STAT20060 - Statistics and Probability
Handout 3 - Probability

Damien McParland

Running example

A fair die is rolled once. The possible outcomes are:

$$\{1, 2, 3, 4, 5, 6\}$$

Preliminary definitions

Experiment

An experiment is an act or process of observation that leads to a single outcome that cannot be predicted with certainty.

- Example: The experiment is rolling the die.

Sample point

A sample point is the most basic outcome of an experiment

- Example: There are 6 sample points in this experiment:

$$1, 2, 3, 4, 5, 6.$$

Preliminary definitions

Sample Space

The sample space of an experiment is the set of ALL its sample points.

- Example: The sample space of this experiment is:

$$\{1, 2, 3, 4, 5, 6\}.$$

Probability Axioms

Let p_i denote the probability of sample point i and let S be the sample space:

- $0 \leq p_i \leq 1$.
- $\mathbb{P}(S) = 1$.
- The probabilities of **all** sample points within a sample space must sum to 1.

$$\sum_i p_i = 1$$

Events

Event

An event is a collection of sample points. It is a subset of the sample space S.

- An event $\mathcal{A}$ occurs if any one of the sample points in $\mathcal{A}$ occur.

Probability of an event.

The probability of an event $\mathcal{A}$ is calculated by summing the probabilities of the sample points in $\mathcal{A}$.

Example: Loaded Die

- A die has been *loaded* so that the probability of side i coming up is proportional to i.
- If $\mathcal{A}$ is the event that either a 2 or a 3 comes up.
- What is $\mathbb{P}(\mathcal{A})$?

Set notation

An event that can be viewed as the composition as 2 or more events is called a compound event.

Union

The union of two events $\mathcal{A}$ and $\mathcal{B}$ is the event that occurs if either $\mathcal{A}$ or $\mathcal{B}$ (or both) occur. Denoted $\mathcal{A} \cup \mathcal{B}$.

Intersection

The intersection of two events $\mathcal{A}$ and $\mathcal{B}$ is the event that occurs if both $\mathcal{A}$ and $\mathcal{B}$ occur. Denoted $\mathcal{A} \cap \mathcal{B}$.

Picture:

Additive law

The probability of the union of events may be calculated without knowing the individual sample point probabilities.

Additive law

- The probability of the union of two events $\mathcal{A}$ and $\mathcal{B}$ is:

$$\mathbb{P}(\mathcal{A} \cup \mathcal{B}) = \mathbb{P}(\mathcal{A}) + \mathbb{P}(\mathcal{B}) - \mathbb{P}(\mathcal{A} \cap \mathcal{B})$$

- Two events $\mathcal{A}$ and $\mathcal{B}$ are mutually exclusive if they cannot occur at the same time.
- We can write $\mathcal{A} \cap \mathcal{B} = \emptyset$ when the events are mutually exclusive.
- If $\mathcal{A}$ and $\mathcal{B}$ are mutually exclusive events, $\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = 0$ and so

$$\mathbb{P}(\mathcal{A} \cup \mathcal{B}) = \mathbb{P}(\mathcal{A}) + \mathbb{P}(\mathcal{B})$$

Complementary events

Complement

The complement of an event $\mathcal{A}$ is the event that $\mathcal{A}$ does not occur. Denoted $\mathcal{A}^c$

Note: All the sample points in S are either in $\mathcal{A}$ or $\mathcal{A}^c$, no sample point can be in both. Thus,

$$\begin{aligned} & \mathbb{P}(\mathcal{A}) + \mathbb{P}(\mathcal{A}^c) = 1 \\ \Rightarrow \quad & \mathbb{P}(\mathcal{A}) = 1 - \mathbb{P}(\mathcal{A}^c) \end{aligned}$$

This is a useful formula for computation.

Example: Dice

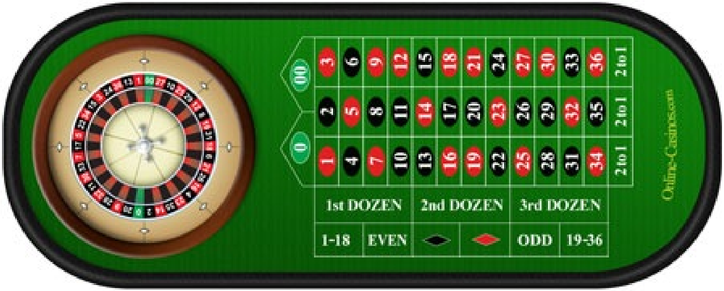
Two fair dice are rolled. Event $\mathcal{A}$ is that we observe a 5. Event $\mathcal{B}$ is that the dice sum to 7. Calculate:

- $\mathbb{P}(\mathcal{A} \cap \mathcal{B})$ and $\mathbb{P}(\mathcal{A} \cup \mathcal{B})$.
- $\mathbb{P}(\mathcal{A}^c)$.

S:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Example: Roulette



- 38 slots, 18 red, 18 black, 2 green, 18 even, 18 odd.

Example: Roulette

- A – outcome is an odd number (0 and 00 are neither odd nor even).
- B – outcome is a red number.
- C – outcome is in the first dozen (1-12).

1. Define the events $A \cap B$ and $A \cup B$ as a specific sets of sample points.
2. Find $\mathbb{P}(A)$, $\mathbb{P}(B)$, $\mathbb{P}(A \cap B)$, $\mathbb{P}(A \cup B)$ and $\mathbb{P}(C)$ by summing the probabilities of the appropriate sample points.
3. Find $\mathbb{P}(A \cup B)$ using the additive rule. Are events A and B mutually exclusive?
4. Find $\mathbb{P}(A \cap B \cap C)$.

Conditional Probability

- Sometimes we are aware of extra information which might affect the outcome of an experiment. This extra information may then alter the probability of a particular event of interest.

- Suppose we are interested in evaluating the probability that event $\mathcal{B}$ happens given that we know that event $\mathcal{A}$ has happened.

- We write $\mathbb{P}(\mathcal{B}|\mathcal{A})$ for this.

- It is called the *conditional probability of $\mathcal{B}$ given $\mathcal{A}$.*

Multiplication Rule and Bayes Theorem

- The multiplication rule of probabilities states that

$$\begin{aligned}\mathbb{P}(\mathcal{B} \cap \mathcal{A}) &= \mathbb{P}(\mathcal{B}|\mathcal{A})\mathbb{P}(\mathcal{A}) \\ &= \mathbb{P}(\mathcal{A}|\mathcal{B})\mathbb{P}(\mathcal{B})\end{aligned}$$

- Bayes Theorem states that

$$\mathbb{P}(\mathcal{B}|\mathcal{A}) = \frac{\mathbb{P}(\mathcal{B} \cap \mathcal{A})}{\mathbb{P}(\mathcal{A})}$$

- This is just the multiplication rule written in a different way.

Example: Illegal Trading

A trading manager knows that 3 out of 10 traders under her supervision are making illegal trades. If she selects 2 workers at random what is the probability that they have both been trading illegally?

Very Useful Identity

- Let's consider the probability of event $\mathcal{A}$.

$$\begin{aligned}\mathbb{P}(\mathcal{A}) &= \mathbb{P}\{(\mathcal{A}\cap\mathcal{B})\cup(\mathcal{A}\cap\mathcal{B}^c)\}\\ &= \mathbb{P}(\mathcal{A}\cap\mathcal{B})+\mathbb{P}(\mathcal{A}\cap\mathcal{B}^c)\\ &= \mathbb{P}(\mathcal{A}|\mathcal{B})\mathbb{P}(\mathcal{B})+\mathbb{P}(\mathcal{A}|\mathcal{B}^c)\mathbb{P}(\mathcal{B}^c)\end{aligned}$$

- Hence, a very useful form of Bayes Theorem can be written as

$$\mathbb{P}(\mathcal{B}|\mathcal{A}) = \frac{\mathbb{P}(\mathcal{A}|\mathcal{B})\mathbb{P}(\mathcal{B})}{\mathbb{P}(\mathcal{A}|\mathcal{B})\mathbb{P}(\mathcal{B})+\mathbb{P}(\mathcal{A}|\mathcal{B}^c)\mathbb{P}(\mathcal{B}^c)}.$$

- This is particularly useful when turning around probabilities that are *the wrong way around*.

Example: Diagnostic Test

Suppose there is a rare disease which affects 1 person in every 1000 of the population. Fortunately a diagnostic medical test exists for the disease. It is a good test in that, if you have the disease, the test will be positive 95% of the time and if you do not have the disease it will be negative 99% of the time. If a patient tests positive for the disease, what is the probability that they actually have the disease?

Independence

Independence

Two events $\mathcal{A}$ and $\mathcal{B}$ are said to be independent if the occurence of $\mathcal{B}$ does not alter the probability that $\mathcal{A}$ has occurred. i.e. $\mathcal{A}$ and $\mathcal{B}$ are independent if:

$$\mathbb{P}(\mathcal{A}|\mathcal{B}) = \mathbb{P}(\mathcal{A})$$

Events which are not independent are said to be dependent.

- Combining the definition above with the multiplicative rule, it can be seen that if $\mathcal{A}$ and $\mathcal{B}$ are independent then:

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B})$$

  The converse is also true, i.e. if $\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B})$ then the events $\mathcal{A}$ and $\mathcal{B}$ are independent.

- Suppose $\mathcal{A}$ and $\mathcal{B}$ are mutually exclusive events. If $\mathcal{B}$ occurs then $\mathcal{A}$ cannot occur simultaneously so $\mathbb{P}(\mathcal{A}|\mathcal{B}) = 0$.

$\Rightarrow$ Mutually exclusive events are dependent events.

Example: Tyres

Three types of concrete (Types A, B and C) are independently tested for suitability for use in buildings in earthquake risk areas. The probabilities that each passes the test are 0.7, 0.6, and 0.5 respectively.

- What is the probability that they all fail the test?
- What is the probability that at least one passes?
- Granted that at least one passed, what is the probability that type B was the only one to do so?

Example: Corrosion

The independence of corrosion and the functional status of a machine component are to be investigated. Are they independent?

	Functioning	Malfunctioning
Corroded	0.2	0.4
Not corroded	0.3	0.1

Counting Rules

Multiplicative counting rule

Suppose we have k sets with n_1 elements in the first set, n_2 elements in the second, ..., n_k elements in the k^{th} set. If we wish to take a sample of size k consisting of 1 element from each set, the number of ways this sample can be formed is:

$$n_1.n_2.\ldots.n_k$$

e.g. A password consists of 1 letter followed by 3 digits. How many possible passwords are there?

$$\begin{aligned}\#\text{ passwords} &= \boxed{26}.\boxed{10}.\boxed{10}.\boxed{10} \\ &= 26,000\end{aligned}$$

Combinations Rule

Given a set of N elements, an unordered subset of these elements is called a combination.

Combinations rule

The number of combinations of size r which can be formed from a set of size N is:

$$\binom{N}{r} = \frac{N!}{r!(N-r)!}$$

e.g. The number of soccer teams which can be formed from a panel of size 22 is:

$$\binom{22}{11} = \frac{22!}{11!(22-11)!} = 705,432 \text{ teams.}$$

Permutations Rule

The arrangement of elements of a set in a distinct order is called a permutation.

Permutations rule

The number of different permutations of size r which can be formed from a set of size N is:

$$P_r^N = \frac{N!}{(N-r)!}$$

e.g. 50 engineers are available to do 3 jobs. How many ways can the engineers be allocated to the jobs?

$$P_3^{50} = \frac{50!}{(50-3)!} = 117,600$$

Example: Poker

1. How many 5 card hands may be dealt from a deck of 52 cards?
2. What is the probability of being dealt 3 of a kind in poker?
3. What is the probability of being dealt a full house in poker? (2 of one denomination and 3 of another)