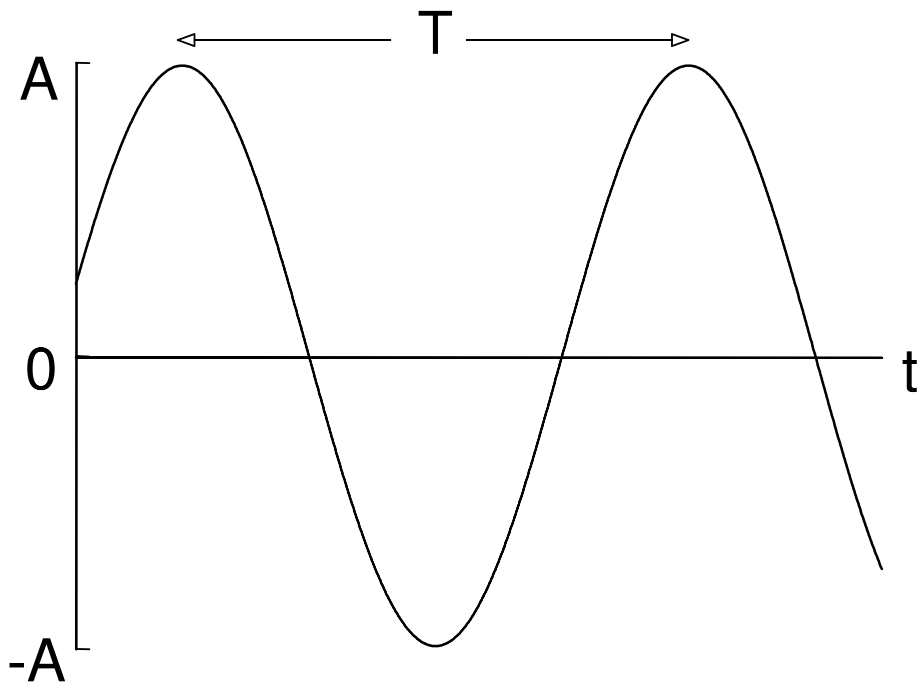
Electrical Circuits Analysis

Phasor Analysis Revision

Sinusoids

A sinusoidal function is given by the equation $A \sin(\omega t + \phi)$



A is the amplitude
ϕ is the phase angle (rad)
ω is the angular frequency (rad/s)

- The frequency, f, is given in cycles per second or Hertz (Hz), where $\omega = 2\pi f$
- The period, T, is the time taken for the waveform to complete a full cycle, where $T = 2\pi/\omega = 1/f$

Root Mean Square (rms)

- To find the root mean square of a function, first square it, find the mean (average) of the result, and finally take the square root
- Considering the sinusoid A sin ωt

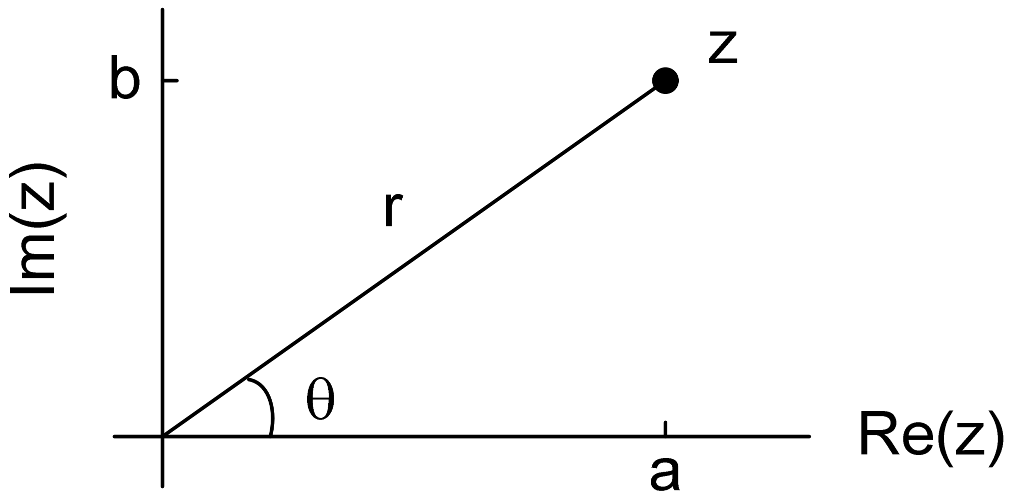
$$\sqrt{\frac{1}{2\pi/\omega}\int_0^{2\pi/\omega} A^2 \sin^2 \omega t\, dt} \quad = \sqrt{\frac{A^2\omega}{4\pi}\int_0^{2\pi/\omega} \left(1-\cos 2\omega t\right) dt}$$

$$= \sqrt{\frac{A^2\omega}{4\pi}\left[t - \frac{1}{2\omega}\sin 2\omega t\right]_0^{2\pi/\omega}} = \sqrt{\frac{A^2}{2}} = \frac{A}{\sqrt{2}} \text{ (rms value)}$$

- *Electrical* engineers measure sinewave amplitude using *rms* values: *electronic* engineers use the *peak* value of the sinewave

Complex Numbers

- A complex number $\mathbf{z}$ can be written in rectangular form $a + \mathrm{j}b$
- a is the real part of $\mathbf{z}$, denoted $\mathrm{Re}(\mathbf{z})$
- b is the imaginary part of $\mathbf{z}$, denoted $\mathrm{Im}(\mathbf{z})$
- $\bar{\mathbf{z}}$ is the complex conjugate of $\mathbf{z}$, equals $a - \mathrm{j}b$



By elementary trigonometry,

$$\mathrm{Re}(\mathbf{z}) = a = r\cos\theta$$

$$\mathrm{Im}(\mathbf{z}) = b = r\sin\theta$$

Complex Numbers

- $\mathbf{z}$ can be written as $r\cos\theta + \mathrm{j}\, r\sin\theta = r\,\mathrm{e}^{\,\mathrm{j}\theta}$ (polar form)
- $\bar{\mathbf{z}}$, the complex conjugate of $\mathbf{z}$, equals $r\,\mathrm{e}^{-\mathrm{j}\theta}$
- We can convert between rectangular and polar forms as,

$$a = r\cos\theta \qquad\qquad b = r\sin\theta$$

$$r = \sqrt{a^2 + b^2} \qquad\qquad \theta = \tan^{-1} b/a$$

- $\tan^{-1}$ function will only give a result between $-\pi/2$ and $\pi/2$

Complex Arithmetic

If $\mathbf{z}_1 = a + \mathrm{j}\, b$ and $\mathbf{z}_2 = c + \mathrm{j}\, d$

$$\mathbf{z}_1 + \mathbf{z}_2 = a + c + \mathrm{j}\,(b + d)$$

$$\mathbf{z}_1 - \mathbf{z}_2 = a - c + \mathrm{j}\,(b - d)$$

$$\mathbf{z}_1\mathbf{z}_2 = (a + \mathrm{j}\, b)\,(c + \mathrm{j}\, d) = ac - bd + \mathrm{j}\,(ad + bc)$$

$$\frac{\mathbf{z}_1}{\mathbf{z}_2} = \frac{a + \mathrm{j}b}{c + \mathrm{j}d} = \frac{a + \mathrm{j}b}{c + \mathrm{j}d} \times \frac{c - \mathrm{j}d}{c - \mathrm{j}d} = \frac{ac + bd}{c^2 + d^2} + \mathrm{j}\frac{bc - ad}{c^2 + d^2}$$

If $\mathbf{z}_1 = r_1 e^{\mathrm{j}\theta_1}$ and $\mathbf{z}_2 = r_2 e^{\mathrm{j}\theta_2}$

$$\mathbf{z}_1\mathbf{z}_2 = r_1 e^{\mathrm{j}\theta_1}\, r_2 e^{\mathrm{j}\theta_2} = r_1 r_2\, e^{\mathrm{j}(\theta_1 + \theta_2)}$$

$$\frac{\mathbf{z}_1}{\mathbf{z}_2} = \frac{r_1 e^{\mathrm{j}\theta_1}}{r_2 e^{\mathrm{j}\theta_2}} = \frac{r_1}{r_2} e^{\mathrm{j}(\theta_1 - \theta_2)}$$

Impedance and Admittance

- *Impedance*, $\mathbf{Z}$, is the phasor domain equivalent of resistance
- The impedance of a capacitor C is $1/\mathrm{j}\omega C$, while the impedance of an inductor L is $\mathrm{j}\omega L$, where ω is the supply frequency (rad/s)

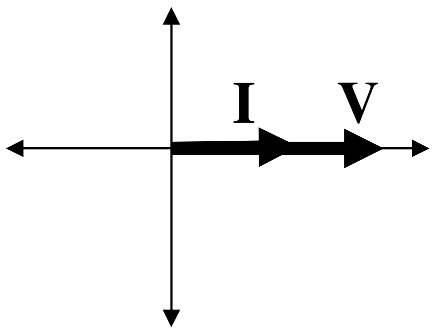
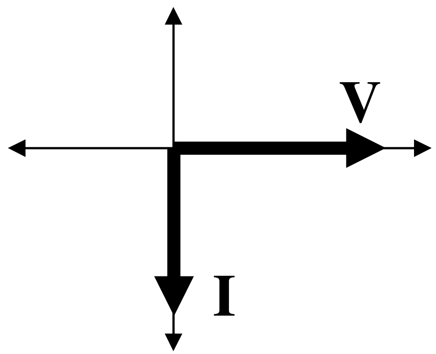
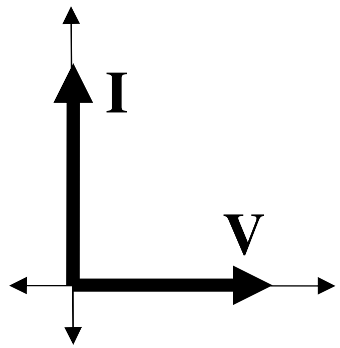
$$\mathbf{I} = \mathrm{j}\omega C\mathbf{V} \quad \text{or} \quad \mathbf{V} = \frac{1}{\mathrm{j}\omega C}\mathbf{I} \qquad\qquad \mathbf{V} = \mathrm{j}\omega L\mathbf{I} \quad \text{or} \quad \mathbf{I} = \frac{1}{\mathrm{j}\omega L}\mathbf{V}$$

- The reciprocal of impedance, $\mathbf{Y}$, is called *admittance*

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = R + \mathrm{j}X \quad \text{and} \quad \mathbf{Y} = \frac{1}{\mathbf{Z}} = G + \mathrm{j}B$$

where X and B are the *reactance* and *susceptance*

Phasor Domain

	Resistor	**Inductor**	**Capacitor**
Time domain	$v = R\, i$	$v(t) = L\frac{di(t)}{dt}$	$i(t) = C\frac{dv(t)}{dt}$
Phasor domain	$\mathbf{V} = R\,\mathbf{I}$	$\mathbf{V} = j\omega L\mathbf{I}$	$\mathbf{I} = j\omega C\mathbf{V}$
		Current through *inductor* is said to *lag* the voltage	*Current* through *capacitor* is said to *lead* the voltage
Phasor diagram CIVIL mnemonic			

Impedances in Series & Parallel

- $\mathbf{Z}_1$ and $\mathbf{Z}_2$ connected in series are equivalent to a single impedance $(\mathbf{Z}_1 + \mathbf{Z}_2)$

$$\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2$$
$$R + \mathrm{j}X = \left(R_1 + \mathrm{j}X_1\right) + \left(R_2 + \mathrm{j}X_2\right)$$

- $\mathbf{Z}_1$ and $\mathbf{Z}_2$ connected in parallel are equivalent to a single impedance, $\mathbf{Z}$, such that

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$
$$\frac{1}{R + \mathrm{j}X} = \frac{1}{R_1 + \mathrm{j}X_1} + \frac{1}{R_2 + \mathrm{j}X_2}$$

Average Power

- If the voltage and current for a circuit element are

$$v(t) = V_m \sin(\omega t + \phi_v) \quad \text{and} \quad i(t) = I_m \sin(\omega t + \phi_i)$$

where V_m and I_m are the peak values of the sinewaves

$\rightarrow$ the associated power at time instant t is

$$p(t) = v(t)\, i(t) = V_m I_m \sin(\omega t + \phi_v) \sin(\omega t + \phi_i)$$
$$= (V_m I_m/2)\,[\cos(\phi_v - \phi_i) - \cos(2\omega t + \phi_v + \phi_i)]$$

- The *average* power follows as

$$P = \tfrac{1}{2} V_m I_m \cos(\phi_v - \phi_i)$$

- $\cos(\phi_v - \phi_i)$ is called the *power factor*

Phasor Representation

- The rms value of the sinusoid $V_\mathrm{m}\sin(\omega t + \theta_\mathrm{v})$ is $V_\mathrm{rms} = V_\mathrm{m}/\sqrt{2}$
- Average power

$$= \tfrac{1}{2} V_\mathrm{m} I_\mathrm{m} \cos(\phi_\mathrm{v} - \phi_\mathrm{i}) = V_\mathrm{rms} I_\mathrm{rms} \cos(\phi_\mathrm{v} - \phi_\mathrm{i})$$

- For a resistor, $\mathbf{V} = R\,\mathbf{I}$

$$\Rightarrow \theta_\mathrm{v} = \theta_\mathrm{i} \rightarrow P = V_\mathrm{rms} I_\mathrm{rms} = V^2_\mathrm{rms}/R = I^2_\mathrm{rms} R$$

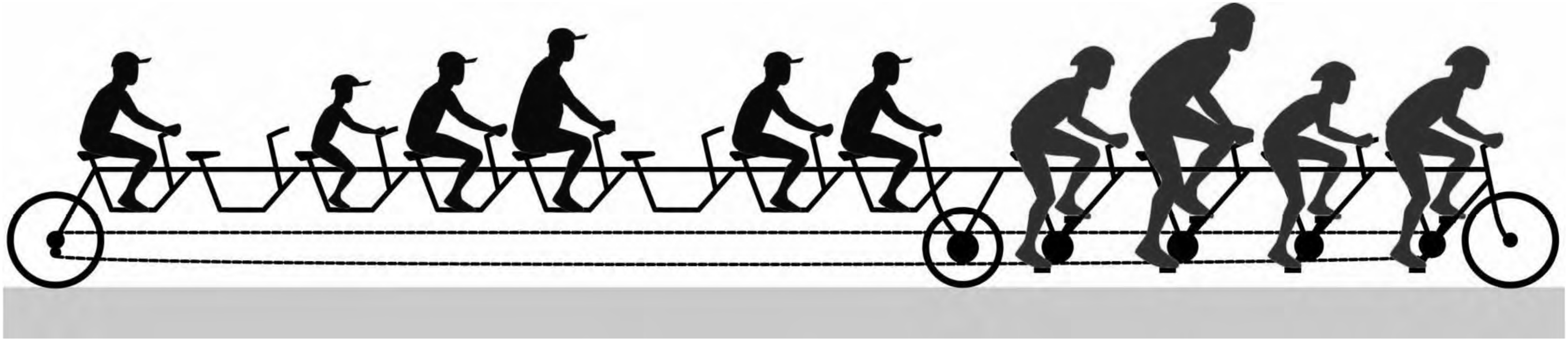
- For an inductor, $\mathbf{V} = \mathrm{j}\,\omega L\,\mathbf{I}$

$$\Rightarrow (\theta_\mathrm{v} - \theta_\mathrm{i}) = 90^\circ \rightarrow P = 0$$

- For a capacitor, $\mathbf{V} = \mathbf{I}/\mathrm{j}\,\omega C$

$$\Rightarrow (\theta_\mathrm{v} - \theta_\mathrm{i}) = -90^\circ \rightarrow P = 0$$

Reactive Power analogy



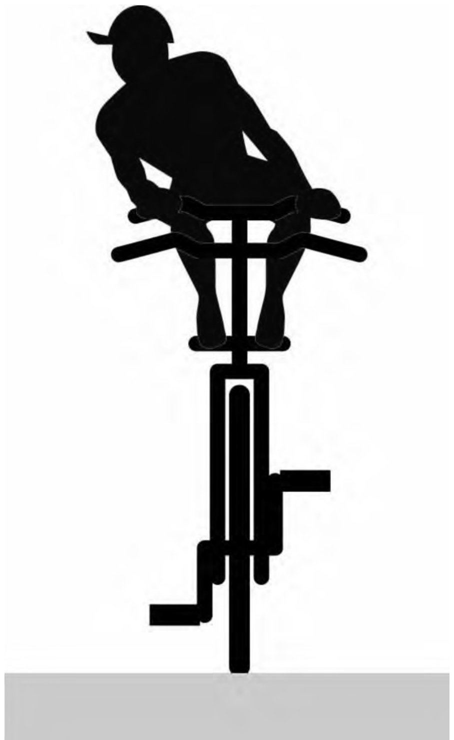
Riders at the back are passengers

Loads

Riders at the front *drive* the bike

Power stations

Reactive Power analogy



Effort required to drive bike is unaffected

… BUT bike might fall over

(A *reactive* load)

Riders at front must compensate (reactive generation)

Pedalling becomes more difficult (reduced capability)

Bicycle drag increases (more losses)

Reactive Power

- Reactive power refers to the energy storage part of a load
- Reactive power is exchanged *back and forth* between inductive and capacitive parts of a *reactive load* (one containing inductors and capacitors) during each ac cycle
- Inductors store energy in their magnetic field while capacitors store energy in their electric field
- There is no net transfer of energy – no work is done
- Loads are generally made up of two components
  - Energy dissipated (active power)
  - Energy stored (reactive power)

Active, Reactive & Apparent Power

- The *average (active) power, P*, is given as

$$P = V_{\text{rms}} \times I_{\text{rms}} \cos\theta \quad \text{W}$$

- The *reactive power, Q,* is then given as

$$Q = V_{\text{rms}} \times I_{\text{rms}} \sin\theta \quad \text{VAr}$$

- The *apparent power*, *S,* is given as

$$S = V_{\text{rms}} \times I_{\text{rms}} \quad \text{VA}$$

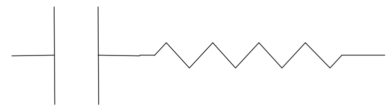
$$\left|\mathbf{S}\right| = \sqrt{P^2 + Q^2}\ , \quad \mathbf{S} = P + \text{j}Q$$

- Equipment ratings are expressed in terms of their apparent power (kVA / MVA) loading

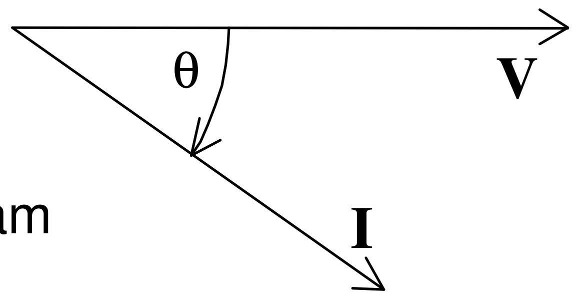
Power Triangles

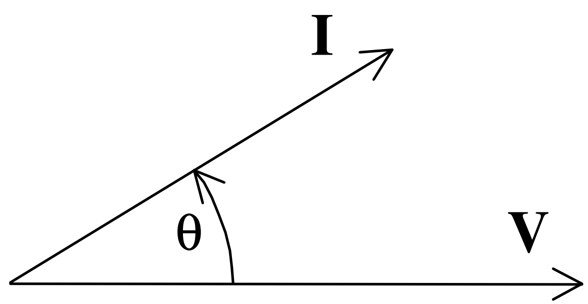
Lagging (inductive) load | Leading (capacitive) load

Circuit element

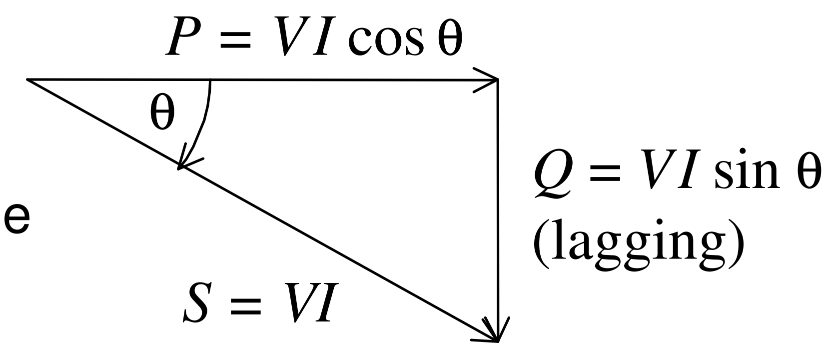


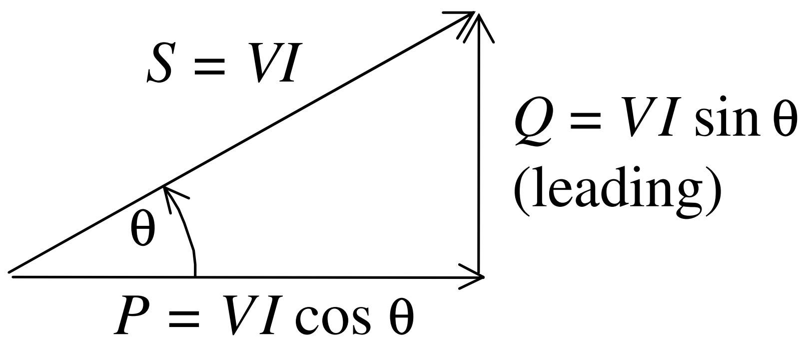
Phasor diagram





Power triangle





Example

Determine the currents flowing in the following circuit