CIRCUIT THEORY EEEN30020

Tutorial 1: Two-Ports and Their Representations

Problem Set, Useful Formulae and Workbook

The aim of this tutorial is to reinforce your knowledge on two-ports. The two-port matrix representations are used to obtain practical information about two-port circuits (input and output impedances, transfer function and dissipated power) and analyse their interconnections.

At the end of the tutorial, you will be given a short **test.** You must answer the test and return it to the Module Coordinator or Teaching Assistants. You can use any materials, for example, the lecture notes. You can answer the test with your Homework/Lab team partner. Allow ten minutes for the test.

Problem Set

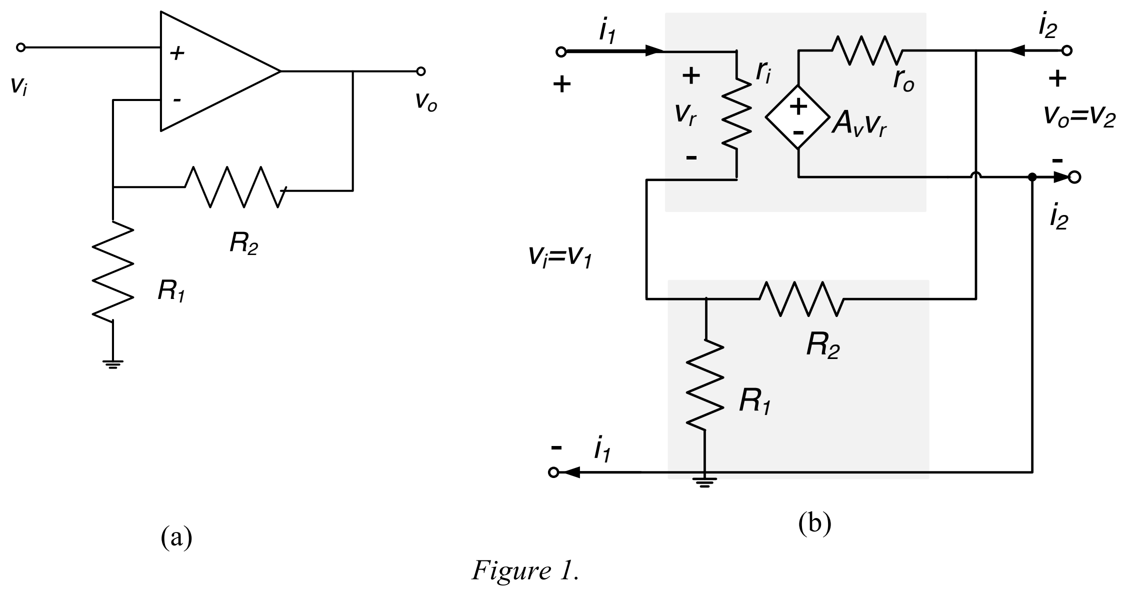
(1) A two-port, whose admittance matrix Y is known, has port 2 terminated in an impedance Z_L. Find the input impedance at port 1 in terms of the *y*-parameters and Z_L. Using the expressions relating the parameters A, B, C and D of the transmission matrix and the y-parameters, check that you obtain the same formula for the input impedance in terms of A, B, C, D and Z_L as was obtained in class.

(2) A non-inverting amplifier that employs an op amp is shown in fig. 1(a). In the case of the ideal op amp with infinite gain, the resulting voltage gain A_v of the circuit is

$$A_v = \frac{v_o}{v_i} = \frac{R_1 + R_2}{R_1} \quad (1)$$

However, a more realistic model of the non-inverting amplifier is shown in fig. 1(b). In this case, the op amp is modelled using a voltage controlled voltage source and two resistances (r_i and r_o) and has a finite gain. For the circuit of fig. 1(b):

- Identify the interconnection of the two two-ports (highlighted) that form the circuit;
- Find the relevant matrix representation of each two-port and of the entire circuit;
- Show that if $r_i = \infty$ (i.e., the two-port that models the op amp has an infinite gain) you obtain the well-known formula (1).



(a) (b)

Figure 1.

(3) Find the input impedance of the circuit of Figure 2 at frequency 1 rad/s.

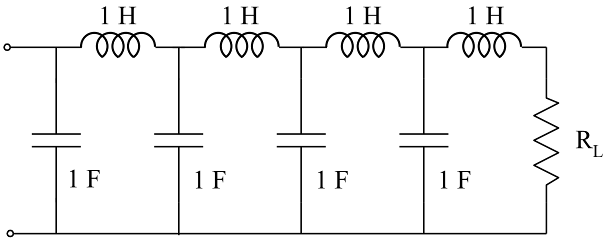


Figure 2.

Useful Formulae

Imedance Matrix

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \qquad \begin{matrix} V_1 = z_{11} I_1 + z_{12} I_2 \\ V_2 = z_{21} I_1 + z_{22} I_2 \end{matrix}$$

$$z_{11} = \left.\frac{\mathrm{V}_1}{\mathrm{I}_1}\right|_{\mathrm{I}_2=0} \qquad z_{21} = \left.\frac{\mathrm{V}_2}{\mathrm{I}_1}\right|_{\mathrm{I}_2=0}$$

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} \qquad z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

Admittance Matrix

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \qquad \begin{matrix} I_1 = y_{11} V_1 + y_{12} V_2 \\ I_2 = y_{21} V_1 + y_{22} V_2 \end{matrix}$$

$$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} \qquad y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$$

$$y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} \qquad y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

Hybrid Matrix

$$\begin{pmatrix} V_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ V_2 \end{pmatrix} \qquad \begin{matrix} V_1 = h_{11} I_1 + h_{12} V_2 \\ I_2 = h_{21} I_1 + h_{22} V_2 \end{matrix}$$

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} \qquad h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} \qquad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$$

Inverse Hybrid Matrix

$$\begin{pmatrix} I_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ I_2 \end{pmatrix} \qquad \begin{matrix} I_1 = g_{11} V_1 + g_{12} I_2 \\ V_2 = g_{21} V_1 + g_{22} I_2 \end{matrix}$$

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0} \qquad g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1=0}$$

$$g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0} \qquad g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0}$$

Transmission Matrix

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V_2 \\ -I_2 \end{pmatrix} \qquad \begin{matrix} V_1 = AV_2 - BI_2 \\ I_1 = CV_2 - DI_2 \end{matrix}$$

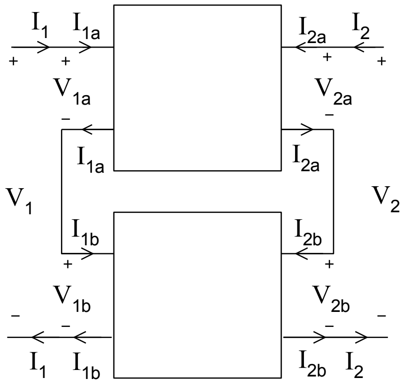
$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} \qquad B = -\left.\frac{V_1}{I_2}\right|_{V_2=0}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} \qquad D = -\left.\frac{I_1}{I_2}\right|_{V_2=0}$$

Inverse Transmission Matrix:

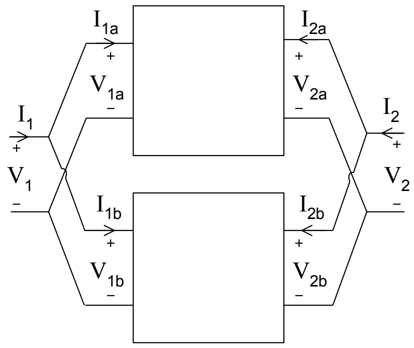
$$\begin{pmatrix} V_2 \\ -I_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \begin{pmatrix} V_1 \\ I_1 \end{pmatrix}$$

Two-Port Interconnections



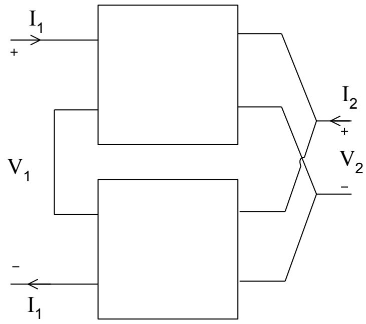
series connection ⇒

add Z matrices



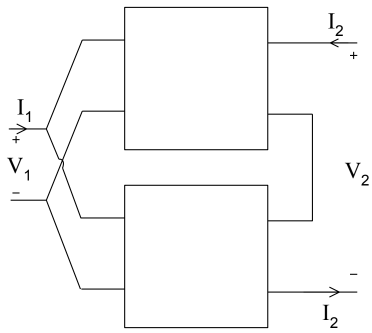
parallel connection ⇒

add Y matrices



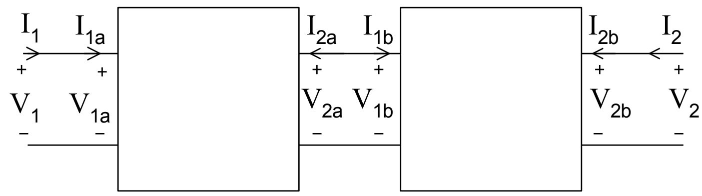
series-parallel connection ⇒

add H matrices



parallel-series connection ⇒

add G matrices

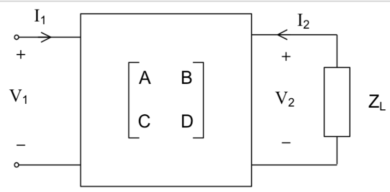


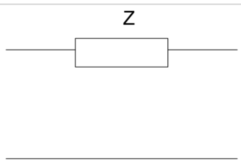
cascade connection ⇒

multiply T matrices

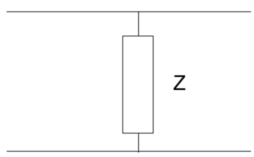
Input impednace in terms of the ABCD parameters:

$$Z_{in} = \frac{V_1}{I_1} = \frac{AZ_L + B}{CZ_L + D}$$





$$T_a = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}$$



$$T_b = \begin{pmatrix} 1 & 0 \\ 1/Z & 1 \end{pmatrix}$$

Workbook

(1) A two-port, whose admittance matrix Y is known, has port 2 terminated in an impedance Z_L. Find the input impedance at port 1 in terms of the *y*-parameters and Z_L. Using the expressions relating the parameters A, B, C and D of the transmission matrix and the y-parameters, check that you obtain the same formula for the input impedance in terms of A, B, C, D and Z_L as was obtained in class.

Solution:

We know the *y*-parameters, or the elements of the admittance matrix. If so, we start by expanding the admittance matrix equation, i.e., presenting it in the form where V_1 and V_2 are expressed in terms of I_1 and I_2 by two linear equations:

Use the definition of the input and output (load) impedances

Z_{in} = Z_L =

to express the current I_1 in terms of V_1 and Z_{in} and the current I_2 in terms of V_2 and Z_L. Substitute them (I_1 and I_2) in the admittance matrix equations – this allows you to exclude the currents I_1 and I_2 from the equations. You will obtain two equations in terms of V_1 and V_2. (Note that these equations will also contain y-parameters, Z_{in} and Z_L).

Express V_2 from the first equation:

Then express V_2 from the second equation:

Clearly, the two expressions must be equal. By equating the two expressions, we find Z_{in}:

Recall the relation between the ABCD and Y matrices:

$$y_{11} = \frac{D}{B} \qquad y_{12} = \frac{BC - AD}{B}$$
$$y_{21} = -\frac{1}{B} \qquad y_{22} = \frac{A}{B}$$

These express y_{11}, y_{12}, y_{21} and y_{22} in terms of the ABCD parameters. Substitute them in the expression for Z_{in} you just obtained. Simplify the expression and check that you receive the correct formula (see page 3 of this tutorial):

(2) A non-inverting amplifier that employs an op amp is shown in fig. 1(a). In the case of the ideal op amp with infinite gain, the resulting voltage gain A_v of the circuit is

$$A_v = \frac{v_o}{v_i} = \frac{R_1 + R_2}{R_1} \quad (1)$$

However, a more realistic model of the non-inverting amplifier is shown in fig. 1(b). In this case, the op amp is modelled using a voltage controlled voltage source and two resistances (r_i and r_o) and has a finite gain. For the circuit of fig. 1(b):

- Identify the interconnection of the two two-ports that form the circuit;
- Find the relevant matrix representation of each two-port and of the circuit;
- Show that if $r_i = \infty$ (i.e., the two-port that models the op amp has an infinite gain) you obtain the well-known formula (1).

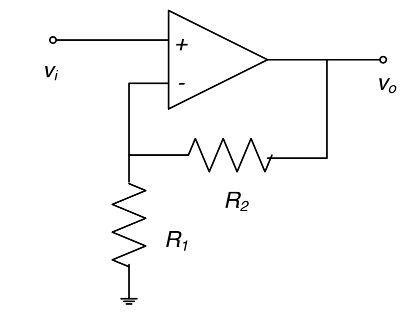


fig.1(a)

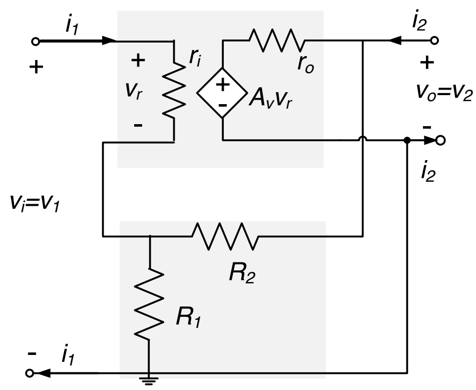


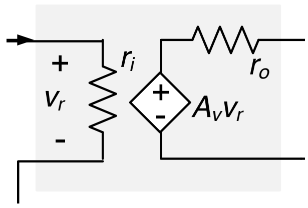
fig.1(b)

Solution:

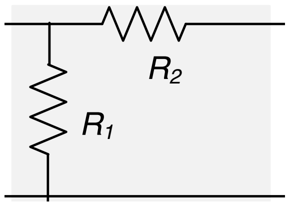
Firstly, note the difference between the two circuits. In the first circuit, we use an ideal op amp with infinite gain. This means that no currents flow to/from the "+" and "–" terminals of the op amp, and the node voltage at "+" terminal is equal to the node voltage at the "–" terminal. This allows us to find formula (1). (See Lecture Notes, p1-6.)

However, realistic op amps are transistor based devices with large (but not infinite gain), as shown in fig. 1b. The gain of an op amp is defined, among other parameters, by the internal resistance r_i. If the internal resistance $r_i \to \infty$, the op amp gain $A_v \to \infty$. In a more realistic case, $r_i \neq \infty$, and so $A_v \neq \infty$. This will result in a current, flowing to the op amp, and in a decrease in the gain of the entire circuit.

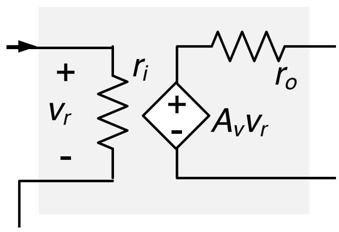
By inspecting the circuit in fig. 1b, we can identify two two-ports:



and



These two two-ports are connected ***in series*** at the input ports and ***in parallel*** at the output ports. Check Lecture Notes or this tutorial (page 3). Thus, their interconnection and the relevant matrices are



Hint: this circuit is similar to the example solved in class where we found the hybrid matrix of a transistor model, page p2-24.

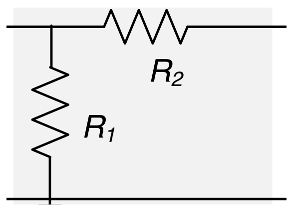
First, set v_2 to zero and find v_1 in terms of i_1 and i_2 in terms of i_1. Then, set i_1 to zero and find v_1 in terms of v_2 and i_2 in terms of v_2.

FYI, the H matrix of the two-port is

$$H_1 = \begin{pmatrix} r_i & 0 \\ -A_v \dfrac{r_i}{r_o} & \dfrac{1}{r_o} \end{pmatrix}$$

Note that this two-port contains a VCVS and thus is not *reciprocal*. If so, $h_{21} \neq -h_{12}$. Clearly, this is the case for this hybrid matrix.

The hybrid matrix of the following two-port:



Again, set v_2 to zero and find v_1 in terms of i_1 and i_2 in terms of i_1. Then, set i_1 to zero and find v_1 in terms of v_2 and i_2 in terms of v_2.

The $\underline{H}$ matrix of this two-port is:

$$H_2 = \begin{pmatrix} \frac{R_1R_2}{R_1+R_2} & \frac{R_1}{R_1+R_2} \\ -\frac{R_1}{R_1+R_2} & \frac{1}{R_1+R_2} \end{pmatrix} = \begin{pmatrix} R_2\gamma & \gamma \\ -\gamma & \frac{\gamma}{R_1} \end{pmatrix} \quad \text{where} \quad \gamma = \frac{R_1}{R_1+R_2}$$

Note that this two-port is linear resistive with no dependent or independent sources and thus is *reciprocal*. If so, $h_{21} = -h_{12}$. Clearly, this is the case for this hybrid matrix.

The resulting hybrid matrix is

$$H_1 + H_2 =$$

And thus the non-inverting amplifier with a finite gain op amp (realistic op amp) is described through the matrix equation:

$$\begin{pmatrix} v_i \\ i_2 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} i_1 \\ v_o \end{pmatrix}$$

Answering the last question in this problem, you should recall that if $r_i = \infty$, there will be no current flowing through such a resistor. And so we state that $i_1 = 0$ in this case. In addition, $v_1 = v_i$ and $v_2 = v_o$ (see fig. 1b). Writing down the hybrid matrix equations:

$$v_1 = h_{11}i_1 + h_{12}v_2$$
$$i_2 = h_{21}i_1 + h_{22}v_2$$

and noting that $i_1 = 0$, from the first equation we obtain the voltage gain:

$$A_v = \frac{v_o}{v_i} = \frac{v_2}{v_1} =$$

Did you obtain the same formula as (1)?

(3) Find the input impedance of the circuit of Figure 2 at frequency 1 rad/s.

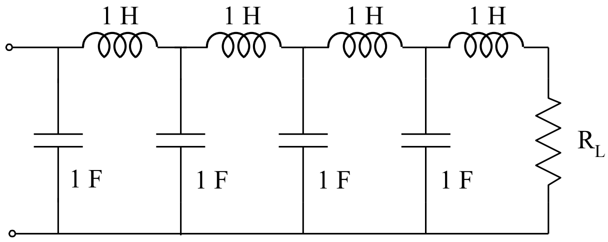


Figure 2.

Solution:

Recall the expression of the input impedance in terms of the ABCD parameters and the load impedance:

$Z_{in} =$

Thus we are to find the transmission matrix of the entire circuit. The individual $\underline{T}$ matrices are

$T_C =$ $T_L =$

This is a cascade connection and so we multiply

$T_{1-2} =$

$T_{1-3} =$

Hint: check that you after multiplying the three matrices you obtained the following result:

$$T_{1-3} = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}$$

$T_{1-4} =$

$T_{1-5} =$

Hint: check that you after multiplying the five matrices you obtained the following result:

$$T_{1-5} = \begin{pmatrix} -1 & j \\ 0 & -1 \end{pmatrix}$$

$T_{1-6} =$

T_{1-7} =

The resulting T matrix is

T_{1-8} =

The input input impedance is: