Signals and Systems

Stage 3: Electrical & Electronic and Eng.

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Note

The Fourier series provides a description of the frequency content of periodic signals. It enables us to express general periodic signals as weighted sums of simpler periodic signals (namely sinusoids). These signals are simpler in the sense that it is simpler to analyse systems whose inputs are signals of this type.

In practice of course true periodic signals cannot be generated. Many important signals (steps and pulses for example) cannot even be said to be approximately periodic. The third theme of the module then is the search for a method for describing the frequency content of general (periodic or non-periodic) signals.

Fourier's limit argument

Fourier engages with this issue more or less head on. Fourier's conjecture can be stated as: for any periodic signal *f* of period *T*

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\frac{2\pi}{T}t}$$

$$c_n = \tfrac{1}{T}\int_0^T f(t) e^{-jn\frac{2\pi}{T}t} dt$$

where we have employed: $\omega_0 = \frac{2\pi}{T}$

Fourier's limit argument

Now as the signal $f(t)$ is periodic of period T we may alternatively write:

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-jn\frac{2\pi}{T}t}dt$$

for:

$$c_n = \frac{1}{T}\int_{0}^{T} f(t)e^{-jn\frac{2\pi}{T}t}dt = \frac{1}{T}\int_{0}^{\frac{T}{2}} f(t)e^{-jn\frac{2\pi}{T}t}dt + \frac{1}{T}\int_{\frac{T}{2}}^{T} f(t)e^{-jn\frac{2\pi}{T}t}dt$$

Fourier's limit argument

$$c_n = \frac{1}{T}\int_{0}^{\frac{T}{2}} f(t)e^{-jn\frac{2\pi}{T}t}dt + \frac{1}{T}\int_{-\frac{T}{2}}^{0} f(t+T)e^{-jn\frac{2\pi}{T}(t+T)}dt$$

$$c_n = \frac{1}{T}\int_{0}^{\frac{T}{2}} f(t)e^{-jn\frac{2\pi}{T}t}dt + \frac{1}{T}\left(\int_{-\frac{T}{2}}^{0} f(t)e^{-jn\frac{2\pi}{T}t}dt\right)e^{-jn2\pi}$$

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{0} f(t)e^{-jn\frac{2\pi}{T}t}dt + \frac{1}{T}\int_{0}^{\frac{T}{2}} f(t)e^{-jn\frac{2\pi}{T}t}dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-jn\frac{2\pi}{T}t}dt$$

Fourier's limit argument

So Fourier's conjecture becomes that for any periodic signal f of period T

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\frac{2\pi}{T}t} \qquad c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-jn\frac{2\pi}{T}t} dt$$

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T}\left(\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-jn\frac{2\pi}{T}t} dt \right) e^{jn\frac{2\pi}{T}t}$$

Fourier imagines a general signal to be a periodic signal of period T where T goes to infinity.

Fourier's limit argument

Consider the double integral:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt\right)e^{j\omega t}d\omega$$

We might consider the Riemann sum associated with the outer integral.

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-jn\Delta\omega t}dt\right)e^{jn\Delta\omega t}\Delta\omega$$

Fourier's limit argument

Let $\Delta\omega = 2\pi/T$ as T becomes very large.

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-jn\frac{2\pi}{T}t}\,dt\right)e^{jn\frac{2\pi}{T}t}\,\frac{2\pi}{T}$$

$$=\frac{1}{T}\sum_{n=-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-jn\frac{2\pi}{T}t}\,dt\right)e^{jn\frac{2\pi}{T}t}$$

The integral is recovered as the limit as T approaches infinity.

Fourier's limit argument

Fourier deduces that:

$$\lim_{T\to\infty} \frac{1}{T} \sum_{n=-\infty}^{\infty} \left(\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-jn\frac{2\pi}{T}t} dt \right) e^{jn\frac{2\pi}{T}t} =$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right) e^{j\omega t} d\omega$$

i.e.

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right) e^{j\omega t} d\omega$$

Fourier's limit argument

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right) e^{j\omega t} d\omega$$

Fourier cannot prove this integral formula in general (mainly because it does not hold in general) but he does confirm that it holds for a good number of special cases. He effectively conjectures that it holds in general and he goes on to offer a very important interpretation of what it is saying.

Fourier Transform

Fourier considers the inner integral. Let:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\exp(-j\omega t)dt$$

then the limit formula asserts that:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\exp(j\omega t)d\omega$$

This formula essentially expresses the signal $f(t)$ as "sum" of complex exponentials. Granted exponentials at all frequencies are now present so the sum is no longer called a sum, it is called an integral.

Fourier Transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) \exp(j\omega t) d\omega$$

We interpret this formula as being the description of the frequency content of the signal, where on this occasion the signal may contain all frequencies. The complex number $F(j\omega)$ tells us what is the frequency content of the signal at the frequency ω. If the modulus of this number is large (relative to the value at other frequencies) then this frequency is strongly present. If the modulus is small or zero this frequency is absent.

Fourier Transform

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(j\omega)\exp(j\omega t)d\omega$$

The formula works for general signals in exactly the same manner that the Fourier series works for periodic signals. The important complex numbers $F(j\omega)$ can be calculated for each frequency ω using the formula:

$$F(j\omega)=\int_{-\infty}^{\infty}f(t)\exp(-j\omega t)dt$$

Fourier Transform

Given a signal $f(t)$ the associated signal $F(j\omega)$ defined by:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\exp(-j\omega t)dt = \mathcal{F}(f(t))$$

is called the *Fourier transform* of the signal $f(t)$.

Given a signal $F(j\omega)$ the associated signal $f(t)$ defined by:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\exp(j\omega t)d\omega = \mathcal{F}^{-1}(F(j\omega))$$

is called the *inverse Fourier transform* of the signal $F(j\omega)$.

Fourier Transform

Given these definitions Fourier's limiting formula asserts that:

$$f(t) = \mathcal{F}^{-1}(\mathcal{F}(f(t)))$$

i.e. that the nomenclature of the previous slide is valid, the inverse Fourier transform is indeed the inverse of the Fourier transform. It is possible to prove that this is so provided certain conditions are imposed upon the signal $f(t)$.

Fourier Transform

The formula $$F(j\omega) = \int_{-\infty}^{\infty} f(t)\exp(-j\omega t)dt = \mathcal{F}(f(t))$$

together with the formula

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\exp(j\omega t)d\omega = \mathcal{F}^{-1}(F(j\omega))$$

indicates that the Fourier transform of a signal $f(t)$ describes the frequency content of the signal, i.e. the modulus of $F(j\omega)$ describes to what extent the frequency ω is present in the signal for every real frequency ω.

Fourier Transform

One could disapprove of the formula:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) \exp(j\omega t) d\omega$$

on the grounds that it expresses the signal $f(t)$ as a "sum" of complex exponentials. Since the signals which we encounter in engineering are almost invariably real that seems somewhat incongruous.

Fourier Transform: Properties 1

If f is real then $F(-j\omega) = \overline{F(j\omega)}$ for all ω.

This is hermitian symmetry.

Proof: $$F(-j\omega) = \int_{-\infty}^{\infty} f(t)\exp(j\omega t)\,dt$$

$$= \overline{\int_{-\infty}^{\infty} f(t)\exp(-j\omega t)\,dt}$$

$$= \overline{F(j\omega)}$$

▲

Fourier Transform

Now we may rewrite:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\exp(j\omega t)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} F(j\omega)\exp(j\omega t)d\omega + \frac{1}{2\pi}\int_{0}^{\infty} F(j\omega)\exp(j\omega t)d\omega$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} F(-j\omega)\exp(-j\omega t)d\omega + \frac{1}{2\pi}\int_{0}^{\infty} F(j\omega)\exp(j\omega t)d\omega$$

Fourier Transform

Now we may rewrite:

$$= \frac{1}{2\pi}\int_0^\infty \left|F(j\omega)\right| \exp\left(-jArg\left(F(j\omega)\right)\right)\exp\left(-j\omega t\right)d\omega +$$

$$\frac{1}{2\pi}\int_0^\infty \left|F(j\omega)\right| \exp\left(jArg\left(F(j\omega)\right)\right)\exp\left(j\omega t\right)d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \left|F(j\omega)\right| \cos\left(\omega t + Arg\left(F(j\omega)\right)\right)d\omega$$

Fourier Transform

So when the signal is real the idea that we have expressed it as a "sum" of complex signals is really a mirage. In fact all of the imaginary terms cancel out and we are left with an expression as a "sum" of sinusoids.

$$= \frac{1}{\pi}\int_0^{\infty} |F(j\omega)| \cos(\omega t + Arg(F(j\omega)))d\omega$$

Hermitian symmetry explains why we do not experience negative frequency. It is because negative frequencies always arise as part of a plus/minus pair of equally weighted frequencies.

Fourier Transform: Properties 2

If $\mathcal{F}\left(f_1(t)\right) = F_1(j\omega)$ and $\mathcal{F}\left(f_2(t)\right) = F_2(j\omega)$

then

$$\mathcal{F}\left(\alpha_1 f_1(t) + \alpha_2 f_2(t)\right) = \alpha_1 F_1(j\omega) + \alpha_2 F_2(j\omega)$$

This property is called *linearity*.

Fourier Transform: Properties 3

If $\mathcal{F}\left(f\left(t\right)\right) = F\left(j\omega\right)$ then

$$\mathcal{F}\left(f\left(\alpha t\right)\right) = \frac{1}{|\alpha|} F\left(j\frac{\omega}{\alpha}\right)$$

This property is called *scaling*.

Note

If one speeds up a tape of music the frequencies present in the music are increased. If the Fourier transform yields the frequency description of a signal then it must explain this phenomenon. The scaling property encapsulates this explanation. Speeding up the tape implies that the signal at the magnetic read heads is compressed into a shorter period of time, although it is otherwise unchanged. Mathematically this is equivalent to time-scaling with parameter α greater than one (for example if α=7 then what previously happened in a week now happens in a day). The scaling property indicates that the affect on the frequency content is, essentially, to frequency-scale with parameter $1/\alpha$. If the original signal had a frequency component at 1,000 rad/sec then the new signal has a frequency component at ω/α=1,000, i.e. at ω =1,000 α rad/sec. If α is greater than one then this is at a higher frequency than the original.

Fourier Transform: Properties 4

If $\mathcal{F}\left(f\left(t\right)\right)=F\left(j\omega\right)$ then

$$\mathcal{F}\left(\exp\left(-\alpha t\right)f\left(t\right)\right)=F\left(j\omega+\alpha\right)$$

If $\mathcal{F}\left(f\left(t\right)\right)=F\left(j\omega\right)$ then

$$\mathcal{F}\left(f\left(t-t_0\right)\right)=\exp\left(-j\omega t_0\right)F\left(j\omega\right)$$

These properties are called *shifting*.

Note

Delaying a signal should have little impact upon its frequency content (since it is essentially the same signal). If I play a piece of music tomorrow it is the same piece of music so it should sound the same. The second shifting property confirms this. The “magnitude spectrum” is unchanged (i.e. the delayed signal has the same frequencies present and to the same relative extent).

Fourier Transform: Properties 5

If $\mathcal{F}(f(t)) = F(j\omega)$ and f is n-times differentiable then

$$\mathcal{F}\left(\frac{d^n}{dt^n} f(t)\right) = (j\omega)^n F(j\omega).$$

This is called the *differentiation* property.

Note

We will start, as we did with the Laplace transform, to talk about two domains: the time-domain and the frequency-domain. When I have a description of a signal as a function of time $f(t)$ I say that I have a *time-domain description*. On the other hand when I have a description of the frequency content $F(j\omega)$, i.e. when I have instead the Fourier transform of the signal I say that I have a *frequency-domain description*. Because the Fourier transform and the inverse Fourier transform permit me to flip back and forth at will between this two descriptions I take the view that neither is "the" correct description, but rather that they are both correct.

Note

Just as with the Laplace transform the value of flipping from time-domain to frequency-domain descriptions is, in part, because the undertaking of a mathematical operation in the time-domain will correspond under the mapping to the undertaking of a different mathematical operation in the frequency-domain. Critically the companion operations in the frequency-domain are sometimes mathematically simpler. So for example the previous differentiation property says that the operation in the frequency domain which is the companion to the operation of differentiation in the time-domain is the operation of just multiplying by $j\omega$.

Note

I should note that having simpler companion mathematical operations is only one of the reasons for flipping from the time-domain to the frequency-domain. Unlike the Laplace transform of a signal which just was what it was, we have an important interpretation of the frequency-domain description of a signal, namely that it describes the frequency content. So the Fourier transform has two uses. Firstly it will be useful for analysing systems (just as the Laplace transform was and for the same reason: it replaces difficult mathematical operations with easier mathematical operations). Secondly it will be useful for analysing signals.

Fourier Transform: Properties 6

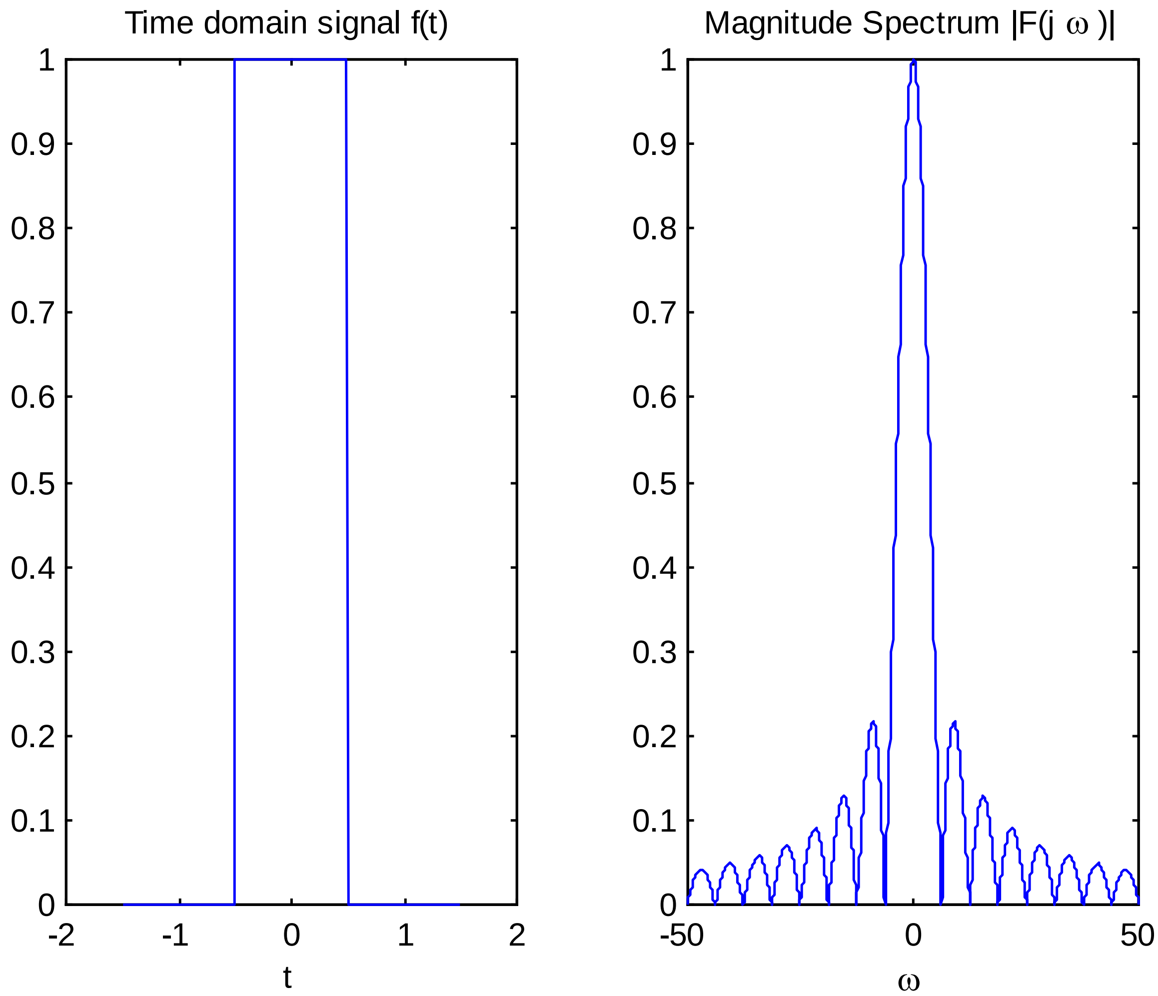
If $\mathcal{F}(f(t)) = F(j\omega)$ and f is integrable, then

$$\mathcal{F}\left(\int_0^t f(t)\,dt\right) = \frac{F(j\omega)}{j\omega}.$$

This is called the *integration* property.

Example 4.1

$$f(t) = \begin{cases} 1 & \text{if } -0.5 \le t \le 0.5 \\ 0 & \text{otherwise} \end{cases}$$



Note

This signal is called the unit pulse.

It has duration of 1 sec.

Much, but not all of the frequency content of the signal lies in the *centre lobe* (frequencies -2π to 2π rad/sec).

Numerical Fourier Transform

Assume signal $f(t)$ is of finite duration T_d and commences at $t = 0$.

$$F(j\omega)=\sum_{k=0}^{N-1}\int_{\frac{kT_d}{N}}^{\frac{(k+1)T_d}{N}} f(t)\exp(-j\omega t)dt$$

$$\cong\sum_{k=0}^{N-1} f\left(\frac{kT_d}{N}\right)\int_{\frac{kT_d}{N}}^{\frac{(k+1)T_d}{N}}\exp(-j\omega t)dt$$

Numerical Fourier Transform

$$F\left(\frac{j2n\pi}{T_d}\right) \cong$$

$$\left(\frac{1-\exp\left(-j2n\pi/N\right)}{j2n\pi/T_d}\right)\sum_{k=0}^{N-1} f\left(\frac{kT_d}{N}\right)\exp\left(-\frac{j2nk\pi}{N}\right)$$

$$\text{for } n = 0,1,\ldots,N-1$$

Note

As it is impossible to numerically evaluate $F(j\omega)$ for all ω we must content ourselves with approximately evaluating for a finite number of ω. We elect to approximately evaluate at $\omega = 0, 2\pi/T_d, 4\pi/T_d, \ldots, 2(N\text{-}1)\pi/T_d$.

Numerical Fourier Transform

$$\text{Let } F_n = \sum_{k=0}^{N-1} f\left(\frac{kT_d}{N}\right)\exp\left(-\frac{j2nk\pi}{N}\right)$$

$$\text{Let } W_n = \begin{cases} \left(\dfrac{1-\exp(-j2n\pi/N)}{j2n\pi/N}\right) & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases}$$

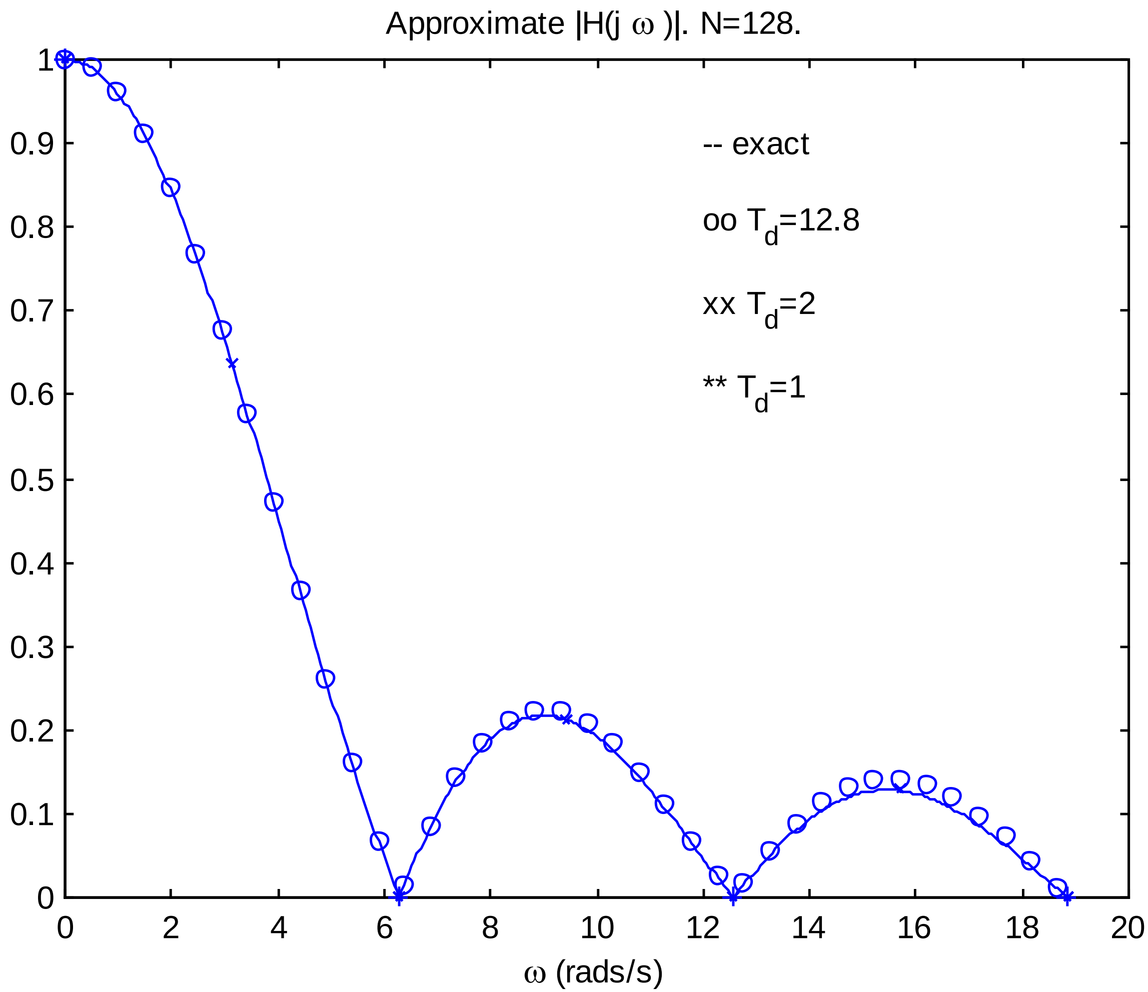
$$F\left(\frac{j2n\pi}{T_d}\right) \cong \frac{T_d W_n F_n}{N} \quad \text{for } n = 0, 1, \ldots, N-1$$

Note

Note that the numbers F_n are the same as those which appeared in the numerical evaluation of the Fourier coefficients of a periodic signal. Hence the same algorithm, the FFT, is employed to calculate them. The workings of the FFT are such that it is better to select N equal to a power of 2 if possible. Note again that the sequence F_n is periodic in n with period N. As the Fourier transform is manifestly not periodic with period $2N\pi\ /T_d$ in general there is clearly some problem. A rigorous investigation of this matter offers the solution employed in practice that only the first $N/2$ samples of the approximate Fourier transform are retained. If N is fairly large the first several terms of the sequence W_n are approximately equal to 1 and the simpler approximation $F(\mathrm{j}2n\pi/T_d) \simeq T_d F_n \,/\, N$ is employed.

Example 4.2

$$h(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$



Note

In approximately evaluating the Fourier transform of this pulse by the method outlined above we have two parameters to select: N and T_d. We choose $N = 128$. One might be forgiven for assuming that T_d must be selected equal to 1 (since this is clearly the duration of the signal). In fact a moments thought reveals that one can select T_d to equal any value greater than or equal to 1 for this signal. It transpires that the quality of the approximation achieved can be influenced by the choice of this parameter. We show frequency components in the range 0 to 6π (rads/sec). Note that the number of frequency samples lying in this range varies with the choice of T_d. In the example the simpler approximation $F(\mathrm{j}2n\pi/T_d) \cong T_d F_n / N$ is employed. Note that the approximation of the low frequencies is good, but that the level of approximation deteriorates at higher frequencies.

Numerical Fourier Transform

It is more often the case that one only has samples of the signal at uniformly spaced sample times such as 0, T_s, $2T_s$, ... where T_s is the sampling period. If we have N samples then $T_d = NT_s$, so that:

$$\frac{kT_d}{N} = kT_s \quad \text{being the sample times.}$$

The frequencies at which the FFT yields the approximate value of the Fourier Transform are:

$$\frac{2n\pi}{T_d} = n\frac{2\pi}{NT_s} \ \text{rad/sec} \ = \frac{n}{NT_s} \ \text{Hz}$$

Finally these frequencies are expressed in terms of the sampling frequency $f_s = 1/T_s$ rather than the sampling period:

$$\frac{n}{N} f_s \ \text{Hz}$$

Note

A .wav audio file can be imported into Matlab by the **wavread** command and played by the **sound** command.

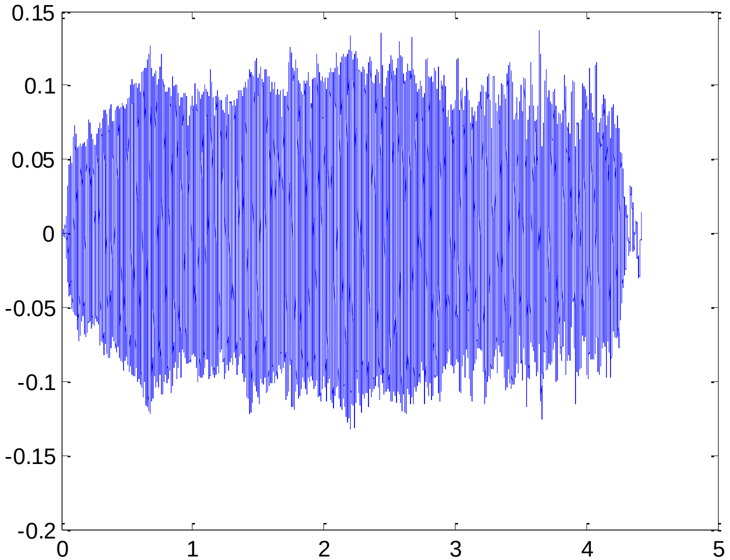
```
>>  [y,fs,nbits] = wavread('Whistle.wav');
```
reads a .wav file into workspace. y is vector of samples. fs is sampling frequency (i.e. number of samples per second), nbits is number of bits per sample.

```
>>  sound(y,fs)
```
plays samples at sample rate fs

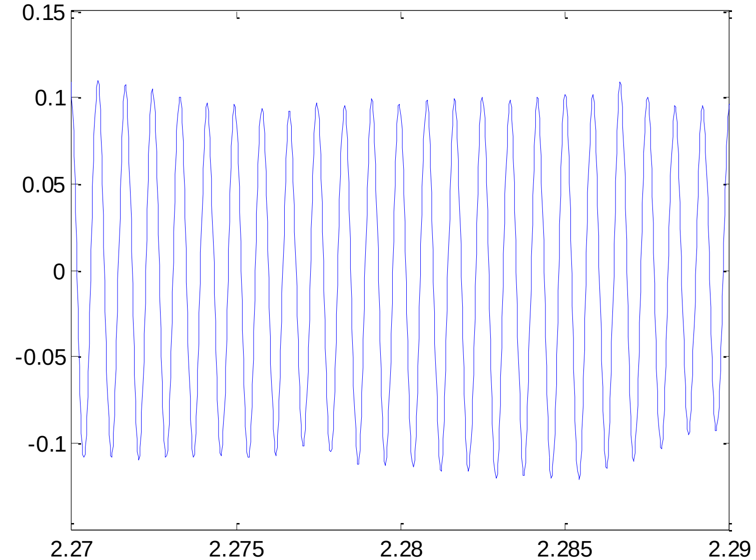
```
>>  FF = fft(y)/fs;
```
finds approximate Fourier transform

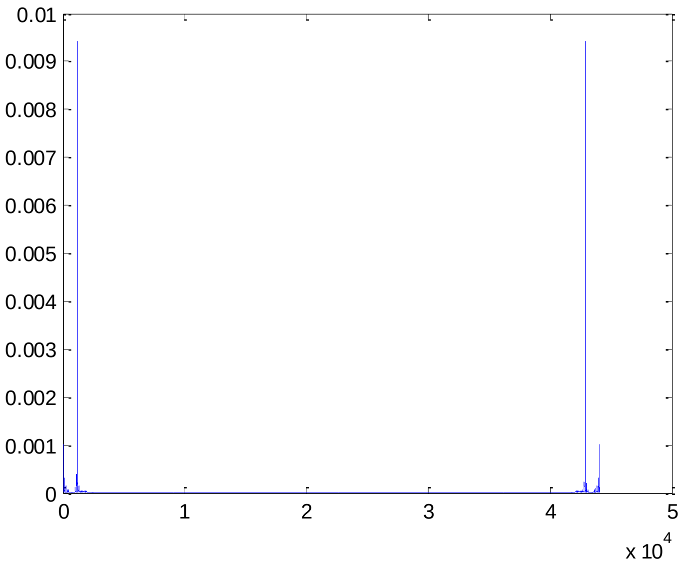
The standard audio format for CD quality has a sampling frequency of 44100 samples per second and 16 bits per sample.

Human Whistle.

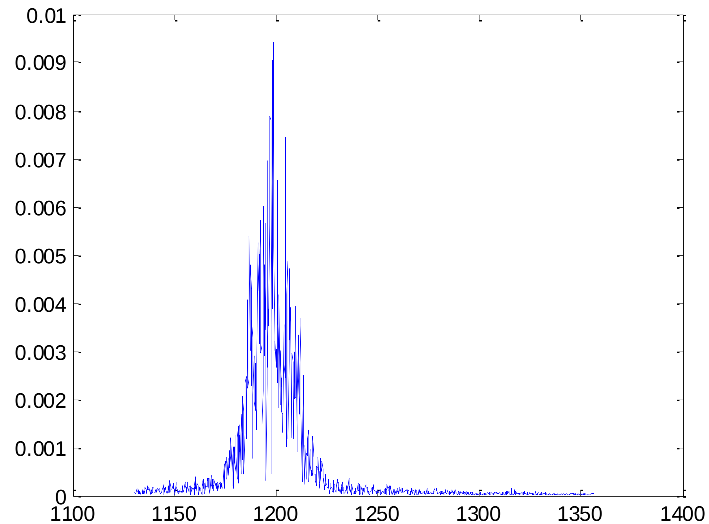


Zoom: time 2.27 sec to 2.29 sec.





Magnitude Spectrum as given by fft. Note symmetry.



Zoom: frequency 1125 Hz to 1350 Hz.

Note

```
>> length(y)
```

ans =

195072

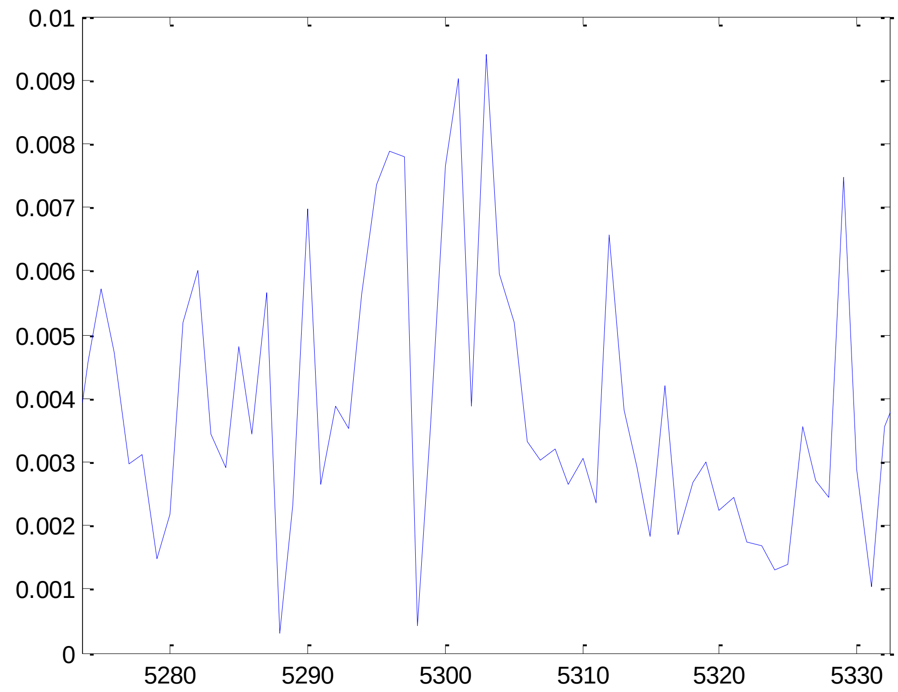
There are 195072 uniformly-spaced samples taken at 44100 samples/sec. The duration of the signal is therefore 195072/44100 = 4.4234 sec. I reflect this in the first subplot horizontal axis. We can convert from index n of **fft**-calculated F_n to corresponding frequency using:

$$\frac{n}{N} f_s \ \text{Hz} = \frac{44100}{195072} n \ \text{Hz}$$

Note

```
>> plot(abs(FF))
```

Select from the toolbar **tools/options** and then select **horizontal zoom**. By zooming in we find the peak at about the 5302nd value, i.e. *n* = 5301. This gives the corresponding frequency:



$$\frac{44100}{195072}(5301)\ \text{Hz} = 1198\ \text{Hz}$$

Note

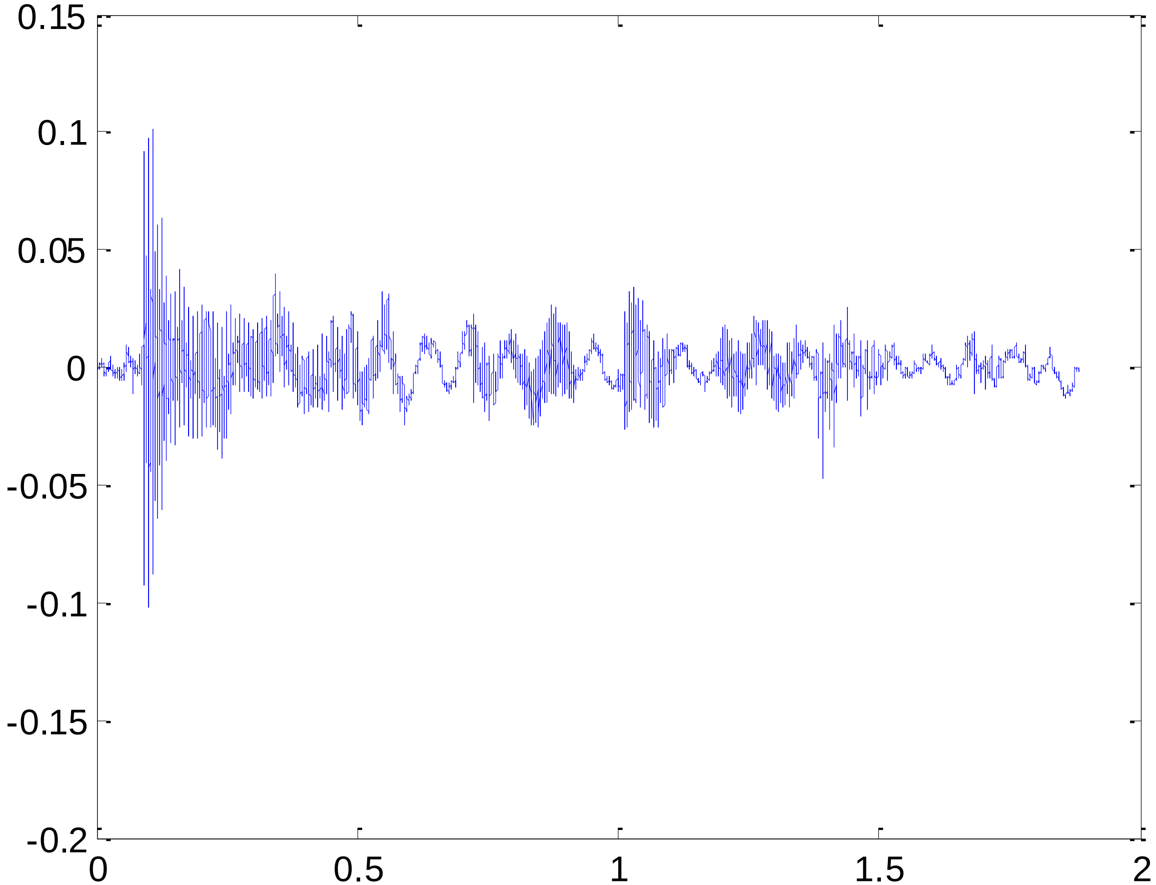
```
>> t1= [2.27:1/44100:2.29];

>> y1 = y(100107:100989);   100107 = (2.27)(44100) and 100989 = (2.29)(44100) are indices corresponding to samples at sampling times 2.27 sec and 2.29 sec.

>> plot(t1,y1)
```

The signal segment from 2.27 sec to 2.29 sec shows 24 cycles of an underlying somewhat periodic signal. The corresponding frequency is 24/0.02 = 1200 Hz. This is consistent with the fairly strong peak in the frequency response at about 1198 Hz.

Note

The signal segment from 2.27 sec to 2.29 sec also shows a lower frequency modulation of the amplitude. We will see later how this causes a spreading of the frequency content producing lower level frequency content at a collection of frequencies just above and just below the peak.

“Finally a Speech Sample”



Note

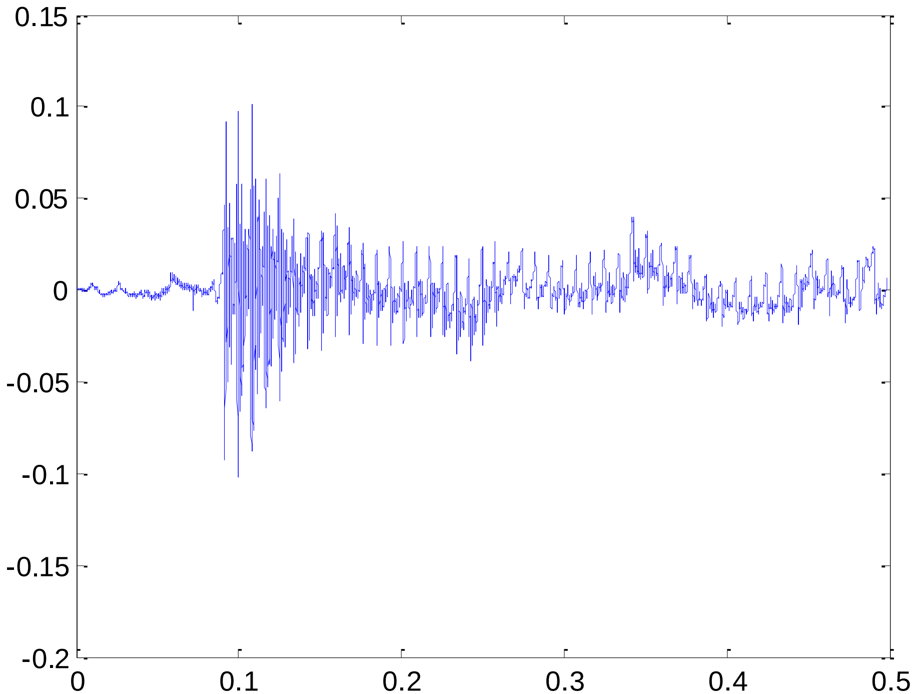
>> [y,fs,nbits] = wavread('Speech.wav'); *reads a .wav file into workspace. y is vector of samples. fs is sampling frequency (i.e. number of samples per second), nbits is number of bits per sample.*

>> sound(y,fs) *plays samples at sample rate fs*

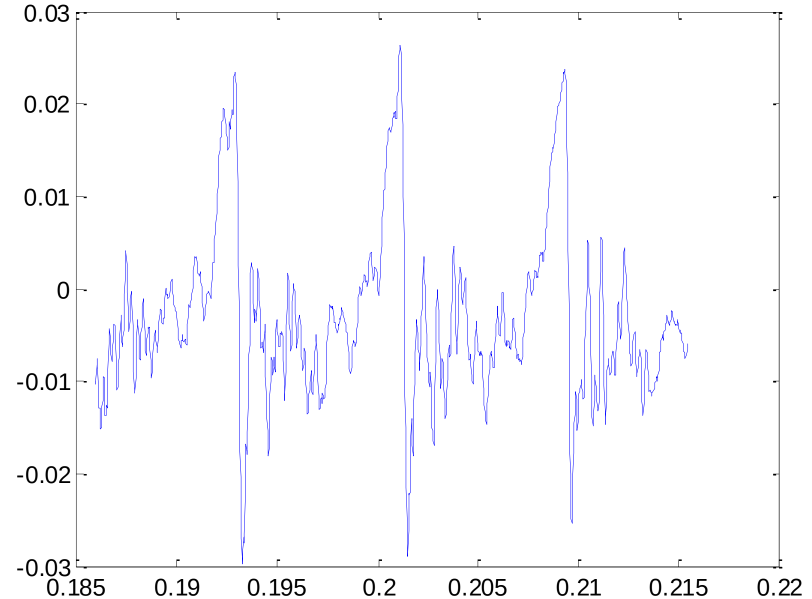
>> FF = fft(y)/fs; *finds approximate Fourier transform*

"Finally a Speech Sample"

Zoom: "Finally"



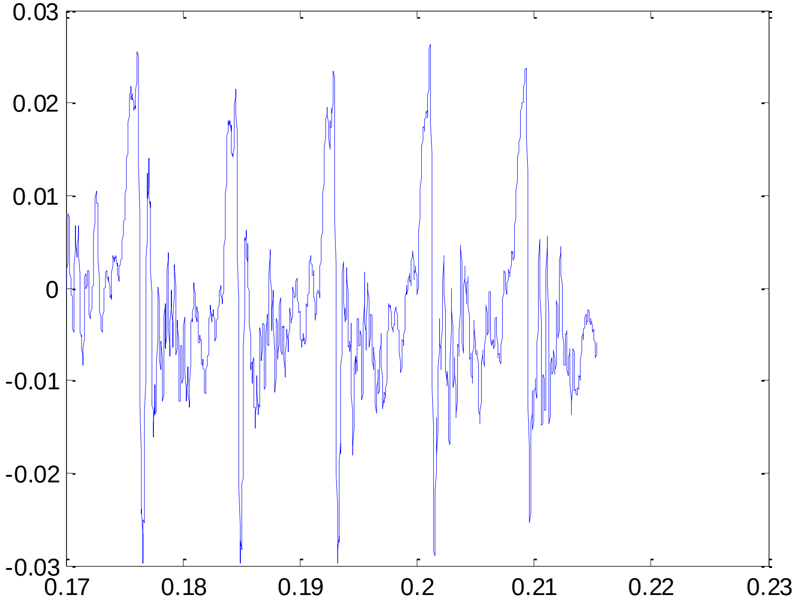
Zoom: 0.185 sec to 0.215 sec



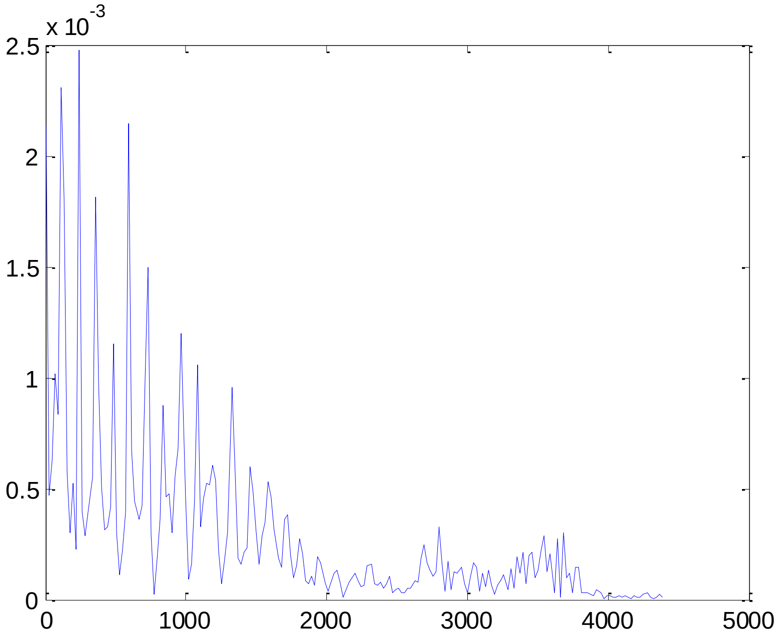
We see a series of *impulse responses*. From minima at 0.1932 sec and 0.2015 sec we determine that pulse frequency (F_0) is about 120.5 Hz, implying adult male. Adult male is about 120 Hz, adult female about 210 Hz and small child about 300 Hz.

"Finally a Speech Sample"

Zoom: 0.17 sec to 0.215 sec



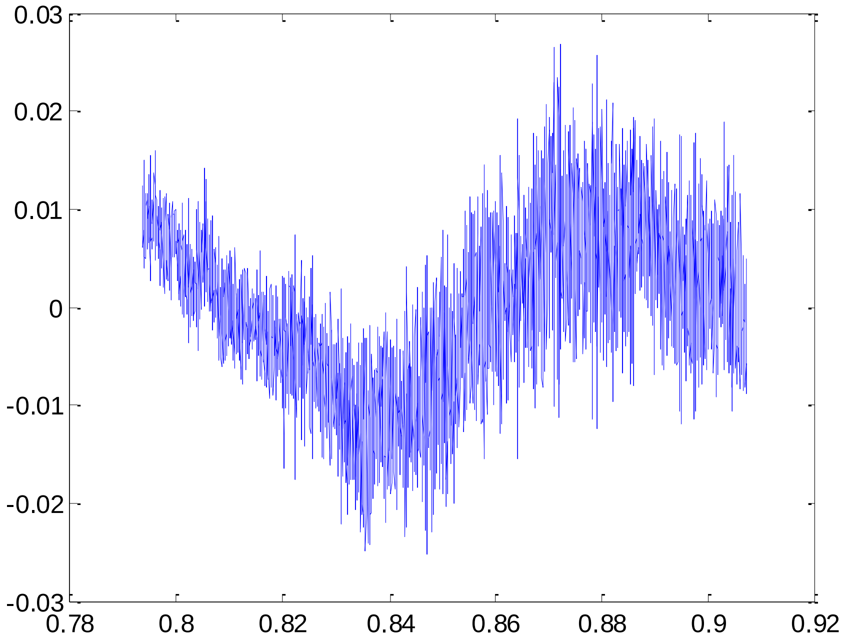
Magnitude spectrum with high frequency removed.



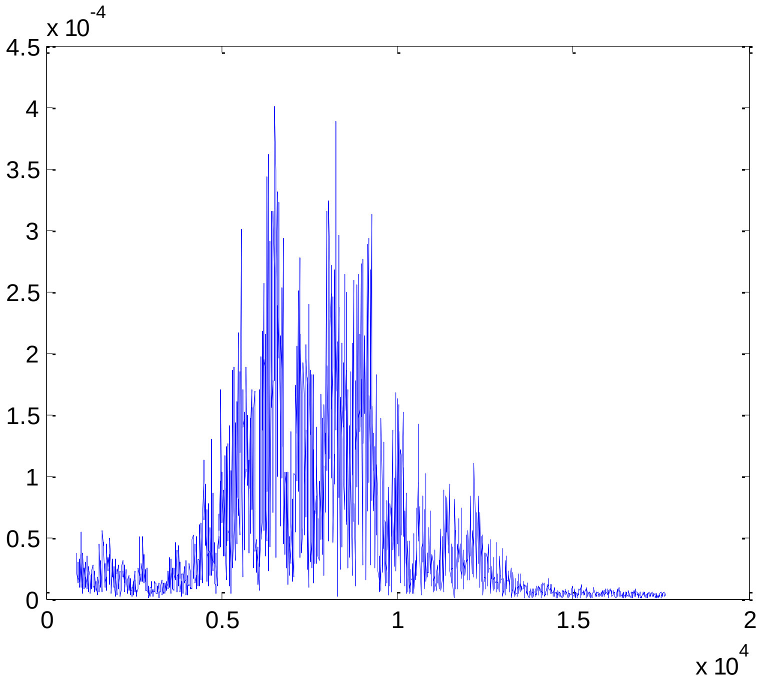
Note that spectrum has strong peaks. Note in particular peak at about 120 Hz and integer multiples thereof.

"Finally a Speech Sample"

Zoom: "S" of "Speech"



Magnitude spectrum with low frequency modulation removed.



Note the rather noisy quality of the s sound, reflected in the *broadband* nature of the spectrum.

Note

In the case of periodic signals we now have two mathematical techniques, the Fourier series and the Fourier transform, which purport to give the same information, namely the frequency content of a the signal. The two techniques will have to be compatible with one another.

Fourier Transform of Periodic Signals

$$\mathcal{F}^{-1}\left(2\pi\delta(\omega)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega)e^{j\omega t}d\omega$$

$$= \int_{-\infty}^{\infty}\delta(\omega)e^{j\omega t}d\omega = e^{j\omega t}\Big|_{\omega=0} = 1$$

$$\mathcal{F}(1) = 2\pi\delta(\omega)$$

Note

The Fourier transform of a constant is a weighted delta function $\delta(\omega)$. Since the delta function is zero for all ω not equal to zero, the interpretation of the Fourier transform leads to the conclusion that the only frequency present in a constant or DC signal is 0 rad/sec or DC. This does make sense and is consistent with the Fourier series. The Fourier series of 1 is 1 being $\cos(0t)$, so again we deduce that the only frequency present in a constant signal is 0 rad/sec or DC.

Fourier Transform of Periodic Signals

$$\mathcal{F}\left(A\sin\left(\omega_0 t\right)\right)=\frac{A\pi}{j}\left(\delta\left(\omega-\omega_0\right)-\delta\left(\omega+\omega_0\right)\right)$$

$$\mathcal{F}\left(A\cos\left(\omega_0 t\right)\right)=A\pi\left(\delta\left(\omega-\omega_0\right)+\delta\left(\omega+\omega_0\right)\right)$$

Note

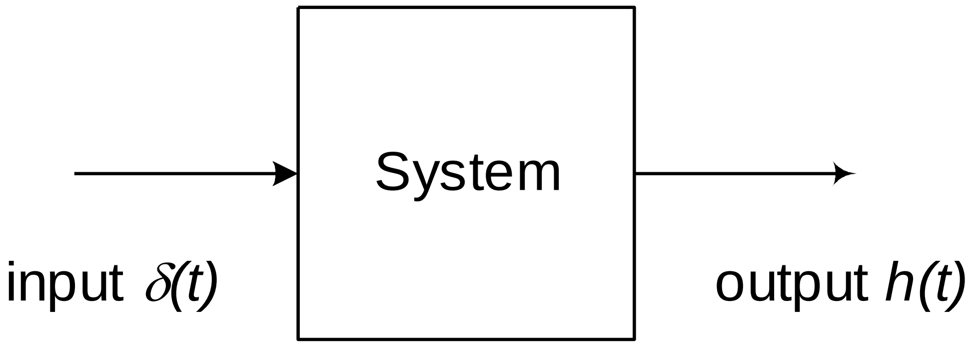
The Fourier transform asserts that the only frequencies present in $A\sin(\omega_0 t)$ or $A\cos(\omega_0 t)$ are +/- ω_0 with these frequencies being equally weighted. The Fourier series of these signals agree:

$$A\sin(\omega_0 t) = \frac{A}{2j}e^{j\omega_0 t} - \frac{A}{2j}e^{-j\omega_0 t}$$

$$A\cos(\omega_0 t) = \frac{A}{2}e^{j\omega_0 t} + \frac{A}{2}e^{-j\omega_0 t}$$

Recall: Impulse Response

The response of a SISO, linear time-invariant (LTI) system to an input equal to the delta function $\delta(t)$ is called the *impulse response* of the system.



Note

As noted above if the impulse response is non-zero for some $t < 0$ then the system is *anticipatory* or *non-causal*. If the system is causal, which we generally assume to be the case, then the impulse response is causal, i.e. $h(t) = 0$ for all $t < 0$. We defined the transfer function of the system to be the Laplace transform of the impulse response, i.e. essentially

$$H(s) = \int_{0^-}^{\infty} h(t) e^{-st} dt$$

Note

Assuming the system to be causal this in turn equals:

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

Formally replacing s by $j\omega$ yields:

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

which of course equals the Fourier transform of the impulse response.

Note

This very much retrospectively justifies the slightly odd notation chosen for the Fourier transform. It follows more generally that if a signal $f(t)$ is causal, i.e. if it equals zero for negative t, then its Laplace transform and its Fourier transform are related by

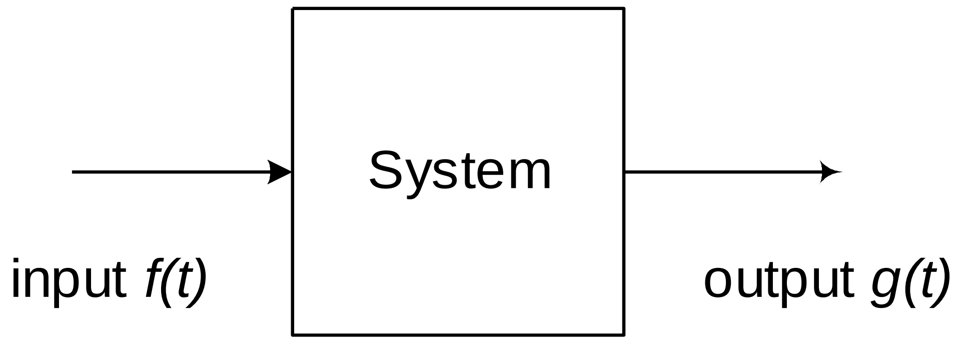
$$\mathcal{L}\left(f\left(t\right)\right)\Big|_{s=j\omega} = \mathcal{F}\left(f\left(t\right)\right)$$

provided of course both transforms actually exist.

Recall: The Convolution Integral

Let the impulse response of a SISO, LTI ,causal system be $h(t)$. Then, under mild conditions, the response $g(t)$ to a causal excitation $f(t)$ is given by Dirichlet's formula

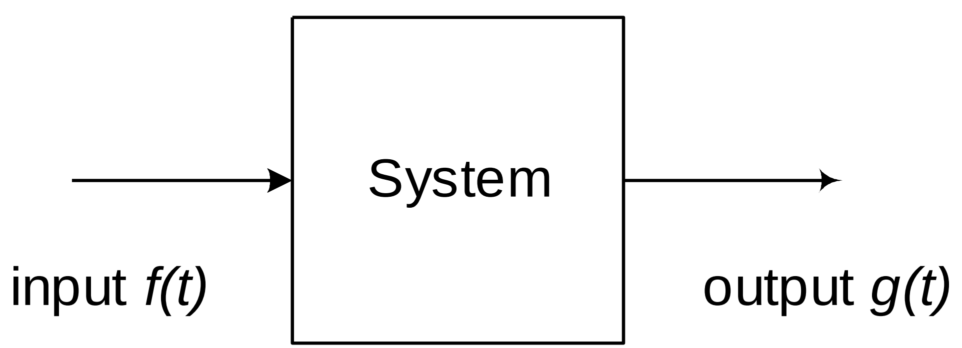
$$g(t)=\int_0^t f(\tau)h(t-\tau)d\tau$$



Recall: The Convolution Integral

By observing that the system is causal and that the input *f*(*t*) is also causal I note that it is possible to express this in a different way:

$$g(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$



Fourier Transform: Properties 7

Convolution

If $\mathcal{F}\left(f_1(t)\right) = F_1(j\omega)$ and $\mathcal{F}\left(f_2(t)\right) = F_2(j\omega)$

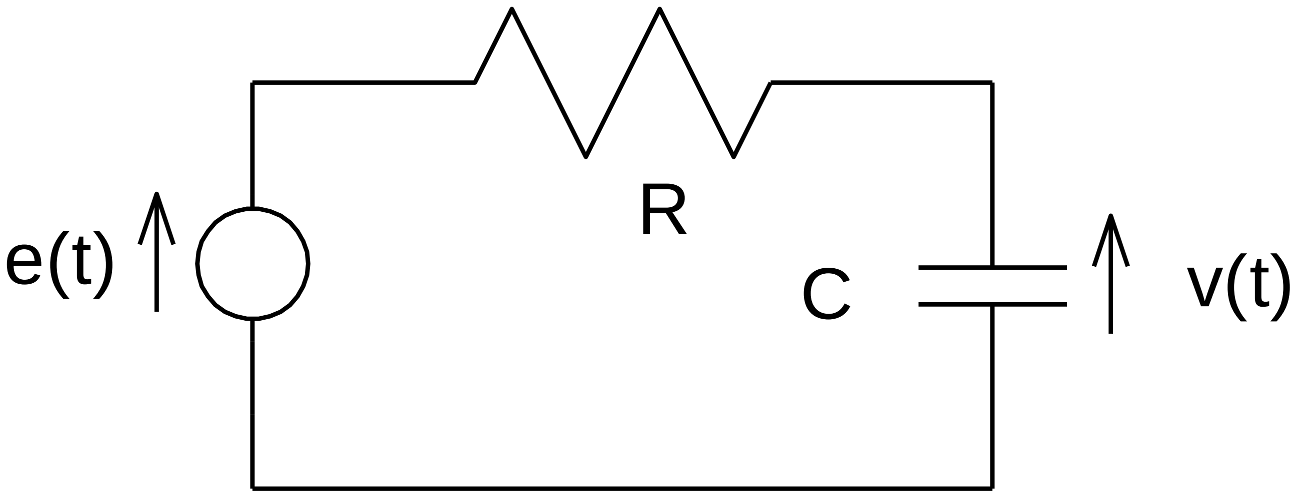
then $\mathcal{F}\left(\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\,d\tau\right) = F_1(j\omega)F_2(j\omega).$

The Transfer Function

Let the impulse response of a SISO, LTI, causal system be $h(t)$, let the input be $f(t)$ and let the output be $g(t)$, then, under mild conditions

$$G(j\omega) = F(j\omega)H(j\omega)$$

Example 4.3



$$e(t) = RC\frac{d}{dt}v(t) + v(t)$$

Note

Classifying voltage signal $e(t)$ as the input and voltage signal $v(t)$ as the output the circuit may be regarded as a SISO system. A question arises as to whether the system is LTI. The answer to this question is not absolutely clear, mainly because the question is not clear. There are at least two mathematical problems which may be implied by this example:

(i) The differential equation holds for all time and a solution is sought which is valid for all time.

(ii) The differential equation holds after a particular time, which we may as well call $t = 0$. Moreover there is a further specification that $v(0) = V_0$ must hold.

Note

The first problem assumes that the circuit has existed and will continue to exist in this exact form for all time. No initial condition is directly specified. The solution to this problem is called the *steady state solution*.

The second assumes that the circuit is switched on at a particular time $t = 0$. The solution then depends, not only on the differential equation, but also on the state of the capacitor (that is to say the charge on the capacitor) at this initial time.

Example 4.3

$$e(t) = RC\frac{d}{dt}v(t) + v(t)$$

$$E(j\omega) = RC(j\omega V(j\omega)) + V(j\omega)$$

$$V(j\omega) = \frac{E(j\omega)}{(1 + j\omega RC)} = H(j\omega)E(j\omega)$$

Note

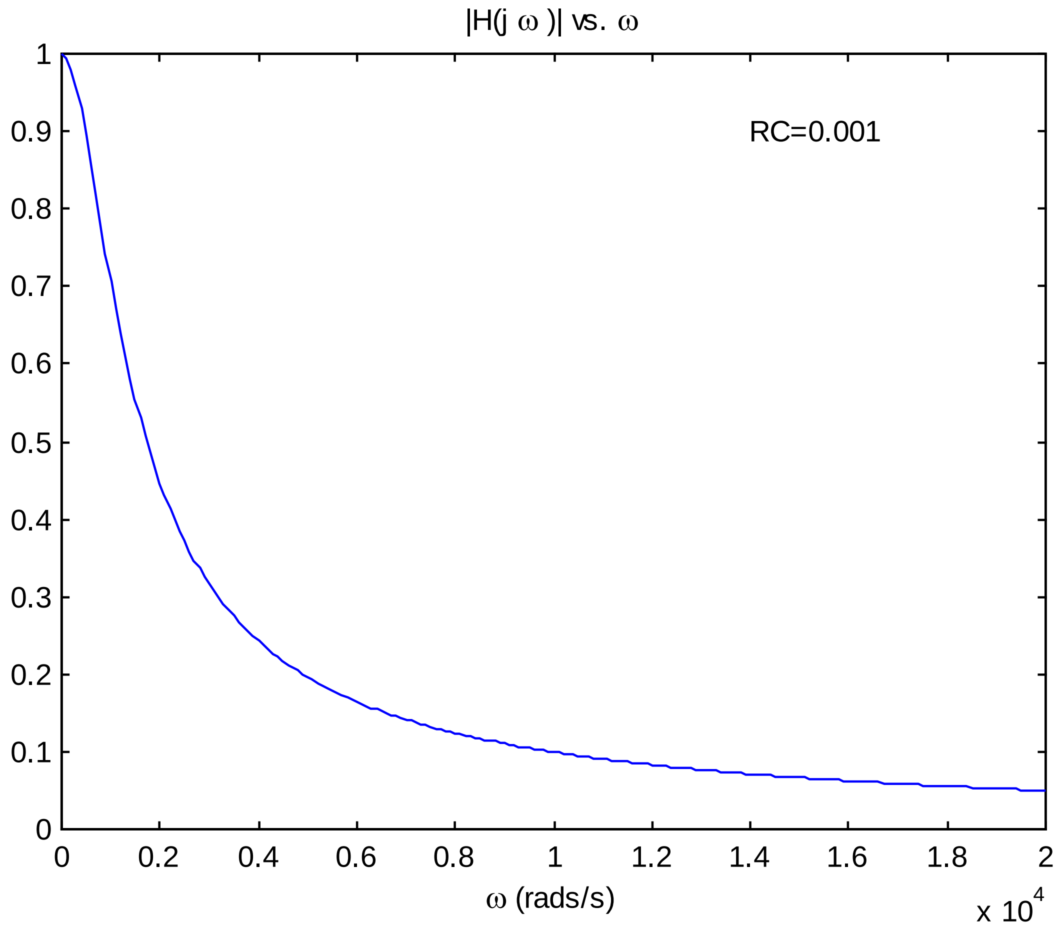
We employ the linearity and differentiation properties of the Fourier transform. The calculation establishes that the Fourier transform of the output equals a multiplier (the transfer function $H(j\omega)$) times the Fourier transform of the input, as we would expect if the system were LTI. We note that the analysis permits us to calculate the transfer function $H(j\omega)$ relatively easily. Note also that two assumptions have crept into the analysis. Firstly we assume that the differential equation holds for all time (since the Fourier transform is based upon the description of the signal for all time). In other words we are focussing on the first mathematical problem, i.e. the Fourier transform permits us to determine the steady state solution.

Note

Secondly we assume that the Fourier transforms of *e* and *v* exist and are well-behaved. In particular this means that both signals must approach zero as time goes to +/- infinity. The significant assumption here is that the voltage across the capacitor is zero at $t = -\infty$. The method is effectively assuming an initial state of zero for the capacitor, albeit at time $t = -\infty$. It is only with the addition of these rather technical assumptions that the circuit can be considered to be a SISO, LTI system.

Example 4.3

$$H(j\omega) = \frac{1}{(1 + j\omega RC)}$$



Note

A second advantage of the transfer function description of a SISO, LTI system is that it permits us to get a feel for what the system does. Recall that the Fourier transform provides a frequency description of the signal. If frequency ωRC is large (i.e. if ω is large relative to $1/RC$) the transfer function $H(j\omega)$ of the RC circuit is nearly zero and if it is small (i.e. if w is small relative to $1/RC$) the transfer function is nearly one. Hence, even if the input signal e has significant high frequency components (i.e. $E(j\omega)$ is non-zero for certain large ω), the output signal v does not have a significant component at these frequencies (since $V(j\omega) = H(j\omega)E(j\omega)$ is nearly zero). On the other hand if the input signal e has significant low frequency components (i.e. $E(j\omega)$ is non-zero for certain small ω), the output signal v also has a significant component at these frequencies (since $V(j\omega) = H(j\omega)E(j\omega)$ is nearly equal to $E(j\omega)$).

Note

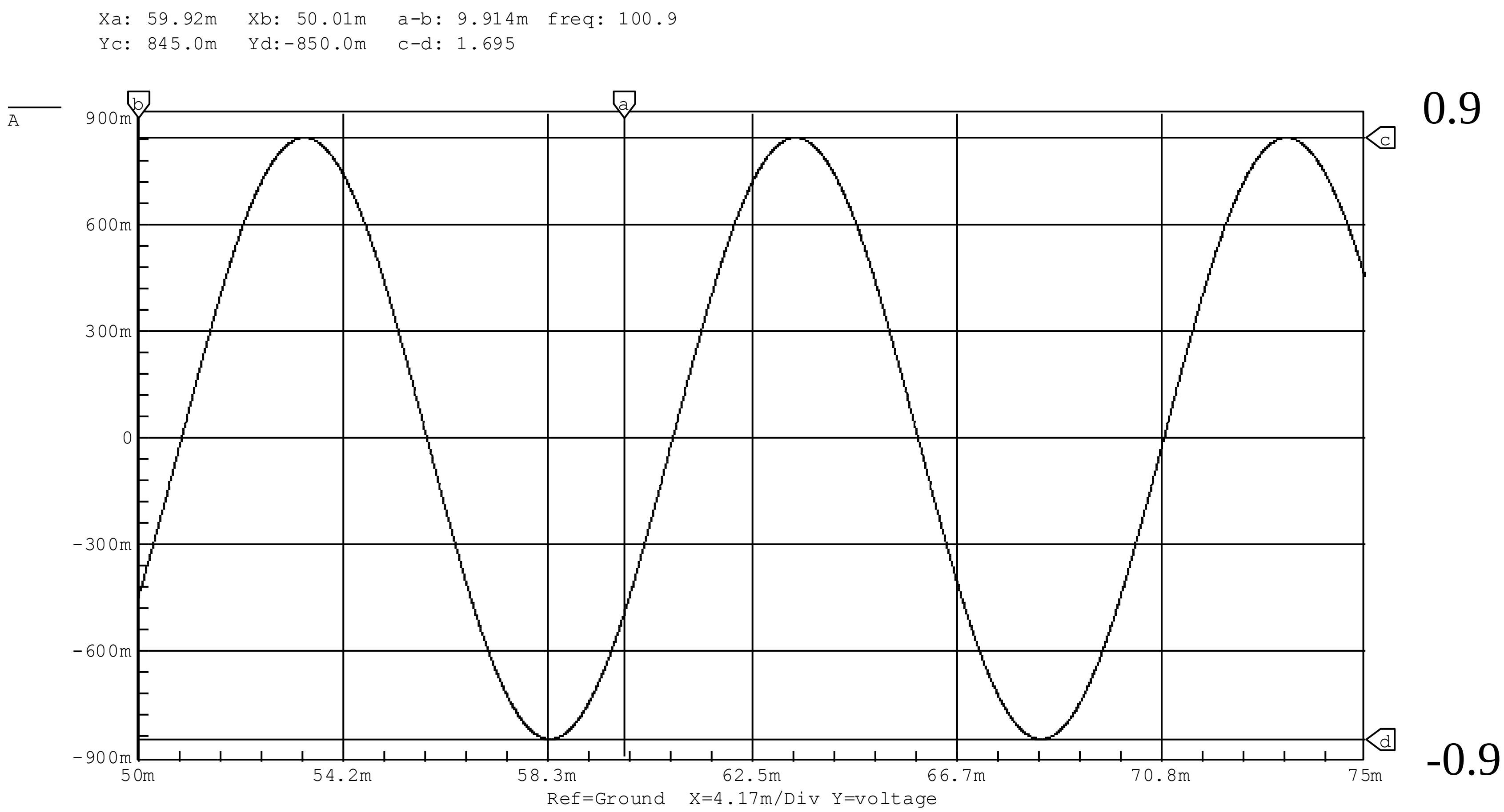
In short the RC circuit above is a *low pass filter*. It filters the frequencies present in the input signal e, suppressing the high frequency components and passing the low frequency components.

Note

One use of filters in general is to clean up a noisy measurement signal. The signal is regarded as a sum of a perfect measurement signal and a noise signal. The noise signal will have a lot of frequency components, a broadband spectrum. Although we have not discussed it the squared-modulus of the Fourier transform of a signal indicates the frequency distribution of the signal energy. A noise signal therefore has its energy spread across many frequencies and consequently may have relatively low energy at any given frequency (i.e. within a small range of frequencies around the given frequency). If the perfect measurement signal has a more compressed spectrum then a filter which passes those frequencies present in the signal but blocks all others will pass the measurement signal and some noise but will block most of the noise. It is quite common for the measurement signal to contain low frequencies only. In this case a low pass filter will clean up the signal in general.

Example 4.3

$$H(j\omega) = \frac{1}{(1 + j\omega RC)}$$

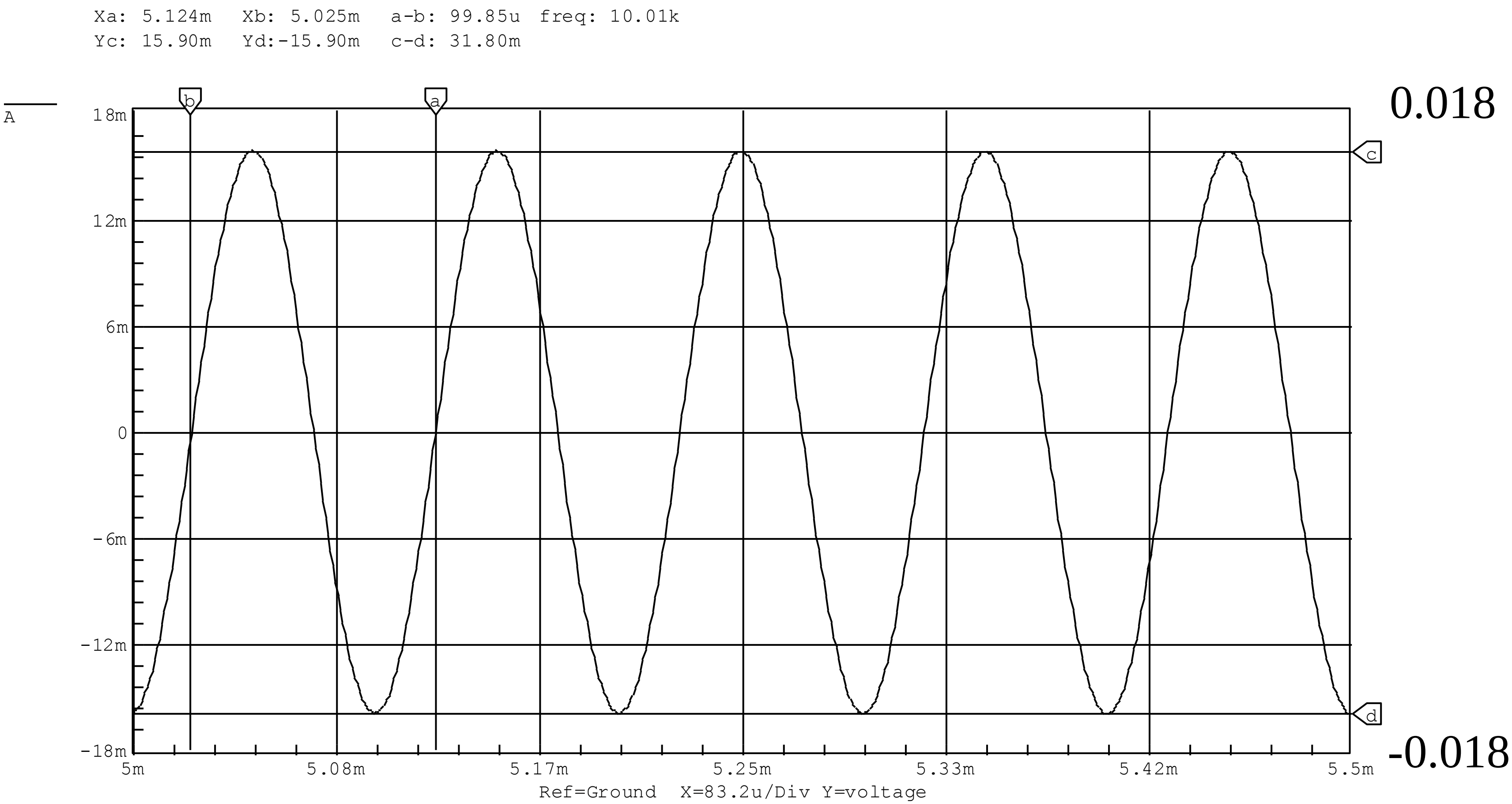


Note

To illustrate the filtering properties of the RC circuit: Simulation above is for $e(t) = \sin(200\pi t)$, $RC = 1$msec. Resulting $v(t)$ has amplitude approximately equal to 0.85V, period is approximately 10msec (i.e. frequency approximately 100 Hz). Note that $|H(j200\pi)| \simeq 0.8467$. The circuit is essentially passing the frequency component $\omega = 200\pi$ rad/sec. This is because 200π is less than $1/RC = 1{,}000$ rad/sec.

Example 4.3

$$H(j\omega) = \frac{1}{(1 + j\omega RC)}$$



Note

Simulation above is for $e(t) = \sin(20{,}000\pi t)$, RC = 1msec. Resulting $v(t)$ has amplitude approximately equal to 15.9 mV, period is approximately 0.1msec (i.e. frequency approximately 10 kHz). Note that $|H(j20{,}000\pi)| \cong 0.0159$. The circuit is essentially blocking the frequency component $\omega = 20{,}000\pi$ rad/sec. This is because $20{,}000\pi$ is more than 1/RC = 1,000 rad/sec.

Example 4.3

$$e(t) = RC\frac{d}{dt}v(t) + v(t)$$

$$e(t) = \begin{cases} 1 & \text{for } 0 \le t \le 1 \\ 0 & \text{for } t < 0 \text{ or } t > 1 \end{cases}$$

$$V(s) = \frac{E(s)}{(sRC+1)} + \frac{RCv_0}{(sRC+1)}$$

Example 4.3

$$E(s) = \int_0^1 e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right|_0^1 = \frac{1-e^{-s}}{s}$$

$$V(s) = \frac{\frac{1}{RC}}{s\left(s+\frac{1}{RC}\right)} - \frac{\frac{1}{RC}e^{-s}}{s\left(s+\frac{1}{RC}\right)} + \frac{v_0}{\left(s+\frac{1}{RC}\right)}$$

$$V(s) = \frac{1}{s} - \frac{1}{\left(s+\frac{1}{RC}\right)} - \frac{e^{-s}}{s} + \frac{e^{-s}}{\left(s+\frac{1}{RC}\right)} + \frac{v_0}{\left(s+\frac{1}{RC}\right)}$$

Example 4.3

The inverse Laplace transform can be found in spite of the presence of the exponential exp(-*s*) by applying the shifting property.

$$v(t) =$$

$$1 - \exp\left(-\tfrac{t}{RC}\right) - u(t-1) + \exp\left(-\tfrac{(t-1)}{RC}\right)u(t-1) + v_0 \exp\left(-\tfrac{t}{RC}\right)$$

$$\text{for} \quad t \geq 0$$

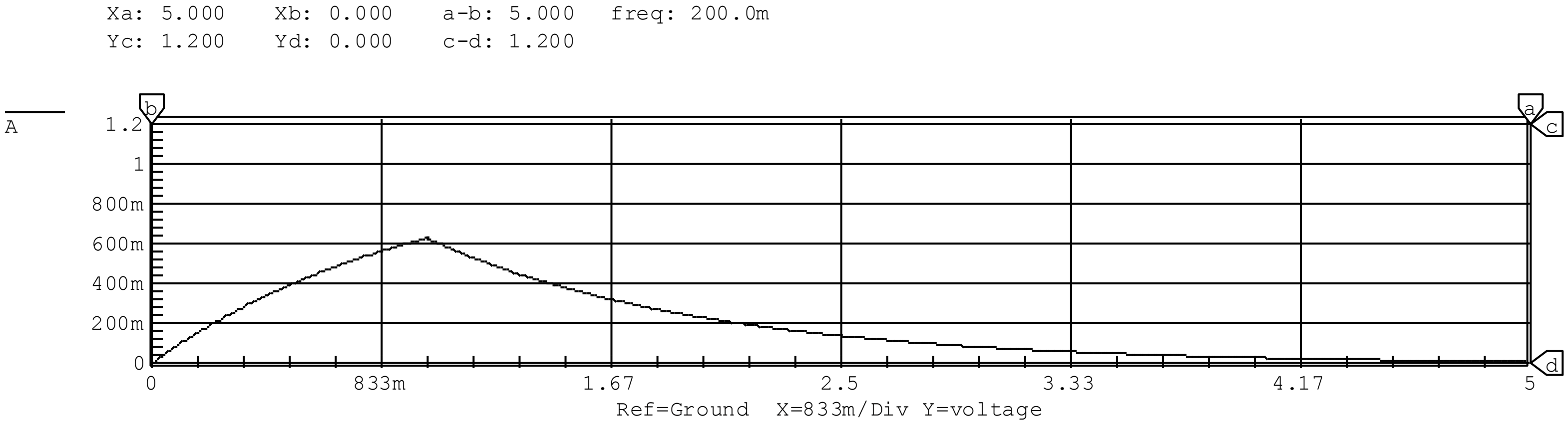
The solution emerges essentially as a sum of exponentials. The Fourier transform offers additional information.

Note

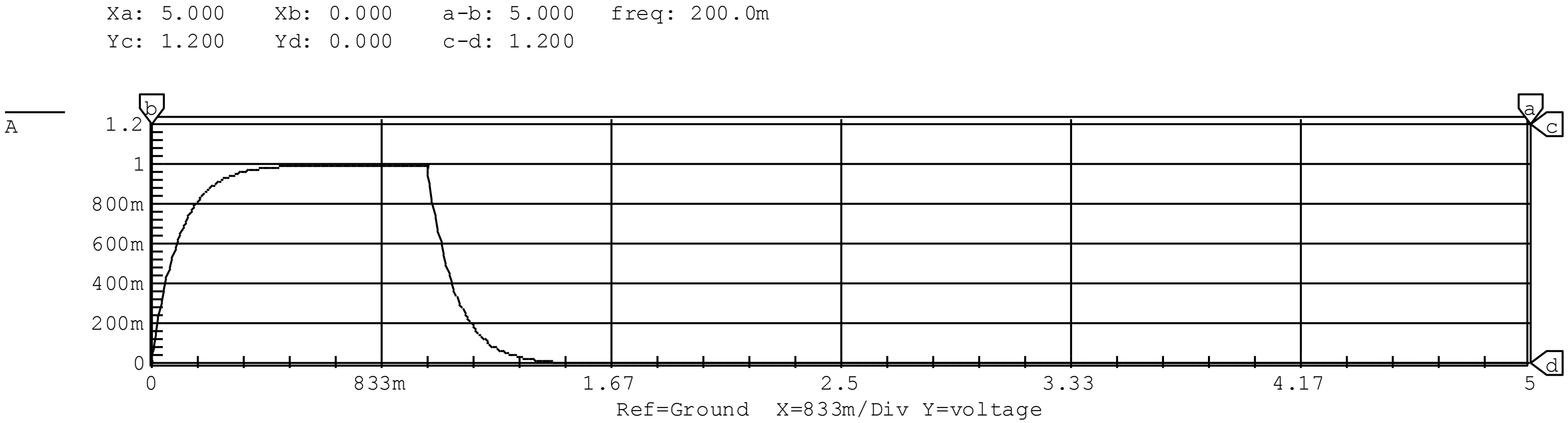
We consider the response of the RC circuit to a unit pulse having pulse width of 1 sec. Since a unit pulse contains components at all frequencies we expect *distortion* (as the RC circuit suppresses the high frequency components). The RC circuit essentially passes frequencies below $1/RC$ and blocks frequencies above $1/RC$. Since most of the significant frequency components of the unit pulse are at lower frequencies, most of these frequencies are passed if $1/RC$ becomes large. Consequently the distortion should become *less* as $1/RC$ becomes larger. This conclusion is confirmed by the simulation results.

Section: 4

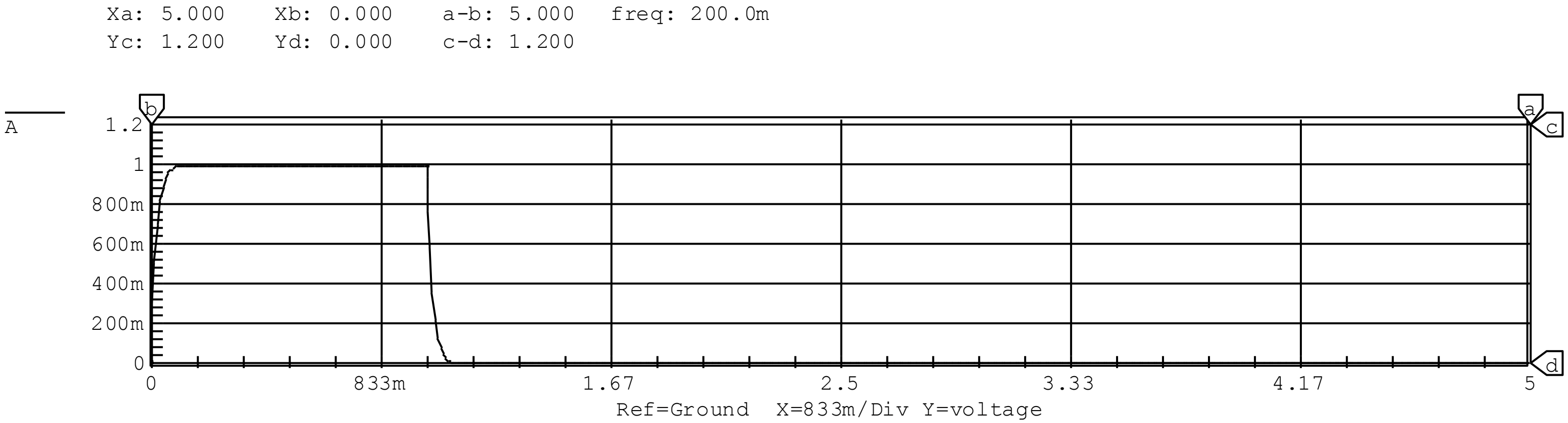
1/RC = 1



1/RC = 10



1/RC = 50



Note

Because a low pass filter tends to compress a signal in the frequency domain, it follows by the scaling property that it tends to broaden the extent of the signal in the time domain. As $1/RC$ decreases the filter removes more of the frequency components of e producing a greater compression in the frequency domain and correspondingly a greater broadening in the time domain. Again the simulations confirm this conclusion. This broadening is such that the time-domain signal starts to occupy several 1 sec blocks of time (about 3 of them for 1/RC = 1).

Note

Suppose the RC circuit models a digital communications channel. The *bandwidth* is a measure of the maximum frequency which the channel transmits or can adequately react to. In this case 1/RC gives a fair measure of this maximum frequency in the sense that this frequency is low if 1/RC is low and high if 1/RC is high. Accordingly 1/RC is also indicative of the bandwidth. The bandwidth is low if 1/RC is low and high if it is high. Digital communications is based upon transmitting symbols, commonly **0** or **1** in a binary system. What we actually transmit is a voltage signal. A voltage of approximately 3.5 V might correspond to or be interpreted as the symbol **1**.

Note

A voltage of approximately 0 V might correspond to or be interpreted as the symbol **0**. Of course we wish to send more than one symbol. One way of doing this is to set the transmit voltage to about 3.5 V for a short time in order to transmit a **1** and to set it to 0 V for a short time in order to transmit a **0**. This is essentially *time-division multiplexing*. Each symbol is allotted a certain time-frame which belongs to it alone. Obviously if the time-frame is long then we cannot transmit at a very high rate. For high rates of transmission of symbols we need rather sort allotted time-frames.

Note

If the allotted time-frame is 1 sec (a ridiculously large value I grant) then the Fourier transform is predicting (and simulation is confirming) that when the symbol **1** is transmitted (i.e. the transmit voltage is set to 1V for 1 sec) at the receiving end the received voltage never climbs as high as 1V when 1/RC is low but remains appreciably above 0V for several consecutive 1 sec time frames. In order that this received signal should not overlap with and corrupt the next received symbol I would need to leave a rather significant extra time-frame to allow the received signal to drop back to 0V.

Note

Two received symbols overlapping each other is called *inter-symbol interference*. The Fourier transform, with simulation confirming, is predicting that the likelihood of inter-symbol interference increases as the *bandwidth* of the system decreases for this communication channel. Indeed this is true of any communication channel. The number of symbols which I can transmit without inter-symbol interference (or other errors) being such as to compromise the received symbols to such an extent that I cannot with better than 50/50 odds predict what symbol was actually sent depends upon the bandwidth. In short the *channel capacity* depends upon the *channel bandwidth*.

Example 4.4: Amplitude Modulation

The efficiency of transmission and reception of an electromagnetic signal by an antenna of length l depends monotonically on l/λ, where λ is the wavelength of the signal. To receive an audio signal (max. frequency 20 kHz, min. wavelength 15 km approximately) one would require a very long antenna. The first viable solution to this technical problem was *amplitude modulation* (AM). Instead of transmitting the signal directly, it is used to turn up or down the amplitude of another signal called the *carrier*.

Example 4.4: Amplitude Modulation

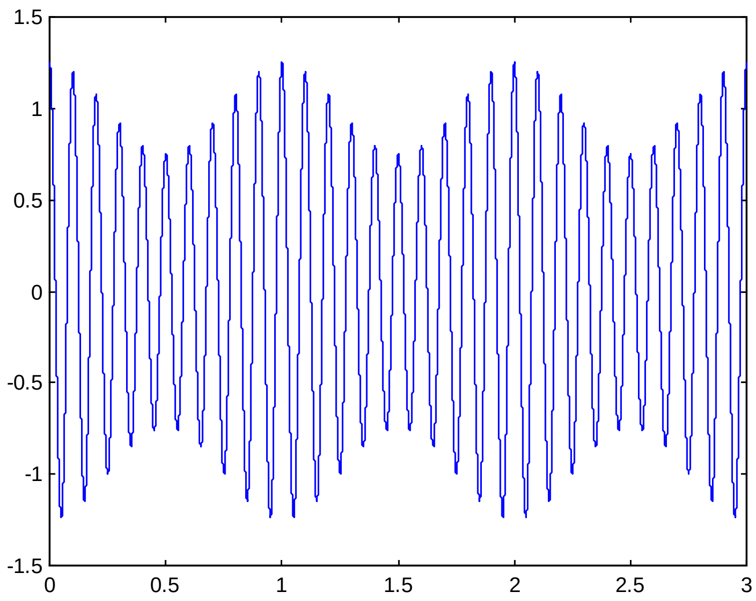
Given audio signal $s(t)$ transmit amplitude modulated signal $p(t)$.

$$p(t) = A(1 + ms(t))\cos(\omega_c t + \Phi_c)$$

e.g. $s(t) = \cos(2\pi t)$

$$m = 0.25$$

$$\omega_c = 20\pi, \quad \Phi_c = 0$$



Example 4.4: Amplitude Modulation

The constant m (sometimes called the modulation depth) is chosen such that the amplitude remains positive. The information signal $s(t)$ is not transmitted directly, but rather indirectly, as a modulation of the amplitude of a much higher frequency carrier signal. Amplitude Modulation was use for a very long time as the method of modulation for radio signals on the long wave (LW) band.

Example 4.4: AM Frequency Domain

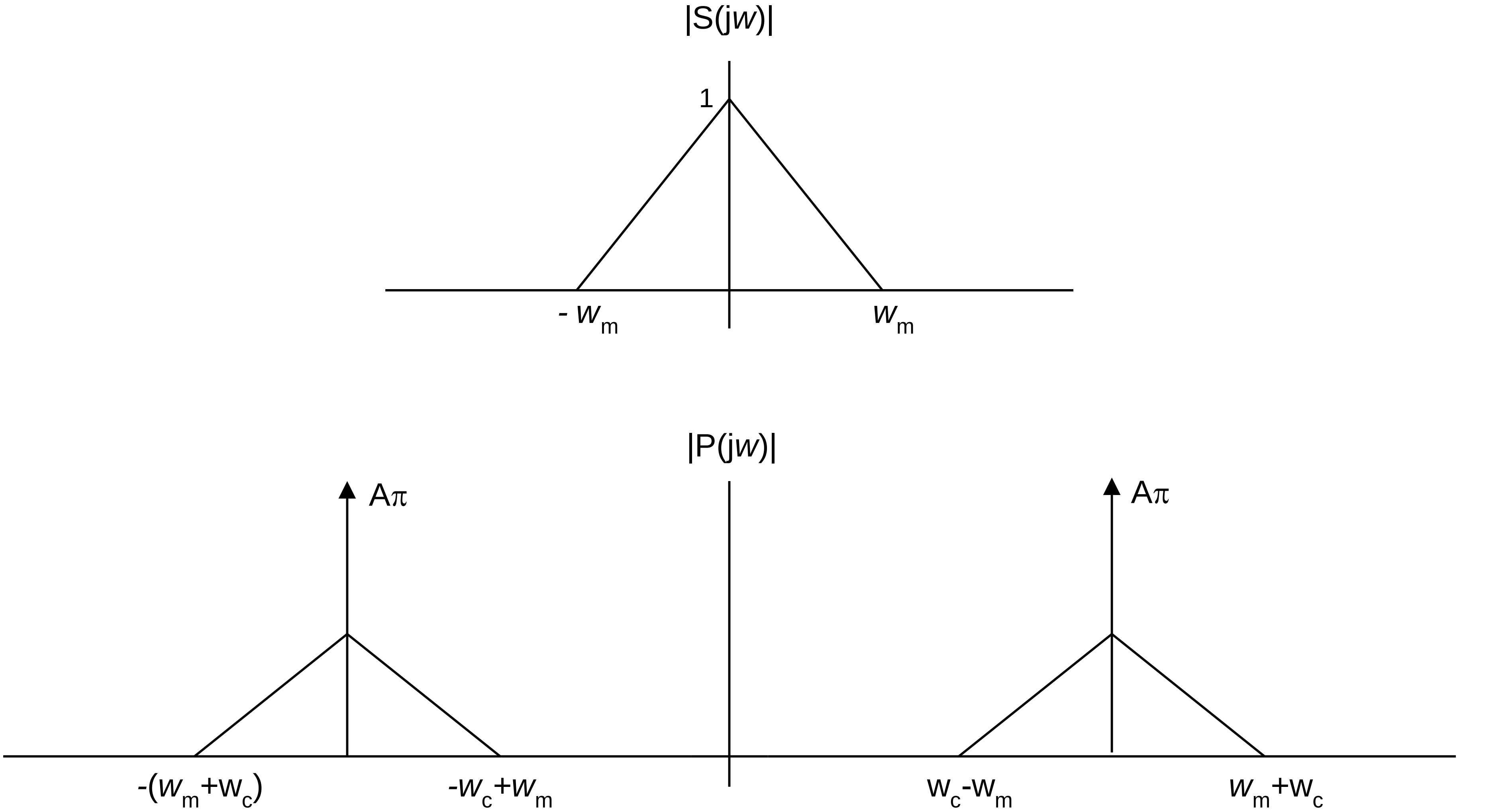
$$P(j\omega) =$$

$$A\pi \exp(j\Phi_c)\delta(\omega-\omega_c) + A\pi \exp(-j\Phi_c)\delta(\omega+\omega_c)$$

$$+ Am\frac{\exp(j\Phi_c)}{2}S(j(\omega-\omega_c))$$

$$+ Am\frac{\exp(-j\Phi_c)}{2}S(j(\omega+\omega_c))$$

Example 4.4: Amplitude Modulation



Note

The amplitude modulated signal clearly contains much higher frequencies than the audio signal, provided ω_c is much greater than ω_m. Different broadcasters can be assigned different carrier frequencies so that the signals which they transmit do not interfere with one another. AM therefore solved two problems for the price of one. It overcame the need to employ huge antennas (particularly on the receive side) and it provided a mechanism by which several signals could be broadcast simultaneously, namely *frequency-division multiplexing*. Pirate stations may broadcast in the gaps resulting in an overlap so that the listener hears a part of both transmissions and the efficacy of multiplexing is compromised.

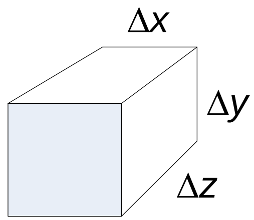
Example 4.5

From first law of thermodynamics:

$$\Delta(\text{heat in}) + \Delta(\text{work in}) + \Delta(\text{other energy conversion})$$
$$= \Delta(\text{heat out}) + \Delta(\text{work out}) + \Delta(\text{internal energy stored})$$

If no net work done:

$$\Delta(\text{work in}) = \Delta(\text{work out})$$

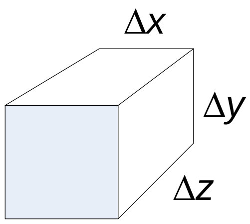


If no "other" energy conversion:

$$\Delta(\text{heat in}) = \Delta(\text{heat out}) + \Delta(\text{internal energy stored})$$

Let $\quad q_x = \dfrac{\Delta(\text{heat flow through } x\text{-face})}{\Delta t}$

Example 4.5



$$q_{x_1} + q_{y_1} + q_{z_1} = q_{x_1+\Delta x} + q_{y_1+\Delta y} + q_{z_1+\Delta z} + \frac{\Delta U}{\Delta t}$$

Conduction: $$q = -kA\frac{\partial T}{\partial n} \qquad q_x = -k\Delta y\Delta z\left(\frac{\partial T}{\partial x}\right)$$

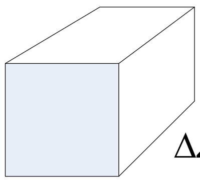
$$q_{x_1} - q_{x_1+\Delta x} = -\Delta y\Delta z\left(k\left(\frac{\partial T}{\partial x}\right)_{x_1} - k\left(\frac{\partial T}{\partial x}\right)_{x_1+\Delta x}\right)$$

$$= \Delta x\Delta y\Delta z\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)$$

$$\Delta x\Delta y\Delta z\left(\left(\frac{\partial}{\partial x}\left(k\left(\frac{\partial T}{\partial x}\right)\right)\right) + \left(\frac{\partial}{\partial y}\left(k\left(\frac{\partial T}{\partial y}\right)\right)\right) + \left(\frac{\partial}{\partial z}\left(k\left(\frac{\partial T}{\partial z}\right)\right)\right)\right) = \frac{\Delta U}{\Delta t}$$

Example 4.5

Assume k is constant:



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k \Delta x \Delta y \Delta z} \frac{\partial U}{\partial t}$$

$$U = (\Delta m) u = (\rho \Delta x \Delta y \Delta z) u$$

where u = internal energy per unit mass and ρ = density.

$c = \dfrac{\partial u}{\partial T}$ specific heat at constant volume (assuming incompressibility).

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{\rho}{k} \frac{\partial u}{\partial t} = \frac{\rho}{k} \frac{\partial u}{\partial T} \frac{\partial T}{\partial t} = \frac{\rho c}{k} \frac{\partial T}{\partial t}$$

Example 4.5

Conduction of heat in infinite one dimensional bar:

$$\frac{\partial T}{\partial t} = \frac{1}{\alpha}\frac{\partial^2 T}{\partial x^2} \qquad T(x,0) = f(x) \ , \ -\infty < x < \infty$$

$$T(x,t) = T_1(x)T_2(t) \qquad \frac{\frac{dT_2(t)}{dt}}{T_2(t)} = \frac{1}{\alpha}\frac{\frac{d^2T_1(x)}{dx^2}}{T_1(x)} = -p^2$$

$$\frac{dT_2(t)}{dt} + p^2 T_2(t) = 0 \ , \quad \frac{d^2T_1(x)}{dx^2} + p^2 \alpha T_1(x) = 0$$

Example 4.5

$$T_2(t) = A_2 \exp\left(-p^2 t\right)$$

$$T_1(x) = A_1 \sin\left(p\sqrt{\alpha}\, x\right) + B_1 \cos\left(p\sqrt{\alpha}\, x\right)$$

$$T(x,0) = f(x) =$$

$$\left(A_1 \sin\left(p\sqrt{\alpha}\, x\right) + B_1 \cos\left(p\sqrt{\alpha}\, x\right)\right) A_2 \exp\left(-p^2 t\right)$$

This only holds for very special $f(x)$ and the method appears to have failed. Again we employ linearity. We regard this method as having found special solutions and consider sums of such solutions.

Example 4.5

$$T(x,t) = \int_0^\infty T_{1p}(x) T_{2p}(t)\, dp =$$

$$\int_0^\infty A_{2p}\left(A_{1p}\sin\left(p\sqrt{\alpha}\,x\right) + B_{1p}\cos\left(p\sqrt{\alpha}\,x\right)\right)\exp\left(-p^2 t\right) dp$$

$$T(x,0) = f(x) = \int_0^\infty A_{2p}\left(A_{1p}\sin\left(p\sqrt{\alpha}\,x\right) + B_{1p}\cos\left(p\sqrt{\alpha}\,x\right)\right) dp$$

Method now works if $f(x)$ can be expressed as an integral of sinusoids, which sounds somewhat familiar but which we have not yet considered.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\exp(j\omega x)d\omega = \frac{1}{\pi}\int_{0}^{\infty} \mathrm{Re}(F(j\omega)\exp(j\omega x))d\omega =$$

$$\frac{1}{\pi}\int_{0}^{\infty} (\mathrm{Re}(F(j\omega))\cos(\omega x) - \mathrm{Im}(F(j\omega))\sin(\omega x))d\omega =$$

$$\frac{1}{\pi}\int_{0}^{\infty} ((A(\omega))\cos(\omega x) + (B(\omega))\sin(\omega x))d\omega$$

where $\quad F(j\omega) = \int_{-\infty}^{\infty} f(x)\exp(-j\omega x)dx$

$$A(\omega) = \mathrm{Re}(F(j\omega)) = \int_{-\infty}^{\infty} f(x)\cos(\omega x)dx$$

$$B(\omega) = -\mathrm{Im}(F(j\omega)) = \int_{-\infty}^{\infty} f(x)\sin(\omega x)dx$$

Example 4.5

$$T(x,0) = f(x) = \int_0^\infty \left(A_{1p}A_{2p}\sin\left(p\sqrt{\alpha}x\right) + B_{1p}A_{2p}\cos\left(p\sqrt{\alpha}x\right)\right)dp$$

$$= \int_0^\infty \left(B(p)\sin\left(p\sqrt{\alpha}x\right) + A(p)\cos\left(p\sqrt{\alpha}x\right)\right)dp \qquad \text{let } \omega = p\sqrt{\alpha}$$

$$= \frac{\pi}{\sqrt{\alpha}}\frac{1}{\pi}\int_0^\infty \left(B\left(\frac{\omega}{\sqrt{\alpha}}\right)\sin(\omega x) + A\left(\frac{\omega}{\sqrt{\alpha}}\right)\cos(\omega x)\right)d\omega$$

$$\frac{\pi}{\sqrt{\alpha}}A\left(\frac{\omega}{\sqrt{\alpha}}\right) = \int_{-\infty}^{\infty} f(x)\cos(\omega x)dx \quad , \quad \frac{\pi}{\sqrt{\alpha}}B\left(\frac{\omega}{\sqrt{\alpha}}\right) = \int_{-\infty}^{\infty} f(x)\sin(\omega x)dx$$

Example 4.5

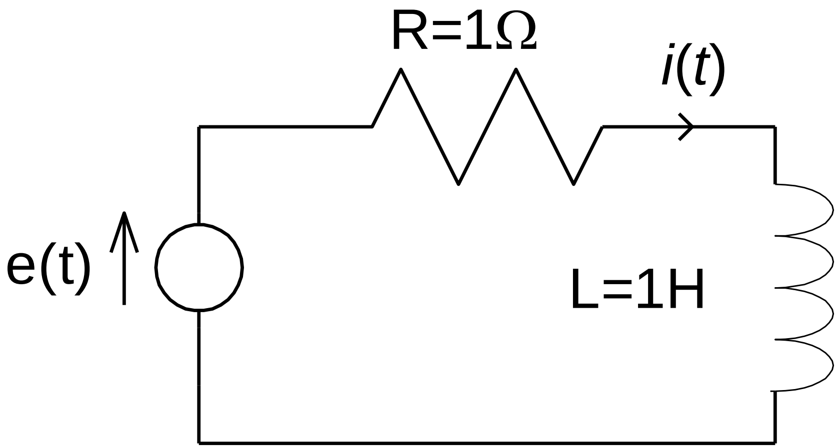
$$T(x,t)=\int_0^\infty\left(B(p)\sin\left(p\sqrt{\alpha}x\right)+A(p)\cos\left(p\sqrt{\alpha}x\right)\right)\exp\left(-p^2t\right)dp$$

$$=\frac{\pi}{\sqrt{\alpha}}\frac{1}{\pi}\int_0^\infty\left(B\left(\frac{\omega}{\sqrt{\alpha}}\right)\sin(\omega x)+A\left(\frac{\omega}{\sqrt{\alpha}}\right)\cos(\omega x)\right)\exp\left(-\frac{\omega^2t}{\alpha}\right)d\omega$$

$$\frac{\pi}{\sqrt{\alpha}}A\left(\frac{\omega}{\sqrt{\alpha}}\right)=\int_{-\infty}^{\infty}f(x)\cos(\omega x)dx \quad , \quad \frac{\pi}{\sqrt{\alpha}}B\left(\frac{\omega}{\sqrt{\alpha}}\right)=\int_{-\infty}^{\infty}f(x)\sin(\omega x)dx$$

Evaluating these integrals is equivalent to finding the Fourier transform of $f(x)$.

Phasor Analysis



$$e(t) - v_R(t) - v_L(t) = 0 \qquad \text{KVL}$$

$$v_L(t) = L\frac{d}{dt}i(t) \qquad \text{Henry's Law}$$

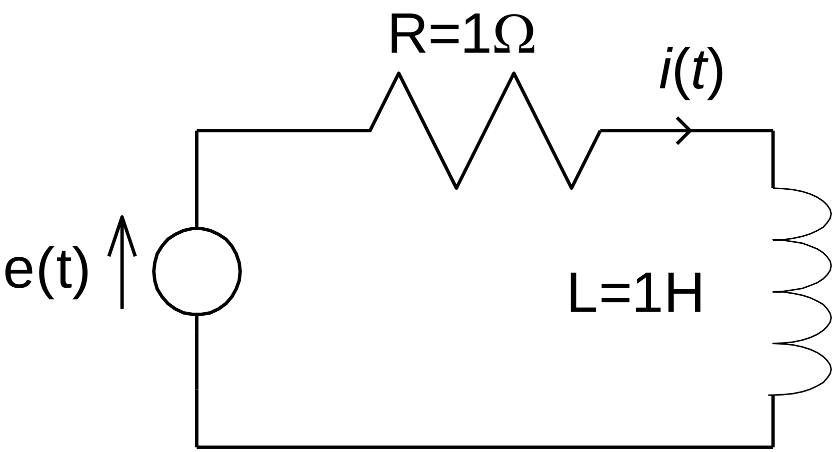
$$v_R(t) = Ri(t) \qquad \text{Ohm's Law}$$

$$L\frac{d}{dt}i(t) + Ri(t) = e(t) \qquad\qquad Lj\omega I(j\omega) + RI(j\omega) = E(j\omega)$$

$$(j\omega L + R)I(j\omega) = E(j\omega)$$

$$\frac{I(j\omega)}{E(j\omega)} = H(j\omega) = \frac{1}{(j\omega L + R)}$$

Phasor Analysis



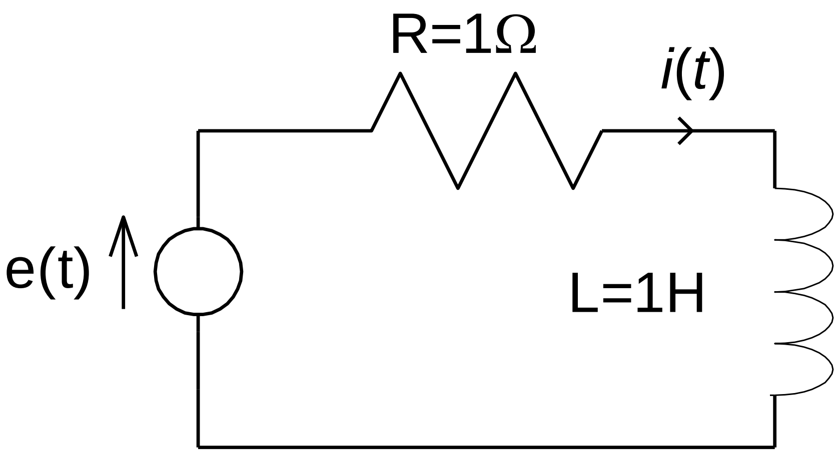
$$e(t) = A\cos(\omega_0 t + \Phi)$$

$$i(t) = A\left|H(j\omega_0)\right|\cos(\omega_0 t + \Phi + Arg(H(j\omega_0)))$$

$$H(j\omega_0) = \frac{1}{(j\omega_0 L + R)}$$

Phasor Analysis

First idea is a neat notation for sinusoidal functions:

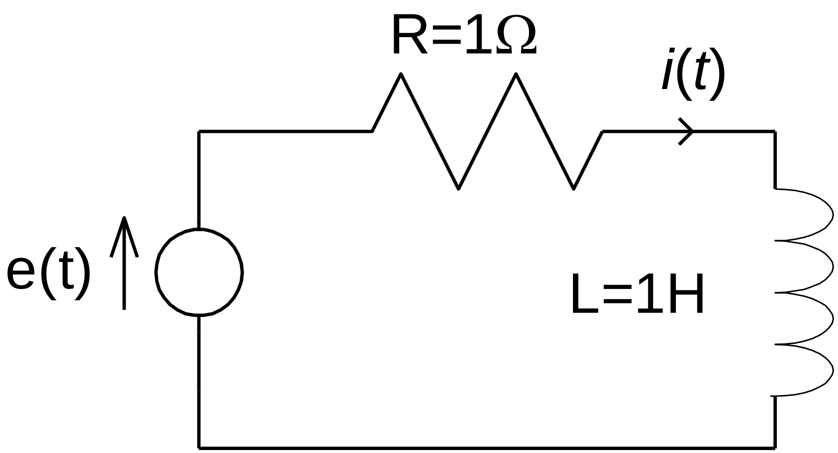


$$e(t) = A\cos(\omega_0 t + \Phi)$$
$$= \mathrm{Re}\left(Ae^{j(\omega_0 t + \Phi)}\right)$$
$$= \mathrm{Re}\left(Ae^{j\Phi}e^{j\omega_0 t}\right) = \mathrm{Re}\left(\dot{E}e^{j\omega_0 t}\right)$$

$$i(t) = \mathrm{Re}\left(A\left|H(j\omega_0)\right|e^{j(\omega_0 t + \Phi + Arg(H(j\omega_0)))}\right)$$
$$= \mathrm{Re}\left(Ae^{j(\omega_0 t + \Phi)}\left|H(j\omega_0)\right|e^{jArg(H(j\omega_0))}\right) = \mathrm{Re}\left(H(j\omega_0)Ae^{j(\omega_0 t + \Phi)}\right)$$
$$= \mathrm{Re}\left(H(j\omega_0)\dot{E}e^{j\omega_0 t}\right) = \mathrm{Re}\left(\dot{I}e^{j\omega_0 t}\right)$$

$$\dot{I} = H(j\omega_0)\dot{E}$$

Phasor Analysis



$$e(t) - v_R(t) - v_L(t) = 0 \qquad \text{KVL}$$

$$v_L(t) = L\frac{d}{dt}i(t) \qquad \text{Henry's Law}$$

$$v_R(t) = Ri(t) \qquad \text{Ohm's Law}$$

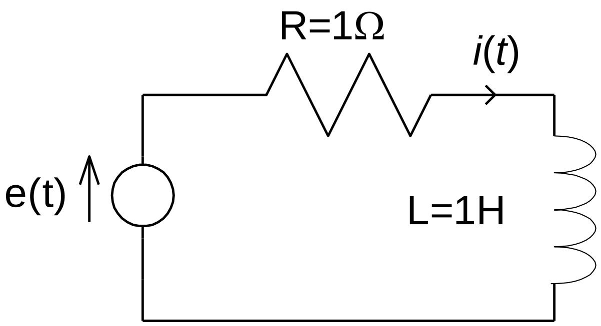
Transform to frequency domain *before* analysis is performed.

$$E(j\omega) - V_R(j\omega) - V_L(j\omega) = 0 \qquad \text{KVL}$$

$$V_L(j\omega) = j\omega L\, I(j\omega) \qquad \text{Henry's Law}$$

$$V_R(j\omega) = RI(j\omega) \qquad \text{Ohm's Law}$$

Phasor Analysis



$$E(j\omega) - V_R(j\omega) - V_L(j\omega) = 0 \qquad \text{KVL}$$

$$V_L(j\omega) = j\omega L\, I(j\omega) \qquad \text{Henry's Law}$$

$$V_R(j\omega) = R I(j\omega) \qquad \text{Ohm's Law}$$

Analyse as if input is DC voltage source with emf $E(j\omega)$ and as if circuit contains resistor R and resistor $j\omega L$.

$$(j\omega L + R) I(j\omega) = E(j\omega)$$

$$H(j\omega) = \frac{1}{(j\omega L + R)}$$

Phasor Analysis

The critical idea here is the capacity of the Fourier transform to change mathematical operations performed in the time-domain into different mathematical operations performed in the frequency-domain. This gives different *characterisations* of circuit elements. An resistor is characterised in the time-domain by:

$$v_L(t) = R\,i(t) \qquad \text{Ohm's Law}$$

In the frequency-domain the same resistor is characterised by:

$$V_L(j\omega) = R\,I(j\omega) \qquad \text{Ohm's Law}$$

Phasor Analysis

An inductor however is characterised in the time-domain by:

$$v_L(t) = L\frac{d\,i(t)}{dt} \qquad \text{Henry's Law}$$

In the frequency-domain the same inductor is characterised by:

$$V_L(j\omega) = j\omega L\, I(j\omega) \qquad \text{Henry's Law}$$

and we have a characterisation not substantially different from the characterisation of the resistor in this domain.

Phasor Analysis

The critical observation is that whereas in the time-domain the inductor is characterised by an equation clearly different from Ohm's law, in the frequency-domain the characterisation is essentially the same as Ohm's law, namely the voltage is something multiplied by the current. The something in question ($j\omega L$) is called the *impedance* (a term invented by Heaviside). So from a certain perspective (namely the frequency-domain perspective) there is really very little difference between a resistor and an inductor. Both are impedances but in the case of the resistor its impedance (R) does not change with frequency whereas in the case of the inductor its impedance ($j\omega L$) increases with frequency.