University College Dublin
An Coláiste Ollscoile, Baile Átha Cliath

SEMESTER 1 EXAMINATION 2013/2014

ACM 30030

Multivariable Calculus Eng II

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Time Allowed: 2 hours

Instructions for Candidates

Full marks will be awarded for complete answers to **three** questions.

Instructions for Invigilators

Candidates are allowed to use non-programmable calculators during this examination.

1. (a) Let C be the parametric curve

$$\mathbf{r}(t) = \frac{t^3}{3}\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}$$

determine the arc length of C between the points $(0,0,0)$ and $(\frac{1}{3},1,2)$.

(b) Determine the equation of the plane that is tangent to the sphere

$$x^2 + y^2 + z^2 = 1$$

at the point $(0,1,0)$. Sketch the sphere and tangent plane on the same diagram.

(c) Find the directional derivative of

$$f(x,y,z) = 2xz - y^3$$

at the point $(1,2,3)$ in the direction $(2,-4,4)$.

(d) Find the curl for the vector field

$$F(x,y,z) = x^3y^2z\mathbf{i} + xy^3z^2\mathbf{j} + x^2yz^3\mathbf{k}.$$

2. (a) Evaluate the double integral

$$\iint_R (x^2 - \frac{y}{2})\, dx\, dy$$

where R is the region in the first quadrant that is enclosed by the curves $y = x, y = x^3$.

(b) State carefully Green's Theorem and use it to evaluate the line integral

$$\int_C ((e^x + x^2y)\mathbf{i} + (e^y - xy^2)\mathbf{j}) \cdot \mathbf{dr}$$

where C is the circle $x^2 + y^2 = 25$ with clockwise rotation.

3. (a) Evaluate the integral

$$\iint_S \sqrt{1 + x^2 + y^2}\, ds$$

where S is the parametric surface

$$S : r(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + v\mathbf{k}, \quad (u,v) \in R$$

where $R = \{(u,v) \mid 0 \le u \le 1,\ 0 \le v \le \pi\}$.

(b) Let $E = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 2\}$ be a solid cube in three dimensional space and let S be the boundary surface of E that has outward orientation. Use the Divergence theorem to evaluate the surface integral

$$\iint_S (e^x \sin y\mathbf{i} + e^x \cos y\mathbf{j} + yz^2\mathbf{k}) \cdot \mathbf{n} ds.$$

4. (a) If $f(x)$ is a periodic function of x of period 2π. then the Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\ dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\ dx.$$

The periodic function f is given by

$$f\ (x) = \begin{cases} 2 + \dfrac{2x}{\pi}, & -\pi < x < 0 \\ 2, \quad 0 \le x \le \pi. \end{cases}$$

and $f(x + 2\pi) = f(x)$. Sketch the graph of the function in the interval $-\pi \le x \le 3\pi$ and obtain a Fourier series representation for $f(x)$.

(b) The Fourier transform of the function $f(x)$ is given by

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\ dx$$

with the inverse transform given by

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\ dk.$$

The following identities may also be useful, $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$.

The non-periodic function f is defined by

$$f\ (x) = \begin{cases} 5, & -2 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

Obtain a Fourier integral representation for $f(x)$.

5. The temperature $u(x,t)$ of a metal rod of length π is a solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

The rod is insulated at both ends $x = 0, \pi$ to give the boundary conditions $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0$.

Construct using separation of variables the temperature $u(x,t)$ and show that it can be written in the form

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos nx.$$

Hence find the temperature in the rod for $t > 0$ when the initial temperature is

$$u(x,0) = 1 - \frac{x}{\pi}.$$

6. Laplace's equation in polar coordinates (r,θ) is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Solve Laplace's equation for:

(a) A disk of radius $r = 1$ with boundary condition $u(1,\theta) = 1 + \cos 2\theta$, for $-\pi \le \theta \le \pi$.

(b) An annular region $\{(r,\theta) \mid a \le r \le b\}$ where $a > 0$ with boundary conditions

$$u(a,\theta) = u_0$$

$$u(b,\theta) = u_1$$

for $-\pi \le \theta \le \pi$ and u_0, u_1 are constants.

Formulæ in the Differential and Integral Calculus

Derivatives

y	$\mathrm{d}y/\mathrm{d}x$	y	$\mathrm{d}y/\mathrm{d}x$	y	$\mathrm{d}y/\mathrm{d}x$
x^n	nx^{n-1}	$\sec x$	$\tan x \sec x$ $= \sin x/\cos^2 x$	$\sec^{-1}\frac{x}{a}$	$\frac{a}{x\sqrt{x^2-a^2}}$
$\sin x$	$\cos x$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2-x^2}}$	e^x	e^x
$\cos x$	$-\sin x$	$\cos^{-1}\frac{x}{a}$	$-\frac{1}{\sqrt{a^2-x^2}}$	e^{ax}	ae^{ax}
$\tan x$	$\sec^2 x$	$\tan^{-1}\frac{x}{a}$	$\frac{a}{a^2+x^2}$	a^x	$a^x \ln a$
$\cot x$	$-\csc^2 x$	$\cot^{-1}\frac{x}{a}$	$-\frac{a}{a^2+x^2}$	$\ln x$	$\frac{1}{x}$
$\csc x$	$-\cot x \csc x$ $= -\cos x/\sin^2 x$	$\csc^{-1}\frac{x}{a}$	$-\frac{a}{x\sqrt{x^2-a^2}}$		

Integrals

y	$\int y\mathrm{d}x$	y	$\int y\mathrm{d}x$	y	$\int y\mathrm{d}x$
x^n	$\frac{x^{n+1}}{n+1}$ $n \neq -1$	$\cot x \csc x$	$-\csc x$	$\frac{1}{x}$	$\ln x$
$\sin x$	$-\cos x$	$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$ or $-\cos^{-1}\frac{x}{a}$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\frac{a}{a^2+x^2}$	$\tan^{-1}\frac{x}{a}$ or $-\cot^{-1}\frac{x}{a}$	$\mathrm{sech}^2 x$	$\tanh x$
$\sec^2 x$	$\tan x$	$\frac{a}{x\sqrt{x^2-a^2}}$	$\sec^{-1}\frac{x}{a}$ or $-\csc^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{x^2+a^2}}$	$\sinh^{-1}\frac{x}{a}$ $= \ln\frac{x+\sqrt{x^2+a^2}}{a}$
$\csc^2 x$	$-\cot x$	e^{ax}	$\frac{e^{ax}}{a}$	$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\frac{x}{a}$ $= \ln\frac{x+\sqrt{x^2-a^2}}{a}$
$\tan x \sec x$	$\sec x$	a^x	$\frac{a^x}{\ln a}$	$\frac{1}{a^2-x^2}$	$\frac{1}{a}\tanh^{-1}\frac{x}{a}$ $= \frac{1}{2a}\ln\frac{a+x}{a-x}$

Other Formulæ

Derivative of Product: $y = uv$, $\frac{\mathrm{d}y}{\mathrm{d}x} = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$.
Derivative of Quotient: $y = u/v$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \left(v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}\right)/v^2$.
Integration by Parts: $\int u\mathrm{d}v = uv - \int v\mathrm{d}u$.
Binomial Theorem: $(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{1\cdot 2}x^2 \pm \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \ldots$
Maclaurin's Theorem: $f(x) = f(0) + xf'(0) + \frac{x^2}{1\cdot 2}f''(0) + \ldots$
Taylor's Theorem: $f(x+h) = f(x) + hf'(x) + \frac{h^2}{1\cdot 2}f''(x) + \ldots$

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