Mean Field Games: Numerical Methods and Applications in Machine Learning

Part 3: Numerical Schemes for MF PDE Systems

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https://mlauriere.github.io/teaching/MFG-PKU-3.pdf

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RECAP

Outline

MFG PDE System

Goal: **(1)** introduce and **(2)** solve a discrete version of the MFG PDE system:

$$\begin{cases} 0 = -\dfrac{\partial u}{\partial t}(t,x) - \nu \Delta u(t,x) + H(x, m(t,\cdot), \nabla u(t,x)), \\ 0 = \dfrac{\partial m}{\partial t}(t,x) - \nu \Delta m(t,x) - \operatorname{div}\left(m(t,\cdot)\partial_p H(\cdot, m(t), \nabla u(t,\cdot))\right)(x), \\ \;u(T,x) = g(x, m(T,\cdot)), \qquad m(0,x) = m_0(x) \end{cases}$$

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Desirable properties for (1):

- **Mass** and **positivity** of distribution: $\int_{\mathcal{S}} m(t,x)dx = 1$, $m \geq 0$
- **Convergence** of discrete solution to continuous solution as mesh step $\to 0$

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For (2): Once we have a discrete system, how can we compute its solution?

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Discretization

Semi-implicit finite difference scheme introduced by [Achdou, Capuzzo-Dolcetta][1]
Discretization:

- For simplicity we consider the domain $\mathbb{T}$ = one-dimensional (unit) torus.
- Let $\nu = \sigma^2/2$.
- We consider N_h and N_T steps respectively in space and time.
- Let $h = 1/N_h$ and $\Delta t = T/N_T$. Let $\mathbb{T}_h$ = discretized torus.
- We approximate $m_0(x_i)$ by ρ_i^0 such that $h\sum_i \rho_i^0 = 1$.

[1] Achdou, Y., & Capuzzo-Dolcetta, I. (2010). Mean field games: numerical methods. *SIAM Journal on Numerical Analysis*, 48(3), 1136-1162.

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Then we introduce the following **discrete operators** : for $\varphi \in \mathbb{R}^{N_T+1}$ and $\psi \in \mathbb{R}^{N_h}$

- **time derivative** : $$(D_t\varphi)^n := \frac{\varphi^{n+1} - \varphi^n}{\Delta t}, \qquad 0 \le n \le N_T - 1$$
- **Laplacian** : $$(\Delta_h\psi)_i := -\frac{1}{h^2}\left(2\psi_i - \psi_{i+1} - \psi_{i-1}\right), \qquad 0 \le i \le N_h$$
- **partial derivative** : $$(D_h\psi)_i := \frac{\psi_{i+1} - \psi_i}{h}, \qquad 0 \le i \le N_h$$
- **gradient** : $$[\nabla_h\psi]_i := \left((D_h\psi)_i, (D_h\psi)_{i-1}\right), \qquad 0 \le i \le N_h$$

[1] Achdou, Y., & Capuzzo-Dolcetta, I. (2010). Mean field games: numerical methods. *SIAM Journal on Numerical Analysis*, 48(3), 1136-1162.

Discrete Hamiltonian

For simplicity, we assume that the drift b and the costs f and g are of the form

$$b(x, m, v) = v, \qquad f(x, m, v) = L(x, v) + \mathtt{f}_0(x, m), \qquad g(x, m) = \mathtt{g}_0(x, m).$$

where $x \in \mathbb{R}^d, v \in \mathbb{R}^d, m \in \mathbb{R}_+$. Then

$$H(x, m, p) = \max_v \{-L(x, v) - \langle v, p \rangle\} - \mathtt{f}_0(x, m) = H_0(x, p) - \mathtt{f}_0(x, m)$$

where H_0 is the convex conjugate (also denoted L^*) of L with respect to v:

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Discrete Hamiltonian: $(x, p_1, p_2) \mapsto \tilde{H}_0(x, p_1, p_2)$ satisfying:

- Monotonicity: decreasing w.r.t. p_1 and increasing w.r.t. p_2
- Consistency with H_0: for every x, p, $\tilde{H}_0(x, p, p) = H_0(x, p)$
- Differentiability: for every x, $(p_1, p_2) \mapsto \tilde{H}_0(x, p_1, p_2)$ is $\mathcal{C}^1$
- Convexity: for every x, $(p_1, p_2) \mapsto \tilde{H}_0(x, p_1, p_2)$ is convex

Example: if $H_0(x, p) = |p|^2$, a possible choice is $\tilde{H}_0(x, p_1, p_2) = ({p_1}^-)^2 + ({p_2}^+)^2$

Discrete HJB

Discrete solution: We replace $u, m : [0,T] \times \mathbb{T} \to \mathbb{R}$ by vectors

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The **HJB equation**

$$\begin{cases} \partial_t u(t,x) + \nu\Delta u(t,x) + H_0(x, \nabla u(t,x)) = \mathtt{f}_0(x, m(t,x)) \\ u(T,x) = \mathtt{g}_0(x, m(T,x)) \end{cases}$$

is discretized as:

$$\begin{cases} -(D_t U_i)^n - \nu(\Delta_h U^n)_i + \tilde{H}_0(x_i, [D_h U^n]_i) = \mathtt{f}_0(x_i, M_i^{n+1}) \\ U_i^{N_T} = \mathtt{g}_0(x_i, M_i^{N_T}) \end{cases}$$

Discrete KFP

The **KFP equation**

$$\partial_t m(t,x) - \nu \Delta m(t,x) + \operatorname{div}\Big(m(t,x) \partial_q H(x, m(t), \nabla u(t,x)) \Big) = 0, \qquad m(0,x) = m_0(x)$$

is discretized as

$$(D_t M_i)^n - \nu(\Delta_h M^{n+1})_i - \mathcal{T}_i(U^n, M^{n+1}) = 0, \qquad M_i^0 = \rho_i^0$$

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Here we use the **discrete transport operator** $\approx -\mathrm{div}(\dots)$

$$\mathcal{T}_i(U,M) := \frac{1}{h}\begin{pmatrix} M_i \partial_{p_1}\tilde{H}_0(x_i, [\nabla_h U]_i) - M_{i-1}\partial_{p_1}\tilde{H}_0(x_{i-1}, [\nabla_h U]_{i-1}) \\ + M_{i+1}\partial_{p_2}\tilde{H}_0(x_{i+1}, [\nabla_h U]_{i+1}) - M_i \partial_{p_2}\tilde{H}_0(x_i, [\nabla_h U]_i) \end{pmatrix}$$

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Intuition: weak formulation & integration by parts

$$\int_{\mathbb{T}} \mathrm{div}\,(m\partial_p H_0(x, \nabla u)) w = -\int_{\mathbb{T}} m \partial_p H_0(x, \nabla u) \cdot \nabla w$$

is discretized as

$$-h \sum_i \mathcal{T}_i(U,M) W_i = h \sum_i M_i \nabla_q \tilde{H}_0(x_i, [\nabla_h U]_i) \cdot [\nabla_h W]_i$$

Discrete System – Properties

Discrete forward-backward system:

$$\begin{cases} -(D_t U_i)^n - \nu(\Delta_h U^n)_i + \tilde{H}_0(x_i, [D_h U^n]_i) = \mathtt{f}_0(x_i, M_i^{n+1}), & \forall n \le N_T - 1 \\ (D_t M_i)^n - \nu(\Delta_h M^{n+1})_i - \mathcal{T}_i(U^n, M^{n+1}) = 0, & \forall n \le N_T - 1 \\ M_i^0 = \rho_i^0, \qquad U_i^{N_T} = \mathtt{g}_0(x_i, M_i^{N_T}), \qquad i = 0, \dots, N_h & \end{cases}$$

[2] Achdou, Y., & Capuzzo-Dolcetta, I. (2010). Mean field games: numerical methods. *SIAM Journal on Numerical Analysis*, 48(3), 1136-1162.

[3] Achdou, Y., Camilli, F., & Capuzzo-Dolcetta, I. (2012). Mean field games: numerical methods for the planning problem. *SIAM Journal on Control and Optimization*, 50(1), 77-109.

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This scheme enjoys many nice properties, among which:

- It yields a monotone scheme for the KFP equation: **mass** and **positivity** are preserved
- **Convergence** to classical solution if monotonicity [Achdou, & Camilli, Capuzzo-Dolcetta][2][3]
- Can sometimes be used to show existence of a **weak** solution [Achdou, Porretta][4]
- The discrete KFP operator is the **adjoint** of the linearized Bellman operator
- **Existence** and **uniqueness** result for the discrete system
- It corresponds to the **optimality condition** of a discrete optimization problem (details later)

[2] Achdou, Y., & Capuzzo-Dolcetta, I. (2010). Mean field games: numerical methods. *SIAM Journal on Numerical Analysis*, 48(3), 1136-1162.

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Outline

Algo 1: Fixed Point Iterations

Input: Initial guess $(\tilde{M}, \tilde{U})$; damping $\delta(\cdot)$; number of iterations $\mathtt{K}$
Output: Approximation of $(\hat{M}, \hat{U})$ solving the finite difference system

Initialize $M^{(0)} = \tilde{M}^{(0)} = \tilde{M}, U^{(0)} = \tilde{U}$
for $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**

Let $U^{(\mathtt{k}+1)}$ be the solution to:

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Remark: the HJB equation is **non-linear**

- **Idea 1:** replace $\tilde{H}_0(x_i, [D_h U^n]_i)$ by $\tilde{H}_0(x_i, [D_h U^{(\mathtt{k}),n}]_i)$

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- **Idea 2:** use non linear solver to find a zero of $\mathbb{R}^{N_h \times (N_T+1)} \ni U \mapsto \varphi(U) \in \mathbb{R}^{N_h \times N_T}$,

$$\varphi(U) = \Big(-(D_t U_i)^n - \nu(\Delta_h U^n)_i + \tilde{H}_0(x_i, [D_h U^n]_i) - \mathtt{f}_0(x_i, \tilde{M}_i^{(\mathtt{k}),n+1}) \Big)_{i=0,\ldots,N_h-1}^{n=0,\ldots,N_T-1}$$

Algo 2: Newton's Method for FD System

Idea: Directly look for a zero of $\varphi = (\varphi_{\mathcal{U}}, \varphi_{\mathcal{M}})^\top$ with $\varphi_{\mathcal{U}}$ and $\varphi_{\mathcal{M}}$ s.t.

$$\begin{cases} \varphi_{\mathcal{U}}(U, M) = 0 & \Leftrightarrow \; (U, M) \text{ solves discrete HJB equation} \\ \varphi_{\mathcal{M}}(U, M) = 0 & \Leftrightarrow \; (U, M) \text{ solves discrete KFP equation} \end{cases}$$

- Let $X^{(k)} = (U^{(k)}, M^{(k)})^\top$
- Iterate: $X^{(k+1)} = X^{(k)} - J_\varphi(X^{(k)})^{-1} \varphi(X^{(k)})$

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- Or rather: $J_\varphi(X^{(k)})Y = -\varphi(X^{(k)})$, then $X^{(k+1)} = Y + X^{(k)}$

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Key step: Solve a linear system of the form

$$\begin{pmatrix} A_{\mathcal{U},\mathcal{U}} & A_{\mathcal{U},\mathcal{M}} \\ A_{\mathcal{M},\mathcal{U}} & A_{\mathcal{M},\mathcal{M}} \end{pmatrix} \begin{pmatrix} U \\ M \end{pmatrix} = \begin{pmatrix} G_{\mathcal{U}} \\ G_{\mathcal{M}} \end{pmatrix}$$

where $A_{\mathcal{U},\mathcal{M}}(U, M) = \nabla_U \varphi_{\mathcal{M}}(U, M), \quad A_{\mathcal{U},\mathcal{U}}(U, M) = \nabla_U \varphi_{\mathcal{U}}(U, M), \quad \ldots$

Newton Method – Implementation

Linear system to be solved: $\begin{pmatrix} A_{\mathcal{U},\mathcal{U}} & A_{\mathcal{U},\mathcal{M}} \\ A_{\mathcal{M},\mathcal{U}} & A_{\mathcal{M},\mathcal{M}} \end{pmatrix} \begin{pmatrix} U \\ M \end{pmatrix} = \begin{pmatrix} G_{\mathcal{U}} \\ G_{\mathcal{M}} \end{pmatrix}$

Structure: $A_{\mathcal{U},\mathcal{M}}, A_{\mathcal{M},\mathcal{U}}$ are block-diagonal, $A_{\mathcal{U},\mathcal{U}} = A_{\mathcal{M},\mathcal{M}}^{\top}$, and

$$A_{\mathcal{U},\mathcal{U}} = \begin{pmatrix} D_1 & 0 & \cdots & \cdots & 0 \\ -\frac{1}{\Delta t}\mathrm{Id}_{N_h} & D_2 & \ddots & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 0 & -\frac{1}{\Delta t}\mathrm{Id}_{N_h} & D_{N_T} \end{pmatrix}$$

where D_n corresponds to the discrete operator

$$Z = (Z_{i,j})_{i,j} \mapsto \left(\frac{1}{\Delta t} Z_{i,j} - \nu(\Delta_h Z)_{i,j} + [\nabla_h Z]_{i,j} \cdot \nabla_p \tilde{H}_0(x_{i,j}, [\nabla_h U^{(k),n}]_{i,j}) \right)_{i,j}$$

[5] Achdou, Y. (2013). Finite difference methods for mean field games. In Hamilton-Jacobi equations: approximations, numerical analysis and applications (pp. 1-47). Springer, Berlin, Heidelberg.

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$$A_{\mathcal{U},\mathcal{U}} = \begin{pmatrix} D_1 & 0 & \cdots & \cdots & 0 \\ -\frac{1}{\Delta t}\mathrm{Id}_{N_h} & D_2 & \ddots & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 0 & -\frac{1}{\Delta t}\mathrm{Id}_{N_h} & D_{N_T} \end{pmatrix}$$

where D_n corresponds to the discrete operator

$$Z = (Z_{i,j})_{i,j} \mapsto \left(\frac{1}{\Delta t} Z_{i,j} - \nu(\Delta_h Z)_{i,j} + [\nabla_h Z]_{i,j} \cdot \nabla_p \tilde{H}_0(x_{i,j}, [\nabla_h U^{(k),n}]_{i,j}) \right)_{i,j}$$

Rem. Initial guess $(U^{(0)}, M^{(0)})$ is important for Newton's method

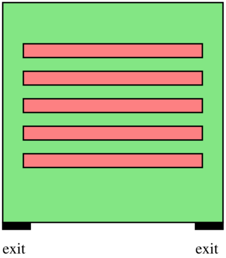
- Idea 1: initialize with the ergodic solution
- Idea 2: continuation method w.r.t. ν (converges more easily with a large viscosity)

See [Achdou'13][5] for more details.

[5] Achdou, Y. (2013). Finite difference methods for mean field games. In Hamilton-Jacobi equations: approximations, numerical analysis and applications (pp. 1-47). Springer, Berlin, Heidelberg.

Example: Exit of a Room – Distribution

Example: evacuation of a room with obstacles and congestion [Achdou, L.'15][6]



Geometry of the room

[6] Achdou, Y., & Laurière, M. (2015). On the system of partial differential equations arising in mean field type control. *Discrete & Continuous Dynamical Systems*, 35(9), 3879.

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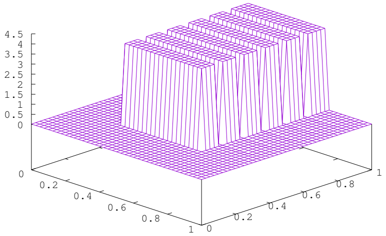

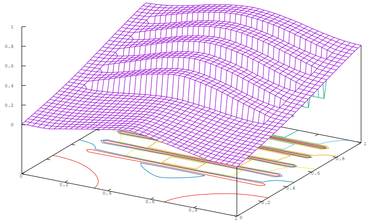

Initial density (left) and final cost (right)

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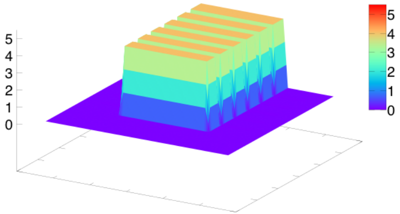

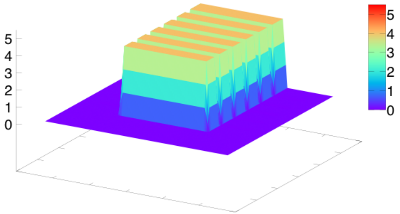

Density in MFGame (left) and MFControl (right)

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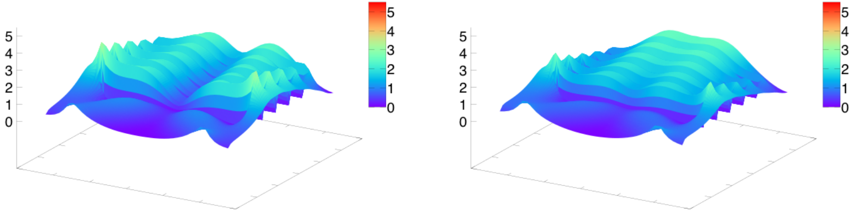

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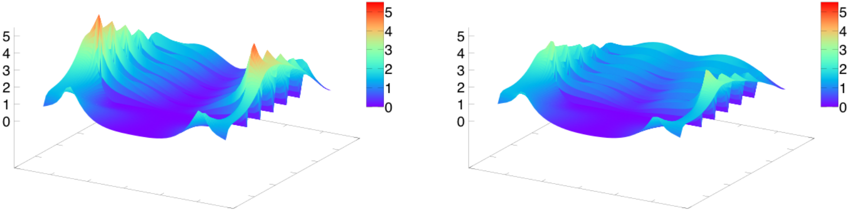

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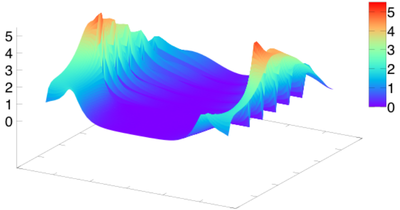

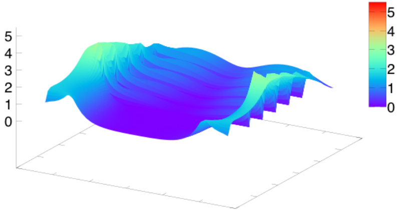

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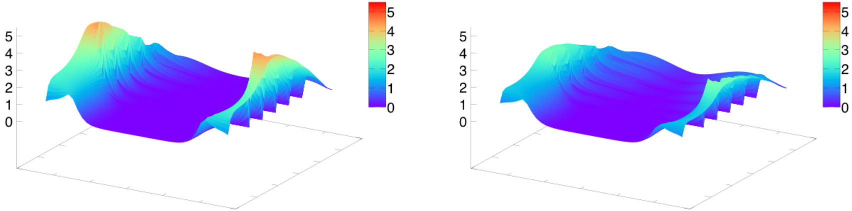

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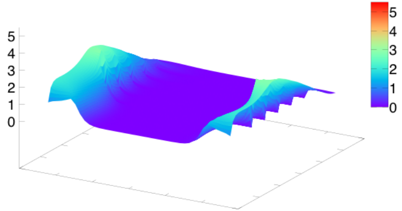

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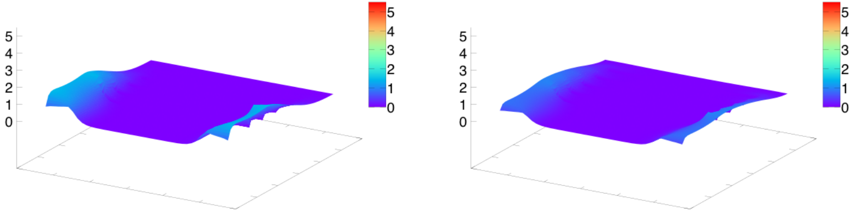

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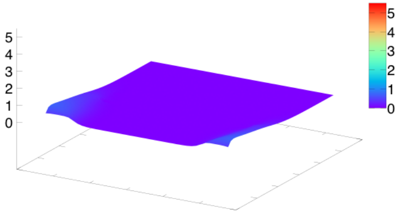

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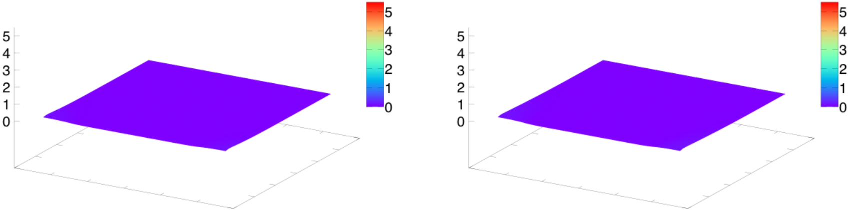

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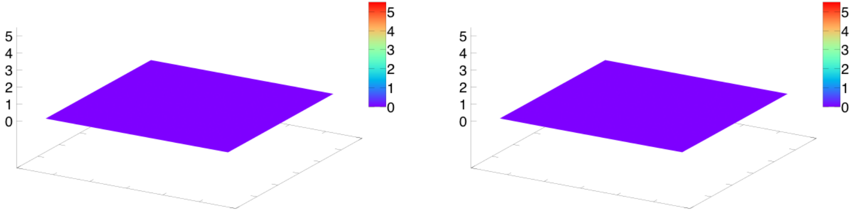

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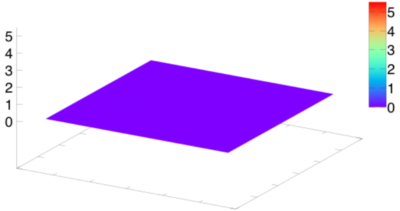

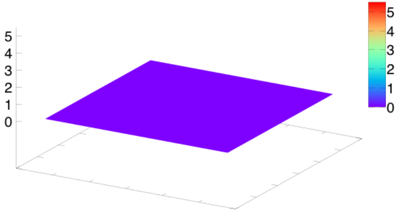

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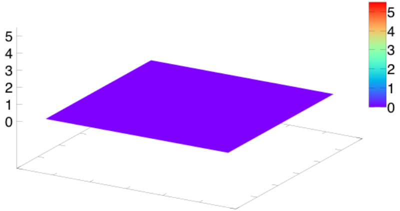

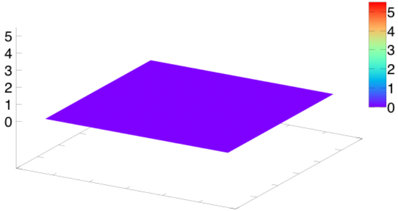

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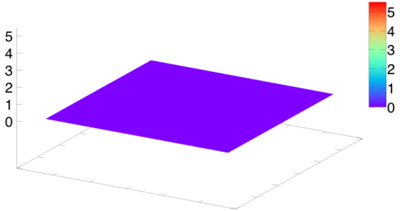

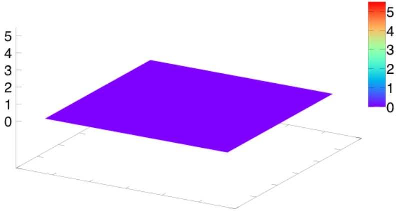

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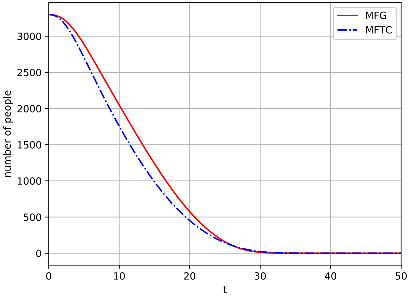

Remaining mass inside the room

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Outline

MFG Setup

- Scheme introduced by [Carlini, Silva][7]
- For simplicity: $d = 1$, domain $\mathcal{S} = \mathbb{R}$, $\mathcal{A} = \mathbb{R}$
- $\nu = 0$ (degenerate second order case also possible; see [Carlini, Silva][8])
- Model:

$$b(x, m, v) = v$$

$$f(x, m, v) = \frac{1}{2}|v|^2 + f_0(x, m), \qquad g(x, m)$$

where f_0 and g depend on $m \in \mathcal{P}_1(\mathbb{R})$ in a potentially non-local way

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- MFG PDE system:

$$\begin{cases} -\dfrac{\partial u}{\partial t}(t,x) + \dfrac{1}{2}|\nabla u(t,x)|^2 = f_0(x, m(t,\cdot)), & \text{in } [0,T) \times \mathbb{R}, \\ \dfrac{\partial m}{\partial t}(t,x) - \operatorname{div}\left(m(t,\cdot)\nabla u(t,\cdot)\right)(x) = 0, & \text{in } (0,T] \times \mathbb{R}, \\ u(T,x) = g(x, m(T,\cdot)), \qquad m(0,x) = m_0(x), & \text{in } \mathbb{R}. \end{cases}$$

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Representation of the Value Function

- Dynamics:

$$X_t^{v} = X_0^{v} + \int_0^t v(s)ds, \qquad t \geq 0.$$

- **Representation formula** for the value function given $m = (m_t)_{t\in[0,T]}$:

$$u[m](t,x) = \inf_{v\in L^2([t,T];\mathbb{R})} \left\{ \int_t^T \left[\frac{1}{2}|v(s)|^2 + f_0(X_s^{v,t,x}, m(s,\cdot))\right] ds + g(X_T^{v,t,x}, m(T,\cdot)) \right\},$$

where $X^{v,t,x}$ starts from x at time t and is controlled by v

Discrete HJB equation

Discrete HJB: Given a flow of densities m,

$$\begin{cases} U_i^n = S_{\Delta t,h}[m](U^{n+1}, i, n), & (n,i) \in [\![N_T - 1]\!] \times \mathbb{Z}, \\ U_i^{N_T} = g(x_i, m(T,\cdot)), & i \in \mathbb{Z}, \end{cases}$$

where

- $S_{\Delta t,h}$ is defined as

$$S_{\Delta t,h}[m](W,n,i) = \inf_{v \in \mathbb{R}} \left\{ \left(\frac{1}{2}|v|^2 + f_0(x_i, m(t_n,\cdot)) \right) \Delta t + I[W](x_i + v\,\Delta t) \right\},$$

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- with $I : \mathcal{B}(\mathbb{Z}) \to \mathcal{C}_b(\mathbb{R})$ is the **interpolation operator** defined as

$$I[W](\cdot) = \sum_{i \in \mathbb{Z}} W_i \beta_i(\cdot),$$

- where $\mathcal{B}(\mathbb{Z})$ is the set of bounded functions from $\mathbb{Z}$ to $\mathbb{R}$
- and $\beta_i = \left[1 - \frac{|x - x_i|}{h}\right]_+$: triangular function with support $[x_{i-1}, x_{i+1}]$ and s.t. $\beta_i(x_i) = 1$.

Discrete HJB equation – cont.

Before moving to the KFP equation:

- **Interpolation:** from $U = (U_i^n)_{n,i}$, construct the function $u_{\Delta t,h}[m](x,t) : [0,T] \times \mathbb{R} \to \mathbb{R}$,

$$u_{\Delta t,h}[m](t,x) = I[U^{[\frac{t}{\Delta t}]}](x), \qquad (t,x) \in [0,T] \times \mathbb{R}.$$

Discrete HJB equation – cont.

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- **Regularization of HJB solution** with a mollifier ρ_ϵ:

$$u^\epsilon_{\Delta t,h}[m](t,\cdot) = \rho_\epsilon * u_{\Delta t,h}[m](t,\cdot), \qquad t \in [0,T].$$

Discrete KFP equation: intuition

- **Eulerian** viewpoint:
  - focus on a location
  - look at the flow passing through it
  - evolution characterized by the velocity at (t, x)
- **Lagrangian** viewpoint:
  - focus on a fluid parcel
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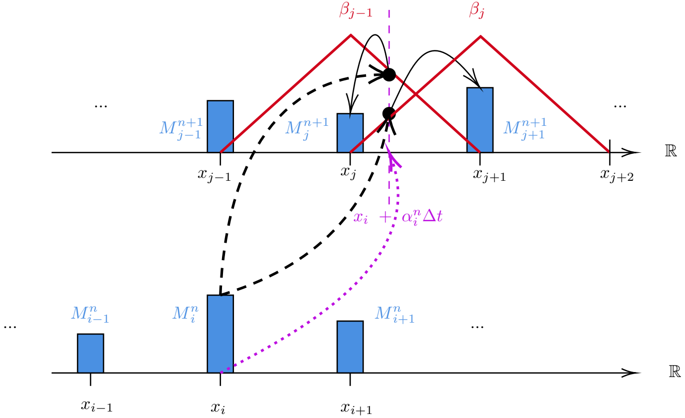
- Here, in our model:

$$X_t^{v} = X_0^{v} + \int_0^t v(s)ds, \qquad t \geq 0.$$

- Time and space discretization?

Discrete KFP equation: intuition – diagram



Movement of the mass when using control $v(t_n, x_i) = \alpha_i^n$.
Bottom: time t_n; top: time t_{n+1}.

Discrete KFP equation

- **Control** induced by value function:

$$\hat{v}^{\epsilon}_{\Delta t,h}[m](t,x) = -\nabla u^{\epsilon}_{\Delta t,h}[m](t,x),$$

and its discrete counter part: $\hat{v}^{\epsilon}_{n,i} = \hat{v}^{\epsilon}_{\Delta t,h}[m](t_n, x_i)$.

- **Discrete flow:**

$$\Phi^{\epsilon}_{n,n+1,i}[m] = x_i + \hat{v}^{\epsilon}_{\Delta t,h}[m](t_n, x_i)\Delta t\,.$$

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- **Discrete KFP equation:** for $M^{\epsilon}[m] = (M^{\epsilon,n}_i[m])_{n,i}$:

$$\begin{cases} M^{\epsilon,n+1}_i[m] = \sum_j \beta_i\left(\Phi^{\epsilon}_{n,n+1,j}[m]\right) M^{\epsilon,n}_j[m], & (n,i) \in [\![N_T - 1]\!] \times \mathbb{Z}, \\ M^{\epsilon,0}_i[m] = \displaystyle\int_{[x_i - h/2, x_i + h/2]} m_0(x)dx, & i \in \mathbb{Z}. \end{cases}$$

Fixed Point Formulation

- **Function** $m^{\epsilon}_{\Delta t,h}[m] : [0,T] \times \mathbb{R} \to \mathbb{R}$ defined as: for $n \in [\![N_T - 1]\!]$, for $t \in [t_n, t_{n+1})$,

$$m^{\epsilon}_{\Delta t,h}[m](t,x) = \frac{1}{h}\left[\frac{t_{n+1}-t}{\Delta t}\sum_{i\in\mathbb{Z}} M_i^{\epsilon,n}[m]\mathbf{1}_{[x_i-h/2,x_i+h/2]}(x) \right.$$
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- **Goal: Fixed-point problem:** Find $\hat{M} = (\hat{M}_i^n)_{i,n}$ such that:

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- **Solution strategy:** Fixed point iterations for example
- See [Carlini, Silva] for more details

Numerical Illustration

Costs:

$$g \equiv 0, \qquad f(x, m, v) = \frac{1}{2}|v|^2 + (x - c^*)^2 + \kappa_{MF} V(x, m),$$

with

$$V(x, m) = \rho_{\sigma_V} * \big(\rho_{\sigma_V} * m\big)(x),$$

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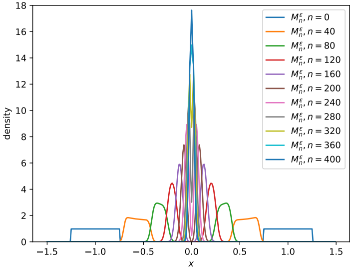
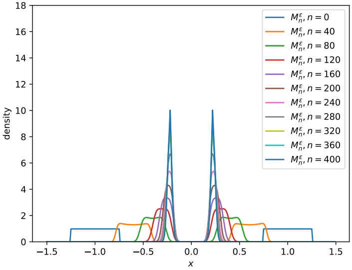
$$V(x, m) = \rho_{\sigma_V} * \big(\rho_{\sigma_V} * m\big)(x),$$

Experiments: target $c^* = 0$, m_0 = unif. on $[-1.25, -0.75]$ and on $[0.75, 1.25]$



$\kappa_{MF} = 0.5$



$\kappa_{MF} = 0.9$

Outline

Outline

1. Introduction

2. A Finite Difference Scheme

3. A Semi-Lagrangian Scheme

4. Optimization Methods for MFC and Variational MFG
- Variational MFGs and Duality
- Alternating Direction Method of Multipliers
- A Primal-Dual Method

5. Conclusion

Variational MFGs

Key ideas:

- Variational MFG
- Duality
- Optimization techniques

A Variational MFG

- $d = 1$, domain = $\mathbb{T}$
- drift and costs:

$$b(x, m, v) = v, \qquad f(x, m, v) = L(x, v) + \mathtt{f}_0(x, m), \qquad g(x, m) = \mathtt{g}_0(x).$$

  where $x \in \mathbb{R}^d, v \in \mathbb{R}^d, m \in \mathbb{R}_+$.
- Then

$$H(x, m, p) = \sup_v \{-L(x, v) - vp\} - \mathtt{f}_0(x, m) = H_0(x, p) - \mathtt{f}_0(x, m)$$

- where H_0 is the convex conjugate (also denoted L^*) of L with respect to v:

$$H_0(x, p) = L^*(x, p) = \sup_v \{ vp - L(x, v)\}$$

- Further assume (for simplicity)

$$L(x, v) = \frac{1}{2}|v|^2, \qquad H_0(x, p) = \frac{1}{2}|p|^2$$

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- Further assume (for simplicity)

$$L(x, v) = \frac{1}{2}|v|^2, \qquad H_0(x, p) = \frac{1}{2}|p|^2$$

- **Claim:**

  MFG PDE system ⇔ optimality condition of two optimization problems in duality

  [Lasry, Lions'07; Cardaliaguet'13; Cardaliaguet, Graber, Porretta, Tonon'15]

A Variational Problem

- At equilibrium, $\mathcal{L}(X_t) = \hat{\mu}_t$ and

$$J(\hat{v}; \hat{m}) = \mathbb{E}\left[\int_0^T f(X_t, \hat{m}(t, X_t), \hat{v}(t, X_t))dt + g(X_T)\right]$$
$$= \int_0^T \int_{\mathbb{T}} \underbrace{f(x, \hat{m}(t,x), \hat{v}(t,x))}_{=L(x,\hat{v}(t,x))+f_0(x,\hat{m}(t,x))} \hat{m}(t,x)dxdt + \int_{\mathbb{T}} g(x)\hat{m}(T,x)dx$$

subject to:

$$0 = \frac{\partial \hat{m}}{\partial t}(t,x) - \nu\Delta\hat{m}(t,x) + \mathrm{div}\left(\hat{m}(t,\cdot)\underbrace{b(\cdot, \hat{m}(t), \hat{v}(t,\cdot))}_{=\hat{v}(t,\cdot)}\right)(x), \qquad \hat{m}_0 = m_0$$

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- Change of variable:

$$\hat{w}(t,x) = \hat{m}(t,x)\hat{v}(t,x)$$

$$\mathcal{B}(\hat{m},\hat{w}) = \int_0^T \int_{\mathbb{T}} \left[L\left(x, \frac{\hat{w}(t,x)}{\hat{m}(t,x)}\right) + \mathtt{f}_0(x, \hat{m}(t,x))\right]\hat{m}(t,x)dxdt + \int_{\mathbb{T}} g(x)\hat{m}(T,x)dx$$

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$$0 = \frac{\partial \hat{m}}{\partial t}(t,x) - \nu\Delta\hat{m}(t,x) + \mathrm{div}\left(\hat{w}(t,\cdot)\right)(x), \qquad \hat{m}_0 = m_0$$

Reformulation

- Reformulation:

$$\mathcal{B}(\hat m, \hat w) = \int_0^T \int_{\mathbb{T}} \Big[\underbrace{L\Big(x, \frac{\hat w(t,x)}{\hat m(t,x)}\Big) \hat m(t,x)}_{\widetilde{L}(x,\hat m(t,x),\hat w(t,x))} + \underbrace{\mathtt{f}_0(x, \hat m(t,x)) \hat m(t,x)}_{\widetilde{F}(x,\hat m(t,x))} \Big] dxdt$$

$$+ \int_{\mathbb{T}} \underbrace{g(x) \hat m(T,x)}_{\widetilde{G}(x,\hat m(t,x))} dx$$

$$= \int_0^T \int_{\mathbb{T}} \Big[\widetilde{L}(x, \hat m(t,x), \hat w(t,x)) + \widetilde{F}(x, \hat m(t,x)) \Big] dxdt + \int_{\mathbb{T}} \widetilde{G}(x, \hat m(t,x)) dx$$

subject to:

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subject to:

$$0 = \frac{\partial \hat{m}}{\partial t}(t,x) - \nu \Delta \hat{m}(t,x) + \mathrm{div}\Big(\hat{w}(t,\cdot)\Big)(x), \qquad \hat{m}_0 = m_0$$

- **Convex** problem under a **linear** constraint, provided $\widetilde{L}, \widetilde{F}, \widetilde{G}$ are convex

Primal Optimization Problem

Primal problem: Minimize over $(m, w) = (m, mv)$:

$$\mathcal{B}(m,w) = \int_0^T \int_{\mathbb{T}} \Big(\widetilde{L}(x, m(t,x), w(t,x)) + \widetilde{F}(x, m(t,x))\Big)\, dx dt + \int_{\mathbb{T}} \widetilde{G}(x, m(T,x)) dx$$

subject to the constraint:

$$\partial_t m - \nu \Delta m + \mathrm{div}(w) = 0, \qquad m(0,x) = m_0(x)$$

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subject to the constraint:

$$\partial_t m - \nu \Delta m + \mathrm{div}(w) = 0, \qquad m(0,x) = m_0(x)$$

where

$$\widetilde{F}(x,m) = \begin{cases} \int_0^m \tilde{f}(x,s)ds, & \text{if } m \geq 0, \\ +\infty, & \text{otherwise,} \end{cases} \qquad \widetilde{G}(x,m) = \begin{cases} m\, \mathtt{g}_0(x), & \text{if } m \geq 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$\widetilde{L}(x,m,w) = \begin{cases} mL\left(x, \frac{w}{m}\right), & \text{if } m > 0, \\ 0, & \text{if } m = 0 \text{ and } w = 0, \\ +\infty, & \text{otherwise} \end{cases}$$

where $\mathbb{R} \ni m \mapsto \tilde{f}(x,m) = \partial_m (m\, \mathtt{f}_0(x,m))$
is non-decreasing (hence $\widetilde{F}$ convex and l.s.c.) provided $m \mapsto m\, \mathtt{f}_0(x,m)$ is convex.

Duality

Dual problem: Maximize over ϕ such that $\phi(T,x) = g_0(x)$

$$\mathcal{A}(\phi) = \inf_{m} \mathcal{A}(\phi, m)$$

with $\mathcal{A}(\phi, m) = \displaystyle\int_0^T \int_{\mathbb{T}} m(t,x)\Big(\partial_t \phi(t,x) + \nu \Delta \phi(t,x) - H(x, m(t,x), \nabla \phi(t,x))\Big) dx dt$

$$+ \int_{\mathbb{T}} m_0(x) \phi(0,x) dx.$$

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Duality relation: $\mathcal{A}$ and $\mathcal{B}$ satisfy: **(A)** $= \sup_{\phi} \mathcal{A}(\phi) = \inf_{(m,w)} \mathcal{B}(m,w) =$ **(B)**

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Duality relation: $\mathcal{A}$ and $\mathcal{B}$ satisfy: **(A)** $= \sup_\phi \mathcal{A}(\phi) = \inf_{(m,w)} \mathcal{B}(m,w) =$ **(B)**

Proof: Fenchel-Rockafellar duality theorem and observe:

$$\textbf{(A)} = -\inf_{\phi} \Big\{\mathcal{F}(\phi) + \mathcal{G}(\Lambda(\phi))\Big\}, \qquad \textbf{(B)} = \inf_{(m,w)} \Big\{\mathcal{F}^*(\Lambda^*(m,w)) + \mathcal{G}^*(-m,-w)\Big\}$$

where $\mathcal{F}^*, \mathcal{G}^*$ are the convex conjugates of $\mathcal{F}, \mathcal{G}$, and Λ^* is the adjoint operator of Λ, and $\Lambda(\phi) = \left(\frac{\partial\phi}{\partial t} + \nu\Delta\phi, \nabla\,\phi\right)$,

$$\mathcal{F}(\phi) = \chi_T(\phi) - \int_{\mathbb{T}^d} m_0(x)\phi(0,x)dx, \qquad \chi_T(\phi) = \begin{cases} 0 & \text{if } \phi|_{t=T} = g_0 \\ +\infty & \text{otherwise,} \end{cases}$$

$$\mathcal{G}(\varphi_1, \varphi_2) = -\inf_{0\le m\in L^1((0,T)\times\mathbb{T}^d)} \int_0^T \int_{\mathbb{T}^d} m(t,x)\left(\varphi_1(t,x) - H(x, m(t,x), \varphi_2(t,x))\right) dxdt.$$

Outline

Augmented Lagrangian

Reformulation of the primal problem:

$$\textbf{(A)} = -\inf_{\phi}\left\{\mathcal{F}(\phi) + \mathcal{G}(\Lambda(\phi))\right\} = -\inf_{\phi}\ \inf_{q}\left\{\mathcal{F}(\phi) + \mathcal{G}(q),\ \text{subj. to } q = \Lambda(\phi)\right\}.$$

- The corresponding **Lagrangian** is

$$\mathcal{L}(\phi, q, \tilde{q}) = \mathcal{F}(\phi) + \mathcal{G}(q) - \langle \tilde{q}, \Lambda(\phi) - q \rangle.$$

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$$\mathcal{L}(\phi, q, \tilde{q}) = \mathcal{F}(\phi) + \mathcal{G}(q) - \langle \tilde{q}, \Lambda(\phi) - q\rangle.$$

- We consider the **augmented Lagrangian** (with parameter $r > 0$)

$$\mathcal{L}^r(\phi, q, \tilde{q}) = \mathcal{L}(\phi, q, \tilde{q}) + \frac{r}{2}\|\Lambda(\phi) - q\|^2$$

- Goal: find a **saddle-point** of $\mathcal{L}^r$.

Alternating Direction Method of Multipliers (ADMM)

Reminder: $\mathcal{L}^r(\phi, q, \tilde{q}) = \mathcal{F}(\phi) + \mathcal{G}(q) - \langle \tilde{q}, \Lambda(\phi) - q \rangle + \frac{r}{2} \|\Lambda(\phi) - q\|^2$

Input: Initial guess $(\phi^{(0)}, q^{(0)}, \tilde{q}^{(0)})$; number of iterations $\mathtt{K}$

Output: Approximation of a saddle point $(\phi, q, \tilde{q})$ solving the finite difference system

1 Initialize $(\phi^{(0)}, q^{(0)}, \tilde{q}^{(0)})$

2 **for** $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**

3 (a) Compute

$$\phi^{(\mathtt{k}+1)} \in \operatorname*{argmin}_{\phi} \left\{ \mathcal{F}(\phi) - \langle \tilde{q}^{(\mathtt{k})}, \Lambda(\phi) \rangle + \frac{r}{2} \|\Lambda(\phi) - q^{(\mathtt{k})}\|^2 \right\}$$

References: ALG2 in the book of [Fortin, Glowinski];

$\rightarrow$ in MFG: [Benamou, Carlier'15; Andreev'17]; in MFC: [Achdou, L.'16]

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(b) Compute

$$q^{(\mathtt{k}+1)} \in \operatorname*{argmin}_{q} \left\{ \mathcal{G}(q) + \langle \tilde{q}^{(\mathtt{k})}, q \rangle + \frac{r}{2}\|\Lambda(\phi^{(\mathtt{k}+1)}) - q\|^2 \right\}$$

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4 (b) Compute

$$q^{(\mathtt{k}+1)} \in \operatorname*{argmin}_{q} \left\{ \mathcal{G}(q) + \langle \tilde{q}^{(\mathtt{k})}, q \rangle + \frac{r}{2}\|\Lambda(\phi^{(\mathtt{k}+1)}) - q\|^2 \right\}$$

5 (c) Compute

$$\tilde{q}^{(\mathtt{k}+1)} = \tilde{q}^{(\mathtt{k})} - r\left(\Lambda(\phi^{(\mathtt{k}+1)}) - q^{(\mathtt{k}+1)}\right)$$

6 **return** $(\phi^{(\mathtt{K})}, q^{(\mathtt{K})}, \tilde{q}^{(\mathtt{K})})$

References: ALG2 in the book of [Fortin, Glowinski];
→ in MFG: [Benamou, Carlier'15; Andreev'17]; in MFC: [Achdou, L.'16]

ADMM: Discrete Primal Problem

Notation: N_h, N_T steps resp. in space and time, $N = (N_T + 1)N_h$, $N' = N_T N_h$.

Recall: $H_0(x,p) = \frac{1}{2}|p|^2$. We take $\tilde{H}_0(x, p_1, p_2) = \frac{1}{2}|(p_1^-, p_2^+)|^2$.

Discrete version of the **dual** convex problem:

$$(\mathbf{A_h}) = -\inf_{\phi \in \mathbb{R}^N} \left\{ \mathcal{F}_h(\phi) + \mathcal{G}_h(\Lambda_h(\phi)) \right\},$$

where $\Lambda_h : \mathbb{R}^N \to \mathbb{R}^{3N'}$ is defined by : $\forall\, n \in \{1, \dots, N_T\}, \forall\, i \in \{0, \dots, N_h - 1\}$,

$$(\Lambda_h(\phi))_i^n = \left((D_t \phi_i)^n + \nu \left(\Delta_h \phi^{n-1} \right)_i, [\nabla_h\, \phi^{n-1}]_i \right),$$

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Discrete version of the **dual** convex problem:

$$(\mathbf{A_h}) = - \inf_{\phi \in \mathbb{R}^N} \left\{ \mathcal{F}_h(\phi) + \mathcal{G}_h(\Lambda_h(\phi)) \right\},$$

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where $\mathcal{F}_h, \mathcal{G}_h$ are the l.s.c. proper functions defined by:

$$\mathcal{F}_h : \mathbb{R}^N \ni \phi \mapsto \chi_T(\phi) - h \sum_{i=0}^{N_h - 1} \rho_i^0 \phi_i^0 \in \mathbb{R} \cup \{+\infty\},$$

$$\mathcal{G}_h : \mathbb{R}^{3N'} \ni (a, b, c) \mapsto - h\Delta t \sum_{n=1}^{N_T} \sum_{i=0}^{N_h - 1} \mathcal{K}_h(x_i, a_i^n, b_i^n, c_i^n) \in \mathbb{R} \cup \{+\infty\},$$

with

$$\mathcal{K}_h(x, a_0, p_1, p_2) = \min_{m \in \mathbb{R}_+} \left\{ m[a_0 + \tilde{H}_0(x, m, p_1, p_2)] \right\}, \quad \chi_T(\phi) = \begin{cases} 0 & \text{if } \phi_i^{N_T} \equiv \mathrm{g}_0(x_i) \\ +\infty & \text{otherwise.} \end{cases}$$

ADMM with Discretization

Discrete Aug. Lag.: $\mathcal{L}_h^r(\phi, q, \tilde{q}) = \mathcal{F}_h(\phi) + \mathcal{G}_h(q) - \langle \tilde{q}, \Lambda_h(\phi) - q \rangle + \frac{r}{2}\|\Lambda(\phi) - q\|^2$

Input: Initial guess $(\phi^{(0)}, q^{(0)}, \tilde{q}^{(0)})$; number of iterations $\mathtt{K}$

Output: Approximation of a saddle point $(\phi, q, \tilde{q})$

Initialize $(\phi^{(0)}, q^{(0)}, \tilde{q}^{(0)})$

for $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**

(a) Compute $\phi^{(\mathtt{k}+1)} \in \operatorname{argmin}_\phi \left\{ \mathcal{F}_h(\phi) - \langle \tilde{q}^{(\mathtt{k})}, \Lambda_h(\phi) \rangle + \frac{r}{2}\|\Lambda_h(\phi) - q^{(\mathtt{k})}\|^2 \right\}$

(b) Compute $q^{(\mathtt{k}+1)} \in \operatorname{argmin}_q \left\{ \mathcal{G}_h(q) + \langle \tilde{q}^{(\mathtt{k})}, q \rangle + \frac{r}{2}\|\Lambda_h(\phi^{(\mathtt{k}+1)}) - q\|^2 \right\}$

(c) Compute $\tilde{q}^{(\mathtt{k}+1)} = \tilde{q}^{(\mathtt{k})} - r\left(\Lambda_h(\phi^{(\mathtt{k}+1)}) - q^{(\mathtt{k}+1)}\right)$

return $(\phi^{(\mathtt{K})}, q^{(\mathtt{K})}, \tilde{q}^{(\mathtt{K})})$

ADMM with Discretization

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Input: Initial guess $(\phi^{(0)}, q^{(0)}, \tilde{q}^{(0)})$; number of iterations $\mathtt{K}$
Output: Approximation of a saddle point $(\phi, q, \tilde{q})$

Initialize $(\phi^{(0)}, q^{(0)}, \tilde{q}^{(0)})$
for $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**
- (a) Compute $\phi^{(\mathtt{k}+1)} \in \operatorname{argmin}_\phi \left\{ \mathcal{F}_h(\phi) - \langle \tilde{q}^{(\mathtt{k})}, \Lambda_h(\phi) \rangle + \frac{r}{2}\|\Lambda_h(\phi) - q^{(\mathtt{k})}\|^2 \right\}$
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return $(\phi^{(\mathtt{K})}, q^{(\mathtt{K})}, \tilde{q}^{(\mathtt{K})})$

First-order Optimality Conditions:

Step (a): finite-difference equation

Step (b): minimization problem at each point of the grid

ADMM with Discretization

Discrete Aug. Lag.: $\mathcal{L}_h^r(\phi, q, \tilde{q}) = \mathcal{F}_h(\phi) + \mathcal{G}_h(q) - \langle \tilde{q}, \Lambda_h(\phi) - q \rangle + \frac{r}{2} \|\Lambda(\phi) - q\|^2$

Input: Initial guess $(\phi^{(0)}, q^{(0)}, \tilde{q}^{(0)})$; number of iterations $\mathtt{K}$
Output: Approximation of a saddle point $(\phi, q, \tilde{q})$

1 Initialize $(\phi^{(0)}, q^{(0)}, \tilde{q}^{(0)})$
2 **for** $\mathtt{k} = 0, 1, 2, \dots, \mathtt{K} - 1$ **do**
3 $\quad(a)$ Compute $\phi^{(\mathtt{k}+1)} \in \operatorname{argmin}_\phi \left\{ \mathcal{F}_h(\phi) - \langle \tilde{q}^{(\mathtt{k})}, \Lambda_h(\phi) \rangle + \frac{r}{2} \|\Lambda_h(\phi) - q^{(\mathtt{k})}\|^2 \right\}$
4 $\quad(b)$ Compute $q^{(\mathtt{k}+1)} \in \operatorname{argmin}_q \left\{ \mathcal{G}_h(q) + \langle \tilde{q}^{(\mathtt{k})}, q \rangle + \frac{r}{2} \|\Lambda_h(\phi^{(\mathtt{k}+1)}) - q\|^2 \right\}$
5 $\quad(c)$ Compute $\tilde{q}^{(\mathtt{k}+1)} = \tilde{q}^{(\mathtt{k})} - r \left(\Lambda_h(\phi^{(\mathtt{k}+1)}) - q^{(\mathtt{k}+1)} \right)$
6 **return** $(\phi^{(\mathtt{K})}, q^{(\mathtt{K})}, \tilde{q}^{(\mathtt{K})})$

First-order Optimality Conditions:

Step (a): finite-difference equation

Step (b): minimization problem at each point of the grid

Rem.: For (a): discrete PDE

- if $\nu = 0$, a direct solver can be used
- if $\nu > 0$, PDE with 4^{th} order linear elliptic operator $\Rightarrow$ needs preconditioner

(See e.g. [Achdou, Perez'12, Andreev'17, Briceño-Arias et al.'19])

Numerical Example: Congestion Without Viscosity

- Domain $\Omega = [0,1]^2 \backslash [0.4, 0.6]^2$ (obstacle at the center)
- Define the Hamiltonian by duality (on $\partial\Omega$ the vector speed is towards the interior)

$$H(x, m, p) = \begin{cases} \sup_{\xi \in \mathbb{R}^2} \left\{ -\xi \cdot p - L(x, m, \xi) \right\} = m^{-\alpha} |p|^{\beta} - \ell(x, m), & \text{if } x \in \Omega, \\ \sup_{\xi \in \mathbb{R}^2 \,:\, \xi \cdot n \leq 0} \left\{ -\xi \cdot p - L(x, m, \xi) \right\}, & \text{if } x \in \partial\Omega. \end{cases}$$

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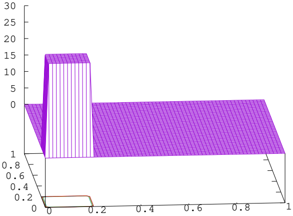
- Ex.: m_0 : & u_T : opposite corners; $\alpha = 0.01, \beta = 2, \ell(x, m) = 0.01m$.
- Results for the mean field control (MFC) problem, with $\nu = 0$



Density at time $t = 0$

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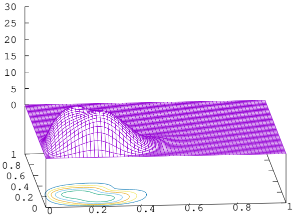

Density at time $t = T/8$

Numerical Example: Congestion Without Viscosity

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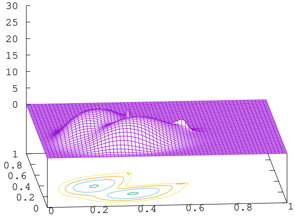

Density at time $t = T/4$

Numerical Example: Congestion Without Viscosity

- Domain $\Omega = [0,1]^2 \backslash [0.4, 0.6]^2$ (obstacle at the center)
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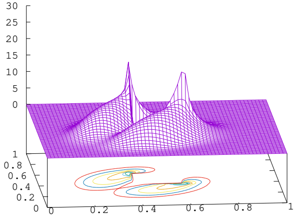

Density at time $t = 3T/8$

Numerical Example: Congestion Without Viscosity

- Domain $\Omega = [0,1]^2 \backslash [0.4, 0.6]^2$ (obstacle at the center)
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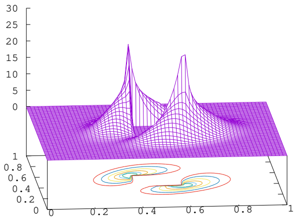

Density at time $t = T/2$

Numerical Example: Congestion Without Viscosity

- Domain $\Omega = [0,1]^2 \backslash [0.4, 0.6]^2$ (obstacle at the center)
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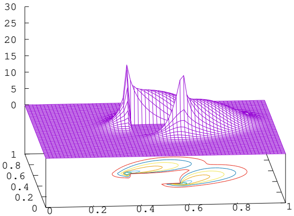

Density at time $t = 5T/8$

Numerical Example: Congestion Without Viscosity

- Domain $\Omega = [0,1]^2 \backslash [0.4, 0.6]^2$ (obstacle at the center)
- Define the Hamiltonian by duality (on $\partial\Omega$ the vector speed is towards the interior)

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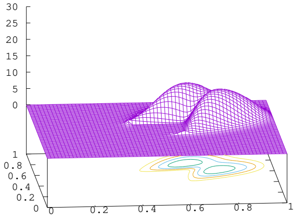

Density at time $t = 3T/4$

Numerical Example: Congestion Without Viscosity

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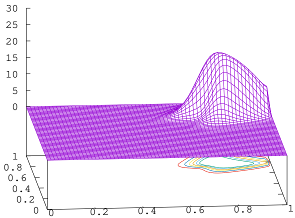

Density at time $t = 7T/8$

Numerical Example: Congestion Without Viscosity

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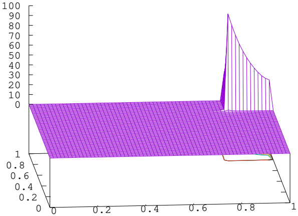

Density at time $t = T$

Outline

Optimality Conditions and Proximal Operator

- Let $\varphi, \psi \colon \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be convex l.s.c. proper functions.
- Consider the optimization problem

$$\min_{y \in \mathbb{R}^N} \varphi(y) + \psi(y),$$

and its dual

$$\min_{\sigma \in \mathbb{R}^N} \varphi^*(-\sigma) + \psi^*(\sigma).$$

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- The 1^{st}-order opt. cond. satisfied by a solution $(\hat{y}, \hat{\sigma})$ are

$$\begin{cases} -\hat{\sigma} \in \partial\varphi(\hat{y}) \\ \hat{y} \in \partial\psi^*(\hat{\sigma}) \end{cases} \Leftrightarrow \begin{cases} \hat{y} - \tau\hat{\sigma} \in \tau\partial\varphi(\hat{y}) + \hat{y} \\ \hat{\sigma} + \gamma\hat{y} \in \gamma\partial\psi^*(\hat{\sigma}) + \hat{\sigma} \end{cases} \Leftrightarrow \begin{cases} \mathrm{prox}_{\tau\varphi}(\hat{y} - \tau\hat{\sigma}) = \hat{y} \\ \mathrm{prox}_{\gamma\psi^*}(\hat{\sigma} + \gamma\hat{y}) = \hat{\sigma}, \end{cases}$$

where $\gamma > 0$ and $\tau > 0$ are arbitrary and

- The **proximal operator** of a l.s.c. convex proper $\phi \colon \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is:

$$\mathrm{prox}_{\gamma\phi}(x) := \operatorname*{argmin}_{y \in \mathbb{R}^N} \left\{ \phi(y) + \frac{|y-x|^2}{2\gamma} \right\} = (I + \partial(\gamma\phi))^{-1}(x), \qquad \forall\, x \in \mathbb{R}^N.$$

Chambolle-Pock's Primal-Dual Algorithm

The following algorithm has been proposed by [Chambolle & Pock][9] It has been proved to converge when $\tau\gamma < 1$.

Input: Initial guess $(\sigma^{(0)}, y^{(0)}, \bar{y}^{(0)})$; $\theta \in [0,1]$; $\gamma > 0, \tau > 0$; number of iterations $\mathtt{K}$
Output: Approximation of $(\hat{\sigma}, \hat{y})$ solving the optimality conditions

1 Initialize $(\sigma^{(0)}, y^{(0)}, \bar{y}^{(0)})$
2 **for** $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K}-1$ **do**
3 (a) Compute

$$\sigma^{(\mathtt{k}+1)} = \mathrm{prox}_{\gamma\psi^*}(\sigma^{(\mathtt{k})} + \gamma\bar{y}^{(\mathtt{k})}),$$

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6 **return** $(\sigma^{(\mathtt{K})}, y^{(\mathtt{K})}, \bar{y}^{(\mathtt{K})})$

[9] Chambolle, A. & Thomas P.. A first-order primal-dual algorithm for convex problems with applications to imaging. *Journal of mathematical imaging and vision* 40.1 (2011): 120-145.

Dual of Discrete Problem $(\mathbf{A_h})$

By Fenchel-Rockafellar theorem, the dual problem of $(\mathbf{A_h})$ is:

$$(\mathbf{B_h}) = \min_{(m,w_1,w_2)=\sigma\in\mathbb{R}^{3N'}} \Big\{\mathcal{F}_h^*(\Lambda_h^*(\sigma)) + \mathcal{G}_h^*(-\sigma)\Big\},$$

where $\mathcal{G}_h^*$ and $\mathcal{F}_h^*$ are respectively the Legendre-Fenchel conjugates of $\mathcal{G}_h$ and $\mathcal{F}_h$, defined by:

$$\mathcal{F}_h^*(\mu) = \sup_{\phi\in\mathbb{R}^N} \Big\{\langle\mu,\phi\rangle_{\ell^2(\mathbb{R}^N)} - \mathcal{F}_h(\phi)\Big\}, \qquad \forall\,\mu\in\mathbb{R}^N$$

$$\mathcal{G}_h^*(-\sigma) = \max_{q\in\mathbb{R}^{3N'}} \Big\{-\langle\sigma,q\rangle_{\ell^2(\mathbb{R}^{3N'})} - \mathcal{G}_h(q)\Big\} = h\Delta t\sum_{n=1}^{N_T}\sum_{i=0}^{N_h-1}\tilde{L}_h(x_i,\sigma_i^n), \qquad \forall\,\sigma\in\mathbb{R}^{3N'}$$

with $\tilde{L}_h(x,\sigma_0) = \max_{p_0\in\mathbb{R}^3}\Big\{-\sigma_0\cdot p_0 + \mathcal{K}_h(x,q_0)\Big\}, \quad \forall\sigma_0\in\mathbb{R}^3.$

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Rem.: The max can be costly to compute but in some cases $\tilde{L}_h$ has a **closed-form** expression.

Finally $\Lambda_h^* : \mathbb{R}^{3N'} \to \mathbb{R}^N$ denotes the adjoint of Λ_h: for all $(m, y, z) \in \mathbb{R}^{3N'}, \phi \in \mathbb{R}^N$:

$$\langle \Lambda_h^*(m,y,z), \phi \rangle_{\ell^2(\mathbb{R}^N)} = \langle (m,y,z), \Lambda_h(\phi) \rangle_{\ell^2(\mathbb{R}^{3N'})}$$

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Rem.: We have $\mathcal{F}_h^*(\Lambda_h^*(m, y, z)) = \begin{cases} h \sum_{i=0}^{N_h - 1} m_i^{N_T} \mathfrak{g}_0(x_i), & \text{if } (m, y, z) \text{ satisfies } (\star) \text{ below}, \\ +\infty, & \text{otherwise}, \end{cases}$

with $\forall\, i \in \{0, \dots, N_h - 1\}$, $m_i^0 = \rho_i^0$, and $\forall\, n \in \{0, \dots, N_T - 1\}$:

$$(D_t m_i)^n - \nu \left(\Delta_h m^{n+1} \right)_i + \frac{y_i^{n+1} - y_{i-1}^{n+1}}{h} + \frac{z_{i+1}^{n+1} - z_i^{n+1}}{h} = 0. \qquad (\star)$$

Reformulation

The discrete dual problem can be recast as:

$$\inf_{(m,w)} \underbrace{\mathbb{B}_h(m,w) + \mathbb{F}_h(m)}_{\varphi(m,w)} + \underbrace{\iota_{\mathbb{G}^{-1}(\rho^0,0)}(m,w)}_{\psi(m,w)} \qquad (P_h)$$

with the costs

$$\mathbb{F}_h(m) := \sum_{i,n} \widetilde{F}(x_i, m_i^n) + \frac{1}{\Delta t} \sum_i \widetilde{G}(x_i, m_i^{N_T}), \qquad \mathbb{B}_h(m,w) := \sum_{i,n} \hat{b}(m_i^n, w_i^{n-1}),$$

$$\hat{b}(m,w) := \begin{cases} mL\left(x, -\frac{w}{m}\right), & \text{if } m > 0, w \in K = \mathbb{R}_- \times \mathbb{R}_+, \\ 0, & \text{if } (m,w) = (0,0), \\ +\infty, & \text{otherwise,} \end{cases}$$

and $\mathbb{G}(m,w) := (m_0, (Am^{n+1} + Bw^n)_{0 \le n \le N_T - 1})$ with

$$(Am)_i^{n+1} := (D_t m)_i^n - \nu(\Delta_h m)_i^{n+1}, \qquad (Bw)_i^n := (D_h w^1)_{i-1}^n + (D_h w^2)_i^n.$$

[10] Briceno-Arias, L., Kalise, D. & Silva, J.. Proximal methods for stationary mean field games with local couplings. *SIAM Journal on Control and Optimization* 56.2 (2018): 801-836.

[11] Briceño-Arias, L, et al. On the implementation of a primal-dual algorithm for second order time-dependent mean field games with local couplings. *ESAIM: Proceedings and Surveys* 65 (2019): 330-348.

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Rem.: The optimality conditions of this problem correspond to the **finite-difference system**

So we can apply **Chambolle-Pock**'s method for (P_h) with

$$y=(m,w), \qquad \varphi(m,w) = \mathbb{B}_h(m,w) + \mathbb{F}_h(m), \qquad \psi(m,w) = \iota_{\mathbb{G}^{-1}(\rho^0,0)}(m,w)$$

See [Briceño-Arias et al.'18][10] and [Briceño-Arias et al.'19][11] in the stationary and dynamic cases.

[10] Briceno-Arias, L., Kalise, D. & Silva, J.. Proximal methods for stationary mean field games with local couplings. *SIAM Journal on Control and Optimization* 56.2 (2018): 801-836.

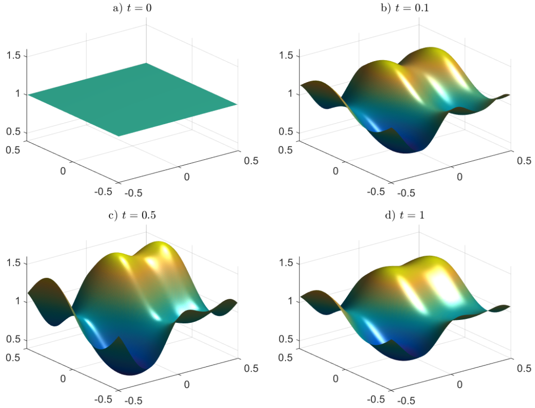
[11] Briceño-Arias, L, et al. On the implementation of a primal-dual algorithm for second order time-dependent mean field games with local couplings. *ESAIM: Proceedings and Surveys* 65 (2019): 330-348.

Numerical Example

Setting: $g \equiv 0$ and $\mathbb{R}^2 \times \mathbb{R} \ni (x, m) \mapsto f(x, m) := m^2 - \overline{H}(x)$, with

$$\overline{H}(x) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(2\pi x_1)$$

We solve the corresponding MFG and obtain the following evolution of the density:



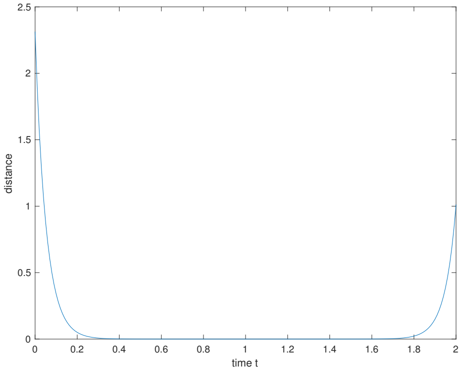
Evolution of the density

(More details in [Briceño-Arias et al.'19])

Turnpike phenomenon

This example also illustrates the **turnpike phenomenon**, see e.g. [Porretta, Zuazua]

- the mass starts from an initial density;
- it **converges to a steady state**, influenced only by the running cost;
- as $t \to T$, the mass is influenced by the final cost and **converges to a final state**.



L^2 **distance between dynamic and stationary solutions**

(More details in [Briceño-Arias et al.'19])

Outline

Summary