Numerical Methods for Mean Field Games

Lecture 6
Reinforcement Learning Methods

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Outline

Motivations

- In the methods discussed so far, the algorithm uses the full knowledge of the model
  - to write the ODEs or PDEs (lectures 2, 3 and 5)
  - to write the FBSDEs (lecture 4)
  - to compute the gradient in the direct approach (lecture 4)

- Can we learn the solution without using the full knowledge the model and by instead relying on a simulator? $\rightarrow$ model-free reinforcement learning (RL)

- Motivations
  - sometimes we really do not know the model and we only have a simulator (e.g., nature)
  - sometimes we do know the model, but using an exact method is too costly (e.g., very large spaces / complex models)

Motivations

(Reinforcement) Learning in games: many recent successes, e.g.:

Go [Silver et al., 2016, Silver et al., 2017, Silver et al., 2018], Chess [Campbell et al., 2002], Checkers [Schaeffer et al., 2007], Hex [Anthony et al., 2017], Starcraft II [Vinyals et al., 2019], poker games [Brown and Sandholm, 2017, Brown and Sandholm, 2019, Moravčík et al., 2017, Bowling et al., 2015], Stratego [McAleer et al., 2020], [Perolat et al., 2022] . . .

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Motivations for combining RL and MFGs:

- Scaling up **population size** $\rightarrow$ **Mean Field Games**
- Scaling up **environment complexity** $\rightarrow$ (model-free) **Reinforcement Learning**

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Go [Silver et al., 2016, Silver et al., 2017, Silver et al., 2018], Chess [Campbell et al., 2002], Checkers [Schaeffer et al., 2007], Hex [Anthony et al., 2017], Starcraft II [Vinyals et al., 2019], poker games [Brown and Sandholm, 2017, Brown and Sandholm, 2019, Moravčík et al., 2017, Bowling et al., 2015], Stratego [McAleer et al., 2020], [Perolat et al., 2022] . . .

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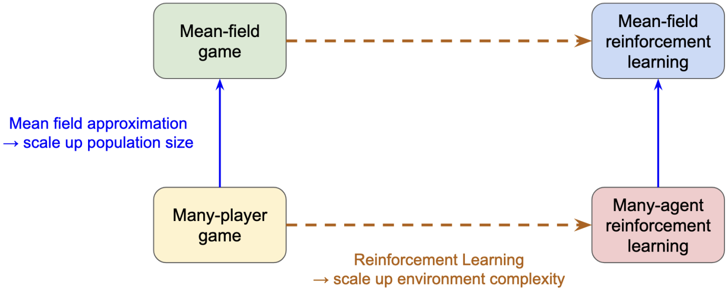

Reinforcement Learning – Setup

- **Markov Decision Process (MDP):** $(\mathcal{S}, \mathcal{A}, p, r, \gamma)$, where:
  - $\mathcal{S}$: state space, $\mathcal{A}$: action space,
  - $p : \mathcal{S} \times \mathcal{A} \to \mathcal{P}(\mathcal{S})$: transition kernel, $p(\cdot|s, a)$ gives next state's distribution
  - $r : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$: reward function, $\gamma \in (0, 1)$: discount factor
- **Goal:** Find (stationary, mixed) policy $\pi^* : \mathcal{S} \to \mathcal{P}(\mathcal{A})$ maximizing:

$$R(\pi) = \mathbb{E}\left[\sum_{n \geq 0} \gamma^n r(s_n, a_n)\right], \qquad \text{with } a_n \sim \pi(\cdot|s_n), s_{n+1} \sim p(\cdot|s_n, a_n)$$

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- **Model:** p, r
- **Two settings:**

  (1) **Known model** : Optimal control theory & methods

  (2) **Sample transitions & rewards**: Reinforcement Learning (RL) framework

We want to **learn** the best control by performing **experiments** of the form:

Given the current state S_t*,*

(1) *Take an action* A_t

(2) *Observe reward* R_{t+1} *& new state* S_{t+1}

Reinforcement Learning – Paradigm

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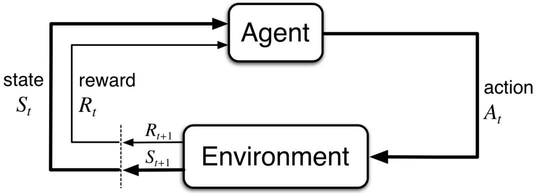

Source: [Sutton and Barto, 2018]

- **Learning the policy:**
  - Policy Gradient

$$\theta^{(\mathtt{k+1})} = \theta^{(\mathtt{k})} - \eta^{(\mathtt{k})} \nabla J(\theta^{(\mathtt{k})}), \qquad \pi^{(\mathtt{k})}(a|s) = \pi(s|a, \theta^{(\mathtt{k})})$$

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  - PPO, TRPO
  - ...

Reinforcement Learning – Methods

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  - PPO, TRPO
  - ...
- **Learning the value function:**
  - Q-learning

$$Q^*(s, a) = r(s, a) + \gamma \max_{\pi} \mathbb{E}_{s' \sim p(\cdot|s,a), a' \sim \pi(\cdot|s')} \left[Q^*(s', a') \right]$$

Note: $V^*(s) = \max_{a \in \mathcal{A}} Q^*(s, a)$, $\alpha^*(s) = \operatorname{argmax}_{a \in \mathcal{A}} Q^*(s, a)$

Reinforcement Learning – Methods

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  - Deep Q-neural network (DQN)
  - ...

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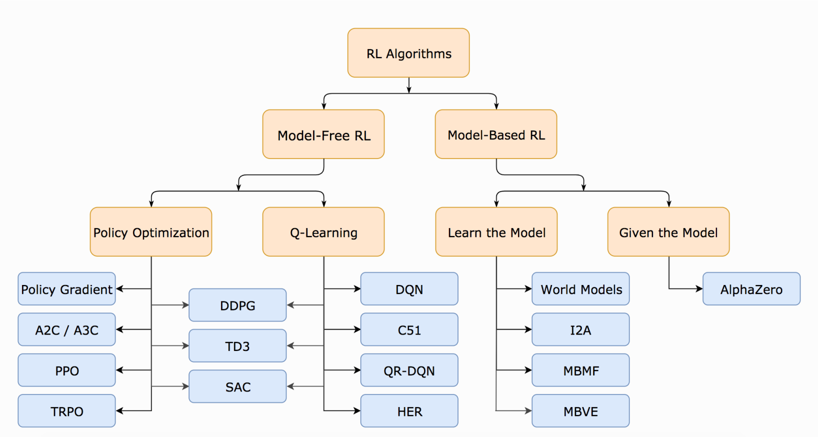
  - Deep Q-neural network (DQN)
  - ...
- **Hybrid:**
  - Deep Deterministic Policy Gradient (DDPG)
  - Soft Actor Critic (SAC)
  - ...

RL Taxonomy



Source: [OpenAI Spinning Up][1]

[1] https://spinningup.openai.com/en/latest/spinningup/rl_intro2.html

DQN

Algorithm 1 Deep Q-learning with Experience Replay

Initialize replay memory $\mathcal{D}$ to capacity N
Initialize action-value function Q with random weights
for episode $= 1, M$ **do**
  Initialise sequence $s_1 = \{x_1\}$ and preprocessed sequenced $\phi_1 = \phi(s_1)$
  for $t = 1, T$ **do**
    With probability ϵ select a random action a_t
    otherwise select $a_t = \max_a Q^*(\phi(s_t), a; \theta)$
    Execute action a_t in emulator and observe reward r_t and image x_{t+1}
    Set $s_{t+1} = s_t, a_t, x_{t+1}$ and preprocess $\phi_{t+1} = \phi(s_{t+1})$
    Store transition $(\phi_t, a_t, r_t, \phi_{t+1})$ in $\mathcal{D}$
    Sample random minibatch of transitions $(\phi_j, a_j, r_j, \phi_{j+1})$ from $\mathcal{D}$
    Set $y_j = \begin{cases} r_j & \text{for terminal } \phi_{j+1} \\ r_j + \gamma \max_{a'} Q(\phi_{j+1}, a'; \theta) & \text{for non-terminal } \phi_{j+1} \end{cases}$
    Perform a gradient descent step on $(y_j - Q(\phi_j, a_j; \theta))^2$ according to equation 3
  end for
end for

Source: [Mnih et al., 2013]

DDPG

Algorithm 1 DDPG algorithm

Randomly initialize critic network $Q(s,a|\theta^Q)$ and actor $\mu(s|\theta^\mu)$ with weights θ^Q and θ^μ.
Initialize target network Q' and μ' with weights $\theta^{Q'} \leftarrow \theta^Q$, $\theta^{\mu'} \leftarrow \theta^\mu$
Initialize replay buffer R
for episode = 1, M **do**
  Initialize a random process $\mathcal{N}$ for action exploration
  Receive initial observation state s_1
  for t = 1, T **do**
    Select action $a_t = \mu(s_t|\theta^\mu) + \mathcal{N}_t$ according to the current policy and exploration noise
    Execute action a_t and observe reward r_t and observe new state s_{t+1}
    Store transition (s_t, a_t, r_t, s_{t+1}) in R
    Sample a random minibatch of N transitions (s_i, a_i, r_i, s_{i+1}) from R
    Set $y_i = r_i + \gamma Q'(s_{i+1}, \mu'(s_{i+1}|\theta^{\mu'})|\theta^{Q'})$
    Update critic by minimizing the loss: $L = \frac{1}{N}\sum_i (y_i - Q(s_i, a_i|\theta^Q))^2$
    Update the actor policy using the sampled policy gradient:

$$\nabla_{\theta^\mu} J \approx \frac{1}{N}\sum_i \nabla_a Q(s,a|\theta^Q)|_{s=s_i,a=\mu(s_i)} \nabla_{\theta^\mu}\mu(s|\theta^\mu)|_{s_i}$$

    Update the target networks:

$$\theta^{Q'} \leftarrow \tau\theta^Q + (1-\tau)\theta^{Q'}$$

$$\theta^{\mu'} \leftarrow \tau\theta^\mu + (1-\tau)\theta^{\mu'}$$

  end for
end for

Source: [Lillicrap et al., 2016]

SAC

Algorithm 1 Soft Actor-Critic

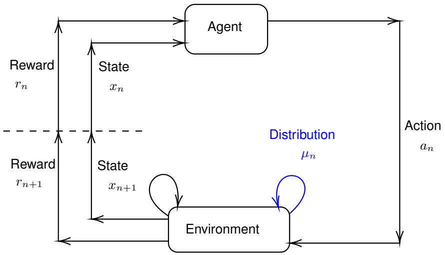
Initialize parameter vectors ψ, $\bar{\psi}$, θ, ϕ.
for each iteration **do**
  for each environment step **do**
    $\mathbf{a}_t \sim \pi_\phi(\mathbf{a}_t|\mathbf{s}_t)$
    $\mathbf{s}_{t+1} \sim p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t)$
    $\mathcal{D} \leftarrow \mathcal{D} \cup \{(\mathbf{s}_t, \mathbf{a}_t, r(\mathbf{s}_t, \mathbf{a}_t), \mathbf{s}_{t+1})\}$
  end for
  for each gradient step **do**
    $\psi \leftarrow \psi - \lambda_V \hat{\nabla}_\psi J_V(\psi)$
    $\theta_i \leftarrow \theta_i - \lambda_Q \hat{\nabla}_{\theta_i} J_Q(\theta_i)$ for $i \in \{1, 2\}$
    $\phi \leftarrow \phi - \lambda_\pi \hat{\nabla}_\phi J_\pi(\phi)$
    $\bar{\psi} \leftarrow \tau\psi + (1 - \tau)\bar{\psi}$
  end for
end for

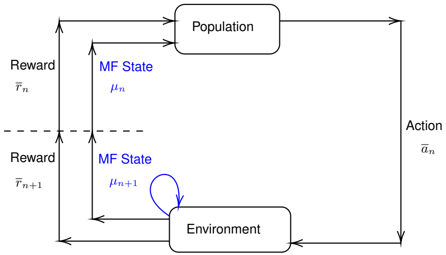
Source: [Haarnoja et al., 2018]

RL Setting for MFG and MFC

Intuitively:

- MFG: a representative agent learns by interacting with an environment, which depends on the population distribution
- MFC: the whole population learns





Population Distribution Approximation

How to deal with the population distribution μ?

- Empirical distribution μ^N
- Histogram (discrete state space)
- ϵ-net in $\mathcal{P}(\mathcal{X})$
- Function approximation for the density:
  - Kernels
  - Neural nets: normalizing flows, ...
  - ...
- ...

So far, most of the literature focuses on finite state space models

But see e.g. [Perrin et al., 2021b] in continuous space using normalizing flows

A (Non-exhaustive) Glance at the literature: RL for MFG

- MARL with mean field approximation: [Yang et al., 2018]
- Inverse RL: [Yang et al., 2017], [Chen et al., 2021]
- Multi-time scales: [Subramanian and Mahajan, 2019a], [Angiuli et al., 2020b, Angiuli et al., 2020a, Angiuli and Hu, 2021]
- Fictitious Play with tabular RL: [Perrin et al., 2020], with deep RL: [Elie et al., 2020a, Cui and Koeppl, 2021a] and distribution embedding: [Perrin et al., 2021c]
- Fixed point iterations with Q-learning and variants: [Guo et al., 2019a, Guo et al., 2020], [Anahtarci et al., 2019, Anahtarcı et al., 2021], [Xie et al., 2021]
- Entropy regularization: [Anahtarci et al., 2020a], [Cui and Koeppl, 2021a]
- LQ MFG with actor-Critic: [Fu et al., 2019, uz Zaman et al., 2020], or policy gradient: [Wang et al., 2021]
- RL for partially observable MFG: [Subramanian et al., 2020b]
- Mean field RL for multiple types: [Subramanian et al., 2020a, uz Zaman et al., 2022]
- Learning Master policies with deep RL: [Perrin et al., 2021a]
- ...

A (Non-exhaustive) Glance at the literature: RL for MFC

- Early works on MDP viewpoint: [Gast and Gaujal, 2011, Gast et al., 2012a]
- Policy optimization for stationary MFC: [Subramanian and Mahajan, 2019a]
- Policy gradient for LQ MFC [Carmona et al., 2019a, Wang et al., 2021] and zero sum mean field type game [Carmona et al., 2020]
- Multi-time scale for MFC (and MFG): [Angiuli et al., 2020b, Angiuli et al., 2020a, Angiuli and Hu, 2021]:
- Mean field MDP: dynamic programming and RL [Carmona et al., 2019b, Gu et al., 2019, Motte and Pham, 2019a, Gu et al., 2020a, Cui et al., 2021]
- Decentralized network approach [Gu et al., 2021]
- Model based RL for MFC: [Pasztor et al., 2021]
- ...

Several talks on this topic are available here:
`https://sites.google.com/view/mlmfgseminar/past-talks`

Survey on this topic: [Laurière et al., 2022a] (updated version soon)

Three Settings [Laurière et al., 2022a]

Intuitively, at least 3 different settings:

- Static:
  - **No states** (normal-form game): each player chooses an **action** $a \sim \pi(\cdot)$
  - Reward: depends on own action & population's action distribution
  - Examples: towel on the beach, urban settlement, ...
- Stationary:
  - **Infinite horizon**: learns a **stationary policy** $\pi(\cdot|x)$
  - Reward: similar than Evolutive case.
  - Initial state distribution = stationary distribution induced by the population's policy or gamma discounted distribution.
  - Examples: player joining a crowd already in a steady state
- Evolutive:
  - **(In)Finite horizon**: each player learns a **time-dependant policy** $\pi_n(\cdot|x)$
  - Reward: depends on own state, action & population's (state,action) distribution.
  - Fixed initial state distribution
  - Examples: crowd motion, traffic routing, ...
- Other settings: asymptotic, γ-discounted, ergodic, ...

In the sequel we mostly stick to the evolutive setting.

Outline

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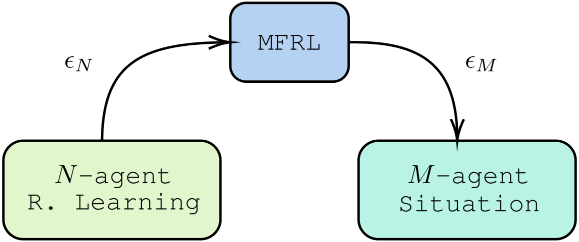
From Optimal Control to MFRL

$$\begin{tikzcd}
\text{Mean-Field Control} \arrow[rr, "\textit{unknown model}"] & & \text{Mean-Field Reinforcement Learning} \\
\text{Optimal Control} \arrow[u, "\substack{N\to\infty \\ MKV}"'] \arrow[rr, "\textit{unknown model}"'] & & \text{Reinforcement Learning} \arrow[u, "\substack{N\to\infty \\ MKV}"]
\end{tikzcd}$$


MFRL
ϵ_N
ϵ_M
N-agent R. Learning
M-agent Situation

Dynamics and cost

- **Dynamics:** discrete time

$$X_{n+1}^{\alpha,\mu} = F(X_n^{\alpha,\mu}, \alpha_n, \mu_n, \epsilon_{n+1}, \epsilon_{n+1}^0), \quad n \geq 0, \qquad X_0^{\alpha,\mu} \sim \mu_0$$

  - $X_n^{\alpha,\mu} \in \mathcal{S} \subseteq \mathbb{R}^d$: state, $\alpha_n \in \mathcal{U} \subseteq \mathbb{R}^k$: action
  - $\epsilon_n \sim \nu$: idiosyncratic noise, $\epsilon_n^0 \sim \nu^0$: common noise (random env.)
  - $p(x'|x, a, \mu)$: corresponding transition probability distribution
  - $\mu_n \in \mathcal{P}(\mathcal{S} \times \mathcal{A})$: a state-action distribution
  - π_n: a policy; randomized actions: $\alpha_n \sim \pi_n(\cdot|s_n, \mu_n)$

- **Cost:** $\mathbb{J}(\pi; \mu) = \mathbb{E}_{\epsilon,\epsilon^0}\left[\sum_{n=0}^{\infty} \gamma^n f\left(X_n^{\alpha,\mu}, \alpha_n, \mu_n\right)\right]$

Two scenarios:

- **Cooperative (MFC):** Find π^* s.t.

$$\pi^* \text{ minimizes } \pi \mapsto J^{MFC}(\pi) = \mathbb{J}(\pi; \mu^\pi) \text{ where } \mu_n^\pi = \mathbb{P}^0_{X_n^{\alpha,\mu^\pi}}$$

- **Non-Cooperative (MFG):** Find $(\hat{\pi}, \hat{\mu})$ s.t.

$$\begin{cases} \hat{\pi} \text{ minimizes } \pi \mapsto J^{MFG}(\pi; \hat{\mu}) = \mathbb{J}(\pi; \hat{\mu}) \\ \hat{\mu}_n = \mathbb{P}^0_{X_n^{\hat{\alpha},\hat{\mu}}} \end{cases}$$

In this section we focus on the MFC case.

MFG in the next section

MFMDP with Common Noise & Randomization

- **Key Remark:**

$$\alpha^* \in \underset{\alpha}{\operatorname{argmin}}\, J^{MFC}(\alpha) = \mathbb{E}_{\epsilon,\epsilon^0}\left[\sum_{n=0}^{\infty} \gamma^n f\left(X_n^{\alpha}, \alpha_n, \mu_n^{\pi}\right)\right], \qquad \mu_n^{\pi} = \mathbb{P}^0_{X_n^{\alpha}}$$

MFMDP with Common Noise & Randomization

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$$= \mathbb{E}_{\epsilon^0}\left[\sum_{n=0}^{\infty} \gamma^n \underbrace{\int_{\mathcal{S}\times\mathcal{U}} f\left(x, a, \mu_n^{\pi}\right) \nu_n^{\pi}(dx, da)}_{\text{function of } \nu_n^{\pi}}\right]$$

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- **Lifted problem:** population / social planner's optimization problem:
  - $\rightarrow$ state = population distribution μ_n^{π}
  - $\rightarrow$ value function = function of the distribution μ

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- **Mean Field Markov Decision Process (MFMDP):** $(\bar{\mathcal{S}}, \bar{\mathcal{A}}, \bar{p}, \bar{r}, \gamma)$, where:
  - State space: $\bar{\mathcal{S}} = \mathcal{P}(\mathcal{S})$
  - Action space: $\bar{\mathcal{A}} = \mathcal{P}(\mathcal{S} \times \mathcal{U})$ with constraint: $pr_1(\bar{a}) = \mu$
  - Transition function: $\mu' = \bar{F}(\mu, \bar{a}, \epsilon^0) \sim \bar{p}(\mu, \bar{a})$
  - Reward function: $\bar{r}(\mu, \bar{a}) = -\int_{\mathcal{S}\times\mathcal{U}} f(x, a, \mu)\bar{a}(dx, da)$

MFMDP with Common Noise & Randomization

- **Key Remark:**

$$\alpha^* \in \underset{\alpha}{\operatorname{argmin}}\, J^{MFC}(\alpha) = \mathbb{E}_{\epsilon,\epsilon^0}\left[\sum_{n=0}^{\infty} \gamma^n f\left(X_n^\alpha, \alpha_n, \mu_n^\pi\right)\right], \qquad \mu_n^\pi = \mathbb{P}^0_{X_n^\alpha}$$

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  - Transition function: $\mu' = \bar{F}(\mu, \bar{a}, \epsilon^0) \sim \bar{p}(\mu, \bar{a})$
  - Reward function: $\bar{r}(\mu, \bar{a}) = -\int_{\mathcal{S}\times\mathcal{U}} f(x, a, \mu)\bar{a}(dx, da)$
- **Goal:** max. $\bar{J}^{\bar{\pi}}(\mu) = \mathbb{E}\left[\sum_{n=0}^{\infty} \gamma^n \bar{r}\left(\mu_n^{\bar{\pi}}, \bar{a}_n\right)\right]$, $\bar{a}_n \sim \bar{\pi}(\cdot|\mu_n^{\bar{\pi}})$, $\mu_{n+1}^{\bar{\pi}} \sim \bar{p}(\cdot|\mu_n^{\bar{\pi}}, \bar{a}_n)$,

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- **Mean field policy:** $\bar{\pi}$ kernel $\bar{\mathcal{S}} \rightarrow \mathcal{P}(\bar{\mathcal{A}})$, *randomized population-strategies* $\bar{a}$

Dynamic Programming Principle (DPP)

Theorem: DPP for MFMDP [Carmona et al., 2019c]

Under suitable conditions,

$$\bar{J}^*(\mu) := \sup_{\bar{\pi}} \bar{J}^{\bar{\pi}}(\mu) = \sup_{\bar{\pi}} \left\{ \int_{\bar{\mathcal{A}}} \left[\bar{r}(\mu, \bar{a}) + \gamma \mathbb{E}\left[\bar{J}^*\left(\bar{F}(\mu, \bar{a}, \epsilon^0) \right) \right] \right] \bar{\pi}(d\bar{a}|\mu) \right\},$$

where the sup is over a subset of $\{\bar{\pi} : \bar{\mathcal{S}} \to \mathcal{P}(\bar{\mathcal{A}})\}$

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Here: discrete time, infinite horizon, common noise, feedback controls, ...

→ well-suited for **RL**

→ Mean-field Q-learning algorithm

Mean Field Learning Settings

Hierarchy of settings:

- **Setting 1: known model**: computational method based on knowledge of MFMDP
  (a) Gradient based methods
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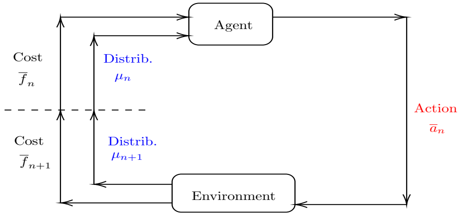

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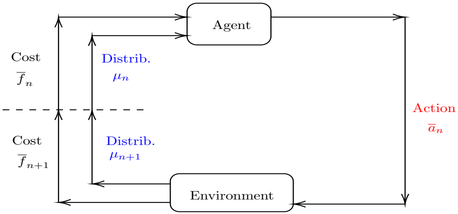
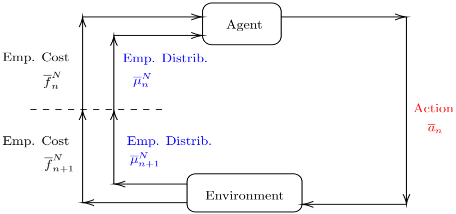


- **Setting 3:** unknown model but **samples from N-agent MDP**: approx. MF learning



Outline

Algorithm

Idea 1: *Make the "policy gradient" approach model-free*

Policy Gradient (PG) to minimize $J(\theta)$

- Control $\approx$ parameterized function (analog to the "direct approach" in lecture 4)
- Look for the optimal parameter θ^*
- Perform gradient descent on the space of parameters

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Theorem: For **Linear-Quadratic** MFC [Carmona et al., 2019c]

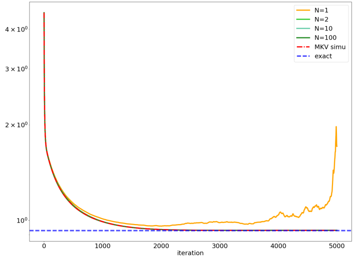
In each case, convergence holds at a linear rate:

Taking $k \approx \mathcal{O}\Big(\log(1/\epsilon)\Big)$ is sufficient to ensure $J(\theta^{(k)}) - J(\theta^*) < \epsilon$.

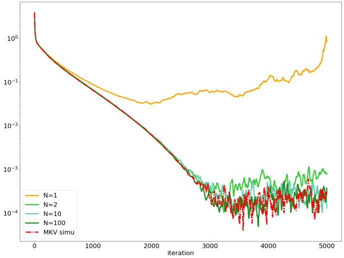
Proof: builds on [Fazel et al., 2018], analysis of perturbation of Riccati equations

Example: Linear dynamics, quadratic costs of the type:

$$f(x, \mu, \alpha) = \underbrace{(\bar{\mu} - x)^2}_{\text{distance to mean position}} + \underbrace{\alpha^2}_{\text{cost of moving}}, \qquad \bar{\mu} = \underbrace{\int \mu(\xi) d\xi}_{\text{mean position}},$$



Value of the MF cost



Rel. err. on MF cost

MF cost = cost in the mean field problem

Numerical Illustration

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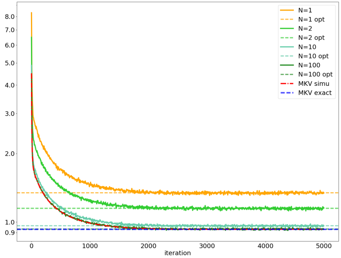
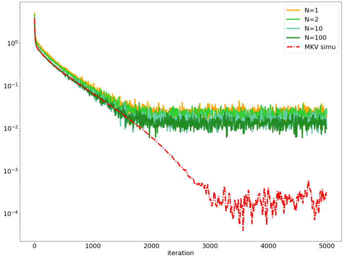


Value of the social cost



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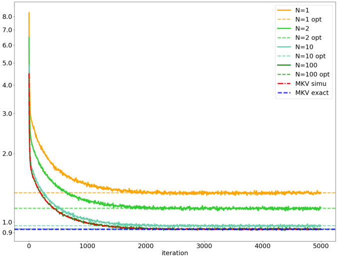

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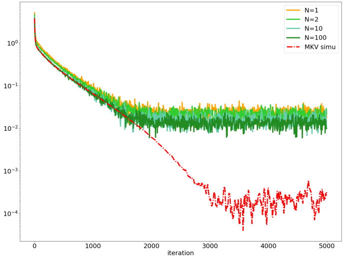

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Main take-away:

Trying to learn the mean-field regime solution can be efficient even for N small

Outline

1. Introduction

2. RL for MFC (MFRL)
   - Setting
   - Model-Free Policy Gradient for MFC
   - Q-Learning for MFC

3. RL for MFGs

4. MFGs in OpenSpiel

5. Conclusion

Mean Field Q-Function

Idea 2: *Generalize Q-learning to Mean-Field Control*

Reminder:

- **Mean Field Markov Decision Process (MFMDP):** $(\bar{\mathcal{S}}, \bar{\mathcal{A}}, \bar{p}, \bar{r}, \gamma)$, where:
  - State space: $\bar{\mathcal{S}} = \mathcal{P}(\mathcal{S})$
  - Action space: $\bar{\mathcal{A}} = \mathcal{P}(\mathcal{S} \times \mathcal{U})$ with constraint: $pr_1(\bar{a}) = \mu$
  - Transition function: $\mu' = \bar{F}(\mu, \bar{a}, \epsilon^0) \sim \bar{p}(\mu, \bar{a})$
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Q-function associated to a policy π:

$$Q^{\pi}(s, a) = r(s, a) + \gamma \mathbb{E}_{s' \sim p(\cdot|s,a), a' \sim \pi(\cdot|s')}\left[Q^{\pi}(s', a')\right]$$

Mean Field Q-function associated to a mean field policy $\bar{\pi}$:

$$\bar{Q}^{\bar{\pi}}(\bar{s}, \bar{a}) = \bar{r}(\bar{s}, \bar{a}) + \gamma \mathbb{E}_{\bar{s}' \sim \bar{p}(\cdot|\bar{s},\bar{a}), \bar{a}' \sim \bar{\pi}(\cdot|\bar{s}')}\left[\bar{Q}^{\bar{\pi}}(\bar{s}', \bar{a}')\right]$$

MF Q-Learning

- **Optimal MF Q-function:**

$$\bar{Q}^*(\bar{s}, \bar{a}) = \bar{r}(\bar{s}, \bar{a}) + \gamma \sup_{\bar{\pi}} \mathbb{E}_{\bar{a}' \sim \bar{\pi}(\cdot|\bar{s}), \bar{s}' \sim \bar{p}(\cdot|\bar{s}, \bar{a}')} \left[\bar{Q}^*(\bar{s}', \bar{a}') \right]$$

- **Algorithm:**
  - Idealized version (synchronous):

$$\bar{Q}^{(k+1)}(\bar{s}, \bar{a}) = \bar{r}(\bar{s}, \bar{a}) + \gamma \sup_{\bar{\pi}} \mathbb{E}_{\bar{s}' \sim \bar{p}(\cdot|\bar{s}, \bar{a}), \bar{a}' \sim \bar{\pi}(\cdot|\bar{s}')} \left[\bar{Q}^{(k)}(\bar{s}', \bar{a}') \right], \qquad (\bar{s}, \bar{a}) \in \bar{\mathcal{S}} \times \bar{\mathcal{A}}$$
$$= [\bar{T}^* \bar{Q}^{(k)}](\bar{s}, \bar{a})$$

  - Following a trajectory (async.): $\bar{s}^{(k+1)} \sim p(\cdot|\bar{s}^{(k)}, \bar{a}^{(k)})$, $\bar{a}^{(k+1)} \sim \bar{\pi}^{(k+1)}(\cdot|\bar{s}^{(k)})$,

$$\begin{cases} \bar{Q}^{(k+1)}(\bar{s}, \bar{a}) = \bar{Q}^{(k)}(\bar{s}, \bar{a}), \qquad (\bar{s}, \bar{a}) \in \bar{\mathcal{S}} \times \bar{\mathcal{A}} \\ \bar{Q}^{(k+1)}(\bar{s}^{(k+1)}, \bar{a}^{(k+1)}) \leftarrow \bar{r}(\bar{s}^{(k+1)}, \bar{a}^{(k+1)}) + \gamma \max_{\bar{a}'} \bar{Q}^{(k)}(\bar{s}^{(k+1)}, \bar{a}') \end{cases}$$

- **Implementation:** several possibilities (can be combined):
  - pure (population and individual) strategies
  - discretization of $\bar{\mathcal{S}} = \mathcal{P}(\mathcal{S}), \bar{\mathcal{A}} = \mathcal{P}(\mathcal{S} \times \mathcal{U})$
  - deep Reinforcement Learning

MF Q-Learning: Numerical Illustration

Cyber-security example of [Kolokoltsov and Bensoussan, 2016] (see also lecture 5)

- MFC viewpoint, MF Q-learning
- pure (population and individual) strategies
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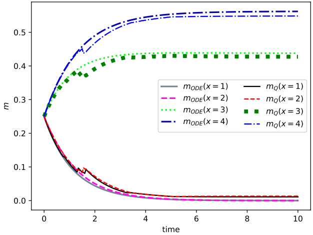
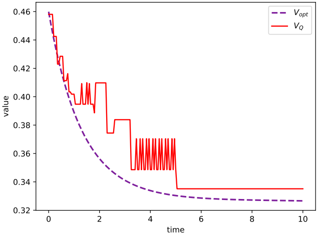
Test 1: $m_0 = (1/4, 1/4, 1/4, 1/4)$



Evolution of m^{m_0} optimally controlled (m_{ODE}) or controlled using the approximate Q-function (m_Q)



V function (V_{opt}) and approximate Q-function (V_Q) along the optimal flow.

(See section 8.1 of [Laurière, 2021] and section 6.1 of [Carmona et al., 2019c])

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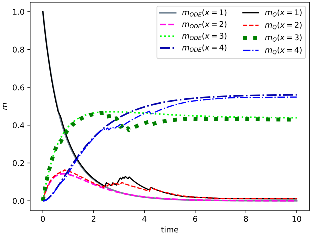

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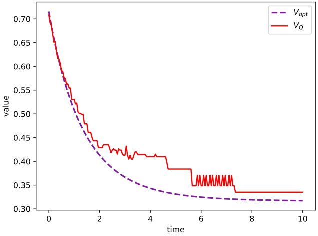

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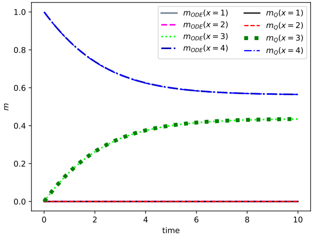

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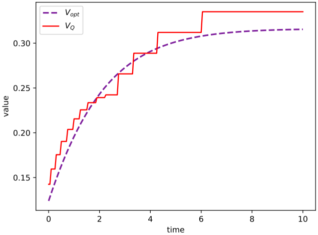

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Deep RL for MFC

- Instead of discretizing the distribution, we can train a parameterized function to approximate the Q-function
- For instance: neural network trained by DDPG
- Note: We do not need to randomize the policy at the population level, but we do allow randomization at the agent level
- See sections 6.2 and 6.3 of [Carmona et al., 2019c]

Outline

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- In game theory, economics, . . . :
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- In machine learning, RL, . . . :
  [Mitchell et al., 1997]: *"A computer program is said to learn from experience E with respect to some class of tasks T and performance measure P if its performance at tasks in T, as measured by P, improves with experience E."*

Learning/Optimization Algorithms in Games

Learning/optimization methods:

- Fixed point iteration
  - Banach-Picard iterations
  - idem + damping/mixing/smoothing
  - Fictitious Play (FP)
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in

- Games, particularly in economics, see e.g. [Fudenberg et al., 1998]
- Non-atomic games. see e.g. [Hadikhanloo et al., 2021]
- Mean Field Games, see e.g. [Hadikhanloo, 2018]

Learning in MFGs

Generic structure: repeated game (iterations)

- Update the representative agent behavior
  - value function
  - policy (control)
- Update the population behavior

$$\dots \quad \mapsto \pi^{(k)} \mapsto \mu^{(k)} \mapsto \pi^{(k+1)} \mapsto \quad \dots$$

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*Where is there **learning**?*

→ First type of “Learning”: meta-algorithm / outside loop

→ Second type of “Learning”: agent’s viewpoint / inner loop

Best Response and Population Behavior Maps

We focus on MFG and write $J = J^{MFG}$. For simplicity let's forget the common noise.

Two important functions:

- **Best Response map:**

$$\texttt{BR} : \mu \mapsto \pi \in \operatorname{argmax} J^{MFG}(\cdot; \mu)$$

- **Population Behavior** induced when everyone using a policy:

$$\texttt{PB} : \pi \mapsto \mu : \mu_{n+1} = \Phi(\mu_n, \pi_n)$$

where:

$$\Phi(\mu, \pi)(x) := \sum_{x \in \mathcal{X}} \sum_{a \in \mathcal{A}} p(x|x_0, a, \mu)\pi(a|x_0, \mu)\mu(x_0), \qquad x \in \mathcal{X}$$

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represents a one-step transition of the population distribution

Mean Field Nash equilibrium: $(\hat{\mu}, \hat{\pi})$ such that

$$\begin{cases} \hat{\mu} = \mathtt{PB}(\pi) \\ \hat{\pi} = \mathtt{BR}(\hat{\mu}) \end{cases}$$

$\hat{\mu}$ can be unique without $\hat{\pi}$ being unique!

Outline

Learning in MFGs

Generic structure: repeated game (iterations)

- Update the representative agent behavior
  - value function
  - policy (control)
- Update the population behavior

*Where is there **learning**?*

$\rightarrow$ First type of “Learning”: meta-algorithm / outside loop

$\rightarrow$ Second type of “Learning”: agent’s viewpoint / inner loop

Banach-Picard Fixed Point Iterations

Fixed point method

- Update agent's policy: $\pi^{(\mathtt{k+1})} \in \mathtt{BR}(\mu^{(\mathtt{k})})$
- Update population's behavior: $\mu^{(\mathtt{k+1})} = \mathtt{Pop}(\pi^{(\mathtt{k+1})})$

- **Convergence:** holds under strict contraction property for the map:

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  are Lipschitz with **small enough** Lipschitz constants

- See e.g. [Huang et al., 2006], [Guo et al., 2019b]

- Can be relaxed with entropy regularization [Anahtarci et al., 2020b], [Cui and Koeppl, 2021b], [Yardim et al., 2022], . . .

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- Can be modified with damping/mixing/smoothing; e.g. Fictitious Play

Fictitious Play

Fictitious Play method

- Update agent's policy: $\pi^{(k+1)} \in \mathtt{BR}(\overline{\mu}^{(k)})$
- Update population's behavior: $\mu^{(k+1)} = \mathtt{Pop}(\pi^{(k+1)})$
- Update population's average behavior: $\overline{\mu}^{(k+1)} = \frac{k}{k+1}\overline{\mu}^{(k+1)} + \frac{1}{k+1}\mu^{(k+1)}$

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  - $\tilde{r}$ is monotone: $\langle \tilde{r}(x, \mu) - \tilde{r}(x, \mu'), \mu - \mu' \rangle \leq 0$

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- Consequence: the exploitability is a Lyapunov function
- where the **exploitability** of π facing μ is:

$$\mathcal{E}(\pi;\mu) = \sup J(\cdot;\mu) - J(\pi;\mu) \geq 0$$

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- See e.g., [Cardaliaguet and Hadikhanloo, 2017], [Hadikhanloo and Silva, 2019], [Elie et al., 2020b], [Perrin et al., 2020], [Geist et al., 2022], . . .

Value iteration VS policy iteration

Reminder:

Fixed point method

- Update agent's policy: $\pi^{(k+1)} \in \mathtt{BR}(\mu^{(k)})$
- Update population's behavior: $\mu^{(k+1)} = \mathtt{Pop}(\pi^{(k+1)})$

- Requires computation of a best response $\Rightarrow$ fully solving an MDP
- This is analogous to value iteration
- An alternative method is policy iteration: greedy update & evaluation

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- An alternative method is policy iteration: greedy update & evaluation
- Note: these are not "standard" VI and PI because we need to intertwine updates of the mean field and the policy/value function

Policy iteration method

- Update agent's Q-function: $Q^{(\mathtt{k+1})} = Q_{\pi^{(\mathtt{k})},\mu^{(\mathtt{k})}}$
- Update agent's policy: $\pi^{(\mathtt{k+1})}(x) = \mathrm{argmax}_{a \in \mathcal{A}} Q^{(\mathtt{k+1})}(x, a), x \in \mathcal{X}$
- Update population's behavior: $\mu^{(\mathtt{k+1})} = \mathtt{Pop}(\pi^{(\mathtt{k+1})})$

- where the representative agent's Q-function, given μ, is:

$$
\begin{aligned}
&Q_{\pi,\mu}(x, a) \\
&= \mathbb{E}\left[\sum_{n \geq 0} \gamma^n r(x_n, a_n, \mu)\right], \; x_{n+1} \sim p(\cdot | x_n, a_n, \mu), a_{n+1} \sim \pi(\cdot | x_{n+1}), x_0 = x, a_0 = a \\
&= r(x, a, \mu) + \gamma \mathbb{E}[Q_{\pi,\mu}(x', a')], \quad x' \sim p(\cdot | x, a, \mu), a' \sim \pi(\cdot | x')
\end{aligned}
$$

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$$= \mathbb{E}\left[\sum_{n\geq 0}\gamma^n r(x_n,a_n,\mu)\right], x_{n+1}\sim p(\cdot|x_n,a_n,\mu), a_{n+1}\sim\pi(\cdot|x_{n+1}), x_0 = x, a_0 = a$$
$$= r(x,a,\mu) + \gamma\mathbb{E}[Q_{\pi,\mu}(x',a')], \quad x'\sim p(\cdot|x,a,\mu), a'\sim\pi(\cdot|x')$$

- Note: Here, no need to compute a BR; just evaluate a Q function & argmax
- See [Cacace et al., 2021], , [Camilli and Tang, 2022], [Tang and Song, 2022], [Laurière et al., 2023] in the continuous setting, and [Cui and Koeppl, 2021b] in the discrete setting.

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- The updates can be "smoothed" by averaging $\rightarrow$ Online Mirror Descent

Online Mirror Descent method

- Update agent's Q-function: $Q^{(\mathtt{k+1})} = Q_{\pi^{(\mathtt{k})},\mu^{(\mathtt{k})}}$
- Update agent's average Q-function: $\overline{Q}^{(\mathtt{k+1})} = \overline{Q}^{(\mathtt{k})} + \eta Q^{(\mathtt{k+1})}$
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where

$$\Gamma(y) := \nabla h^*(y) = \operatorname*{argmax}_{p \in \mathcal{P}(\mathcal{A})}[\langle y, p\rangle - h(\pi)].$$

with a regularizer $h : \mathcal{P}(\mathcal{A}) \to \mathbb{R}$ and $h^* : \mathbb{R}^{|\mathcal{A}|} \to \mathbb{R}$ its convex conjugate defined by $h^*(y) = \max_{p \in \mathcal{P}(\mathcal{A})} [\langle y, p\rangle - h(\pi)]$

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- See e.g., [Hadikhanloo, 2018] in the continuous setting, and [Pérolat et al., 2022], [Geist et al., 2022], ... in the discrete setting

Summary for FP and OMD

Algorithm: Fixed point iter.

input : Initial policy π^0

$\mu^0 := \mu^{\pi^0}$;
for $k = 1, \dots, K$: **do**
  $\pi^k :=$ BR against μ^{k-1};
  $\mu^k := \mu^{\pi^k}$;
return π^K, μ^K

↓

Algorithm: Fictitious Play

input : Initial policy π^0

$\bar{\pi}^0 := \pi^0$;
$\bar{\mu}^0 := \mu^{\bar{\pi}^0}$;
for $k = 1, \dots, K$: **do**
  $\pi^k :=$ BR against $\bar{\mu}^{k-1}$;
  $\bar{\mu}^k := \frac{k}{k+1}\bar{\mu}^{k-1} + \frac{1}{k+1}\mu^{\pi^k}$;
  $\bar{\pi}^k :=$ policy giving $\bar{\mu}^k$;
return $\bar{\pi}^K$, $\bar{\mu}^K$

Algorithm: Policy iter.

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$\mu^0 := \mu^{\pi^0}$;
for $k = 1, \dots, K$: **do**
  $Q^k :=$ Q-func. for π^{k-1} given μ^{k-1};
  $\pi^k := \operatorname{argmax} Q^k$;
  $\mu^k := \mu^{\pi^k}$;
return π^K, μ^K

↓

Algorithm: OMD

input : Initial policy π^0

$\mu^0 := \mu^{\pi^0}$;
for $k = 1, \dots, K$: **do**
  $Q^k :=$ Q-func. for π^{k-1} given μ^{k-1};
  $\bar{Q}^k := \bar{Q}^{k-1} + \alpha Q^k$;
  $\pi^k := \mathrm{softmax}_\tau\, \bar{Q}^k$;
  $\mu^k := \mu^{\pi^k}$;
return π^K, μ^K

Other Variations and improvements

Possible ways to fix lack of convergence issues:

- Damping / smoothing: e.g.,

$$\mu^{k+1} \leftarrow \text{average of past mean fields}, \pi^{k+1} \leftarrow \text{average of past BR}, \dots$$

- Softmax policy, e.g.

$$\operatorname{argmax} Q(x,\cdot) \leftarrow \operatorname{softmax}_\tau Q(x,\cdot)$$

- Entropy regularization, e.g.

$$r(x,a,\mu) \leftarrow r(x,a,\mu) - \eta \log\left(\frac{\pi(a|x)}{\tilde{\pi}(a|x)}\right)$$

- ...

$\rightarrow$ Encompasses many possible variants

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$\rightarrow$ Second type of “Learning”: agent’s viewpoint / inner loop

Given the mean field, the problem faced by a representative player is a *standard* MDP

$\Rightarrow$ We can use any RL algorithm from the literature

Next, we provide some examples

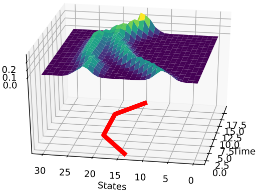
Example (Systemic risk model of [Carmona et al., 2015])

$$J((a_n)_n;(m_n)_n) = -\mathbb{E}\Big[\sum_{n=0}^{N_T}\Big(a_n^2 \underbrace{-q a_n(m_n - X_n)}_{\substack{\text{borrow if } X_n < m_n\\ \text{lend if } X_n > m_n}} + \kappa(m_n - X_n)^2\Big) + c(m_{N_T} - X_{N_T})^2\Big]$$

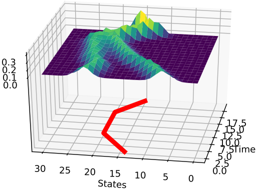
Subj. to: $X_{n+1} = X_n + [K(m_n - X_n) + a_n] + \epsilon_{n+1} + \epsilon^0_{n+1}$

At equilibrium: $m_n = \mathbb{E}[X_n|\epsilon^0],\ n \geq 0$

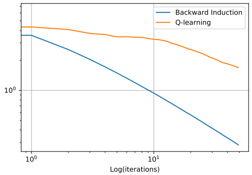
[Perrin et al., 2020]: Fictitious Play with Backward Induction or tabular Q-learning



Exact solution



Fictitious Play & RL



Exploitability

Example (Ergodic crowd aversion model of [Almulla et al., 2017])

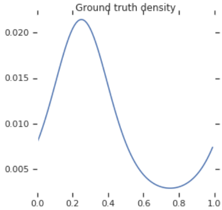
MFG on $\mathbb{T}$,

$$f(x, m, \alpha) = \frac{1}{2}|\alpha|^2 + \tilde{f}(x) + \ln(m(x)),$$

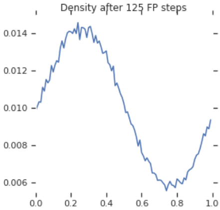
with $\tilde{f}(x) = 2\pi^2 \left[-\sum_{i=1}^d c\sin(2\pi x_i) + \sum_{i=1}^d |c\cos(2\pi x_i)|^2 \right] - 2\sum_{i=1}^d c\sin(2\pi x_i)$,

then the solution is given by $u(x) = c\sum_{i=1}^d \sin(2\pi x_i)$ and $m(x) = e^{2u(x)} / \int e^{2u}$

[Elie et al., 2020b]: Fictitious Play & DDPG (continuous spaces)



Analytical m



m Learnt by Deep RL

Flocking

Example (Flocking aversion model of [Nourian et al., 2011])

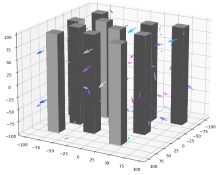
state = (position, velocity) = $(x, v) \in \mathbb{R}^{2d}$,
$$\begin{cases} x_{n+1} = x_n + v_n \Delta t, \\ v_{n+1} = v_n + a_n \Delta t + \epsilon_{n+1}, \end{cases}$$

with running cost:
$$f_\beta^{\text{flock}}(x, v, \mu) = \left\| \int_{\mathbb{R}^{2d}} \frac{(v - v')}{(1 + \|x - x'\|^2)^\beta} \, d\mu(x', v') \right\|^2,$$

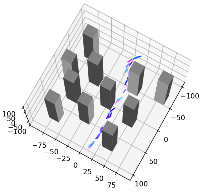
where $\beta \geq 0$, and μ is the position-velocity distribution.

[Perrin et al., 2021d]: For continuous space problems: **Deep RL**

- Deep RL (SAC) for the policy ($\approx$ control)
- Deep NN (normalizing flow) for the population distribution



Initial distribution



At convergence

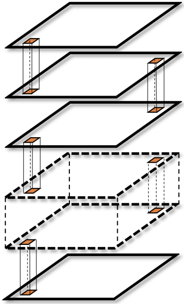
Video: `https://www.youtube.com/watch?v=TdXysW_FA3k`

Example (Crowd motion during building evacuation)

Grid world with movement to neighboring cells, and reward:

$$r(x, a, \mu) = -\eta \log(\mu(x)) + 10 \times \mathbb{1}_{floor=0}$$

Inspired by [Djehiche et al., 2017]



Initial distribution

Building Evacuation [Pérolat et al., 2022]

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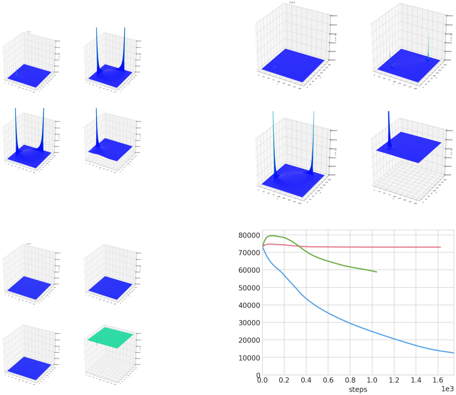
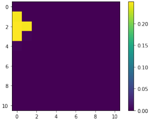
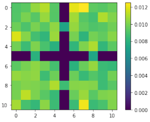


FP (red, $\alpha = 10^{-5}$), FP damped (green, $\alpha = 10^{-3}$) and OMD (blue, $\alpha = 10^{-4}$)

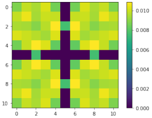
Crowd motion in 2D grid world, $r(x, a, \mu) = -\log(\mu(x))$. (See also lecture 1)



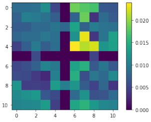
Fixed point



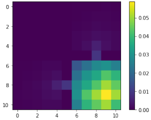
Fictitious Play



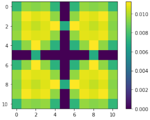
OMD



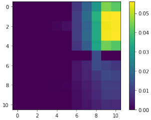
Damped Fixed Point



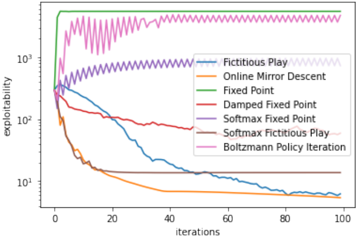
Softmax Fixed Point



Softmax FP

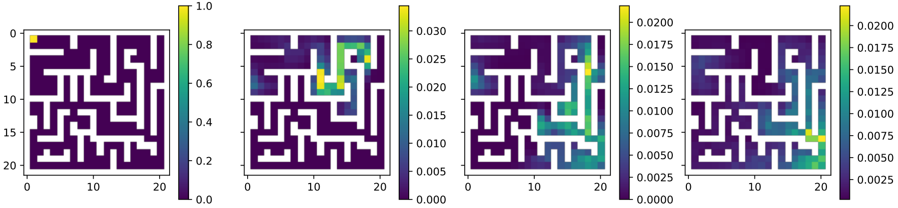


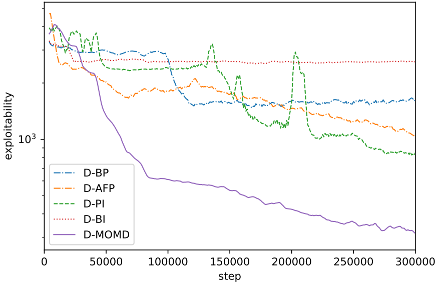
Boltzmann PI



Crowd exiting a maze, with congestion effects in the reward

Deep RL combined with Online Mirror Descent & Fictitious Play





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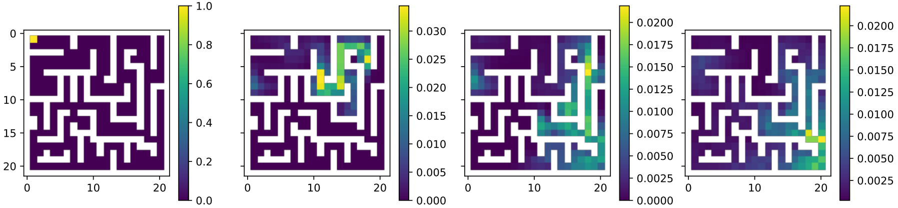

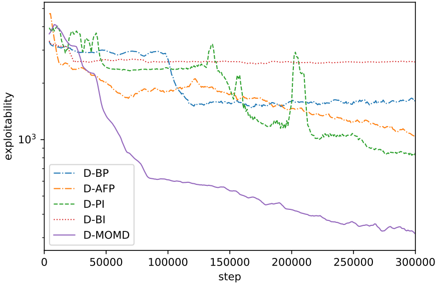

You can reproduce this experiment in OpenSpiel! (see next section)

Outline

MFControl: Fix a control α, compute induced distribution μ^{α}, update α, ...

MFGame: Fix a distribution μ, compute best response α^{μ}, update μ, ...

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Unification: update both α, μ simultaneously but at different rates ρ^α, ρ^μ

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Q-learning: Given μ, **optimal** cost-to-go when starting at x using action a

$$Q(x,a) = f(x,\mu,a) + \sum_{x' \in \mathcal{X}} p(x'|x,\mu,a) \underbrace{\min_{a'} Q(x',a')}_{=V(x')}.$$

Note: optimal control is $\hat{\alpha}_Q(x) = \operatorname{argmin}_a Q(x,a)$.

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where $\begin{cases} \mathcal{T}(Q,\mu)(x,a) = f(x,a,\mu) + \gamma \sum_{x'} p(x'|x,a,\mu) \min_{a'} Q(x',a') - Q(x,a), \\ \mathcal{P}(Q,\mu)(x) = (\mu P^{Q,\mu})(x) - \mu(x), \qquad \text{with } P^{Q,\mu}(x,x') = p(x'|x,\hat{\alpha}_Q(x),\mu) \end{cases}$

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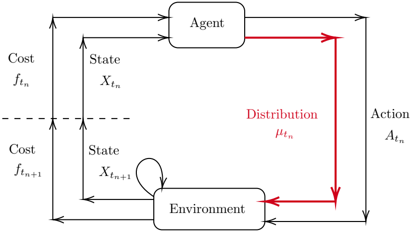
Convergence: based on Borkar's **two timescale** approach (includes sto. approx.)

Rem.: For MFG only see e.g. [Mguni et al., 2018], [Subramanian and Mahajan, 2019b]

Extra difficulty: the agent needs to **estimate** the distribution



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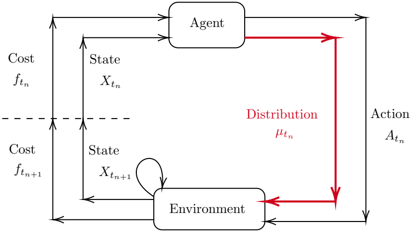
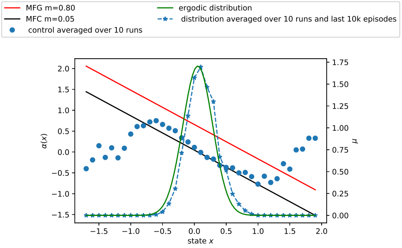
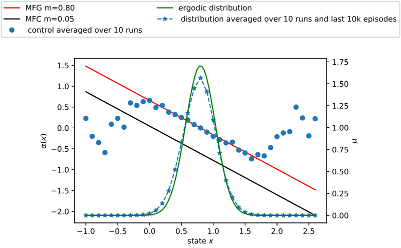


Numerical illustration: Linear-quadratic example



MFC solution ($\rho^Q < \rho^\mu$)



MFG solution ($\rho^Q > \rho^\mu$)

Extensions

- Tuning properly the two learning rates is not trivial
- Proof of convergence (ongoing work with Andrea Angiuli, Jean-Pierre Fouque, and Mengrui Zhang)
- Application to other models, such as mean field control games [Angiuli et al., 2022b, Angiuli et al., 2022a]: mean field of players in a Nash equilibrium, where each agent is of mean field type (solves an MFC) $\rightarrow$ 3 time scales
- Continuous setting (ongoing work of Andrea Angiuli, Jean-Pierre Fouque, Ruimeng Hu et al.)
- RL for MFG without oracle for the distribution [Zaman et al., 2023]

Outline

OpenSpiel

- Open source framework for research in learning in games
- Main motivation: multi-agent reinforcement learning (MARL)
- Marc Lanctot (Google DeepMind) + many contributors
- Mostly in C++ and Python; APIs in Julia, . . .
- Various games including zero-sum games, N-player games, imperfect information, . . .
- Chess, Blackjack, Atari, Kuhn poker, Go, . . .
- And also: Mean field games

OpenSpiel

Introduction to OpenSpiel:

- `https://openspiel.readthedocs.io/en/latest/intro.html`

- Python notebook:
  `https://colab.research.google.com/github/deepmind/open_spiel/blob/master/open_spiel/colabs/OpenSpielTutorial.ipynb`

- Tutorials by Marc Lanctot available online:
  `https://www.youtube.com/watch?v=8NCPqtPwlFQ`

- Paper [Lanctot et al., 2019]

- Two big components:
  - Games
  - Algorithms

MFG in OpenSpiel

- Julien Pérolat, Raphael Marinier, Sertan Girgin & growing number of contributors
  Théophille Cabannes, Sarah Perrin, Paul Muller, ...

- For today, two main questions:
  - How to define a new MFG model (environment)?
  - How to define a new algorithm to learn the MFG solution?

Existing codes for MFG in OpenSpiel

- MFG models in C++: `https://github.com/deepmind/open_spiel/tree/master/open_spiel/games/mfg`
- MFG models in Python: `https://github.com/deepmind/open_spiel/tree/master/open_spiel/python/mfg/games`
  - Crowd modeling 1D illustrated in [Perrin et al., 2020]
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  - Deep fictitious play [Laurière et al., 2022b]
  - Boltzmann policy iteration [Cui and Koeppl, 2021b]
  - Fictitious play [Perrin et al., 2020], ...
  - Fixed point
  - Mirror descent [Pérolat et al., 2022]
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  as well as codes for policies and an evaluation metric: exploitability (nash_conv)

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  as well as codes for policies and an evaluation metric: exploitability (nash_conv)
- Some examples: `https://github.com/deepmind/open_spiel/tree/master/open_spiel/python/mfg/examples`

More to come soon. Contributions are welcome!

MFG model in OpenSpiel: State

Q1. *How to define a new MFG model?*

- State of the game = all the information required to describe the current stage
- In an MFG: representative player's state and mean field state
- Evolution of the state:
  - Players play in turn
  - Every change to the state occurs through a node
  - Each node has a set of possible actions and a probability to pick each action

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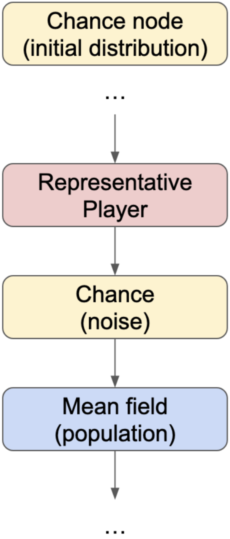
  - So: the representative player is a node
  - the "mean field" is viewed as a node
  - and the "noise" is viewed as a node too
  - Time is part of the state: (t, x)
- The state evolves along a tree of possibilities

MFG model in OpenSpiel: State evolution



MFG model in OpenSpiel: State types

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  - probabilities: distribution of the noise values
- **Mean field:** no actions

MFG in OpenSpiel: Distribution

- The distribution is something specific to MFGs (compared with other games in OpenSpiel)
- Remember that time is part of the state object. Evaluating the distribution at a given state means evaluating the distribution at (t, x).
- `master/open_spiel/python/mfg/algorithms/distribution.py`
  - Computes the distribution of a policy
  - `DistributionPolicy`
    - `evaluate`: based on the logic behind nodes
    - `_one_forward_step`
- `master/open_spiel/python/mfg/distribution.py`
  - Representation of a distribution for a game
  - `Distribution`
- `master/open_spiel/python/mfg/tabular_distribution.py`
  - Tabular representation of a distribution for a game
  - `TabularDistribution`

MFG model in OpenSpiel: Example

We take a concrete example: crowd modeling in 1D with a grid world

`master/open_spiel/python/mfg/games/crowd_modelling.py`

3 main classes

- `MFGCrowdModellingGame`:
  - `__init__`: initialization
  - `new_initial_state`: generate new initial state
- `MFGCrowdModellingState`:
  - `__init__`: initialization
  - `_legal_actions`: actions that are valid
  - `chance_outcomes`: distribution over values of the noise in the dynamics
  - `_apply_action`: will be called at each node to modify the state based on the action
  - `_rewards`: representative player's reward
- `Observer`:
  - defines an observation, here basically t and x

MFG algorithms in OpenSpiel: Principles

Q2. *How to define a new algorithm?*

Simplest one: Fixed point
`master/open_spiel/python/mfg/algorithms/fixed_point.py`

A bit more involved: Fictitious play
`master/open_spiel/python/mfg/algorithms/fictitious_play.py`

- Main class `FictitiousPlay`
- Main method `iteration`
  - Compute the distribution (sequence) associated to the current policy
  - Update the policy (using fictitious play rule); this uses an auxiliary class `MergedPolicy` to mix the previous policy and the new one
- `get_policy`: returns the current policy

MFG algorithms in OpenSpiel: Reinforcement Learning

Two building blocks:

- Environment (in the sense of RL): in charge of updating the State based on the based on the Game
- Agent: block in charge of training the policy by interacting with the environment

Example of DQN (fixed distribution):

`master/open_spiel/python/mfg/examples/mfg_dqn_jax.py`

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Example of DQN (fixed distribution):

`master/open_spiel/python/mfg/examples/mfg_dqn_jax.py`

Example of DQN embedded in Fictitious Play (updating the distribution):

`master/open_spiel/python/mfg/examples/mfg_dqn_fp_jax.py`

Key steps:

- `fp.iteration(br_policy=joint_avg_policy)`: performs one iteration of fictitious play (updates the policy and the distribution)
- `distrib = distribution.DistributionPolicy(game, fp.get_policy())`: get the distribution induced by the new policy, just computed by fictitious play iteration
- `env.update_mfg_distribution(distrib)`: update the environment's distribution using the one obtained from the fictitious play iteration
- `agents[p].step(time_step)`: train the agent

Sample code

Code

Sample code to illustrate: IPython notebook

`https://colab.research.google.com/drive/1HyDFqZ-qMW25sL1zyR2qYv86f_ldrm5g?usp=sharing`

- MFG example in OpenSpiel

Outline

Summary (of this lecture)

- Background on RL
- RL for MFC
  - Mean Field MDP viewpoint
- RL for MFG
  - Meta-algorithm to update the mean field
  - RL algorithm to update the policy
- Open Spiel
- Survey paper: [Laurière et al., 2022a]

Summary of this course

Some References

- Introduction to Mean Field Games:
  - Pierre-Louis Lions' lectures at Collège de France (`https://www.college-de-france.fr/`)
  - Pierre Cardaliaguet's notes (2013): `https://www.ceremade.dauphine.fr/ cardaliaguet/MFG20130420.pdf`
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Thank you for your attention

Questions?

Feel free to reach out: `mathieu.lauriere@nyu.edu`

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