Mean Field Games: Numerical Methods and Applications in Machine Learning

Part 4: Methods Based on the Probabilistic Approach

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https://mlauriere.github.io/teaching/MFG-PKU-3.pdf

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RECAP

MKV FBSDE System

- Recall: generic form:

$$\begin{cases} dX_t = B(X_t, \mathcal{L}(X_t), Y_t, Z_t)dt + \sigma dW_t, & 0 \le t \le T \\ dY_t = -F(X_t, \mathcal{L}(X_t), Y_t, Z_t)dt + Z_t dW_t, & 0 \le t \le T \\ X_0 \sim m_0, \qquad Y_T = G(X_T, \mathcal{L}(X_T)) \end{cases}$$

- Decouple:
  - Given $(\mathcal{L}(X), Y, Z)$, solve for X
  - Given $(X, \mathcal{L}(X))$ solve for (Y, X)
- Iterate
- Algorithm proposed by [Chassagneux et al.'19][1], [Angiuli et al.'19][2]

[1] Chassagneux, J.-F., Crisan, D., & Delarue, F.. Numerical method for FBSDEs of McKean–Vlasov type. *The Annals of Applied Probability* 29.3 (2019): 1640-1684.

[2] Angiuli, A., et al. Cemracs 2017: numerical probabilistic approach to MFG. *ESAIM: Proceedings and Surveys* 65 (2019): 84-113.

Picard Scheme for MKV FBSDE System

Input: Initial guess (ξ, ζ); initial condition ξ; terminal condition ζ; time horizon T; number of iterations $\mathtt{K}$

Output: Approximation of (X, Y, Z) solving the MKV FBSDE system

Initialize $X_t^{(0)} = \xi, Y_t^{(0)} = 0, Z_t^{(0)} = 0, 0 \leq t \leq T$

for $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**

Let $X^{(\mathtt{k}+1)}$ be the solution to:

$$\begin{cases} dX_t = B(X_t^{(\mathtt{k})}, \mathcal{L}(X_t^{(\mathtt{k})}), Y_t^{(\mathtt{k})}, Z_t^{(\mathtt{k})})dt + \sigma dW_t, & 0 \leq t \leq T \\ X_0 = \xi \end{cases}$$

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2 **for** $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**

3 Let $X^{(\mathtt{k}+1)}$ be the solution to:

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4 Let $(Y^{(\mathtt{k}+1)}, Z^{(\mathtt{k}+1)})$ be the solution to:

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5 **return** $\mathtt{Picard}[T](\xi, \zeta) = (X^{(\mathtt{K})}, Y^{(\mathtt{K})}, Z^{(\mathtt{K})})$

Notation: $\Phi_{\xi,\zeta} : (X^{(\mathtt{k})}, \mathcal{L}(X^{(\mathtt{k})}), Y^{(\mathtt{k})}, Z^{(\mathtt{k})}) \mapsto (X^{(\mathtt{k}+1)}, \mathcal{L}(X^{(\mathtt{k}+1)}), Y^{(\mathtt{k}+1)}, Z^{(\mathtt{k}+1)})$

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Contraction? Small T or small Lipschitz constants for B, F, G

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- Grid: $0 = T_0 < T_1 < \cdots < T_{M-1} < T_M = T$
- Subproblem: Given $(\xi_{T_m}, \mathcal{L}(\xi_{T_m}))$ and $\zeta_{T_{m+1}}$, solve:

$$\begin{cases} dX_t = B(X_t, \mathcal{L}(X_t), Y_t, Z_t)dt + \sigma dW_t, & T_m \leq t \leq T_{m+1} \\ dY_t = -F(X_t, \mathcal{L}(X_t), Y_t, Z_t)dt + Z_t dW_t, & T_m \leq t \leq T_{m+1} \\ X_{T_m} = \xi_{T_m}, \qquad Y_{T_{m+1}} = \zeta_{T_{m+1}} \end{cases}$$

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- How to find ξ_{T_m} and $\zeta_{T_{m+1}}$?
  - $\rightarrow$ ξ_{T_m} from previous problem’s solution (or initial condition)
  - $\rightarrow$ $\zeta_{T_{m+1}}$ from next problem’s solution (or terminal condition)

Global Solver for MKV FBSDE System

Following [Chassagneux et al.'19], a global solver can be defined recursively, and then we call:

$$\texttt{Solver}[m](\xi_0, \mu_0)$$

with ξ_0 a random variable with distribution μ_0

Input: Initial guess $(\xi, \mathcal{L}(\xi))$; time step index m; number of iterations $\texttt{K}$
Output: Approximation of Y_{T_m} where (X, Y, Z) solves the MKV FBSDE system on $[T_m, T]$ starting with $(\xi, \mathcal{L}(\xi))$ at time T_m

Initialize $X_t^{(0)} = \xi, \mathcal{L}(X_t^{(0)}) = \mathcal{L}(\xi)$ for all $T_m \leq t \leq T_{m+1}$
for $\texttt{k} = 0, 1, 2, \ldots, \texttt{K} - 1$ **do**
    Compute

$$Y_{T_{m+1}}^{(\texttt{k}+1)} = \texttt{Solver}[m+1](X_{T_{m+1}}^{(\texttt{k})}, \mathcal{L}(X_{T_{m+1}}^{(\texttt{k})}))$$

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1. Initialize $X_t^{(0)} = \xi, \mathcal{L}(X_t^{(0)}) = \mathcal{L}(\xi)$ for all $T_m \le t \le T_{m+1}$
2. **for** $\texttt{k} = 0, 1, 2, \ldots, \texttt{K} - 1$ **do**
3. Compute
$$Y_{T_{m+1}}^{(\texttt{k}+1)} = \texttt{Solver}[m+1](X_{T_{m+1}}^{(\texttt{k})}, \mathcal{L}(X_{T_{m+1}}^{(\texttt{k})}))$$
4. Compute:
$$(X_t^{(\texttt{k}+1)}, \mathcal{L}(X_t^{(\texttt{k}+1)}), Y_t^{(\texttt{k}+1)}, Z_t^{(\texttt{k}+1)})_{T_m \le t \le T_{m+1}} = \texttt{Picard}[T_{m+1} - T_m](X_{T_m}^{(\texttt{k})}, Y_{T_{m+1}}^{(\texttt{k}+1)})$$
5. **return** $\texttt{Solver}[m](\xi, \mathcal{L}(\xi)) := Y_{T_m}^{(\texttt{K})}$

Implementation: Discretizations

Following [Angiuli et al.'19][3]

- Tree algorithm:
  - Time discretization
  - Space discretization: binomial tree structure
- Grid algorithm:
  - Time and space discretization on a grid

[3] Angiuli, A., et al. Cemracs 2017: numerical probabilistic approach to MFG. *ESAIM: Proceedings and Surveys* 65 (2019): 84-113.

Outline

1. A Picard Scheme for MKV FBSDE

2. A Class of Finite Dimensional MKV Control Problems