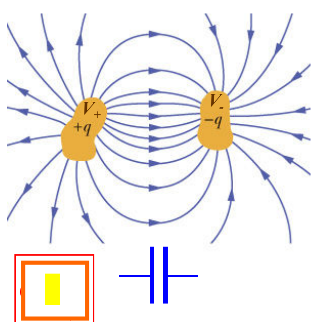
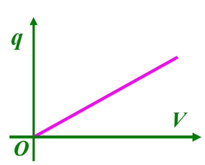
Capacitance

-Capacitance C of a system of two isolated conductors.
-Calculation of the capacitance for some simple geometries.
-Methods of connecting capacitors (in series, in parallel).
-Equivalent capacitance.
-Energy stored in a capacitor.
-Behavior of an insulator (a.k.a. dielectric) when placed in the electric field created in the space between the plates of a capacitor.
-Gauss' law in the presence of dielectrics.



Capacitance

A system of two isolated conductors, one with a charge $+q$ and the other $-q$, separated by an insulator (this can be vacuum or air) is known as a "capacitor." The symbol used to indicate a capacitor is two parallel lines. We refer to the conductors as "plates." We refer to the "charge" of the capacitor as the absolute value of the charge on either plate.



As shown in the figure, the charges on the capacitor plates create an electric field in the surrounding space. The electric potential of the positive and negative plate are V_+ and V_-, respectively. We use the symbol V for the potential difference $V_+ - V_-$ between the plates (ΔV would be more appropriate).

If we plot the charge q as a function of V we get the straight line shown in the figure. The capacitance C is defined as the ratio

SI Unit : Farad (symbol F) We define a capacitor of $C = 1$ F as one that acquires a charge $q = 1$ C if we apply a voltage difference $V = 1$ V between its plates.

Q. How much charge is on the plates of a 11-μF capacitor that has been connected to a 120 V dc power supply for a long time?

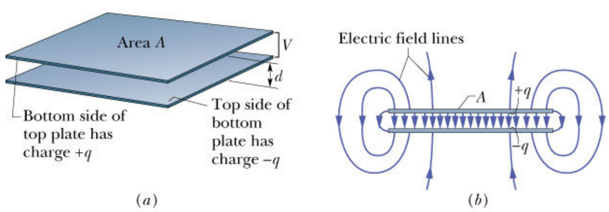
a) 1.3×10^{-3} C

b) 9.2×10^{-2} C

c) 1.1×10^{-4} C

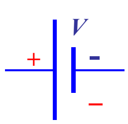
d) 1.3×10^{-6} C

e) 1.2×10^{-1} C



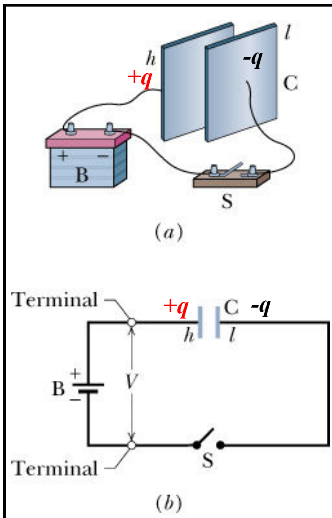
Parallel Plate Capacitor

A parallel plate capacitor is defined as a capacitor made up from two parallel plane plates of area A separated by a distance d. The electric field between the plates and away from the plate edges is uniform. Close to the plates' edges the electric field (known as "fringing field") becomes nonuniform.



Batteries

A battery is a device that maintains a constant potential difference V between its two terminals. These are indicated in the battery symbol using two parallel lines unequal in length. The longer line indicates the terminal at higher potential while the shorter line denotes the lower-potential terminal.



Charging a Capacitor

One method to charge a capacitor is shown in the figure. When the switch S is closed, the electric field of the battery drives electrons from the battery negative terminal to the capacitor plate connected to it (labeled "ℓ" for low). The battery positive terminal removes an equal number of electrons from the plate connected to it (labeled "h" for high). Initially the potential difference V between the capacitor plates is zero. The charge on the plates as well as the potential difference between the plates increase, and the charge movement from the battery terminals to and from the plates decreases. All charge movement stops when the potential difference between the plates becomes equal to the potential difference between the battery terminals.

Q. The plates of an isolated parallel plate capacitor with a capacitance C carry a charge Q. What is the capacitance of the capacitor if the charge is increased to $4Q$?

a) $C/2$

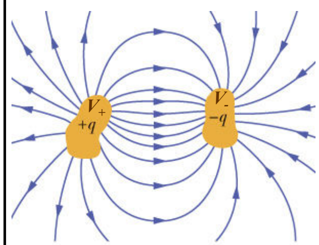
b) $C/4$

c) $4C$

d) $2C$

e) C

Calculating the Capacitance



The capacitance depends on the geometry of the plates

Recipe :

1. Assume that the plates have charges $+q$ and

2. Use to determine the electric field $\vec{E}$ between the plates $\left(\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{\text{enc}}\right)$.

3. Determine the potential difference V between the plates using the equation

$V = \int_{-}^{+}$ along any path that connects the negative with the positive plate.

4. The capacitance C is given by the equation

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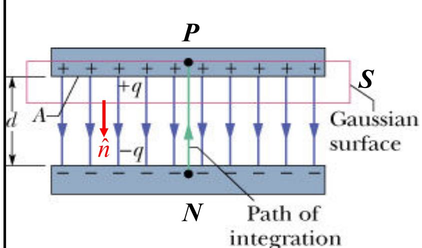
Capacitance of a Parallel Plate Capacitor

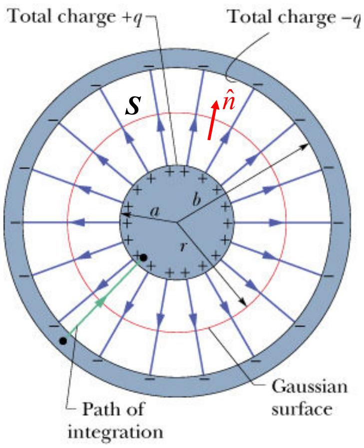


From Gauss' law we have: $\Phi = \frac{q}{\varepsilon_0} \rightarrow EA = \frac{q}{\varepsilon_0} \rightarrow E = \frac{q}{A}$

The potential difference V between the positive and the negative plate is given by: $V = \int_{-}^{+} Eds\cos 0 = E\int_{-}^{+} ds = Ed =$

The capacitance $C = \frac{q}{V} = \frac{q}{qd / A\varepsilon_0} = \frac{A\varepsilon_0}{d}$.

Cylindrical Capacitor



The flux of the electric field through S is

$\Phi = \boxed{\quad\quad\quad}$

Using Gauss' law we have: $\Phi = \dfrac{q}{\varepsilon_0}$.

If we combine these equations we have:

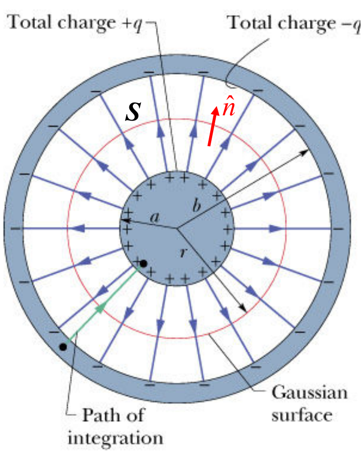
$$E = \frac{q}{\varepsilon_0 2\pi r L}.$$

The potential difference V between the positive and the negative plate is given by:

$$V = \int_-^+ \boxed{\quad} \cos 180 = -\frac{q}{\varepsilon_0 2\pi L}\int_b^a \frac{dr}{r} = -\frac{q}{\varepsilon_0 2\pi L}\boxed{\quad} = \frac{q}{\varepsilon_0 2\pi L}\ln\left(\frac{b}{a}\right).$$

The capacitance $C = \boxed{\quad} = \dfrac{q}{\left(q/2\pi L\varepsilon_0\right)\ln\left(b/a\right)} = \dfrac{2\pi L\varepsilon_0}{\ln\left(b/a\right)}$.

Spherical Capacitor



The flux of the electric field through S is:

$\Phi = 4\pi r^2 E\cos 0 = \boxed{\quad}$.

Using Gauss' law we have: $\Phi = \dfrac{q}{\varepsilon_0}$.

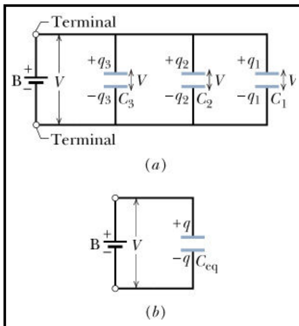
If we combine these equations we have:

$$E = \frac{q}{4\pi\varepsilon_0 r^2}.$$

The potential difference V between the positive and the negative plate is given by:

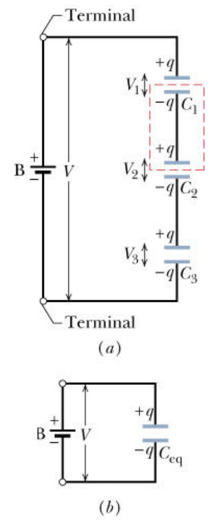
$$V = \int_-^+ E dr \cos 180 = -\frac{q}{4\pi\varepsilon_0}\int_b^a \boxed{\quad} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r}\right]_b^a = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right).$$

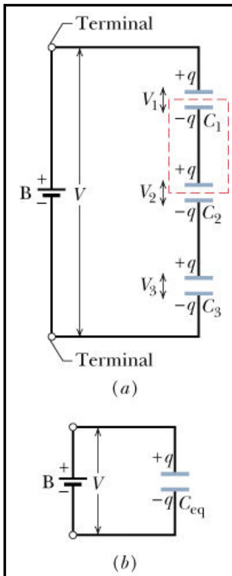
The capacitance $C = \dfrac{q}{V} = \dfrac{q}{\boxed{\quad}\left(\dfrac{1}{a}-\dfrac{1}{b}\right)} = \dfrac{4\pi\varepsilon_0}{\left(\dfrac{1}{a}-\dfrac{1}{b}\right)} = 4\pi\varepsilon_0\left(\dfrac{ab}{b-a}\right)$.



Equivalent Capacitor

Consider the combination of capacitors shown in the figure to the left and to the right (upper part). We will substitute these combinations of capacitor with a single capacitor C_{eq} that is "electrically equivalent" to the capacitor group it substitutes.





Capacitors in Series

We will substitute the series combination of fig. *a* with a single equivalent capacitor shown in fig. *b*, which is also connected to an identical battery.

The three capacitors have the **same charge *q*** on their plates.

The voltage across C_1 is $V_1 = q / C_1$.

The voltage across C_2 is $V_2 = q / C_2$.

The voltage across C_3 is $V_3 = q / C_3$.

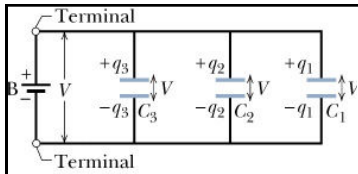
The net voltage across the combination $V = V_1 + V_2 + V_3$.

Thus we have: $V = q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)$.

The equivalent capacitance $C_{eq} = \frac{q}{V} = \frac{q}{q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)} \rightarrow$

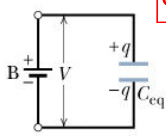
$\frac{1}{C_{eq}} = \boxed{\quad}$

Capacitors in Parallel



(a)

$C_{\text{eq}} = \boxed{\quad\quad\quad}$



(b)

We will substitute the parallel combination of fig. *a* with a single equivalent capacitor shown in fig. *b*, which is also connected to an identical battery.

The three capacitors have the **same potential difference** V across their plates.
The charge on C_1 is $q_1 = C_1 V$. The charge on C_2 is $q_2 = C_2 V$.
The charge on C_3 is $q_3 = C_3 V$. The net charge $q = q_1 + q_2 + q_3 = (C_1 + C_2 + C_3)V$.

The equivalent capacitance $C_{\text{eq}} = \dfrac{q}{V} = \dfrac{(C_1 + C_2 + C_3)V}{V} = C_1 + C_2 + C_3$.

For a parallel combination of n capacitors it is given by the expression:

$$C_{\text{eq}} = C_1 + C_2 + \ldots + C_n = \sum_{j=1}^{n} C_j$$

Q. Capacitor B has one-half the capacitance of capacitor A. How does the charge on capacitor A compare to that on B when the two are connected in series to a battery for a long time?

a) The charge on capacitor A is one-fourth the charge on capacitor B.

b) The charge on capacitor A is one-half the charge on capacitor B.

c) The charge on capacitor A is the same as the charge on capacitor B.

d) The charge on capacitor A is twice the charge on capacitor B.

e) The charge on capacitor A is four times the charge on capacitor B.

Q. Capacitor B has one-half the capacitance of capacitor A. How does the charge on capacitor A compare to that on B when the two are connected in parallel with a battery for a long time?

a) The charge on capacitor A is one-fourth the charge on capacitor B.

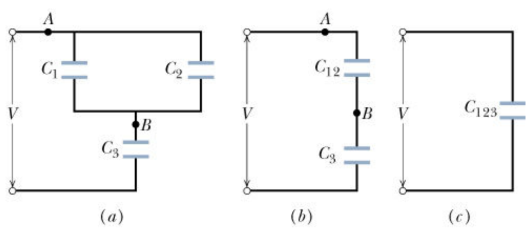
b) The charge on capacitor A is one-half the charge on capacitor B.

c) The charge on capacitor A is the same as the charge on capacitor B.

d) The charge on capacitor A is twice the charge on capacitor B.

e) The charge on capacitor A is four times the charge on capacitor B.

More Complex Capacitor Systems

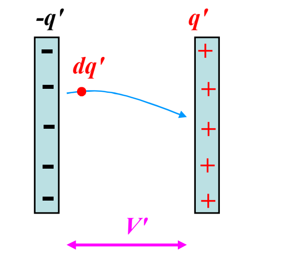


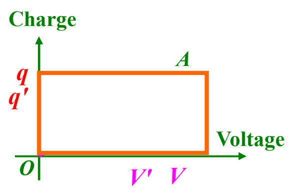
For example, C_1 and C_2 in fig. *a* are connected in

They can be substituted by the equivalent capacitor $C_{12} =$ as shown in fig. *b*. Capacitors C_{12} and C_3 in fig. *b* are connected in

They can be substituted by a single capacitor C_{123} as shown in fig. *c*.

C_{123} is given by the equation $\frac{1}{C_{123}} =$





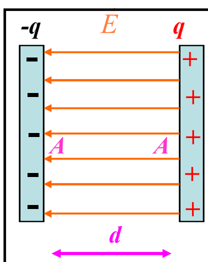
Energy Stored in an Electric Field

We can calculate the work W required to charge the capacitor by assuming that we transfer a charge dq' from the negative plate to the positive plate. We assume that the capacitor charge is q' and the corresponding voltage V'. The work dW required for the charge transfer is given by: $dW = \boxed{}$

We continue this process till the capacitor charge is equal to q. The total work $W = \int V'dq' = \frac{1}{C}\int_0^q q'dq'$.

$W = \boxed{}$ If we substitute $\boxed{}$ we get: $W = \frac{CV^2}{2}$ or $W = \frac{qV}{2}$.

Work W can also be calculated by determining the area A of triangle OAB, which is equal to $\int V'dq'$: Area $= W = \frac{Vq}{2}$.



Potential Energy Stored in a Capacitor

The work W spent to charge a capacitor is stored in the form of potential energy $U = W$ that can be retrieved when the capacitor is discharged. Thus $U = \frac{q^2}{2C} = \frac{CV^2}{2} = \frac{qV}{2}$.

Energy Density

We can ask the question: Where is the potential energy of a charged capacitor stored? The answer is counterintuitive. The energy is stored in the space between the capacitor plates where a uniform electric field $E = V/d$ is generated by the capacitor charges. In other words, the electric field can store energy in empty space!

$U = \boxed{}$

$u = \boxed{}$

We define energy density (symbol u) as the potential energy per unit volume: $u = \frac{U}{V}$.

The volume V between the plates is $V = Ad$, where A is the plate area.

Thus the energy density $u = \frac{U}{Ad} = \frac{CV^2}{2Ad} = \boxed{}$

This result, derived for the parallel plate capacitor, holds in general.

Q. A capacitor has a very large capacitance of 10 F. The capacitor is charged by placing a potential difference of 2 V between its plates. How much energy is stored in the capacitor?

a) 2000 J

b) 500 J

c) 100 J

d) 40 J

e) 20 J

Q. The plates of an isolated parallel plate capacitor are separated by a distance d and carry a fixed charge of magnitude q. The distance between the plates is then reduced to $d/2$. How is the energy stored in the capacitor affected by this change?

a) The energy increases to twice its initial value.

b) The energy increases to four times its initial value.

c) The energy is not affected by this change.

d) The energy decreases to one fourth of its initial value.

e) The energy decreases to one half of its initial value.

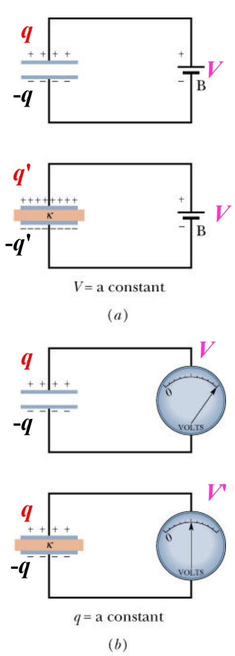
Q. Two parallel conducting plates are connected to a battery for a long time and become fully-charged. How does the capacitance change, if at all, when a conducting slab is inserted in between the plates without touching wither plate?

a) The capacitance will increase.

b) The capacitance will decrease.

c) The capacitance will remain unchanged.

$\boxed{C = \kappa C_{\text{air}}}$



Capacitor with a Dielectric

In 1837 Michael Faraday investigated what happens to the capacitance C of a capacitor when the gap between the plates is completely filled with an insulator (a.k.a. dielectric).

Faraday discovered that the new capacitance is given by $C = \boxed{\quad}$. Here C_{air} is the capacitance before the insertion of the dielectric between the plates. The factor κ is known as the $\boxed{\qquad\qquad}$ of the material.

Faraday's experiment can be carried out in two ways:

1. With the voltage V across the plates remaining constant. In this case a battery remains connected to the plates. This is shown in fig. *a*.

2. With the charge q of the plates remaining constant. In this case the plates are isolated from the battery. This is shown in fig. *b*.

$\boxed{C = \kappa C_{\text{air}}}$

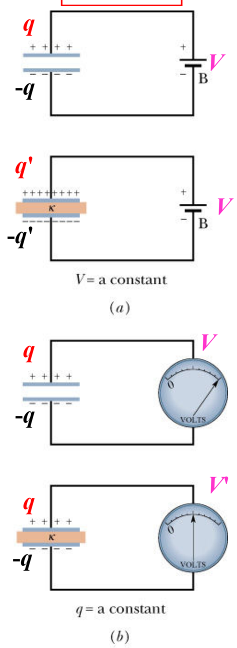


Fig. a: Capacitor voltage V remains constant

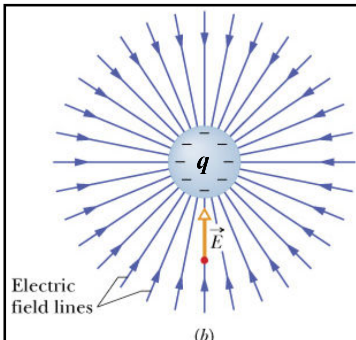
After the dielectric is inserted between the capacitor plates the plate charge changes from q to $q' = \kappa q$.

The new capacitance $C = \dfrac{q'}{V} = \dfrac{\kappa q}{V} = \kappa \dfrac{q}{V} = \kappa C_{\text{air}}$.

Fig. b: Capacitor charge q remains constant

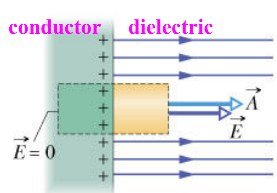
After the dielectric is inserted between the capacitor plates the plate voltage changes from V to $V' = \dfrac{V}{\kappa}$.

The new capacitance $C = \dfrac{q}{V'} = \dfrac{q}{V/\kappa} = \kappa \dfrac{q}{V} = \kappa C_{\text{air}}$.



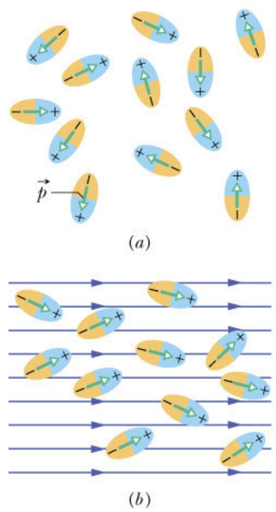
In a region completely filled with an insulator of dielectric constant κ, all electrostatic equations containing the constant ε_0 are to be modified by replacing ε_0 with $\kappa\varepsilon_0$.

Example 1: Electric field of a point charge inside a dielectric is: $E = \boxed{}\dfrac{q}{r^2}$.



Example 2:
The electric field outside an isolated conductor immersed in a dielectric becomes:

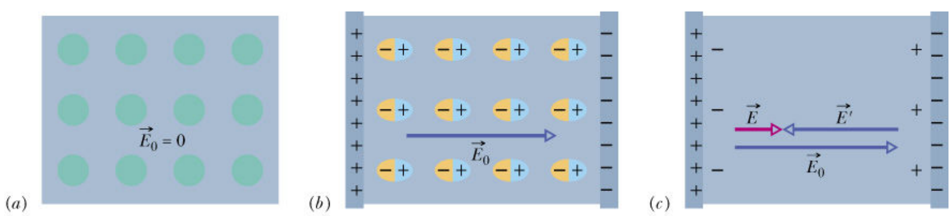
$E = \boxed{}$



$U =$ $\boxed{}$

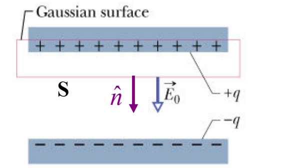
Dielectrics : An Atomic View

Dielectrics are classified as "polar" and "nonpolar." Polar dielectrics consist of molecules that have a nonzero electric dipole moment even at zero electric field due to the asymmetric distribution of charge within the molecule (e.g., H_2O). At zero electric field (see fig. *a*) the electric dipole moments are randomly oriented. When an external electric field $\vec{E}_0$ is applied (see fig. *b*) the electric dipole moments tend to align preferentially along the direction of $\vec{E}_0$ because this configuration corresponds to a minimum of the potential energy and thus is a position of stable equilibrium. Thermal random motion opposes the alignment and thus ordering is incomplete. Even so, the partial alignment produced by the external electric field generates an internal electric field that opposes $\vec{E}_0$. Thus the net electric field $\vec{E}$ is **weaker** than $\vec{E}_0$.



A nonpolar dielectric, on the other hand, consists of molecules that in the absence of an electric field have zero electric dipole moment (see fig. *a*) . If we place the dielectric between the plates of a capacitor the external electric field $\vec{E}_0$ induces an electric dipole moment $\vec{p}$ that becomes aligned with $\vec{E}_0$ (see fig. *b*). The aligned molecules do not create any net charge inside the dielectric. A net charge appears at the left and right surfaces of the dielectric opposite the capacitor plates. These charges come from negative and positive ends of the electric dipoles. These **induced** surface charges have sign **opposite** that of the opposing plate charges. Thus the induced charges create an electric field $\vec{E}'$ that opposes the applied field $\vec{E}_0$ (see fig. *c*). As a result, the net electric field $\vec{E}$ between the capacitor plates is weaker.

$E =$ $\boxed{}$



(a)

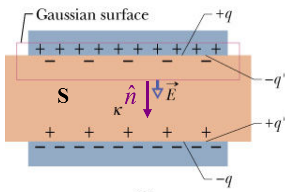
Gauss' Law and Dielectrics

Consider first the parallel plate capacitor shown in fig. *a*.

We will use the Gaussian surface S. The flux $\Phi = E_0 A = \dfrac{q}{\varepsilon_0}$

$\rightarrow E_0 = \dfrac{q}{\varepsilon_0 A}$. Now we fill the space between the plates

with an insulator of dielectric constant κ (see fig. *b*) .



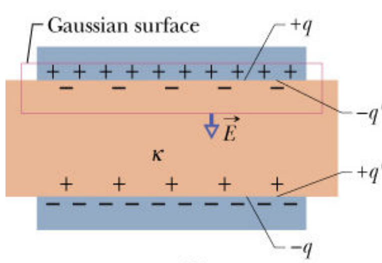
(b)

We will apply Gauss' law for the same surface S. Inside S in addition to the plate charge q

we also have the induced charge q' on the surface of the dielectric: $\Phi = EA = \dfrac{q - q'}{\varepsilon_0} \rightarrow$

$E = \dfrac{q - q'}{A\varepsilon_0}$ (eq. 1). From Faraday's experiments we have: $E = \dfrac{E_0}{\kappa} = \dfrac{q}{\kappa A \varepsilon_0}$ (eq. 2).

If we compare eq. 1 with eq. 2 we have: $q - q' = \dfrac{q}{\kappa} \rightarrow \varepsilon_0 \oint \kappa \vec{E} \cdot d\vec{A} = q.$



(b)

Gauss' Law in the Presence of Dielectrics

$= q$

Even though the equation above was derived for the parallel plate capacitor, it is true in general.

Note 1 : The flux integral now involves $\kappa \vec{E}$.

Note 2 : The charge q that is used is the plate charge, also known as "free charge." Using the equation above we can ignore the induced charge q'.

Note 3 : The dielectric constant κ is kept inside the integral to describe the most general case in which κ is not constant over the Gaussian surface.