Application of Fourier Analysis to Partial Differential Equations.

Aside

2nd Order linear differential equations.

A differential equation relates a function to one or more of its derivatives. e.g.

$$\text{(i)} \quad \frac{dy}{dx} = \lambda y \qquad \lambda \text{ constant}$$

$$\text{(ii)} \quad a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

a, b, c constant

(i) Describes exponential growth. A 1st order equation
A solution is $y = 10e^{\lambda x}$

A General Solution is $y = Ce^{\lambda x}$

(ii) If a, b, c all positive ODE describes a damped linear oscillator. A second order equation

Linear odes

A linear ode includes only terms linear in y and its derivatives.

Ex: $y\frac{dy}{dx} = 10$ nonlinear

$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + y = 0$ linear

$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^2 + y = 0$ nonlinear.

2nd Order, Constant Coefficient, linear equations

Consider

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

General form of the solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where y_1, y_2 are two independent (not proportional to each other) solutions and c_1, c_2 are arbitrary constants.

To determine $y_1(x)$, $y_2(x)$ look for solutions of form

$$y(x) = e^{\lambda x}, \quad \lambda \text{ to be determined.}$$

Substitute $e^{\lambda x}$ into $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

$$a(\lambda^2 e^{\lambda x}) + b\lambda e^{\lambda x} + ce^{\lambda x} = 0$$

$$(a\lambda^2 + b\lambda + c)\underbrace{e^{\lambda x}}_{+ve} = 0$$

$$\Rightarrow \quad a\lambda^2 + b\lambda + c = 0$$ Auxiliary Equation.

Solution of auxiliary equation gives λ values for $y_{1,2} = e^{\lambda x}$

Examples

1. $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0$$

Auxiliary equation $\lambda^2 + 3\lambda + 1 = 0$
(can't factrise)

Use $$\lambda = \frac{-3 \pm \sqrt{9-4}}{2} = -\frac{3}{2} \pm \frac{\sqrt{5}}{2}$$

Two solutions
$$y_1 = e^{\left(\frac{-3}{2} + \frac{\sqrt{5}}{2}\right)x}$$
$$y_2 = e^{\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)x}$$

General solution
$$y(x) = e^{\frac{-3}{2}x}\left(c_1 e^{\frac{\sqrt{5}}{2}x} + c_2 e^{-\frac{\sqrt{5}}{2}x}\right)$$

Sometimes auxiliary equation has complex roots.

Example (i)
$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 25y = 0$$

Auxiliary Equation
$$\lambda^2 + 8\lambda + 25 = 0$$
$$\lambda = \frac{-8 \pm \sqrt{64 - 100}}{2}$$
$$= -4 \pm 3i \qquad i = \sqrt{-1}$$

General solution is
$$y(x) = e^{-4x}\left(c_1 e^{3ix} + c_2 e^{-3ix}\right) \qquad \text{complex form?}$$

Recall Euler's formula
$$e^{i\theta} = \cos\theta + i\sin\theta$$

Write

$$y(x) = e^{-4x}\left[\begin{array}{l} c_1(\cos 3x + i\sin 3x) \\ + c_2(\cos(-3x) + i\sin(-3x)) \end{array}\right]$$

$$= e^{-4x}\left[(c_1+c_2)\cos 3x + i(c_1-c_2)\sin 3x\right]$$

$$= e^{-4x}\left[A\cos 3x + B\sin 3x\right]$$

where A, B are arbitrary constants

Note $\lambda = -4 \pm 3i$

Example (ii)

$$\frac{d^2y}{dx^2} + 4y = 0$$

Auxiliary Equation

$$\lambda^2 + 4 = 0$$

or

$$\lambda = 0 \pm 2i$$

General solution

$$y(x) = A\cos 2x + B\sin 2x$$

A third possibility is that the auxiliary equation has two equal roots.

Example: $$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

Auxiliary equation $$\lambda^2 + 6\lambda + 9 = 0$$
$$(\lambda+3)^2 = 0 \Rightarrow \lambda = -3$$

$y_1(x) = e^{-3x}$ is <u>one</u> solution

(method fails to give two solutions)

It can be shown that a second independent solution can be written in the form
$$y_2(x) = x\,y_1(x) = x e^{-3x}$$

General solution is
$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x}$$
$$= e^{-3x}(c_1 + c_2 x)$$

Summary

To solve $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$

General solution $y = C_1 y_1(x) + C_2 y_2(x)$

Auxiliary equation $a\lambda^2 + b\lambda + c = 0$

Three cases

- Two real roots $\lambda_1, \lambda_2 \in \mathbb{R}$

  $$y_1 = e^{\lambda_1 x}, \qquad y_2 = e^{\lambda_2 x}$$

- Two complex conjugate roots

  $$\lambda_1 = a + ib$$
  $$\lambda_2 = a - ib$$

  $$y_1 = e^{ax}\cos bx, \quad y_2 = e^{ax}\sin bx$$

- One (double) root $\lambda \in \mathbb{R}$.

  $$y_1 = e^{\lambda x}, \qquad y_2 = x e^{\lambda x}$$