Mean Field Games: Numerical Methods and Applications in Machine Learning

Part 5: Deep Learning for MFC and MKV FBSDE

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https://mlauriere.github.io/teaching/MFG-PKU-5.pdf

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RECAP

Numerical Methods for MFG: Some references

Methods based on a deterministic approach:

- Finite diff. & Newton meth.: [Achdou, Capuzzo-Dolcetta'10; Achdou, Capuzzo-Dolcetta, Camilli'13; . . .]
- Gradient descent: [L., Pironneau'14; Pfeiffer'16]
- Semi-Lagrangian scheme: [Carlini, Silva'14; Carlini, Silva'15]
- Augmented Lagrangian & ADMM: [Benamou, Carlier'14; Achdou, L.'16; Andreev'17]
- Primal-dual algo.: [Briceño-Arias, Kalise, Silva'18; BAKS + Kobeissi, L., Mateos González'18]
- Monotone operators: [Almulla *et al.*'17; Gomes, Saúde'18; Gomes, Yang'18]

Methods based on a probabilistic approach:

- Cubature: [Chaudru de Raynal, Garcia Trillos'15]
- Recursion: [Chassagneux *et al.*'17; Angiuli *et al.*'18]
- MC & Regression: [Balata, Huré, L., Pham, Pimentel'18]

Surveys and lecture notes: [Achdou'13 (LNM); Achdou, L.'20 (Cetraro); L.'21 (AMS)]

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Limitations:

- dimensionality (typically: state in dimension ≤ 3)
- structure of the problem (typically: simple costs, dynamics and noises)

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Recent progress: extending the toolbox with tools from **machine learning**:

- approximation without a grid (mesh-free methods): **opt. control** & **distribution**
  $\longrightarrow$ [Carmona, L.; Al-Aradi *et al.*; Fouque *et al.*; Germain *et al.*; Ruthotto *et al.*; Agram *et al.*; . . .]
- even when the **dynamics / cost are not known** (model-free methods)
  $\longrightarrow$ [Guo *et al.*; Subramanian *et al.*; Elie *et al.*; Carmona *et al.*; Pham *et al.*; Yang *et al.*; . . .]

Outline

Ingredient 1: Neural Networks

- **Goal:** Minimize over $\varphi(\cdot)$, $\mathbb{J}(\varphi) := \mathbb{E}_\xi[\mathbb{L}(\varphi, \xi)]$
- **Ex.:** Regression: $\xi = (x, f(x))$ for some f, $\mathbb{L}(\varphi, \xi) = \|\varphi(x) - f(x)\|^2$

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- **Idea:** Instead of min. over all $\varphi(\cdot)$, min. over parameters θ of $\varphi_\theta(\cdot)$
- **Ex.: Feedforward fully-connected neural network:**
  φ_θ with weights & biases $\theta = (\beta^{(k)}, w^{(k)})_{k=1,\dots,\ell}$

$$\underbrace{\varphi_\theta(x)}_{\varphi(\theta,x)} = \psi^{(\ell)}\Big(\beta^{(\ell)} + w^{(\ell)} \dots \psi^{(2)}\Big(\beta^{(2)} + w^{(2)} \underbrace{\psi^{(1)}(\beta^{(1)} + w^{(1)}x)}_{\text{one hidden layer}}\Big) \dots\Big)$$

  where $\psi^{(i)} \in \{$ sigmoid, ReLU, $\dots\}$: non-linear activation functions (coordinate-wise)
- Depth = number of layers; width of a layer = dimension of bias vector

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- Other architectures

Ingredient 1: Neural Networks – Universal Approximation

Differentiation: can compute partial derivatives by automatic differentiation (AD) at every (θ, x):

- With respect to parameters: $\nabla_\theta \varphi(\theta, x)$

$$\nabla_{\beta^{(\ell+1)}} \varphi(\theta, x) = \ldots, \qquad \nabla_{w^{(2)}} \varphi(\theta, x) = \ldots$$

$\Rightarrow$ can perform **SGD** on these parameters

Ingredient 1: Neural Networks – Gradients

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- With respect to state variable: $\nabla_x \varphi(\theta, x)$ can be computed by AD too

$$\partial_{x_1} \varphi(\theta, x) = \dots$$

  $\Rightarrow$ can be used in PDEs

Ingredient 2: Stochastic Gradient Descent

Goal: Minimize over $\varphi(\cdot)$, $\mathbb{J}(\varphi) := \mathbb{E}_\xi[\mathbb{L}(\varphi, \xi)]$

Parameterization: $\widetilde{\mathbb{J}}(\theta) := \mathbb{E}_\xi[\widetilde{\mathbb{L}}(\theta, \xi)]$, where $\widetilde{\mathbb{L}}(\theta, \xi) := \mathbb{L}(\varphi_\theta, \xi)$

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- we have some samples (i.e. random realizations) of ξ
- we know $\mathbb{L}$

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Input: Initial param. θ_0; data $S = (\xi_s)_{s=1,\dots,|S|}$; nb of steps $\mathtt{K}$; learning rates $(\eta^{(\mathtt{k})})_{\mathtt{k}}$

Output: Parameter $\theta^\star$ s.t. $\varphi_{\theta^\star}$ (approximately) minimizes $\widetilde{\mathbb{J}}$

Initialize $\theta^{(0)} = \theta_0$
for $\mathtt{k} = 0, 1, 2, \dots, \mathtt{K} - 1$ **do**
  Pick $s \in S$ randomly
  Compute the gradient $\nabla_\theta \widetilde{\mathbb{L}}(\theta^{(\mathtt{k}-1)}, \xi_s) = \frac{d}{d\theta} \mathbb{L}(\varphi_{\theta^{(\mathtt{k}-1)}}, \xi_s)$
  Set $\theta^{(\mathtt{k})} = \theta^{(\mathtt{k}-1)} - \eta^{(\mathtt{k})} \nabla_\theta \widetilde{\mathbb{L}}(\theta^{(\mathtt{k}-1)}, \xi_s)$
return $\theta^{(\mathtt{K})}$

Ingredient 2: Stochastic Gradient Descent – cont.

- Many variants:
  - Learning rate: `ADAM` (Adaptive Moment Estimation), ...
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Ingredient 2: Stochastic Gradient Descent – cont.

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- Generator for $\xi \Rightarrow$ can generate Monte Carlo samples on the fly
- Robbins-Monro
- Links with convex minimization & stochastic approximation

Analysis: Error Types

- Consider the task: minimize over φ the **population risk**:

$$\mathcal{R}(\varphi) = \mathbb{E}_{x,y}[L(\varphi(x), y)]$$

with $x \sim \mu$ and $y = f(x) + \epsilon$ for some noise ϵ where f is unknown

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- In practice:
  - minimize over a **hypothesis class** $\mathcal{F}$ of φ
  - finite number of samples, $S = (x_m, y_m)_{m=1,\dots,M}$: (regularized) **empirical risk**:

$$\hat{\mathcal{R}}_S(\varphi) = \frac{1}{M} \sum_{m=1}^{M} L(\varphi(x_m), y_m) \qquad (+\text{ regu})$$

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  - finite number of **optimization steps**, say $\mathtt{k}$
- We are interested in:
  - **Approximation error:** Letting $\varphi^* = \operatorname{argmin}_{\varphi\in\mathcal{F}} \operatorname{dist}(\varphi, f)$,

$$\epsilon_{\text{approx}} = \operatorname{dist}(\varphi^*, f)$$

  - **Estimation error:** Letting $\hat{\varphi}_S = \operatorname{argmin}_{\varphi\in\mathcal{F}} \hat{\mathcal{R}}_S(\varphi)$

$$\epsilon_{\text{estim}} = \operatorname{dist}(\hat{\varphi}_S, \varphi^*)$$

  - **Optimization error:** After $\mathtt{k}$ steps, we get $\varphi_S^{(\mathtt{k})}$;

$$\epsilon_{\text{optim}} = \operatorname{dist}(\varphi_S^{(\mathtt{k})}, \hat{\varphi}_S)$$

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  - **Generalization error** of the learnt $\varphi_S^{(\mathtt{k})}$:

$$\epsilon_{\text{gene}} = \epsilon_{\text{approx}} + \epsilon_{\text{estim}} + \epsilon_{\text{optim}}$$

Outline

Stochastic optimal control problem:

Minimize over $v(\cdot,\cdot)$

$$J(v(\cdot,\cdot)) = \mathbb{E}\Big[\int_0^T f(X_t, v(t,X_t))\,dt + g(X_T)\Big],$$

with

$$X_0 \sim m_0\,, \quad dX_t = b(X_t, v(t,X_t))\,dt + \sigma dW_t$$

Stochastic Optimal Control: Approximate Problem

Stochastic optimal control problem: (2) neural network φ_θ,

Minimize over **neural network** parameters θ

$$J(\theta) = \mathbb{E}\Big[\int_0^T f\left(X_t, \varphi_\theta(t, X_t)\right)\, dt + g\left(X_T\right) \Big],$$

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Stochastic Optimal Control: Approximate Problem

Stochastic optimal control problem: (2) neural network φ_θ, (3) time discretization

Minimize over **neural network** parameters θ and N_T time steps

$$J^{N_T}(\theta) = \mathbb{E}\Big[\sum_{n=0}^{N_T-1} f\left(X_n, \varphi_\theta(t_n, X_n)\right) \Delta t + g\left(X_{N_T}\right) \Big],$$

with

$$X_0 \sim m_0\,, \quad X_{n+1} - X_n = b(X_n, \varphi_\theta(t_n, X_n))\Delta t + \sigma \Delta W_n$$

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$\rightarrow$ neural network induces an approximation error

$\rightarrow$ time discretization induce extra errors

MFC: Approximate Problem

MFC problem:

Minimize over $v(\cdot, \cdot)$

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where $\mu_t = \mathcal{L}(X_t)$ with

$$X_0 \sim m_0\,, \quad dX_t = b(X_t, \mu_t, v(t, X_t))\, dt + \sigma dW_t$$

MFC problem: (1) Finite pop.,

Minimize over **decentralized** controls $v(\cdot,\cdot)$ with N agents

$$J^N(v(\cdot,\cdot)) = \mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^{N}\int_0^T f\left(X_t^i, \mu_t^N, v(t, X_t^i)\right)\,dt + g\left(X_T^i, \mu_T^N\right)\Big],$$

where $\mu_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}$, with

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MFC problem: (1) Finite pop., (2) neural network φ_θ,

Minimize over **neural network** parameters θ with N agents

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MFC problem: (1) Finite pop., (2) neural network φ_θ, (3) time discretization

Minimize over **neural network** parameters $\theta \in \Theta$ with N agents and N_T time steps

$$J^{N,N_T}(\theta) = \mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^{N}\sum_{n=0}^{N_T-1} f\left(X_n^i, \mu_n^N, \varphi_\theta(t_n, X_n^i)\right)\Delta t + g\left(X_{N_T}^i, \mu_{N_T}^N\right)\Big],$$

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N.B.: **decentralized** control

Convergence Analysis

- The following kind of convergence result (bound on the **approximation error**) can be proved (see [Carmona, L.'19][1]):
  Under suitable assumptions (in particular regularity of the value function),

$$\left| \inf_{v(\cdot,\cdot)} J(v(\cdot,\cdot)) - \inf_{\theta \in \Theta} J^{N,N_T}(\theta) \right| \leq \epsilon_1(N) + \epsilon_2(\dim(\theta)) + \epsilon_3(N_T)$$

[1] Carmona, R., & Laurière, M. (2019). Convergence Analysis of Machine Learning Algorithms for the Numerical Solution of Mean Field Control and Games: II–The Finite Horizon Case. arXiv preprint arXiv:1908.01613. To appear in *Annals of Applied Probability*

[2] Carmona, R., & Laurière, M. (2021). Convergence Analysis of Machine Learning Algorithms for the Numerical Solution of Mean Field Control and Games I: The Ergodic Case. *SIAM Journal on Numerical Analysis*, 59(3), 1455-1485.

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- Many extensions to be studied

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Approximation Error Analysis: Main Ingredients of the Proof

Proposition 1 (N agents & decentralized controls):

Under suitable assumptions, there exists a decentralized control v^* s.t. *(d = dimension of X_t)*

$$\left| \inf_{v(\cdot)} J(v(\cdot)) - J^N(v^*(\cdot)) \right| \leq \epsilon_1(N) \in \widetilde{O}\left(N^{-1/d}\right).$$

Proof: propagation of chaos type argument [Carmona, Delarue'18]

Approximation Error Analysis: Main Ingredients of the Proof

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There exists a set of parameters $\theta \in \Theta$ for a one-hidden layer $\hat{\varphi}_\theta$ s.t.

$$\left|J^N(v^*(\cdot)) - J^N(\hat{\varphi}_\theta(\cdot))\right| \leq \epsilon_2(\dim(\theta)) \in O\left(\dim(\theta)^{-\frac{1}{3(d+1)}}\right).$$

Proof: Key difficulty: approximate $v^*(\cdot)$ by $\hat{\varphi}_\theta(\cdot)$ while controlling $\|\nabla\hat{\varphi}_\theta(\cdot)\|$ by $\|\nabla v^*(\cdot)\|$

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Proposition 3 (Euler-Maruyama scheme):

For a specific neural network $\hat{\varphi}_\theta(\cdot)$,

$$\left| J^N(\hat{\varphi}_\theta(\cdot)) - J^{N,N_T}(\hat{\varphi}_\theta(\cdot)) \right| \leq \epsilon_3(N_T) \in O\left(N_T^{-1/2}\right).$$

Key point: $O(\cdot)$ independent of N and $\dim(\theta)$

Proof: analysis of **strong error rate** for Euler scheme (reminiscent of [Bossy, Talay'97])

Implementation

- Key idea: replace optimal control problem by (finite dim.) optimization problem:
  - Loss function = cost: $J^{N,N_T}(\theta) = \mathbb{E}[\mathbb{L}(\varphi_\theta, \xi)]$
  - One sample: $\xi = \left(X_0^j, (\Delta W_n^j)_{n=0,\dots,N_T-1}\right)_{j=1,\dots,N}$

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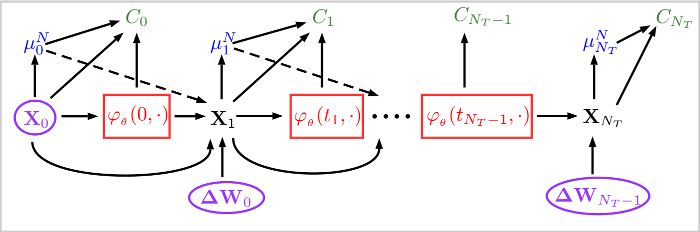

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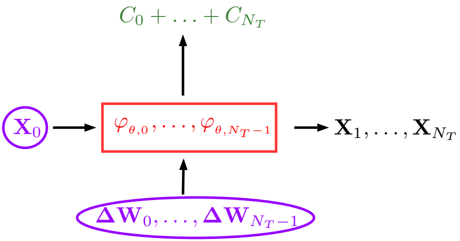


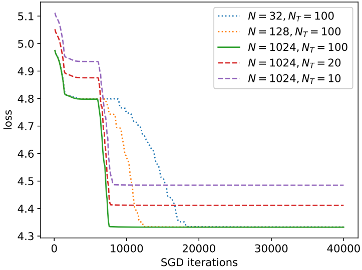
Numerical Illustration 1: LQ MFC

Benchmark to assess **empirical convergence of SGD:** LQ problem with explicit sol.

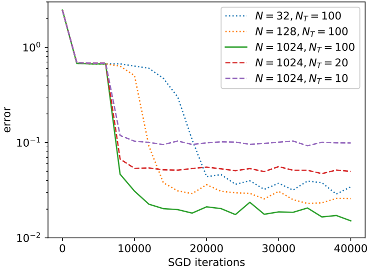
Example: Linear dynamics, quadratic costs of the type

$$f(x,\mu,v) = \underbrace{(\bar\mu - x)^2}_{\text{distance to mean position}} + \underbrace{v^2}_{\text{cost of moving}}, \qquad \bar\mu = \underbrace{\int \mu(\xi)d\xi}_{\text{mean position}}, \qquad g(x) = x^2$$

Numerical example with $d = 10$ (see [Carmona, L.'19]):



total cost (= loss function)



L^2-error on the control

Numerical Illustration 2: min-LQ MFC with common noise

MFC with simple CN (inspired by [Salhab, Malhamé, Le Ny] and [Achdou, Lasry]):

- $dX_t = \phi_t(X_t, \epsilon_t^0)dt + \sigma dW_t$, $\epsilon_t^0 = 0$ until $t = T/2$, and then ξ_1 or ξ_2 w.p. $1/2$
- running cost $|\phi_t(X_t, \epsilon_t^0)|^2$, final cost $(X_T - \epsilon_T^0)^2 + \bar{Q}_T(\bar{m}_T - X_T)^2$

- Ex.: $\sigma = 0.1$, $T = 1$, $\xi_1 = -1.5$, $\xi_2 = +1.5$
- Numerics: **neural network** $\varphi_\theta(t, X_t, \epsilon_t^0)$ VS benchmark with **system of 6 PDEs**

(More details in [Carmona, L.'19])

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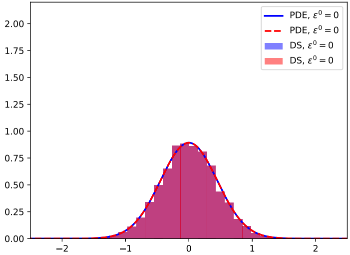

$t = 0$

(More details in [Carmona, L.'19])

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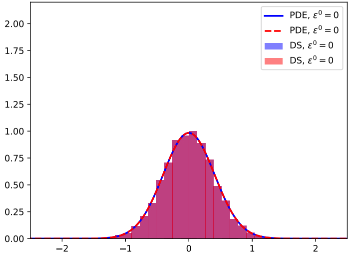

$t = 0.1$

- Until $T/2$: concentrate around mid-point $= 0$

(More details in [Carmona, L.'19])

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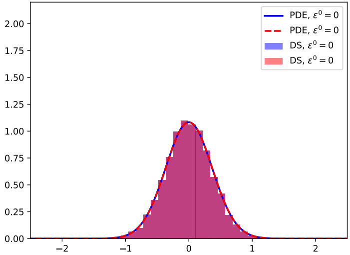

$t = 0.2$

- Until $T/2$: concentrate around mid-point $= 0$

(More details in [Carmona, L.'19])

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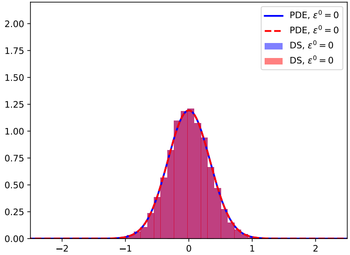

$t = 0.3$

- Until $T/2$: concentrate around mid-point $= 0$

(More details in [Carmona, L.'19])

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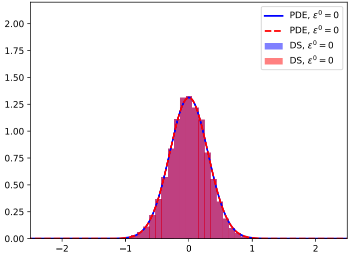

$t = 0.4$

- Until $T/2$: concentrate around mid-point $= 0$

(More details in [Carmona, L.'19])

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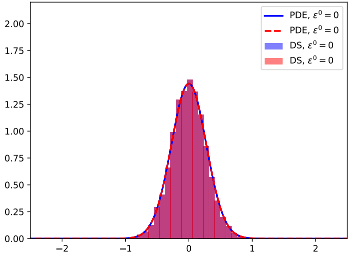

$t = 0.5$

- Until $T/2$: concentrate around mid-point $= 0$

(More details in [Carmona, L.'19])

Numerical Illustration 2: min-LQ MFC with common noise

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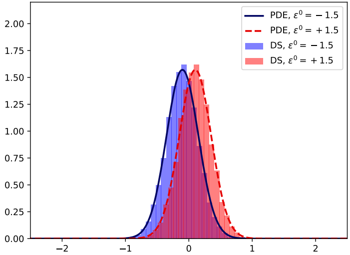

$t = 0.6$

- Until $T/2$: concentrate around mid-point $= 0$
- After $T/2$: move towards the target selected by common noise

(More details in [Carmona, L.'19])

Numerical Illustration 2: min-LQ MFC with common noise

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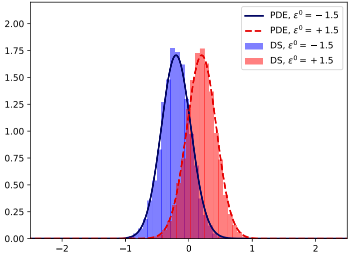

$t = 0.7$

- Until $T/2$: concentrate around mid-point $= 0$
- After $T/2$: move towards the target selected by common noise

(More details in [Carmona, L.'19])

Numerical Illustration 2: min-LQ MFC with common noise

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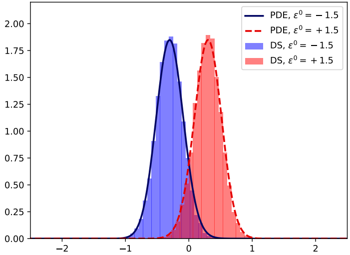

$t = 0.8$

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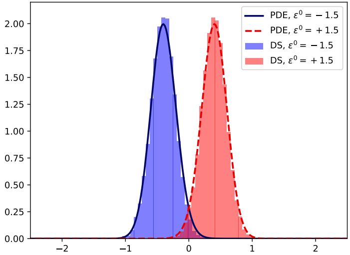

$t = 0.9$

- Until $T/2$: concentrate around mid-point $= 0$
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(More details in [Carmona, L.'19])

Numerical Illustration 2: min-LQ MFC with common noise

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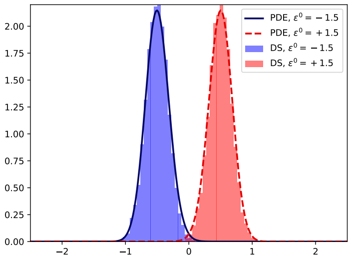

$t = 1$

- Until $T/2$: concentrate around mid-point $= 0$
- After $T/2$: move towards the target selected by common noise

(More details in [Carmona, L.'19])

Numerical Illustration 3: MFC with Interactions Through the Controls

Price Impact Model (see [Carmona, Lacker; Carmona, Delarue]):
Price process: with ν^v = population's distribution over actions,

$$dS_t^v = \gamma \int_{\mathbb{R}} a d\nu_t^v(a) dt + \sigma_0 dW_t^0$$

Typical agent's inventory: $dX_t^v = v_t dt + \sigma dW_t$
Typical agent's wealth: $dK_t^v = -\big(v_t S_t^v + c_v(v_t)\big) dt$
Typical agent's portfolio value: $V_t^v = K_t^v + X_t^v S_t^v$

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Objective: minimize

$$J(v) = \mathbb{E}\Big[\int_0^T c_X(X_t^{v}) dt + g(X_T^{v}) - V_T^{v} \Big]$$

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Equivalent problem:

$$J(v) = \mathbb{E}\Big[\int_0^T \bigg(c_v(v_t) + c_X(X_t^v) - \gamma X_t^v \int_{\mathbb{R}} a d\nu_t^v(a) \bigg) dt + g(X_T^v) \Big]$$

Numerical Illustration 3: MFC with Interactions Through the Controls

Price Impact Model (see [Carmona, Lacker; Carmona, Delarue]):
Price process: with $\nu^v =$ population's distribution over actions,

$$dS_t^v = \gamma \int_{\mathbb{R}} a d\nu_t^v(a) dt + \sigma_0 dW_t^0$$

Typical agent's inventory: $dX_t^v = v_t dt + \sigma dW_t$
Typical agent's wealth: $dK_t^v = -\big(v_t S_t^v + c_v(v_t)\big) dt$
Typical agent's portfolio value: $V_t^v = K_t^v + X_t^v S_t^v$
Objective: minimize

$$J(v) = \mathbb{E}\Big[\int_0^T c_X(X_t^v) dt + g(X_T^v) - V_T^v \Big]$$

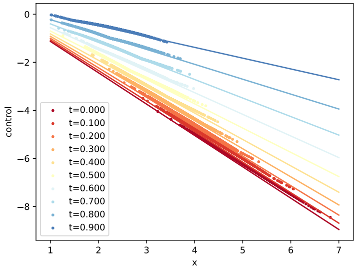
Equivalent problem:

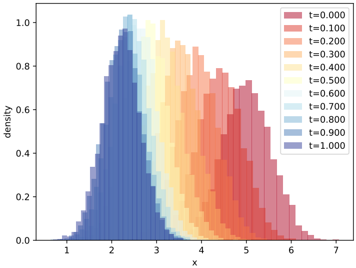
$$J(v) = \mathbb{E}\Big[\int_0^T \left(c_v(v_t) + c_X(X_t^v) - \gamma X_t^v \int_{\mathbb{R}} a d\nu_t^v(a) \right) dt + g(X_T^v) \Big]$$

Take: $c_v(v) = \frac{1}{2} c_v v^2$, $c_X(x) = \frac{1}{2} c_X x^2$ and $g(x) = \frac{1}{2} c_g x^2$

Numerical Illustration 3: MFC with Interactions Through the Controls

Control learnt (left) and associated state distribution (right)





$$T = 1,\ c_X = 2,\ c_v = 1,\ c_g = 0.3,\ \sigma = 0.5,\ \gamma = 0.2$$

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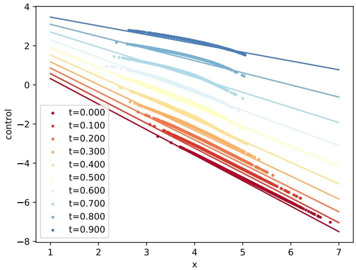

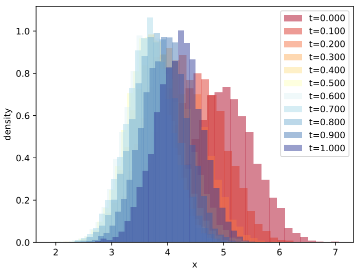

$$T = 1,\ c_X = 2,\ c_v = 1,\ c_g = 0.3,\ \sigma = 0.5,\ \gamma = 1$$

Outline

DeepBSDE: Shooting Method for FBSDE

Solutions of sto. control problems can be characterized by **FBSDEs** of the form

$$\begin{cases} dX_t = B(t, X_t, Y_t)dt + dW_t, & X_0 \sim m_0 & \to \text{state} \\ dY_t = -F(t, X_t, Y_t)dt + Z_t \cdot dW_t, & Y_T = G(X_T) & \to \text{control/cost} \end{cases}$$

(stemming from sto. Pontryagin's or Bellman's principle: $F = f$ or $F = \partial_x H$)

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Reformulation as a new **optimal control problem**

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$$\mathfrak{J}(y_0(\cdot), \mathbf{z}(\cdot)) = \mathbb{E}\left[\|Y_T^{y_0,\mathbf{z}} - G(X_T^{y_0,\mathbf{z}})\|^2 \right],$$

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- Ex. HJB equation. Many variations/extensions

Shooting Method for MKV FBSDE

Solutions of MFG (and MFC) can be characterized by **MKV FBSDEs** of the form

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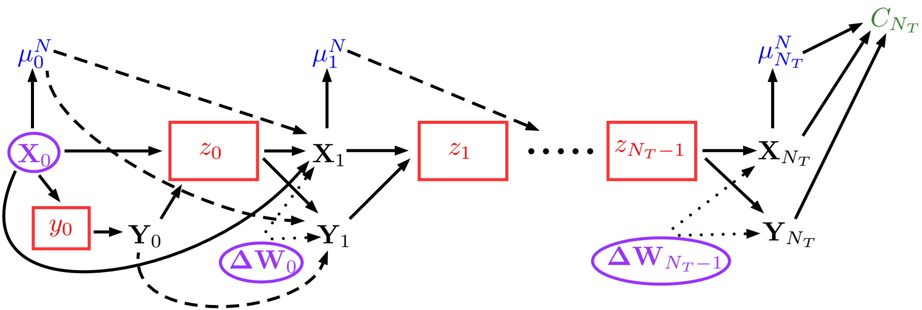
Analysis?

Implementation



- **Inputs:** initial positions $\mathbf{X}_0 = (X_0^i)_i$, BM increments: $\mathbf{\Delta W}_n = (\Delta W_n^i)_i$, for all n
- **Loss function:** total cost = C_{N_T} = terminal penalty; state = (X_n, Y_n)
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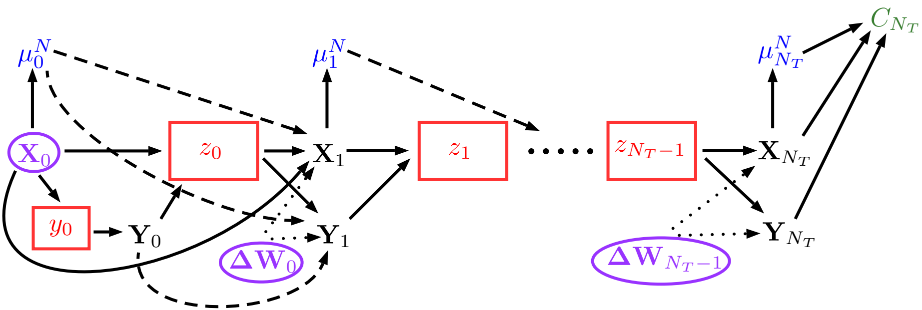

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- Alternative implementation: $1 + N_T$ NNs for $y_0(\cdot), z_0(\cdot), \ldots, z_{N_T-1}(\cdot)$

Numerical Illustration 1: Comparison with Picard Solver

Example of MKV FBSDE from [Chassagneux *et al.*'17] ($\rho =$ coupling parameter)

$$
\begin{aligned}
dX_t &= -\boldsymbol{\rho} Y_t dt + \sigma dW_t, \qquad X_0 = x_0 \\
dY_t &= \operatorname{atan}(\mathbb{E}[X_t])dt + Z_t dW_t, \qquad Y_T = G'(X_T) := \operatorname{atan}(X_T)
\end{aligned}
$$

Comes from the **MFG** defined by $dX_t^v = v_t dt + dW_t$ and

$$
J(v;\mu) = \mathbb{E}\left[G(X_T^v) + \int_0^T \left(\frac{1}{2\boldsymbol{\rho}} v_t^2 + X_t^v \operatorname{atan}\left(\int x \mu_t(dx)\right)\right) dt\right]
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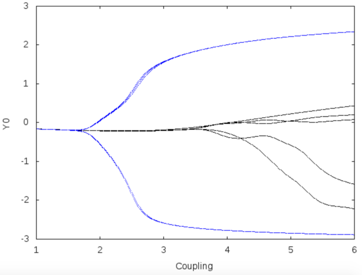
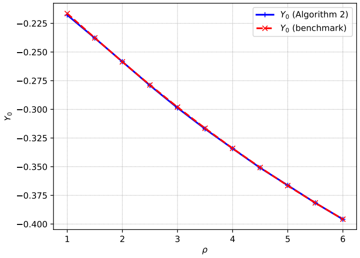


Results from [Chassagneux *et al.*]



NN (FBSDE system)

(More details in [Carmona, L.'19])

Numerical Illustration 2: LQ MFG with Common Noise

Example: MFG for inter-bank borrowing/lending [Carmona, Fouque, Sun]

X = log-monetary reserve, v = rate of borrowing/lending to central bank, cost:

$$J(v;\bar{m}) = \mathbb{E}\left[\int_0^T \left[\frac{1}{2}v_t^2 - q v_t(\bar{m}_t - X_t) + \frac{\epsilon}{2}(\bar{m}_t - X_t)^2\right] dt + \frac{c}{2}(\bar{m}_T - X_T)^2\right]$$

where $\bar{m} = (\bar{m}_t)_{t\geq 0}$ = conditional mean of the population states given W^0, and

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Numerical Illustration 2: LQ MFG with Common Noise

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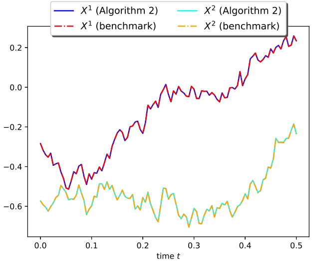
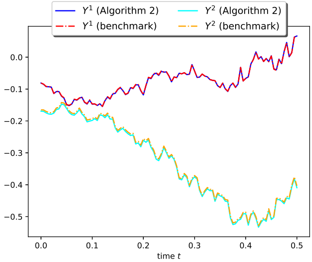
NN for FBSDE system VS (semi) analytical solution (LQ structure)



Samples of X



Samples of Y

(More details in [Carmona, L.'19])

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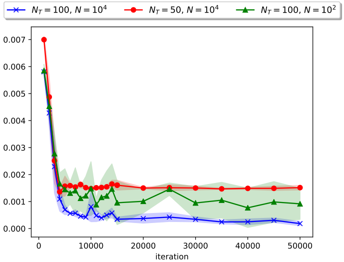
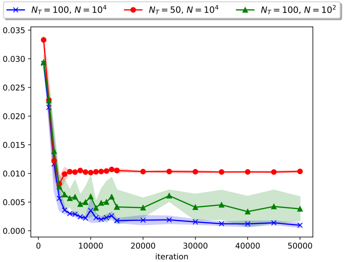


L^2 error on X



L^2 error on Y

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Outline

Methods Based on Dynamic Programming

Method proposed by [Bachouch, Huré, Langrené, Pham'21][4] to minimize:

$$J^{N_T}(v) = \mathbb{E}\left[\sum_{n=0}^{N_T-1} f(X_n, v_n) + g(X_{N_T})\right]$$

where

$$X_{n+1} = X_n + b(X_n, v_n) + \epsilon_{n+1}.$$

[4] Bachouch, A., Huré, C., Langrené, N., & Pham, H. (2021). Deep neural networks algorithms for stochastic control problems on finite horizon: numerical applications. *Methodology and Computing in Applied Probability*, 1-36.

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Input: Training distribution $(\mu_n)_{n=0,\dots,N_T}$

Output: Parameters $(\theta_n^\star)_{n=0,\dots,N_T}$ s.t. $(\varphi_{\theta_n^\star})_{n=0,\dots,N_T}$ (approximately) minimizes J^{N_T}

1 **for** $\mathtt{n} = N_T - 1, N_T - 2, \dots, 1, 0$ **do**

2 Compute (e.g., using SGD) θ_n^* minimizing:

$$\theta \mapsto \mathbb{E}\left[f(X_n^\theta, \varphi_{\theta_n}(X_n^\theta)) + \sum_{n'=n+1}^{N_T-1} f(X_{n'}^\theta, \varphi_{\theta^*_{n'}}(X_{n'}^\theta)) + g(X_{N_T}^v)\right]$$

where $X_n^\theta \sim \mu_n$ and

$$\begin{cases} X_{n+1}^\theta = X_n^\theta + b(X_n^\theta, \varphi_{\theta_n}(X_n^\theta)) + \epsilon_{n+1}, \\ X_{n'+1}^\theta = X_{n'}^\theta + b(X_{n'}^\theta, \varphi_{\theta^*_{n'}}(X_{n'}^\theta)) + \epsilon_{n'+1}, \qquad n' > n. \end{cases}$$

3 **return** $(\theta_n^*)_{n=0,\dots,N_T}$

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Methods Based on Dynamic Programming for MFC