University College Dublin
An Coláiste Ollscoile, Baile Átha Cliath

SEMESTER 1 EXAMINATION 2014/2015

ACM 30030

Multivariable Calculus Eng II

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Time Allowed: 2 hours

Instructions for Candidates

Full marks will be awarded for complete answers to **three** questions.
At least one question must be submitted from both Sections A and B.

Instructions for Invigilators

Candidates are allowed to use non-programmable calculators during this examination.

SECTION A

1. (a) Let C be the parametric curve

$$\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}, \qquad 0 \leq t \leq 4\pi.$$

Determine the arc length of C.

(b) Let $P = (1, 1, 2)$ be a point on the ellipsoid $3x^2 + 2y^2 + z^2 = 9$.

i. Find a unit normal vector to the ellipsoid at the point P.

ii. Determine an equation of the tangent plane to the ellipsoid at the point P.

(c) Find the directional derivative of

$$f(x, y, z) = 3x^2 + 2xy + 3zx^3$$

at the point $(2, 3, 4)$ in the direction of the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

2. (a) Consider the line integral

$$\int_C ((y^2 + 2xy)\mathbf{i} + (x^2 + 2xy)\mathbf{j}) \cdot \mathbf{dr}$$

where C is a curve connecting the points $(-1, 2)$ and $(3, 1)$ in the (x,y) plane.

i. Show that this line integral is independent of the path.

ii. Evaluate the line integral.

(b) Evaluate the double integral

$$\iint_R (2x + y^2)dA$$

where R is the region in the (x, y) plane bounded by the curves $x = y^2$ and $x = y^3$.

(c) Evaluate, using Stokes Theorem, the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{dr}$$

where C is the (counterclockwise) circle $x^2 + y^2 = 9, z = -4$ for

$$\mathbf{F} = y\mathbf{i} + xz^3\mathbf{j} - zy^3\mathbf{k}$$

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3. (a) Let S be the parametric surface

$$\mathbf{r}(u, v) = (v - 1)\mathbf{i} + (2u + 3v)\mathbf{j} + (u - v)\mathbf{k}, \quad (u, v) \in R$$

where $R = \{(u, v) : 0 \leq u \leq 1, 0 \leq v \leq 4\ \}$. Find the surface area of S by evaluating the surface integral

$$\iint_S 1 dS$$

(b) Use the Divergence theorem to evaluate the surface integral

$$\iint_S ((x^3 + \tan(yz))\mathbf{i} + (y^3 - e^{xz})\mathbf{j} + (3z + x^3)\mathbf{k}) \cdot \mathbf{n} dS.$$

where S is the surface of the solid that is bounded by the cylinder $x^2+y^2 = 4$ and the planes $z = 0$ and $z = 3$. Assume that S has an outward orientation.

SECTION B

4. If $f(x)$ is a periodic function of x of period L. then the Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{2\pi nx}{L} + b_n \sin\frac{2\pi nx}{L})$$

where

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x) \cos\frac{2\pi nx}{L} dx$$

$$b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x) \sin\frac{2\pi nx}{L}\ dx.$$

Let the periodic function f be defined by

$$f\ (x) = \begin{cases} 2, & -2 < x \leq 0 \\ x, & 0 < x \leq 2. \end{cases}$$

and $f(x + 4) = f(x)$.

(a) Sketch the function $f(x)$ over two periods.

(b) Determine the Fourier series representation for $f(x)$.

(c) What is the value of the series at $x = 0$.

(d) Use parts (b) and (c) to show that

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

5. Solve the diffusion equation

$$\frac{\partial u}{\partial t} = 50\frac{\partial^2 u}{\partial x^2}$$

subject to the boundary conditions

$$u(0,t) = 0$$
$$u(\pi,t) = 0$$

and initial conditions

$$u(x,0) = \sin(3x) - \sin(7x), \quad 0 \le x \le \pi$$

6. Solve Laplaces equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for $0 < x < \pi$, $0 < y < \pi$. With boundary conditions

$$u(0,y) = u(x,0) = u(x,\pi) = 0, u(\pi,y) = \cos y$$

Note: You can assume the Fourier sine series

$$\cos r = \frac{4}{\pi}\left(\frac{1}{2^2-1}\sin 2r + \frac{1}{4^2-1}\sin 4r + \frac{1}{6^2-1}\sin 6r + \cdots\right), \quad 0 < r < \pi$$

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Formulæ in the Differential and Integral Calculus

Derivatives

y	dy/dx	y	dy/dx	y	dy/dx
x^n	nx^{n-1}	$\sec x$	$\tan x \sec x$ $= \sin x / \cos^2 x$	$\sec^{-1} \frac{x}{a}$	$\frac{a}{x\sqrt{x^2-a^2}}$
$\sin x$	$\cos x$	$\sin^{-1} \frac{x}{a}$	$\frac{1}{\sqrt{a^2-x^2}}$	e^x	e^x
$\cos x$	$-\sin x$	$\cos^{-1} \frac{x}{a}$	$-\frac{1}{\sqrt{a^2-x^2}}$	e^{ax}	ae^{ax}
$\tan x$	$\sec^2 x$	$\tan^{-1} \frac{x}{a}$	$\frac{a}{a^2+x^2}$	a^x	$a^x \ln a$
$\cot x$	$-\csc^2 x$	$\cot^{-1} \frac{x}{a}$	$-\frac{a}{a^2+x^2}$	$\ln x$	$\frac{1}{x}$
$\csc x$	$-\cot x \csc x$ $= -\cos x / \sin^2 x$	$\csc^{-1} \frac{x}{a}$	$-\frac{a}{x\sqrt{x^2-a^2}}$		

Integrals

y	$\int y dx$	y	$\int y dx$	y	$\int y dx$
x^n	$\frac{x^{n+1}}{n+1} \quad n \neq -1$	$\cot x \csc x$	$-\csc x$	$\frac{1}{x}$	$\ln x$
$\sin x$	$-\cos x$	$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a}$ or $-\cos^{-1} \frac{x}{a}$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\frac{a}{a^2+x^2}$	$\tan^{-1} \frac{x}{a}$ or $-\cot^{-1} \frac{x}{a}$	$\text{sech}^2 x$	$\tanh x$
$\sec^2 x$	$\tan x$	$\frac{a}{x\sqrt{x^2-a^2}}$	$\sec^{-1} \frac{x}{a}$ or $-\csc^{-1} \frac{x}{a}$	$\frac{1}{\sqrt{x^2+a^2}}$	$\sinh^{-1} \frac{x}{a}$ $= \ln \frac{x+\sqrt{x^2+a^2}}{a}$
$\csc^2 x$	$-\cot x$	e^{ax}	$\frac{e^{ax}}{a}$	$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1} \frac{x}{a}$ $= \ln \frac{x+\sqrt{x^2-a^2}}{a}$
$\tan x \sec x$	$\sec x$	a^x	$\frac{a^x}{\ln a}$	$\frac{1}{a^2-x^2}$	$\frac{1}{a} \tanh^{-1} \frac{x}{a}$ $= \frac{1}{2a} \ln \frac{a+x}{a-x}$

Other Formulæ

Derivative of Product: $y = uv, \quad \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$.
Derivative of Quotient: $y = u/v, \quad \frac{dy}{dx} = \left(v\frac{du}{dx} - u\frac{dv}{dx}\right)/v^2$.
Integration by Parts: $\int u dv = uv - \int v du$.
Binomial Theorem: $(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{1\cdot 2}x^2 \pm \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \ldots$
Maclaurin's Theorem: $f(x) = f(0) + xf'(0) + \frac{x^2}{1\cdot 2}f''(0) + \ldots$
Taylor's Theorem: $f(x+h) = f(x) + hf'(x) + \frac{h^2}{1\cdot 2}f''(x) + \ldots$