Solutions

1) (a) The given permutation is:

52134

The inversions are:

$(5,2),(5,1),(5,3),(5,2)$

$(2,1)$

Thus there are five (odd) inversion, and so this is an odd permutation.

(b) The given permutation is:

45213

The inversions are:

$(4,2),(4,1),(4,3)$

$(5,2),(5,1),(5,3)$

$(2,1)$

Here, number of permutations are:

$3+3+1$

$=7$

which is odd, thus given permutation is odd permutation.

(c) 42135

The inversions are:

$(4,2),(4,1),(4,3)$

$(2,1)$

Here, the number of conversions are:

$3+1$
$=4$

which is even

Thus, the given permutation is an even permutation.

(d) 54321

The inversions are:

$(5,4),(5,3),(5,2),(5,1)$

$(4,3),(4,2),(4,1)$

$(3,2),(3,1)$

$(2,1)$

Here, the number of conversions are:

$$4+3+2+1$$
$$=10$$

which is even

Thus, the given permutation is an even permutation.

2) (a) 436215 and 416235

The number of inversions in 436215 is given by

$$3+2+3+1+0+0$$
$$=9$$

The number of inversions in 416235 is given by

$$3+0+3+0+0+0$$
$$=6$$

Thus, the number of inversions differ by:

$$9-6=3$$

which is odd. Hence verified.

(b) 623415 and 523416

The number of inversions in 623415 is given by

$$5+0+0+1+0+0$$
$$=6$$

The number of inversions in 523416 is given by

$$4+1+1+1+0+0$$
$$=7$$

Thus, the number of inversions differ by:

$$7-6=1$$

which is odd. Hence verified.

(c) 321564 and 341562

The number of inversions in 321564 is given by

$$\begin{aligned}&2+1+0+1+1+0\\&=5\end{aligned}$$

The number of inversions in 341562 is given by

$$\begin{aligned}&2+2+0+1+1+0\\&=6\end{aligned}$$

Thus, the number of inversions differ by:

$$6-5=1$$

which is odd. Hence verified.

(d) 123564 and 423561

The number of inversions in 123564 is given by

$$\begin{aligned}&0+0+0+1+1+0\\&=2\end{aligned}$$

The number of inversions in 423561 is given by

$$\begin{aligned}&3+1+1+1+1+0\\&=7\end{aligned}$$

Thus, the number of inversions differ by:

$$7-2=5$$

which is odd. Hence verified.

3) (a) $D=\begin{vmatrix}1 & 3\\ 2 & 4\end{vmatrix}$

$$=(1)(4)-(3)(2)$$

$$=4-6$$

$$=-2$$

(b) $D=\begin{vmatrix}1 & 2 & 1\\ 1 & 3 & 2\\ 1 & 0 & 1\end{vmatrix}$

$$= (1)\begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} - (2)\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + (1)\begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix}$$

$$= (1)((3)(1) - (2)(0)) - (2)((1)(1) - (2)(1)) + (1)((1)(0) - (3)(1))$$

$$= (1)(3 - 0) - (2)(1 - 2) + (1)(0 - 3)$$

$$= (1)(3) - (2)(-1) + (1)(-3)$$

$$= 3 + 2 - 3$$

$$= 2$$

(c) $D = \begin{vmatrix} -4 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 1 & 0 & 2 \\ 1 & 3 & 0 & 3 \end{vmatrix}$

$$= (-4)\begin{vmatrix} 3 & 1 & 0 \\ 1 & 0 & 2 \\ 3 & 0 & 3 \end{vmatrix} - (2)\begin{vmatrix} 2 & 1 & 0 \\ 3 & 0 & 2 \\ 1 & 0 & 3 \end{vmatrix} + (0)\begin{vmatrix} 2 & 3 & 0 \\ 3 & 1 & 2 \\ 1 & 3 & 3 \end{vmatrix} - (0)\begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 3 & 0 \end{vmatrix}$$

$$= (-4)(3(0 - 0) - 1(3 - 6) + 0(0 - 0)) - 2(2(0 - 0) - 1(9 - 2) + 0(0 - 0)) + 0 - 0$$

$$= (-4)(0 - 1(-3) + 0) - 2(0 - 1(7) + 0) + 0 - 0$$

$$= (-4)(3) - 2(-7)$$

$$= -12 + 14$$

$$= 2$$

(d) $D = \begin{vmatrix} 4 & 2 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$

$$= (4)\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} - (2)\begin{vmatrix} 2 & 0 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} + (2)\begin{vmatrix} 2 & 0 & 0 \\ 3 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} - (0)\begin{vmatrix} 2 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

If all elements of a row/column are zero, then value of determinant is zero.

$$D = (4)(0) - (2)\begin{vmatrix} 2 & 0 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} + (2)(0) - (0)\begin{vmatrix} 2 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 0 - (2)(2(0-1) - 0 + 0) + (0) - (0)$$

$$= -(2)(2(-1))$$

$$= -4$$

4) $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

Given that,

$$\det(A) = |A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 3 \qquad \ldots \qquad \text{(I)}$$

Consider

$$\det(B) = |B| = \begin{vmatrix} a_1 + 2b_1 - 3c_1 & a_2 + 2b_2 - 3c_2 & a_3 + 2b_3 - 3c_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Interchanging the rows R2 and R3, sign of the determinant gets changed.

$$= -\begin{vmatrix} a_1 + 2b_1 - 3c_1 & a_2 + 2b_2 - 3c_2 & a_3 + 2b_3 - 3c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Splitting the determinant along R1,

$$= -\left(\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} 2b_1 & 2b_2 & 2b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} -3c_1 & -3c_2 & -3c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \right)$$

Taking 2 common from R1 of second determinant, and -3 common from 3^{rd} determinant.

$$= -\left(\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + 2\begin{vmatrix} b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} - 3\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \right)$$

Since R1 and R2 of 2nd determinant are identical, so it is equal to zero

R1 and R3 of 3rd determinant are identical, so it is also zero.

$$= -\left(\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + 2(0) - 3(0) \right)$$

$$= -\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= -\det(A)$$
$$= -3$$

Now, consider

$$\det(C) = \begin{vmatrix} a_1 & 3a_2 & a_3 \\ b_1 & 3b_2 & b_3 \\ c_1 & 3c_2 & c_3 \end{vmatrix}$$

Taking 3 common from C2

$$= 3\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= 3\det(A)$$
$$= 3(3)$$

$$= 9$$

Now, consider

$$\det(D) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \left| A^T \right|$$

Since, determinants of a matrix and its transpose, are same.

$\det(D) = |A|$

$= 3$

5) (a) $D = \begin{vmatrix} 2 & 0 & 1 & 4 \\ 3 & 2 & -4 & -2 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix}$

$R_2 - R_3$

$$D = \begin{vmatrix} 2 & 0 & 1 & 4 \\ 1 & -1 & -3 & -2 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix}$$

$R_2 \leftrightarrow R_1$

$$D = -\begin{vmatrix} 1 & -1 & -3 & -2 \\ 2 & 0 & 1 & 4 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix}$$

$R_2 - 2R_1$
$R_3 - 2R_1$
$R_4 - 11R_1$

$$D = -\begin{vmatrix} 1 & -1 & -3 & -2 \\ 0 & 2 & 7 & 8 \\ 0 & 5 & 5 & 4 \\ 0 & 19 & 29 & 28 \end{vmatrix}$$

Expand along C1

$$D = -(1)\begin{vmatrix} 2 & 7 & 8 \\ 1 & -9 & -12 \\ 19 & 29 & 28 \end{vmatrix}$$

Taking 4 common C3

$$D = -(1)(4)\begin{vmatrix} 2 & 7 & 2 \\ 1 & -9 & -3 \\ 19 & 29 & 7 \end{vmatrix}$$

$R_1 - 2R_2$
$R_3 - 19R_2$

$$D = -4\begin{vmatrix} 0 & 25 & 8 \\ 1 & -9 & -3 \\ 0 & 200 & 64 \end{vmatrix}$$

Taking 8 common from R3

$$D = -4(8)\begin{vmatrix} 0 & 25 & 8 \\ 1 & -9 & -3 \\ 0 & 25 & 8 \end{vmatrix}$$

Since R1 and R3 are identical, so this determinant is zero

Thus given determinant is zero.

$$\text{(b)}\ \ D = \begin{vmatrix} 4 & 2 & 3 & -4 \\ 3 & -2 & 1 & 5 \\ -2 & 0 & 1 & -3 \\ 8 & -2 & 6 & 4 \end{vmatrix}$$

Taking 2 common from R4

$$D = 2\begin{vmatrix} 4 & 2 & 3 & -4 \\ 3 & -2 & 1 & 5 \\ -2 & 0 & 1 & -3 \\ 4 & -1 & 3 & 2 \end{vmatrix}$$

$R_1 + 2R_4$
$R_2 - 2R_4$

$$D = 2\begin{vmatrix} 12 & 0 & 9 & 0 \\ -5 & 0 & -5 & 1 \\ -2 & 0 & 1 & -3 \\ 4 & -1 & 3 & 2 \end{vmatrix}$$

Expand along C2

$$D = 2(-1)\begin{vmatrix} 12 & 9 & 0 \\ -5 & -5 & 1 \\ -2 & 1 & -3 \end{vmatrix}$$

Taking 3 common from R1

$$D = 2(-1)(3)\begin{vmatrix} 4 & 3 & 0 \\ -5 & -5 & 1 \\ -2 & 1 & -3 \end{vmatrix}$$

$R_3 + 3R_1$

$$D = 2(-1)(3)\begin{vmatrix} 4 & 3 & 0 \\ -5 & -5 & 1 \\ 13 & 16 & 0 \end{vmatrix}$$

Expand along C3

$$D = 2(-1)(3)(-1)\begin{vmatrix} 4 & 3 \\ 13 & 16 \end{vmatrix}$$

$$D = 2(-1)(3)(-1)(64-39)$$

$$D = 6(25)$$

$$D = 150$$

(c) $D = \begin{vmatrix} 2 & 0 & 0 & 0 \\ -5 & 3 & 0 & 0 \\ 3 & 2 & 4 & 0 \\ 4 & 2 & 1 & -5 \end{vmatrix}$

The given determinant is in Triangular form, i.e. all the entries above the diagonal are zero

Thus, Determinant is given by multiplying the diagonal entries only

$$D = (2)(3)(4)(-5)$$

$$D = -120$$

6) Consider,

$$D_n = \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & & \alpha_1^{n-2} & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-2} & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & & \alpha_3^{n-2} & \alpha_3^{n-1} \\ & \vdots & & \ddots & \vdots & \\ 1 & \alpha_{n-1} & \alpha_{n-1}^2 & \cdots & \alpha_{n-1}^{n-2} & \alpha_{n-1}^{n-1} \\ 1 & \alpha_n & \alpha_n^2 & & \alpha_n^{n-2} & \alpha_n^{n-1} \end{vmatrix}$$

Subtract R1 from each of the other rows and leave Dn unchanged:

$$D_n = \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & & \alpha_1^{n-2} & \alpha_1^{n-1} \\ 0 & \alpha_2 - \alpha_1 & \alpha_2^2 - \alpha_1^2 & \cdots & \alpha_2^{n-2} - \alpha_1^{n-2} & \alpha_2^{n-1} - \alpha_1^{n-1} \\ 0 & \alpha_3 - \alpha_1 & \alpha_3^2 - \alpha_1^2 & & \alpha_3^{n-2} - \alpha_1^{n-2} & \alpha_3^{n-1} - \alpha_1^{n-1} \\ & \vdots & & \ddots & \vdots & \\ 0 & \alpha_{n-1} - \alpha_1 & \alpha_{n-1}^2 - \alpha_1^2 & \cdots & \alpha_{n-1}^{n-2} - \alpha_1^{n-2} & \alpha_{n-1}^{n-1} - \alpha_1^{n-1} \\ 0 & \alpha_n - \alpha_1 & \alpha_n^2 - \alpha_1^2 & & \alpha_n^{n-2} - \alpha_1^{n-2} & \alpha_n^{n-1} - \alpha_1^{n-1} \end{vmatrix}$$

Subtract in order, α_1 times C-(n-1) from C-(n),

$C_n - \alpha_1 C_{n-1}$
$C_{n-1} - \alpha_1 C_{n-2}$
...
$C_2 - \alpha_1 C_1$

$$D_n = \begin{vmatrix} 1 & 0 & 0 & & 0 & 0 \\ 0 & \alpha_2 - \alpha_1 & (\alpha_2 - \alpha_1)\alpha_2 & \cdots & (\alpha_2 - \alpha_1)\alpha_2^{n-3} & (\alpha_2 - \alpha_1)\alpha_2^{n-2} \\ 0 & \alpha_3 - \alpha_1 & (\alpha_3 - \alpha_1)\alpha_3 & & (\alpha_3 - \alpha_1)\alpha_3^{n-3} & (\alpha_3 - \alpha_1)\alpha_3^{n-2} \\ & \vdots & & \ddots & \vdots & \\ 0 & \alpha_{n-1} - \alpha_1 & (\alpha_{n-1} - \alpha_1)\alpha_{n-1} & \cdots & (\alpha_{n-1} - \alpha_1)\alpha_{n-1}^{n-3} & (\alpha_{n-1} - \alpha_1)\alpha_{n-1}^{n-2} \\ 0 & \alpha_n - \alpha_1 & (\alpha_n - \alpha_1)\alpha_n & & (\alpha_n - \alpha_1)\alpha_n^{n-3} & (\alpha_n - \alpha_1)\alpha_n^{n-2} \end{vmatrix}$$

From (k)th row, extracting $\left(\alpha_k - \alpha_1\right)$ as factors, we get,

$$D_n = \prod_{k=2}^{n} (\alpha_k - \alpha_1) \begin{vmatrix} 1 & 0 & 0 & & 0 & 0 \\ 0 & 1 & \alpha_2 & \cdots & \alpha_2^{n-3} & \alpha_2^{n-2} \\ 0 & 1 & \alpha_3 & & \alpha_3^{n-3} & \alpha_3^{n-2} \\ & \vdots & & \ddots & & \vdots \\ 0 & 1 & \alpha_{n-1} & \cdots & \alpha_{n-1}^{n-3} & \alpha_{n-1}^{n-2} \\ 0 & 1 & \alpha_n & & \alpha_n^{n-3} & \alpha_n^{n-2} \end{vmatrix}$$

Expand along C1

$$D_n = \prod_{k=2}^{n} (\alpha_k - \alpha_1) \begin{vmatrix} 1 & \alpha_2 & \cdots & \alpha_2^{n-3} & \alpha_2^{n-2} \\ 1 & \alpha_3 & & \alpha_3^{n-3} & \alpha_3^{n-2} \\ \vdots & & \ddots & \vdots & \\ 1 & \alpha_{n-1} & \cdots & \alpha_{n-1}^{n-3} & \alpha_{n-1}^{n-2} \\ 1 & \alpha_n & & \alpha_n^{n-3} & \alpha_n^{n-2} \end{vmatrix}$$

It can be easily seen,

$$D_n = \prod_{k=2}^{n} (\alpha_k - \alpha_1) D_{n-1}$$

$$\Rightarrow D_n = \prod_{k=2}^{n} (\alpha_k - \alpha_1) \prod_{m=2}^{n} (\alpha_m - \alpha_2) D_{n-2}$$

and so on,…

Thus, $\Rightarrow D_n = \prod_{1 \le i \langle j \le n} (\alpha_j - \alpha_i)$

Hence proved.