# Numerical Methods for Mean Field Games

## *Lecture 4*
## *Deep Learning Methods: Part I*
## *MFC and MKV FBSDE*

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# Outline

## Summary so far

Numerical methods discussed so far:

- ODE system for LQ setting
- FBPDE system
- FBSDE system

# “Classical” Numerical Methods for MFG: Some references

Some methods based on the deterministic approach to MFG/MFC:

- Finite difference & Newton method: [Achdou and Capuzzo-Dolcetta, 2010], [Achdou et al., 2012], ...
- (Semi-)Lagrangian approach: [Carlini and Silva, 2014, Carlini and Silva, 2015], [Carlini and Silva, 2018], [Calzola et al., 2022], ...
- Augmented Lagrangian & ADMM: [Benamou and Carlier, 2015], [Andreev, 2017a], [Achdou and Laurière, 2016], ...
- Primal-dual algo.: [Briceño Arias et al., 2018], [Briceño Arias et al., 2019], ...
- Gradient descent based methods [Laurière and Pironneau, 2016], [Pfeiffer, 2016], [Lavigne and Pfeiffer, 2022], ...
- Monotone operators [Almulla et al., 2017], [Gomes and Saúde, 2018], [Gomes and Yang, 2020], ...
- Policy iteration [Cacace et al., 2021], [Cui and Koeppl, 2021], [Camilli and Tang, 2022], [Tang and Song, 2022], [Laurière et al., 2023], ...
- Finite elements [Benamou and Carlier, 2015], [Andreev, 2017b], ...

# “Classical” Numerical Methods for MFG: Some references

Some methods based on the probabilistic approach to MFG/MFC:

- Cubature [de Raynal and Trillos, 2015], . . .
- Markov chain approximation: [Bayraktar et al., 2018], . . .
- Probabilistic approach and Picard: [Chassagneux et al., 2019], [Angiuli et al., 2019], . . .
- Probabilistic approach and regression: [Balata et al., 2019], . . .
- . . .

# “Classical” Numerical Methods for MFG: Shortcomings

Many of these methods are very efficient and have been analyzed in detail

However, they are usually limited to problems with:

- (relatively) small dimension
- (relatively) simple structure

$\Rightarrow$ motivations to develop machine learning methods (see lectures 4, 5, 6)

# Deep learning

- In this lecture and the following one, we will use deep learning to solve MFGs
- At a high level, there are two main ingredients:
  - Approximation using deep neural networks
  - Minimization of a loss function using stochastic gradient descent
- Many variants and refinements, ...
- See e.g. [LeCun et al., 2015, Goodfellow et al., 2016], ...

# Ingredient 1: Neural Networks

- **Goal:** Minimize over $\varphi(\cdot)$, $\mathbb{J}(\varphi) := \mathbb{E}_\xi[\mathbb{L}(\varphi, \xi)]$
- Example: Regression: $\xi = (x, f(x))$ for some $f$, $\mathbb{L}(\varphi, \xi) = \|\varphi(x) - f(x)\|^2$

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- Example: Regression: $\xi = (x, f(x))$ for some $f$, $\mathbb{L}(\varphi, \xi) = \|\varphi(x) - f(x)\|^2$
- **Idea:** Instead of min. over all $\varphi(\cdot)$, min. over parameters $\theta$ of $\varphi_\theta(\cdot)$
- Example: Feedforward fully-connected neural network:
  - $\varphi_\theta(\cdot)$
  - with weights & biases $\theta = (\beta^{(k)}, w^{(k)})_{k=1,\dots,\ell}$
  - activation functions $\psi^{(i)}$: sigmoid, tanh, ReLU, . . . ; applied coordinate-wise

$$\underbrace{\varphi_\theta(x)}_{\varphi(\theta, x)} = \psi^{(\ell)}\Big(\beta^{(\ell)} + w^{(\ell)} \dots \psi^{(2)}\Big(\beta^{(2)} + w^{(2)} \underbrace{\psi^{(1)}(\beta^{(1)} + w^{(1)} x)}_{\text{one hidden layer}}\Big) \dots\Big)$$

  - Depth = number of layers; width of a layer = dimension of bias vector

# Ingredient 1: Neural Networks – Comments

- Many other architectures (convolutional neural networks, recurrent neural networks, . . . ), see e.g. [Leijnen and Veen, 2020]
- Successes of deep learning in many fields: natural language processing, computer vision, drug design, . . . and even games!
- Combination with reinforcement learning (see lecture 6)
- Universal approximation theorems [Cybenko, 1989], [Hornik, 1991], . . .
- Connections with numerical analysis, see e.g. [Després, 2022]

# Ingredient 1: Neural Networks – Gradients

**Differentiation:** can compute partial derivatives by automatic differentiation (AD) at every $(\theta, x)$:

- With respect to parameters: $\nabla_\theta \varphi(\theta, x)$

$$\nabla_{\beta^{(\ell)}} \varphi(\theta, x) = \ldots, \qquad \nabla_{w^{(2)}} \varphi(\theta, x) = \ldots$$

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- With respect to state variable: $\nabla_x \varphi(\theta, x)$ can be computed by AD too

$$\partial_{x_1} \varphi(\theta, x) = \dots$$

$\Rightarrow$ can be used in PDEs (see lecture 5)

# Ingredient 2: Stochastic Gradient Descent

- **Goal:** Minimize over $\varphi(\cdot)$, $\mathbb{J}(\varphi) := \mathbb{E}_\xi[\mathbb{L}(\varphi, \xi)]$
- **Parameterization:** $\widetilde{\mathbb{J}}(\theta) := \mathbb{E}_\xi[\widetilde{\mathbb{L}}(\theta, \xi)]$, where $\widetilde{\mathbb{L}}(\theta, \xi) := \mathbb{L}(\varphi_\theta, \xi)$

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- **Setting:** the distribution of $\xi$ is unknown so we cannot compute $\mathbb{E}_\xi$, but
  - we have some samples (i.e. random realizations) of $\xi$
  - we know $\mathbb{L}$

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**Algorithm:** Stochastic Gradient Descent

**Input:** Initial param. $\theta_0$; data $S = (\xi_s)_{s=1,\dots,|S|}$; nb of steps $\mathtt{K}$; learning rates $(\eta^{(\mathtt{k})})_{\mathtt{k}}$

**Output:** Parameter $\theta^\star$ s.t. $\varphi_{\theta^\star}$ (approximately) minimizes $\widetilde{\mathbb{J}}$

Initialize $\theta^{(0)} = \theta_0$

**for** $\mathtt{k} = 0, 1, 2, \dots, \mathtt{K} - 1$ **do**

  Pick $s \in S$ randomly

  Compute the gradient $\nabla_\theta \widetilde{\mathbb{L}}(\theta^{(\mathtt{k}-1)}, \xi_s) = \frac{d}{d\theta} \mathbb{L}(\varphi_{\theta^{(\mathtt{k}-1)}}, \xi_s)$

  Set $\theta^{(\mathtt{k})} = \theta^{(\mathtt{k}-1)} - \eta^{(\mathtt{k})} \nabla_\theta \widetilde{\mathbb{L}}(\theta^{(\mathtt{k}-1)}, \xi_s)$

**return** $\theta^{(\mathtt{K})}$

# Ingredient 2: Stochastic Gradient Descent – Comments

- Many variants:
  - Learning rate: `ADAM` (Adaptive Moment Estimation) [Kingma and Ba, 2014], ...
  - Samples: Mini-batches, ...

- Proofs of convergence e.g. using stochastic approximation [Robbins and Monro, 1951], [Borkar, 2009]

- In practice: many details to be discussed, see e.g.[Bottou, 2012]; choice of hyperparameters
  - architecture
  - initialization
  - learning rate
  - loss function
  - ...

# Analysis: Error Types

- Consider the task: minimize over $\varphi$ the population risk:

$$\mathcal{R}(\varphi) = \mathbb{E}_{x,y}[L(\varphi(x), y)]$$

with $x \sim \mu$ and $y = f(x) + \epsilon$ for some noise $\epsilon$ where $f$ is unknown

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- In practice:
  - minimize over a hypothesis class $\mathcal{F}$ of $\varphi$
  - finite number of samples, $S = (x_m, y_m)_{m=1,\dots,M}$: empirical risk:

$$\hat{\mathcal{R}}_S(\varphi) = \frac{1}{M} \sum_{m=1}^{M} L(\varphi(x_m), y_m) \qquad (+\text{ regu})$$

  - finite number of optimization steps, say `k`

## Analysis: Error Types

We are interested in:

- Approximation error: Letting $\varphi^* = \operatorname{argmin}_{\varphi \in \mathcal{F}} \operatorname{dist}(\varphi, f)$,

$$\epsilon_{\text{approx}} = \operatorname{dist}(\varphi^*, f)$$

- Estimation error: Letting $\hat{\varphi}_S = \operatorname{argmin}_{\varphi \in \mathcal{F}} \hat{\mathcal{R}}_S(\varphi)$

$$\epsilon_{\text{estim}} = \operatorname{dist}(\hat{\varphi}_S, \varphi^*)$$

- Optimization error: After `k` steps, we get $\varphi_S^{(\mathtt{k})}$;

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- Approximation error: Letting $\varphi^* = \operatorname{argmin}_{\varphi \in \mathcal{F}} \operatorname{dist}(\varphi, f)$,

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- Optimization error: After $\mathtt{k}$ steps, we get $\varphi_S^{(\mathtt{k})}$;

$$\epsilon_{\text{optim}} = \operatorname{dist}(\varphi_S^{(\mathtt{k})}, \hat{\varphi}_S)$$

- Generalization error of the learnt $\varphi_S^{(\mathtt{k})}$:

$$\epsilon_{\text{gene}} = \epsilon_{\text{approx}} + \epsilon_{\text{estim}} + \epsilon_{\text{optim}}$$

# Outline

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# From optimal control to optimization

- An optimal control is a "temporally extended" optimization problem
- Numerically, we cannot minimize over all possible controls
- We can parameterize the control function
- and then optimize over the parameters
- See e.g. [Gobet and Munos, 2005], [Han and E, 2016], ...

**Stochastic optimal control problem:**

Minimize over $\alpha(\cdot,\cdot)$

$$J(\alpha(\cdot,\cdot)) = \mathbb{E}\Big[\int_0^T f(X_t, \alpha(t, X_t))\, dt + g(X_T)\Big],$$

with

$$X_0 \sim m_0\,, \quad dX_t = b(X_t, \alpha(t, X_t))\, dt + \sigma dW_t$$

# Stochastic Optimal Control: Approximate Problem

**Stochastic optimal control problem:** (1) neural network $\varphi_\theta$,

Minimize over **neural network** parameters $\theta$

$$J(\theta) = \mathbb{E}\Big[ \int_0^T f\left(X_t, \varphi_\theta(t, X_t)\right)\, dt + g\left(X_T\right) \Big],$$

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**Stochastic optimal control problem:** (1) neural network $\varphi_\theta$, (2) time discretization

Minimize over **neural network** parameters $\theta$ and $N_T$ time steps

$$J^{N_T}(\theta) = \mathbb{E}\Big[ \sum_{n=0}^{N_T-1} f\left(X_n, \varphi_\theta(t_n, X_n)\right) \Delta t + g\left(X_{N_T}\right) \Big],$$

with

$$X_0 \sim m_0 , \quad X_{n+1} - X_n = b(X_n, \varphi_\theta(t_n, X_n))\Delta t + \sigma \Delta W_n$$

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→ neural network induces an approximation error

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To implement SGD, at each iteration we pick a sample $\xi = (X_0, \Delta W_0, \dots, \Delta W_{N_T-1})$

# Outline

# MFC: Approximate Problem

**MFC problem:**

Minimize over $\alpha(\cdot, \cdot)$

$$J(\alpha(\cdot, \cdot)) = \mathbb{E}\Big[ \int_0^T f(X_t, \mu_t, \alpha(t, X_t))\, dt + g(X_T, \mu_T) \Big],$$

where $\mu_t = \mathcal{L}(X_t)$ with

$$X_0 \sim m_0 \,, \quad dX_t = b(X_t, \mu_t, \alpha(t, X_t))\, dt + \sigma dW_t$$

# MFC: Approximate Problem

**MFC problem:** (1) Finite pop.,

Minimize over **decentralized** controls $\alpha(\cdot,\cdot)$ with $N$ agents

$$J^N(\alpha(\cdot,\cdot)) = \mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^{N}\int_0^T f\left(X_t^i, \mu_t^N, \alpha(t, X_t^i)\right)\, dt + g\left(X_T^i, \mu_T^N\right)\Big],$$

where $\mu_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}$, with

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$$J^{N,N_T}(\theta) = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\sum_{n=0}^{N_T-1} f\left(X_n^i, \mu_n^N, \varphi_\theta(t_n, X_n^i)\right)\Delta t + g\left(X_{N_T}^i, \mu_{N_T}^N\right)\right],$$

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Note: we aim for a decentralized control, whereas for a general $N$-agent control problem, the optimal control is not always of this type

# Convergence Analysis

- The following kind of convergence result (bound on the **approximation error**) can be proved, see [Carmona and Laurière, 2022]:

**Approximation theorem**

Under suitable assumptions (in particular regularity of the value function),

$$\left| \inf_{\alpha(\cdot,\cdot)} J(\alpha(\cdot,\cdot)) - \inf_{\theta \in \Theta} J^{N,N_T}(\theta) \right| \leq \epsilon_1(N) + \epsilon_2(\dim(\theta)) + \epsilon_3(N_T)$$

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- The **optimization error** remains to be studied
- Many extensions and refinements to be investigated

# Approximation Error Analysis: Main Ingredients of the Proof

**Proposition 1** ($N$ agents & decentralized controls):

Under suitable assumptions, there exists a decentralized control $\alpha^*$ s.t. *($d$ = dimension of $X_t$)*

$$\left| \inf_{\alpha(\cdot)} J(\alpha(\cdot)) - J^N(\alpha^*(\cdot)) \right| \leq \epsilon_1(N) \in \widetilde{O}\left(N^{-1/d}\right).$$

**Proof: propagation of chaos** type argument [Carmona and Delarue, 2018]

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**Proposition 2** (approximation by neural networks): Under suitable assumptions

There exists a set of parameters $\theta \in \Theta$ for a one-hidden layer $\hat{\varphi}_\theta$ s.t.

$$\left| J^N(\alpha^*(\cdot)) - J^N(\hat{\varphi}_\theta(\cdot)) \right| \leq \epsilon_2(\dim(\theta)) \in O\left(\dim(\theta)^{-\frac{1}{3(d+1)}}\right).$$

**Proof: Key difficulty:** approximate $v^*(\cdot)$ by $\hat{\varphi}_\theta(\cdot)$ while controlling $\|\nabla \hat{\varphi}_\theta(\cdot)\|$ by $\|\nabla v^*(\cdot)\|$

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**Proposition 3** (Euler-Maruyama scheme):

For a specific neural network $\hat{\varphi}_\theta(\cdot)$,

$$\left| J^N(\hat{\varphi}_\theta(\cdot)) - J^{N,N_T}(\hat{\varphi}_\theta(\cdot)) \right| \leq \epsilon_3(N_T) \in O\left(N_T^{-1/2}\right).$$

**Key point:** $O(\cdot)$ independent of $N$ and $\dim(\theta)$

**Proof:** analysis of **strong error rate** for Euler scheme (reminiscent of [Bossy and Talay, 1997])

- Key idea: replace optimal control problem by (finite dim.) optimization problem:
  - Loss function = cost: $J^{N,N_T}(\theta) = \mathbb{E}[\mathbb{L}(\varphi_\theta, \xi)]$
  - One sample: $\xi = \left(X_0^j, (\Delta W_n^j)_{n=0,\dots,N_T-1}\right)_{j=1,\dots,N}$

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# Implementation

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  - One sample: $\xi = \left(X_0^j, (\Delta W_n^j)_{n=0,\dots,N_T-1}\right)_{j=1,\dots,N}$

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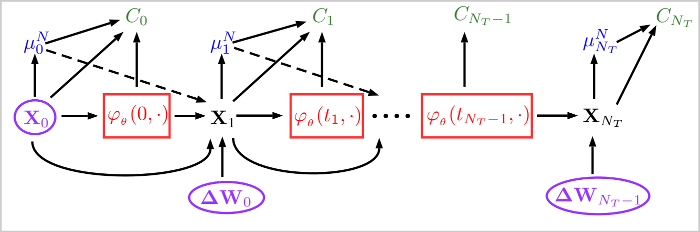

# Implementation

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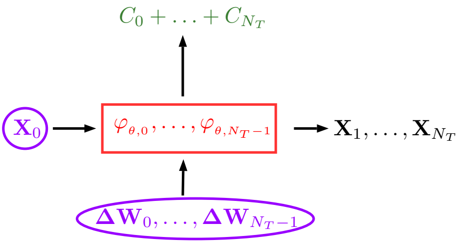


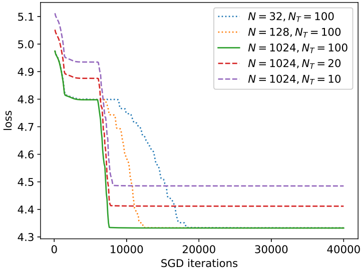
# Numerical Illustration 1: LQ MFC

**Benchmark** to assess **empirical convergence of SGD:** LQ problem with explicit sol.

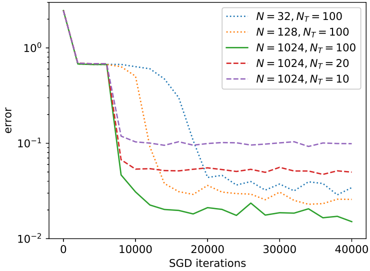
**Example:** Linear dynamics, quadratic costs of the type

$$f(x,\mu,v) = \underbrace{(\bar{\mu} - x)^2}_{\text{distance to mean position}} + \underbrace{v^2}_{\text{cost of moving}}, \qquad \bar{\mu} = \underbrace{\int \mu(\xi)d\xi}_{\text{mean position}}, \qquad g(x) = x^2$$

Numerical example with $d = 10$ (see [Carmona and Laurière, 2022]):




total cost (= loss function)




$L^2$-error on the control

# Numerical Illustration 2: min-LQ MFC with common noise

The following model is inspired by [Salhab et al., 2015] and [Achdou and Lasry, 2019].

**MFC with simple CN:**

Dynamics: $dX_t = \phi_t(X_t, \epsilon^0_t)dt + \sigma dW_t$, $\epsilon^0_t = 0$ until $t = T/2$, and then $\xi_1$ or $\xi_2$ w.p. $1/2$

Running cost $|\phi_t(X_t, \epsilon^0_t)|^2$, final cost $(X_T - \epsilon^0_T)^2 + \bar{Q}_T(\bar{m}_T - X_T)^2$

Parameter values: $\sigma = 0.1$, $T = 1$, $\xi_1 = -1.5$, $\xi_2 = +1.5$

Numerical results:

- neural network $\varphi_\theta(t, X_t, \epsilon^0_t)$, taking as an input the common noise
- benchmark solution computed by solving a **system of 6 PDEs** (see [Achdou and Lasry, 2019])

# Numerical Illustration 2: min-LQ MFC with common noise

Here the common noise takes one among two values, at time $T/2$.

More details in [Carmona and Laurière, 2022]

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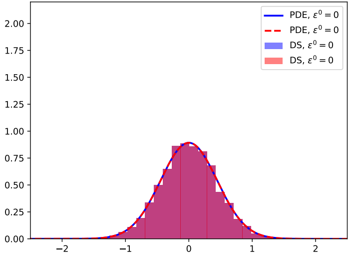


$t = 0$

Until $T/2$: concentrate around mid-point $= 0$

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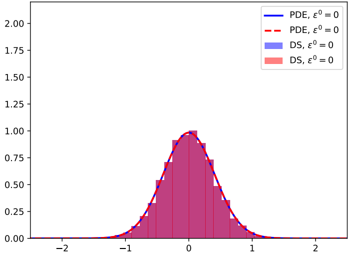


$t = 0.1$

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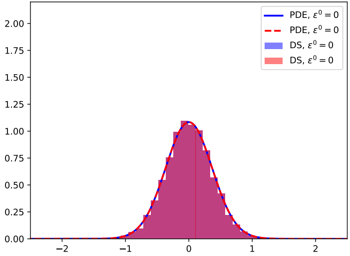


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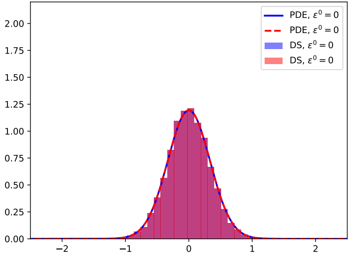


$t = 0.3$

Until $T/2$: concentrate around mid-point $= 0$

More details in [Carmona and Laurière, 2022]

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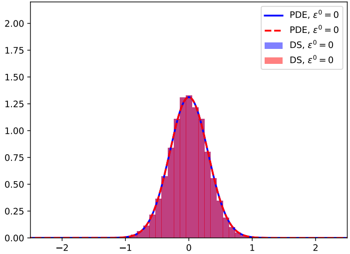


$$t = 0.4$$

Until $T/2$: concentrate around mid-point $= 0$

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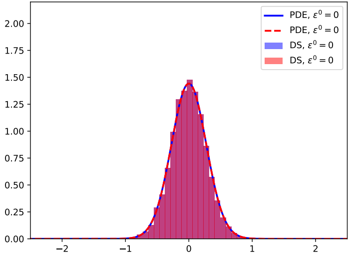


$t = 0.5$

Until $T/2$: concentrate around mid-point $= 0$

More details in [Carmona and Laurière, 2022]

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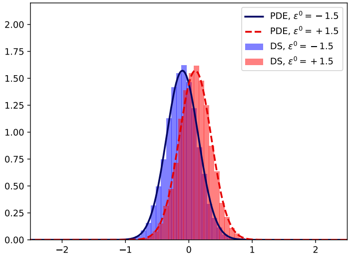


$t = 0.6$

Until $T/2$: concentrate around mid-point $= 0$

After $T/2$: move towards the target selected by common noise

More details in [Carmona and Laurière, 2022]

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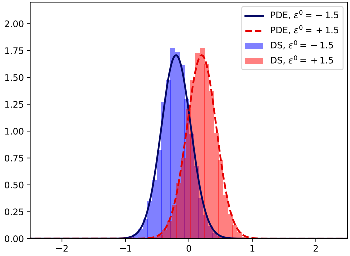


$t = 0.7$

Until $T/2$: concentrate around mid-point $= 0$

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More details in [Carmona and Laurière, 2022]

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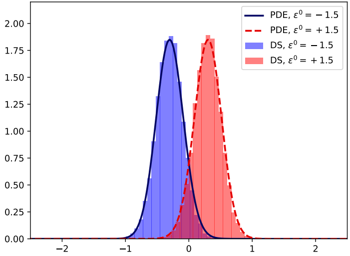


$t = 0.8$

Until $T/2$: concentrate around mid-point $= 0$

After $T/2$: move towards the target selected by common noise

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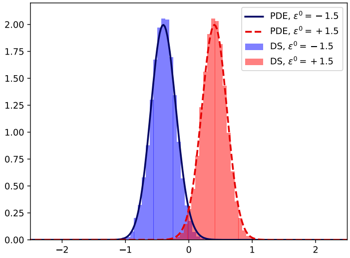


$t = 0.9$

Until $T/2$: concentrate around mid-point $= 0$

After $T/2$: move towards the target selected by common noise

More details in [Carmona and Laurière, 2022]

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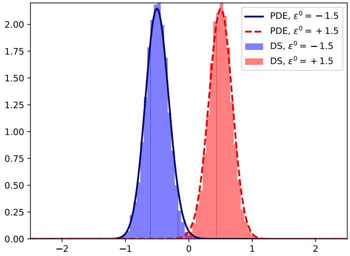


$t = 1$

Until $T/2$: concentrate around mid-point $= 0$

After $T/2$: move towards the target selected by common noise

More details in [Carmona and Laurière, 2022]

# Numerical Illustration 3: MFC with Interactions Through the Controls

**Price Impact Model** [Carmona and Lacker, 2015, Carmona and Delarue, 2018]:

- Price process: with $\nu^{\alpha}$ = population's distribution over actions,

$$dS_t^{\alpha} = \gamma \int_{\mathbb{R}} a d\nu_t^{\alpha}(a) dt + \sigma_0 dW_t^0$$

- Typical agent's inventory: $dX_t^{\alpha} = \alpha_t dt + \sigma dW_t$
- Typical agent's wealth: $dK_t^{\alpha} = -\big(\alpha_t S_t^{\alpha} + c_{\alpha}(\alpha_t)\big) dt$
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**Objective:** minimize

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Equivalent problem:

$$J(\alpha) = \mathbb{E}\Big[ \int_0^T \left( c_\alpha(\alpha_t) + c_X(X_t^\alpha) - \gamma X_t^\alpha \int_{\mathbb{R}} a d\nu_t^\alpha(a) \right) dt + g(X_T^\alpha) \Big]$$

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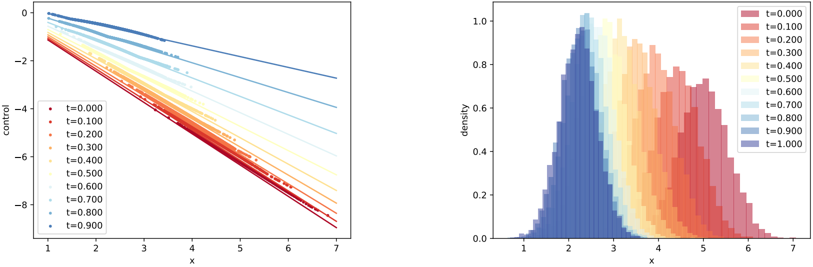
We take: $c_\alpha(v) = \frac{1}{2} c_\alpha v^2$, $c_X(x) = \frac{1}{2} c_X x^2$ and $g(x) = \frac{1}{2} c_g x^2$

# Numerical Illustration 3: MFC with Interactions Through the Controls

Control learnt (left) and associated state distribution (right)




$$T = 1,\ c_X = 2,\ c_\alpha = 1,\ c_g = 0.3,\ \sigma = 0.5,\ \gamma = 0.2$$

See Section 2 in [Carmona and Laurière, 2023] for more details.

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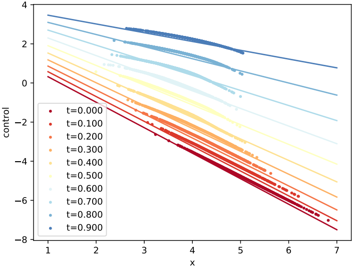


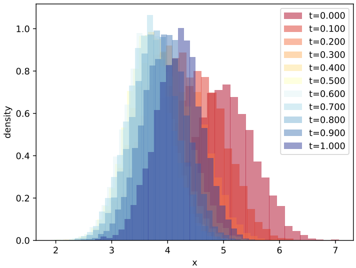


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# Sample code

## Code

Sample code to illustrate: IPython notebook

```
https://colab.research.google.com/drive/1QYWz4Sclw9goRZsbd0uB6wR6a0Uu0a3k?usp=sharing
```

- Deep learning for MFC using a direct approach where the control is parameterized as a neural network
- Applied to the price impact model discussed above

# Related works

- DL for stochastic control [Gobet and Munos, 2005], [Han and E, 2016], . . .
- Various possible implementations; example: 1 NN per time step instead of a single 1 NN as a function of time
- Extensions to finite-player games [Hu, 2021]
- Extension to MFC presented here [Carmona and Laurière, 2022]; see also [Carmona and Laurière, 2023]
- Related works with mean field: [Fouque and Zhang, 2020] (MFC with delay), [Germain et al., 2019], [Agram et al., 2020], [Dayanikli et al., 2023] (with population-dependent controls), . . .

# Outline

# Shooting Method for FBSDE

- Goal: solve an FBSDE system
- The backward process has a value $Y_0$ at time $0$, but it is not known
- Try to guess the correct initial condition so that the terminal condition is satisfied
- This yields a new optimal control problem
- See e.g. [Kohlmann and Zhou, 2000], [Sannikov, 2008], ...
- For the new optimal control problem, use deep learning [E et al., 2017]

## DeepBSDE (E et al.)

Solutions of sto. control problems can be characterized by **FBSDEs** of the form

$$\begin{cases} dX_t = B(t, X_t, Y_t)dt + dW_t, & X_0 \sim m_0 & \to \text{state} \\ dY_t = -F(t, X_t, Y_t)dt + Z_t \cdot dW_t, & Y_T = G(X_T) & \to \text{control/cost} \end{cases}$$

(stemming from sto. Pontryagin's or Bellman's principle: $F = f$ or $F = \partial_x H$)

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**Reformulation** as a new **optimal control problem**

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$$\mathfrak{J}(y_0(\cdot), \mathbf{z}(\cdot)) = \mathbb{E}\left[ \|Y_T^{y_0,\mathbf{z}} - G(X_T^{y_0,\mathbf{z}})\|^2 \right],$$

under the constraint that $(X^{y_0,\mathbf{z}}, Y^{y_0,\mathbf{z}})$ solve: $\forall t \in [0, T]$

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Note: This problem is *not* the original stochastic control problem !

# Application to Solve PDEs

This method can be used to solve PDEs [E et al., 2017]

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- $X$ solves the SDE:

$$dX_t = B(t,x)dt + \sigma dW_t$$

- $(Y, Z)$ solves the BSDE:

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- In fact $Z_t = \sigma \partial_x u(t, X_t)$

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This method can be used to solve PDEs [E et al., 2017]

**Feynman-Kac formula**: correspondence $u(t, X_t) = Y_t$ where

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- Ex. HJB equation. Many variations/extensions

# Deep MKV FBSDE

Solutions of MFG (and MFC) can be characterized by **MKV FBSDEs** of the form

$$\begin{cases} dX_t = B(t, X_t, \mathcal{L}(X_t), Y_t)dt + dW_t, & X_0 \sim m_0 & \rightarrow \text{state} \\ dY_t = -F(t, X_t, \mathcal{L}(X_t), Y_t)dt + Z_t \cdot dW_t, & Y_T = G(X_T, \mathcal{L}(X_T)) & \rightarrow \text{control/cost} \end{cases}$$

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**Reformulation** as a **MFC problem** [Carmona and Laurière, 2022]

**Minimize** over $y_0(\cdot)$ and $\mathbf{z}(\cdot) = (z_t(\cdot))_{t\geq 0}$

$$\mathfrak{J}(y_0(\cdot), \mathbf{z}(\cdot)) = \mathbb{E}\Big[ \|Y_T^{y_0,\mathbf{z}} - G(X_T^{y_0,\mathbf{z}}, \mathcal{L}(X_T^{y_0,\mathbf{z}}))\|^2 \Big],$$

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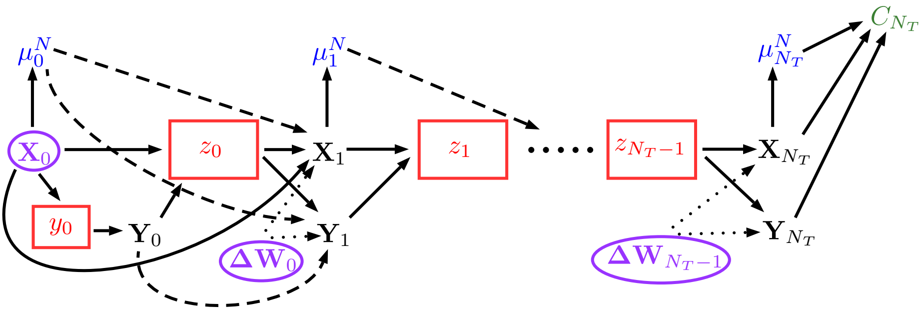
NB: This problem is *not* the original MFG or MFC

# Implementation




- **Inputs:** initial positions $\mathbf{X}_0 = (X_0^i)_i$, BM increments: $\mathbf{\Delta W}_n = (\Delta W_n^i)_i$, for all $n$
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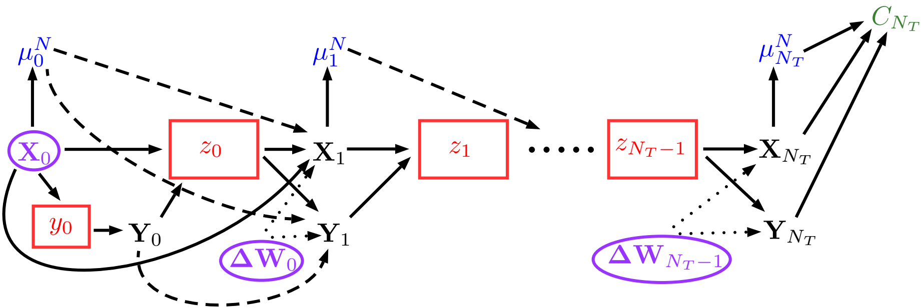


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- Alternative implementation: $1 + N_T$ NNs for $y_0(\cdot), z_0(\cdot), \ldots, z_{N_T-1}(\cdot)$

# Numerical Illustration 1: Comparison with Picard Solver

**Example of MKV FBSDE** from [Chassagneux et al., 2019] ($\boldsymbol{\rho} =$ coupling parameter)

$$dX_t = -\boldsymbol{\rho} Y_t dt + \sigma dW_t, \qquad X_0 = x_0$$

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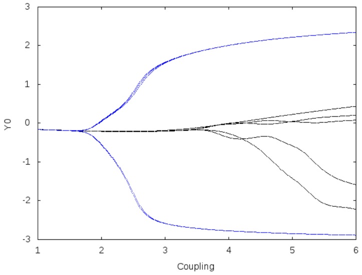
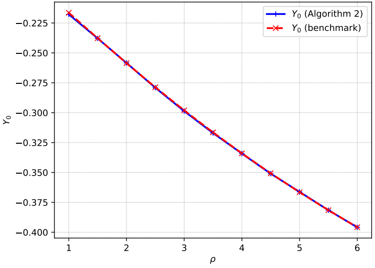



[Chassagneux et al., 2019]




NN (FBSDE system)

More details in [Carmona and Laurière, 2022]

# Numerical Illustration 2: LQ MFG with Common Noise

**Example: MFG for inter-bank borrowing/lending** [Carmona et al., 2015]

$X$ = log-monetary reserve, $\alpha$ = rate of borrowing/lending to central bank, cost:

$$J(\alpha; \bar{m}) = \mathbb{E}\left[\int_0^T \left[\frac{1}{2}\alpha_t^2 - q\alpha_t(\bar{m}_t - X_t) + \frac{\epsilon}{2}(\bar{m}_t - X_t)^2\right] dt + \frac{c}{2}(\bar{m}_T - X_T)^2\right]$$

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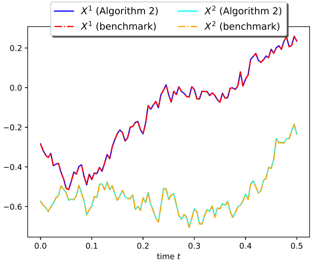
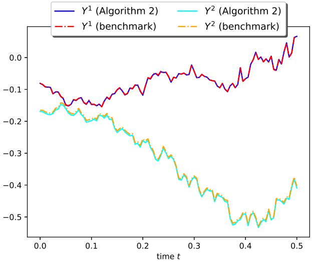
**NN** for FBSDE system VS (semi) analytical solution (LQ structure)




Samples of $X$




Samples of $Y$

More details in [Carmona and Laurière, 2022]

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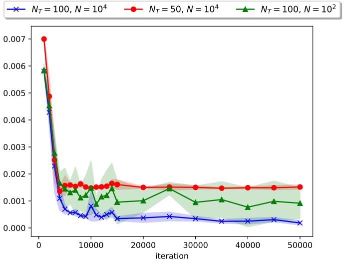
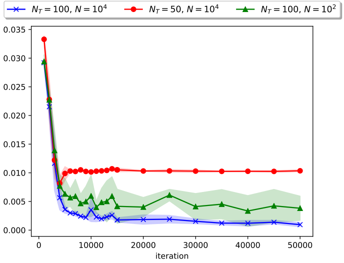



$L^2$ error on $X$




$L^2$ error on $Y$

More details in [Carmona and Laurière, 2022]

# Sample code

## Code

Sample code to illustrate: IPython notebook

https://colab.research.google.com/drive/1RXmRaf2OqtNensLMt0ASn25PfSC58saY?usp=sharing

- Deep learning for MKV FBSDEs
- Applied to the systemic risk model discussed above

# Comments

- Convergence of the DeepBSDE method [Han and Long, 2020]
- Extension to finite-player games [Han et al., 2022]
- Analysis of the different types of errors to be done for MKV case
- The new MFC problem is not standard
- Deep learning of MKV FBSDEs as presented here [Carmona and Laurière, 2022]; see also [Carmona and Laurière, 2023]
- Related works on deep learning for MKV FBSDEs: [Fouque and Zhang, 2020] (MFC with delay), [Germain et al., 2019], [Aurell et al., 2022b], . . .
- Similar "shooting" strategy can be applied to (infinite-dimensional) ODE systems obtained in graphon games [Aurell et al., 2022a]. Code (Gökçe Dayanıklı):

  `https://github.com/gokce-d/GraphonEpidemics`

# Outline

# Outline

# Stackelberg MFG

MFG with a Stackelberg (leader-follower) structure:

- A Principal chooses a policy $\lambda$
- A population of agents react and form a Nash equilibrium:

$$J^{\boldsymbol{\lambda}}(\boldsymbol{\alpha}, \boldsymbol{\mu}) := \mathbb{E}\left[\int_0^T f(t, X_t, \alpha_t, \mu_t; \lambda(t))dt + g(X_T, \mu_T; \lambda(T))\right],$$

- This is an MFG parameterized by $\lambda$
- The resulting mean field flow $\hat{\boldsymbol{\mu}}^{\lambda}$ incurs a cost to the principal

$$J^0(\boldsymbol{\lambda}) := \int_0^T f_0(t, \hat{\mu}_t^{\boldsymbol{\lambda}}, \lambda(t))dt + g_0(\hat{\mu}_T^{\boldsymbol{\lambda}}, \lambda(T))$$

Related works: Holmström-Milgrom (1987), Sannikov (2008, 2013), Djehiche-Helgesson (2014), Cvitanić *et al* (2018), Carmona-Wang (2018), Elie *et al* (2019)

# DL for Stackelberg MFG

Reminder:

- MFG solution can be characterized using a MKV FBSDE system
- This MKV FBSDE can be rewritten as a control problem
  - 2 forward equations
  - terminal cost

Stackelberg MFG:

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For more details, see:

- [Aurell et al., 2022b] with application to epidemics management (finite state MFG): the principal gives guidelines (social distancing, etc.) and the population reacts
- **Code available** ((Gökçe Dayanıklı)):

  `https://github.com/gokce-d/StackelbergMFG`
- Extension to other Stackelberg MFGs: [Dayanikli and Lauriere, 2023]

# Outline

1. Introduction

2. Deep Learning for MFC

3. Deep Learning for MKV FBSDE

4. Two Examples of Extensions
   - Solving Stackelberg MFG with Deep MKV FBSDE
   - Computing MFC Value Function with DBDP

5. Conclusion

# Social optimum: Mean Field Control

Reminder from lecture 2 about mean field (type) control or control of McKean-Vlasov (MKV) dynamics

**Definition (Mean field control (MFC) problem)**

$\alpha^*$ is a solution to the MFC problem if it minimizes

$$J^{MFC}(\alpha) = \mathbb{E}\left[\int_0^T f(X_t^\alpha, \alpha_t, m_t^\alpha)dt + g(X_T^\alpha, m_T^\alpha)\right].$$

Main difference with MFG: here not only $X$ but $m$ too is controlled by $\alpha$.

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Main difference with MFG: here not only $X$ but $m$ too is controlled by $\alpha$.

**Optimality conditions?** Several approaches:

- Dynamic programming value function depending on $m$; value function $V$
- Calculus of variations taking $m$ as a state; adjoint state $u$
- Pontryagin's maximum principle for the (MKV process) $X$; adjoint state $Y$

**Dynamic programming** for MFC [Laurière and Pironneau, 2014], [Bensoussan et al., 2015], [Pham and Wei, 2017], [Djete et al., 2022], ...

$\to$ Algorithm?

# DBDP for Non-Mean Field Control

For standard (non-mean field) stochastic optimal control problems, [Huré et al., 2019] have introduced the Deep Backward Dynamic Programming (DBDP):

**Idea:** learn $Y_n$ and $Z_n$ at each $n$ as functions of $X_n$, backward in time:

- Initialize $\hat{Y}_{N_T} = g$ and then, for $n = N_T - 1, \dots, 0$, either:
- Version 1: Let $(\hat{Y}_n, \hat{Z}_n)$ = minimizer over $(Y_n, Z_n)$ of:

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For more details on deep learning methods for (non-mean field) optimal control problems, see e.g. [Germain et al., 2021b]

# DBDP for MFC

- Can we apply the same idea to MFC, replacing $V$ by a neural network?
- Main challenge: the value function $V$ takes $m \in \mathcal{P}(\mathbb{R}^d)$ as an input
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- Main challenge: the value function $V$ takes $m \in \mathcal{P}(\mathbb{R}^d)$ as an input
- We need to approximate $m$
- One possibility:

$$V(t, m_t) \approx \tilde{V}(t, m_t^N) \approx \tilde{V}_\theta(t, X_t^1, \dots, X_t^N)$$

  where $\tilde{V}_\theta$ is a neural network which is symmetric with respect to the inputs
- See the next lecture for more details
- See [Germain et al., 2021a] for more details about the implementation and [Germain et al., 2022] for the analysis
- See also e.g. [Dayanikli et al., 2023] for different approximations of the population

# Outline

# Summary

- Two algorithms based on the stochastic approach
- Direct approach without any optimality condition
- DeepBSDE: recasting (MKV) FBSDEs as control problems
- Many possible extensions and variations
- Some surveys on DL for control/games:
  [Germain et al., 2021b, Carmona and Laurière, 2023, Hu and Laurière, 2023]

Next lecture: deep learning methods for the PDE approach

Thank you for your attention

Questions?

Feel free to reach out: `mathieu.lauriere@nyu.edu`

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