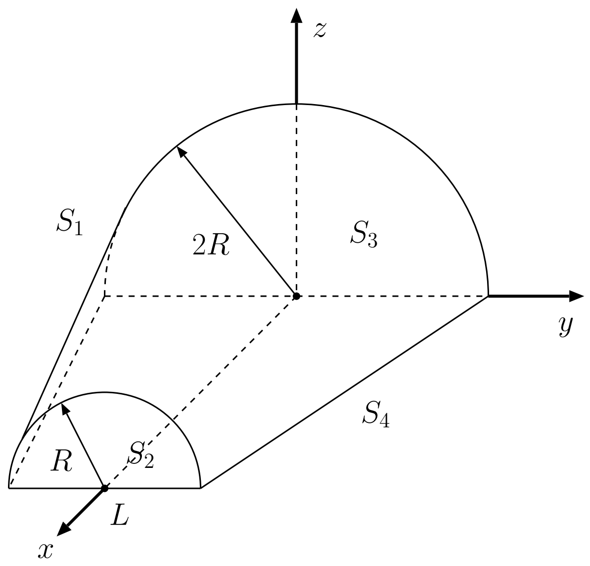
1. Applying Gauss' law $\oint_S \vec{B}\cdot d\vec{S} = 0$ and considering $\vec{B} = 3\hat{x} + 2\hat{y} - \frac{4\pi R}{L}\hat{z}$, we will compute the magnetic flux through the partial cone surface S_1 shown in the next figure.



For this purpose, let us first define the closed surface S composed by the partial cone surface S_1, the semicircle S_2 with radius R, the semicircle S_3 with radius $2R$, and the trapezoid S_4 with bases $2R$ and $4R$ and height L. Since the surface normal vector of the partial cone surface is defined pointing inward, with a $-\hat{z}$ component , we should also define the surface normal vectors of the other three surfaces pointing inward the volume. Now, we can rewrite Gauss' law as follows

$$\oint_S \vec{B}\cdot d\vec{S} = \int_{S_1} \vec{B}\cdot d\vec{S}_1 + \int_{S_2} \vec{B}\cdot d\vec{S}_2 + \int_{S_3} \vec{B}\cdot d\vec{S}_3 + \int_{S_4} \vec{B}\cdot d\vec{S}_4 = 0.$$

On the semicircle S_2, the magnetic flux is

$$\int_{S_2} \vec{B}\cdot d\vec{S}_2 = \int_{S_2} \left(3\hat{x} + 2\hat{y} - \frac{4\pi R}{L}\hat{z}\right)\cdot(-\hat{x})\, dS_2 = -3\cdot \text{Area}_2 = \frac{-3\pi R^2}{2}.$$

On the semicircle S_3, the magnetic flux is

$$\int_{S_3} \vec{B}\cdot d\vec{S}_3 = \int_{S_3} \left(3\hat{x} + 2\hat{y} - \frac{4\pi R}{L}\hat{z}\right)\cdot(\hat{x})\ dS_3 = 3\cdot \text{Area}_3 = 3\frac{\pi\,(2R)^2}{2} = 6\pi R^2.$$

On the trapezoid S_4, the magnetic flux is

$$\int_{S_4} \vec{B}\cdot d\vec{S}_4 = \int_{S_4} \left(3\hat{x} + 2\hat{y} - \frac{4\pi R}{L}\hat{z}\right)\cdot(\hat{z})\ dS_4 = -\frac{4\pi R}{L}\cdot \text{Area}_4 = -\frac{4\pi R}{L}\left(\frac{2R+4R}{2}L\right) = -12\pi R^2.$$

Finally, the magnetic flux through S_1 is

$$\begin{aligned}\int_{S_1} \vec{B}\cdot d\vec{S}_1 &= -\left(\int_{S_2} \vec{B}\cdot d\vec{S}_2 + \int_{S_3} \vec{B}\cdot d\vec{S}_3 + \int_{S_4} \vec{B}\cdot d\vec{S}_4\right)\\ &= -\left(\frac{-3\pi R^2}{2} + 6\pi R^2 - 12\pi R^2\right) = \frac{15\pi R^2}{2}.\end{aligned}$$

2. An infinite current sheet with a uniform current density $J_s = J_s\hat{z}\,\mathrm{A/m}$ produces magnetostatic fields B with $\frac{\mu_o J_s}{2}$ magnitude on both sides of the sheet and with opposing direction.

   a) Magnetic field intensity at the origin due to sheet 1 ($\vec{J}_{s1} = 2\hat{z}\,\frac{\mathrm{A}}{\mathrm{m}}$)

   $$\vec{H}_1 = \frac{1}{2}\vec{J}_{s1} \times \hat{x} = 1\hat{y}\,\frac{\mathrm{A}}{\mathrm{m}}.$$

   Magnetic field intensity at the origin due to sheet 2 ($\vec{J}_{s2} = 2\hat{z}\,\frac{\mathrm{A}}{\mathrm{m}}$)

   $$\vec{H}_2 = \frac{1}{2}\vec{J}_{s2} \times (-\hat{x}) = -1\hat{y}\,\frac{\mathrm{A}}{\mathrm{m}}.$$

   Resultant displacement vector due to both sheets

   $$\vec{H} = \vec{H}_1 + \vec{H}_2 = \vec{0}\,\frac{\mathrm{A}}{\mathrm{m}}.$$

   b) Magnetic field intensity at the origin due to sheet 1 ($\vec{J}_{s1} = 2\hat{z}\,\frac{\mathrm{A}}{\mathrm{m}}$)

   $$\vec{H}_1 = \frac{1}{2}\vec{J}_{s1} \times \hat{x} = 1\hat{y}\,\frac{\mathrm{A}}{\mathrm{m}}.$$

   Magnetic field intensity at the origin due to sheet 2 ($\vec{J}_{s2} = -2\hat{z}\,\frac{\mathrm{A}}{\mathrm{m}}$)

   $$\vec{H}_2 = \frac{1}{2}\vec{J}_{s2} \times (-\hat{x}) = 1\hat{y}\,\frac{\mathrm{A}}{\mathrm{m}}.$$

   Resultant displacement vector due to both sheets

   $$\vec{H} = \vec{H}_1 + \vec{H}_2 = 2\hat{y}\,\frac{\mathrm{A}}{\mathrm{m}}.$$

3. Let us compute the magnetic field $\vec{B}$ generated by a current flowing in a infinite slab of width $W = 3\,\mathrm{m}$ parallel to the y-z plane. The current density in the slab is $\vec{J} = 2\hat{y}\,\frac{\mathrm{A}}{\mathrm{m}^2}$.

   a) Biot-Savart law tells us that the direction of the magnetic field generated by an infinitesimal current element is parallel to the cross product between the direction of the current and the vector joining the current element and the point under consideration. In our case, the current is parallel to $\hat{y}$, then, using the right-hand rule, we can easily verify that, above the slab ($x > 2$), $\vec{B}$ should be parallel to $-\hat{z}$,

   $$\vec{B} \parallel 2\hat{y} \times \hat{x} \implies \vec{B} \parallel -\hat{z},$$

   while, below the slab ($x < -1$), $\vec{B}$ should be parallel to $\hat{z}$,

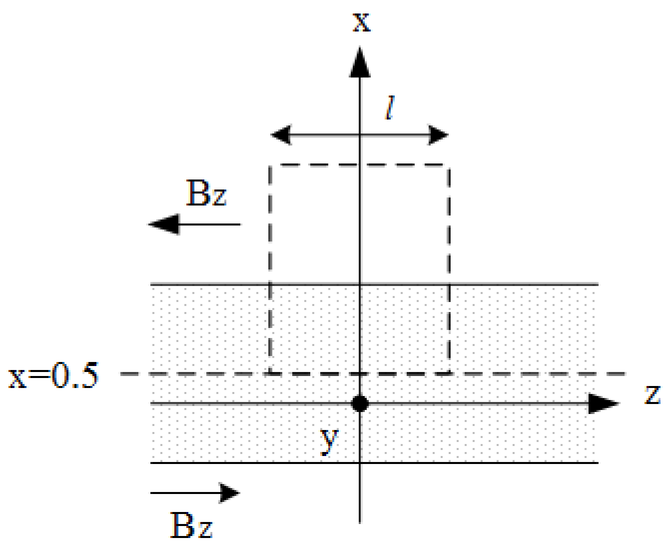
   $$\vec{B} \parallel 2\hat{y} \times -\hat{x} \implies \vec{B} \parallel \hat{z}.$$

   In addition, since the current distribution is symmetric with respect to the plane at $x = 0.5$, the fields at the same distance from above and below the $x = 0.5$ plane should have equal amplitudes.

   b) Same as in part (a), the magnetic fields slightly above and below $x = 0.5$ should have equal amplitudes but point in opposite directions. Exactly at $x = 0.5$ the fields must cancel each other such that

   $$\vec{B}(x = 0.5) = \vec{0}\,\mathrm{T}.$$

c) To compute $\vec{B}$ above the slab using Ampere's law, let us first define a rectangle R placed on the x-z plane as it is shown in the figure below.



Integrating $\vec{B}$ along the perimeter of the rectangle and noting that only the segment above the slab contributes to the integration, we have that -

$$\frac{1}{\mu_o}\oint_R \vec{B}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{S}$$
$$\frac{1}{\mu_o}(-B_z \times l) = 2 \times 1.5 \times l$$
$$B_z = -3\mu_o\,\mathrm{T}.$$

Similarly, we can compute $\vec{B}$ below the slab and obtain

$$\mathbf{B} = \begin{cases} -3\mu_o\,\hat{z}\,\mathrm{T} & \text{for}\, x > 2\,\mathrm{m}, \\ 3\mu_o\,\hat{z}\,\mathrm{T} & \text{for}\, x < -1\,\mathrm{m}. \end{cases}$$

d) Applying the same technique, but considering a rectangle of height shorter than 1.5 m, we can compute $\vec{B}$ inside the slab (i.e., for $-1 \leq x \leq 2\,\mathrm{m}$) as follows. Since the center of the slab is at $x = 0.5\,\mathrm{m}$, to take advantage of symmetry, we will use the distance in $\hat{x}$ direction as the distance to $x = 0.5\,\mathrm{m}$.First, look at the region $0.5\,\mathrm{m} \leq x \leq 2\,\mathrm{m}$.

$$\frac{1}{\mu_o}\oint_R \vec{B}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{S}$$
$$\frac{1}{\mu_o}(-B_z \times l) = 2 \times (x - 0.5) \times l$$
$$B_z = -2\mu_o(x - 0.5).$$

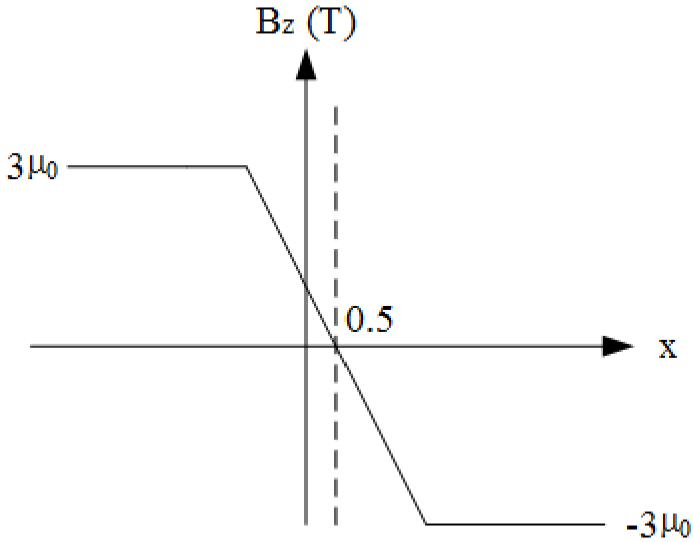
Finally, extending this result to the region below $x = 0.5\,\mathrm{m}$ plane , we obtain

$$\mathbf{B} = -2\mu_o(x - 0.5)\,\hat{z}\,\mathrm{T} \quad \text{for} - 1\,\mathrm{m} \leq x \leq 0.5\,\mathrm{m}.$$

So overall, the expression is

$$\mathbf{B} = -2\mu_o(x - 0.5)\,\hat{z}\,\mathrm{T} \quad \text{for} - 1\,\mathrm{m} \leq x \leq 2\,\mathrm{m}.$$

e) Plotting B_z as function of x.



4. Two concentric circular wire loops of radii $a = 10\,\text{cm}$ and $b = 0.25\,\text{cm}$ are placed on the $x-y$ plane, with their centers at the origin.

   a) The current in the outer loop can be calculated as follows

$$I_a = \frac{V}{R} = \frac{V}{\frac{2\pi a}{\sigma\pi r^2}} = \frac{5}{\frac{2\times 0.1}{4\times 10^7\times 0.001^2}} = 1000\,(\text{A}).$$

   b) Refering to the Lecture 13, the magnetic flux density generated by the outer loop only has z component along the z axis

$$B_z = \frac{\mu_o I_a a^2}{2(a^2+z^2)^{\frac{3}{2}}}.$$

   At the origin, $B_z(z=0) = \frac{\mu_o I_a}{2a}\,(\text{Wb/m}^2)$. Since $b << a$, we assume the B_z across the inner loop is constant. Thus,

$$\Psi_{a\to b} = \int_S B\cdot dS = \int_S B_z(z=0)\hat{z}\cdot\hat{z}dS = \frac{\mu_o I_a}{2a}\pi b^2 = \frac{\mu_o \pi b^2}{2a} I_a\,(\text{Wb}).$$

   c) The numerical value of $L_{a\to b}$ is

$$L_{a\to b} = \frac{\Psi_{a\to b}}{I_a} = \frac{\mu_o\pi b^2}{2a} = 1.23\times 10^{-10}\,(\text{H}).$$