems of equations.

- The methods were applied in the LQ setting only for pedagogical purposes.

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Price of Anarchy – Illustration

- Introduced by [Koutsoupias and Papadimitriou, 1999]

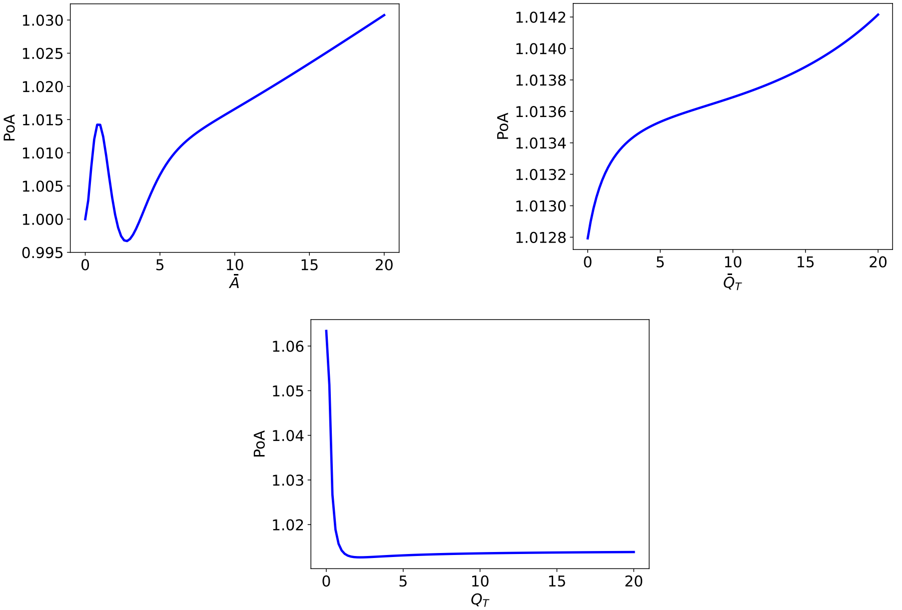
- Extension to MFGs: assuming there exist a unique MFG equilibrium $(\hat{\alpha}, \hat{m})$ and a unique MFC optimum α^*

$$PoA = \frac{J^{MFG}(\hat{\alpha}; \hat{m})}{J^{MFC}(\alpha^*)}$$

- Ratio of the expected cost for a typical player in the MFG by her expected cost in the MFC

- See in particular [Carmona et al., 2019] for explicit computations in the LQ case

Price of Anarchy – Illustration



Sample code

Code

Sample code to illustrate: IPython notebook

https://colab.research.google.com/drive/1a0TKAnc1Ng5LQ36ZqBPTToJX6oOkoSkd?usp=sharing

- ODE system for Linear-quadratic MFG
- Solved by fixed point, damped fixed point, fictitious play and Newton's method

Exercises

Exercise

Modify the previous code to solve the ODE system for MFC.

Compute the price of anarchy.

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MFG PDE System

Recall the MFG PDE system:

$$\begin{cases} 0 = -\dfrac{\partial u}{\partial t}(t,x) - \nu\Delta u(t,x) + H(x, m(t,\cdot), \nabla u(t,x)), \\ 0 = \dfrac{\partial m}{\partial t}(t,x) - \nu\Delta m(t,x) - \operatorname{div}\left(m(t,\cdot)\partial_p H(\cdot, m(t), \nabla u(t,\cdot))\right)(x), \\ \quad u(T,x) = g(x, m(T,\cdot)), \qquad m(0,x) = m_0(x) \end{cases}$$

Goals:

1. introduce a discrete version of this system $\rightarrow$ numerical scheme
2. solve it numerically $\rightarrow$ algorithm

Properties

For (1): some desirable properties:

- **Mass** and **positivity** of distribution: $\int_{\mathcal{S}} m(t,x)dx = 1$, $m \geq 0$
- **Convergence** of discrete solution to continuous solution as mesh step $\to 0$

Properties

For (1): some desirable properties:

- **Mass** and **positivity** of distribution: $\int_{\mathcal{S}} m(t,x)dx = 1$, $m \geq 0$
- **Convergence** of discrete solution to continuous solution as mesh step $\to 0$
- The KFP equation is the **adjoint** of the linearized HJB equation
- Link with **optimality condition** of a discrete problem

$\Rightarrow$ Needs a careful discretization

Properties

For (1): some desirable properties:

- **Mass** and **positivity** of distribution: $\int_{\mathcal{S}} m(t,x)dx = 1$, $m \geq 0$
- **Convergence** of discrete solution to continuous solution as mesh step $\to 0$
- The KFP equation is the **adjoint** of the linearized HJB equation
- Link with **optimality condition** of a discrete problem

$\Rightarrow$ Needs a careful discretization

For (2): Once we have a discrete system, how can we compute its solution?

Two Numerical Schemes

Numerical schemes: We are going to illustrate two approaches:

1. Finite difference scheme introduced in [Achdou and Capuzzo-Dolcetta, 2010]
2. Semi-Lagrangian scheme introduced in [Carlini and Silva, 2014]

There are other options such as finite elements, see e.g. [Benamou and Carlier, 2015, Andreev, 2017].

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Summary

1. Linear-Quadratic MFG and MFC
2. Forward-backward ODE system
3. Several algorithms

Remarks:

1. In the LQ case, these algorithms are just for pedagogical purposes
2. But analogous algorithms can be useful for finite-state MFGs
3. Similarly for continuous-space MFGs up to space-discretization

Thank you for your attention

Questions?

Feel free to reach out: `mathieu.lauriere@nyu.edu`

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