1.

(a) $\mathbf{A} - \mathbf{B} + 3\mathbf{C} = -\hat{x} + 5\hat{y} + 16\hat{z}$

(b) $|\mathbf{A} - \mathbf{B} + 3\mathbf{C}| = \sqrt{(-1)^2 + 5^2 + 16^2} = 16.79$

(c) $\mathbf{A} + 2\mathbf{B} - \mathbf{C} = (3\hat{x} - 3\hat{y} + 3\hat{z}) + 2(\hat{x} - 2\hat{y} - \hat{z}) - (-\hat{x} + 2\hat{y} + 4\hat{z}) = 6\hat{x} - 9\hat{y} - 3\hat{z}$

$$|\mathbf{A} + \mathbf{2B} - \mathbf{C}| = \sqrt{6^2 + (-9)^2 + (-3)^2} = 11.23$$

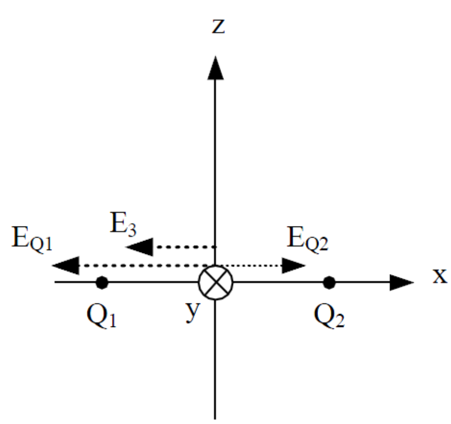
$$\hat{u} = \frac{\mathbf{A} + \mathbf{2B} - \mathbf{C}}{|\mathbf{A} + \mathbf{2B} - \mathbf{C}|} = \frac{6\hat{x} - 9\hat{y} - 3\hat{z}}{11.23} = 0.53\hat{x} - 0.80\hat{y} - 0.27\hat{z}$$

(d) $\mathbf{A} \cdot \mathbf{B} = (3\hat{x} - 3\hat{y} + 3\hat{z}) \cdot (\hat{x} - 2\hat{y} - \hat{z}) = 3 \times 1 + (-3) \times (-2) + 3 \times (-1) = 6$

(e) $\mathbf{B} \times \mathbf{C} = \left(\hat{x} - 2\hat{y} - \hat{z}\right) \times (-\hat{x} + 2\hat{y} + 4\hat{z}) = -6\hat{x} - 3\hat{y}$

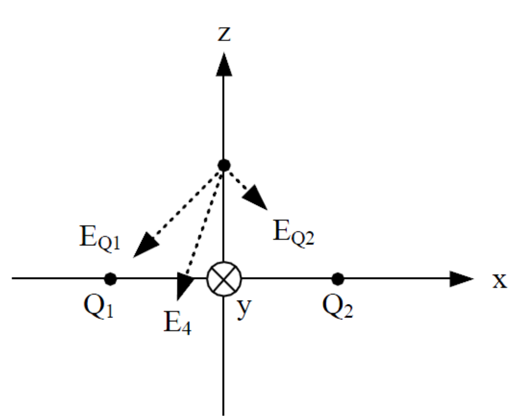
2. We know $Q_1 = -8\pi\varepsilon_0$, so $Q_2 = Q_1/2 = -4\pi\varepsilon_0$. The electric field at the point P_3 is the superposition of those induced by Q_1 and Q_2, namely

$$\mathbf{E}_3 = \sum_{i=1}^{2} \frac{Q_i}{4\pi\varepsilon_0|\mathbf{r}_3 - \mathbf{r}_i|^2} \cdot \frac{\mathbf{r}_3 - \mathbf{r}_i}{|\mathbf{r}_3 - \mathbf{r}_i|} = \frac{-2}{|\hat{x}|^3}(0 - (-\hat{x})) - \frac{1}{|-\hat{x}|^3}(0 - \hat{x}) = -\hat{x} \text{ (V/m)}$$



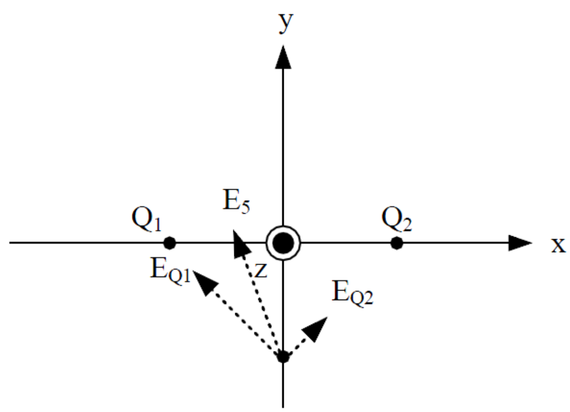
The field at the point P_4 can be obtained in a similar way

$$\mathbf{E}_4 = \sum_{i=1}^{2} \frac{Q_i}{4\pi\varepsilon_0|\mathbf{r}_4 - \mathbf{r}_i|^2} \cdot \frac{\mathbf{r}_4 - \mathbf{r}_i}{|\mathbf{r}_4 - \mathbf{r}_i|} = \frac{-2}{|\hat{x} + \hat{z}|^3}(\hat{x} + \hat{z}) - \frac{1}{|-\hat{x} + \hat{z}|^3}(-\hat{x} + \hat{z}) = \frac{-\hat{x} - 3\hat{z}}{2\sqrt{2}} \text{ (V/m)}$$



The field at the point P_5 can be obtained in a similar way

$$\mathbf{E}_5 = \sum_{i=1}^{2} \frac{Q_i}{4\pi\varepsilon_0 |\mathbf{r}_5 - \mathbf{r}_i|^2} \cdot \frac{\mathbf{r}_5 - \mathbf{r}_i}{|\mathbf{r}_5 - \mathbf{r}_i|} = \frac{-2}{|\hat{x} - \hat{y}|^3} (\hat{x} - \hat{y}) - \frac{1}{|-\hat{x} - \hat{y}|^3} (-\hat{x} - \hat{y}) = \frac{-\hat{x} + 3\hat{y}}{2\sqrt{2}} \text{ (V/m)}$$



3. In the three cases, we have the same $\mathbf{E}$ and $\mathbf{B}$ at the origin, but different particle velocities $\mathbf{v}$. Thus, we have three different Lorentz force vectors $\mathbf{F}$. In each case, $\mathbf{v}$ and $\mathbf{F}$ must satisfy $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Therefore, we have the following three equations:

   $2\hat{y} = \mathbf{E}$

   $0 = \mathbf{E} + (-2\hat{z}) \times \mathbf{B}$

   $2\hat{y} - \hat{z} = \mathbf{E} + \hat{y} \times \mathbf{B}$

   From which we obtain

   $\mathbf{E} = 2\hat{y}$

   $-2\hat{z} \times \mathbf{B} = -2\hat{y}$

   $\hat{y} \times \mathbf{B} = -\hat{z}$

   If we assume $\mathbf{B} = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$, then

   $-2\hat{z} \times (\hat{x}B_x + \hat{y}B_y + \hat{z}B_z) = -2\hat{y}$

   $\hat{y} \times (\hat{x}B_x + \hat{y}B_y + \hat{z}B_z) = -\hat{z}$

   Further

   $\hat{y}(-2)B_x - \hat{x}(-2)B_y = -2\hat{y}$

   $-\hat{z}B_x + \hat{x}B_z = -\hat{z}$

   Therefore, we have

   $B_y = B_z = 0$

   $B_x = 1$

   In summary, $\mathbf{E} = 2\hat{y}$ (V/m) and $\mathbf{B} = \hat{x}$ $\left(\text{Wb/m}^2\right)$.

4.

(a) In the first case $z = 0.5\,\mathrm{m}$. Therefore $\sin(\pi z) = 1$, $\cos(\pi z) = 0$ and the electric and magnetic fields are

$$\mathbf{E} = \hat{y}\,60\pi\cos(\omega t)\ \ \mathrm{V/m},$$

and

$$\mathbf{B} = 0.$$

The total force on the particle is calculated as

$$\mathrm{F} = \mathrm{q}\,(\mathbf{E} + \mathbf{v}\times\mathbf{B}) = \hat{y}\,60\pi q\cos(\omega t).$$

The acceleration of the particle is found from Newton's law of motion $\mathbf{F} = m\mathbf{a}$, yielding

$$\mathbf{a}(t) = \hat{y}\frac{60\pi q\cos(\omega t)}{m},$$

and its velocity is

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{v}(0) + \int \hat{y}\frac{60\pi q\cos(\omega t)}{m}dt \\ &= \hat{y}\,100 + \hat{y}\,\frac{60\pi q}{m\omega}\sin(\omega t) \\ &= \hat{y}\,100 - \hat{y}\,1.922\times10^{-10}\sin(\omega t)\ \ \mathrm{m/s},\end{aligned}$$

which finally yields the position

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}(0) + \int \mathbf{v}(t)dt \\ &= \hat{z}\,0.5 + \hat{y}\,100t + \hat{y}\,1.224\times10^{-19}\cos(\omega t)\ \ \mathrm{m}.\end{aligned}$$

The particle travels in the $\hat{y}$ direction with a small periodic acceleration.

(b) At $z = 0.75$ m, both the electric and magnetic fields will exert Lorentz forces on the charged particle as follows:

$$\mathbf{F} = q\left(\hat{y}\,60\pi\sin(\frac{3}{4}\pi)\cos(\omega t) + \mathbf{v}\times\hat{x}(2\pi\times10^{-7})\cos(\frac{3}{4}\pi)\sin(\omega t)\right)$$

With an initial velocity in the $\hat{y}$ direction, the particle is subject to both electric Lorentz force that oscillates in the $\pm\hat{y}$ direction as described above, along with a magnetic Lorentz force initially in the $-\hat{z}$ direction (since $q < 0$ and $\cos(\frac{3}{4}\pi) < 0$ and $\hat{y}\times\hat{x} = -\hat{z}$), which causes the particle trajectory to bend slightly towards $-\hat{z}$. Since the magnetic Lorentz force is always perpendicular to the direction of particle motion, however, the force continually changes direction with changing particle motion in response to the combined forces. Overall, the particle orbits counterclockwise around the direction of positive $\mathbf{B}$ (which in turn oscillates as a function of time) while it accelerates periodically in the $-\hat{y}$ direction.

5.

(a) From Newton's force law, the acceleration of a particle is $\mathbf{a} = \frac{\mathbf{F}}{\mathrm{m}} = \frac{\mathrm{q}}{\mathrm{m}}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\ \mathrm{m/s^2}$. Thus, the acceleration of an electron in a static electric field $\mathbf{E} = E_0\hat{z}$ [V/m] is along the $-\hat{z}$ direction as follows:

$$\mathbf{a_e} = \frac{-eE_0}{m_e}\hat{z}$$

while an ion experiences an acceleration in the $+\hat{z}$ direction as follows:

$$\mathbf{a_i} = \frac{eE_0}{m_i}\hat{z}$$

The top panel in the figure depicts the resulting particle motions, which are in opposite directions.

(b) From dimensional analysis, we can see that volumetric current density (in $\mathrm{A/m^2}$) is the product of volumetric charge density (in $\mathrm{C/m^3}$) and velocity (in m/s), since an Ampere is a Couloumb per second:

$$\frac{\mathrm{A}}{\mathrm{m}^2} = \frac{\mathrm{C}}{\mathrm{m}^3}\frac{\mathrm{m}}{\mathrm{s}}$$

So we can derive the volumetric current density carried by the ions and electrons as:

$$\mathbf{J}_\mathrm{e} = \rho_\mathrm{e}\mathbf{v}_\mathrm{e} = -eN_e(-v_e\hat{z}) = eN_ev_e\hat{z}$$

$$\mathbf{J}_\mathrm{i} = \rho_\mathrm{i}\mathbf{v}_\mathrm{i} = eN_iv_i\hat{z}$$

$$\mathbf{J} = \mathbf{J}_\mathrm{e} + \mathbf{J}_\mathrm{i} = (eN_ev_e + eN_iv_i)\hat{z}$$

Thus, although the electrons move in the opposite direction to the ions, the current carried by both species is in the same direction, along $+\hat{z}$.

(c) At $\mathrm{t} = \mathrm{t}_0$, an electron will experience a Lorentz force that includes a magnetic force term:

$$\mathbf{F_e} = -e[E_0\hat{\mathrm{z}} + (-v_\mathrm{e}\hat{\mathrm{z}}) \times (B_0\hat{\mathrm{x}})] = -e[E_0\hat{\mathrm{z}} - v_\mathrm{e}B_0\hat{\mathrm{y}})]$$

such that the resulting electron acceleration is

$$\mathbf{a}_e = -\frac{eE_0}{m_e}\hat{z} + \frac{ev_\mathrm{e}B_0}{m_e}\hat{y}$$

with a magnitude $|\mathbf{a_e}| = \sqrt{(\frac{e}{m_e})^2(E_0^2 + v_e^2B_0^2)}$ along the unit direction

$$\mathbf{u_e} = \frac{\mathbf{a}_e}{|\mathbf{a}_e|} = \frac{-\hat{z} + v_eB_0\hat{y}}{\sqrt{E_0^2 + v_e^2B_0^2}}$$

As for an ion, it will experience a Lorentz force given by

$$\mathbf{F_i} = e[E_0\hat{\mathrm{z}} + (v_\mathrm{i}\hat{\mathrm{z}}) \times (B_0\hat{\mathrm{x}})] = e[\mathrm{E}_0\hat{\mathrm{z}} + v_\mathrm{i}B_0\hat{\mathrm{y}})]$$

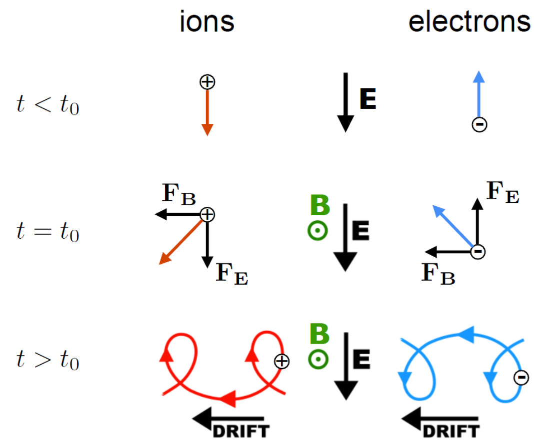
and resulting acceleration

$$\mathbf{a}_i = \frac{eE_0}{m_i}\hat{z} + \frac{ev_iB_0}{m_i}\hat{y}$$

along the unit direction given by:

$$\mathbf{u_i} = \frac{\mathbf{a}_i}{|\mathbf{a}_i|} = \frac{\hat{z} + v_iB_0\hat{y}}{\sqrt{E_0^2 + v_i^2B_0^2}}$$

Since the ions and the electrons are initially moving in opposite directions in response to the electric Lorentz force, the magnetic Lorentz force is in the same direction (along $\hat{y}$) for both species (see the second panel in the figure). Since magnetic fields do no work (the magnetic Lorentz force is perpendicular to the direction of particle motion), the magnetic field cannot increase the particle velocities relative to one another and thus cannot increase the magnitude of current density, which is at a maximum under electric Lorentz force at time $t > t_0$. Instead, the addition of the magnetic field at $t = t_0$ serves to decrease the current density, since the particles now have a component of acceleration in the same direction as one another. The magnetic force causes the charges to orbit the magnetic field for $t > t_0$, and the combined electric and magnetic forces are parallel during one portion of the orbital motion around **B** (giving a larger velocity and thus larger orbital radius) and antiparallel on the other side (giving a smaller velocity and thus smaller orbital radius). As a result, in steady state, the particles will drift in a direction perpendicular to both **E** and **B** at equal speeds, resulting in no net current (see the last panel in the figure).



(a) If the magnetic field is in the $\hat{z}$ direction, then the magnetic field will not exert any force on the electrons and ions because $\hat{z} \times \hat{z} = 0$. The current density will not be affected by this magnetic field.

6. Let us use S_1, S_2, S_3, S_4, S_5 and S_6 to denote the surfaces $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$, respectively. We consider S_1 and S_2 first. The unit vector on S_1 pointing away from the volume is along $-\widehat{x}$ direction, so we have

   (a) $\int_{S_1} \mathbf{J} \cdot d\mathbf{S} = \int_0^1 \int_0^1 \left[x^2 y\,(\hat{x} + \hat{y} + \hat{z})\right] |_{x=0} \cdot (-\hat{x})\, dydz = 0$
   For S_2, the unit vector pointing away from the volume is along $+\widehat{x}$ direction, therefore
   $\int_{S_2} \mathbf{J} \cdot d\mathbf{S} = \int_0^1 \int_0^1 \left[x^2 y\,(\hat{x} + \hat{y} + \hat{z})\right] |_{x=1} \cdot (+\hat{x})\, dydz = \frac{1}{2}$
   Similarly, for S_3, S_4, S_5 and S_6, we can obtain
   $\int_{S_3} \mathbf{J} \cdot d\mathbf{S} = \int_0^1 \int_0^1 \left[x^2 y\,(\hat{x} + \hat{y} + \hat{z})\right] |_{y=0} \cdot (-\hat{y})\, dxdz = 0$
   $\int_{S_4} \mathbf{J} \cdot d\mathbf{S} = \int_0^1 \int_0^1 \left[x^2 y\,(\hat{x} + \hat{y} + \hat{z})\right] |_{y=1} \cdot (+\hat{y})\, dxdz = \frac{1}{3}$
   $\int_{S_5} \mathbf{J} \cdot d\mathbf{S} = \int_0^1 \int_0^1 \left[x^2 y\,(\hat{x} + \hat{y} + \hat{z})\right] |_{z=0} \cdot (-\hat{z})\, dxdy = -\frac{1}{6}$
   $\int_{S_6} \mathbf{J} \cdot d\mathbf{S} = \int_0^1 \int_0^1 \left[x^2 y\,(\hat{x} + \hat{y} + \hat{z})\right] |_{z=1} \cdot (+\hat{z})\, dxdy = \frac{1}{6}$
   Thereafter,
   $\oint_S \mathbf{J} \cdot d\mathbf{S} = \sum_{i=1}^{6} \int_{S_i} \mathbf{J} \cdot d\mathbf{S} = \frac{5}{6}$ (A)

   (b) Since the current flux through the closed surface is positive, the positive current flux must be leaving the volume. This positive current flux can be carried by positive charges flowing outwards through the closed surface or by negative charges flowing inwards. Regardless, the total electrical charge inside the volume must be decreasing.

   (c) Gauss's law $\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \iiint_V \rho dV$ indicates that the total electric flux flowing out from a volume V surrounded by a surface S, $\oint_S \mathbf{E} \cdot d\mathbf{S}$ is proportional to the total charge contained by the volume, i.e. $\iiint_V \rho dV$. Hence, if the total charge contained in a volume V decreases, then the electric field flux out of the same closed surface S will decrease.