1.

(a) According to Gauss's law,

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\int_V \rho dV = \frac{1}{\varepsilon_0}\int_V (-3)dV = \frac{-3}{\epsilon_0}\int_V dV = -\frac{3}{\epsilon_0}L^3 \text{ V·m}$$

where $\int_V dV = L^3 = (10^{-2})^3 = 10^{-6}$ m^3 is the volume of the cube.

(b)

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\int_V \rho dV = \frac{1}{\varepsilon_0}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}} (x^2+y^2+z^2)dxdydz$$

By symmetry,

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{3}{\varepsilon_0}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 dxdydz = \frac{3}{\varepsilon_0}(1)\cdot(1)\cdot\left[\frac{1}{3}x^3\right]\Big|_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{L^3}{4\varepsilon_0} = \frac{10^{-6}}{4\epsilon_0} \text{ V·m.}$$

(c) Taking advantage of the symmetry of the charge distribution, it can be easily verified that the electric flux through each of the six faces of the cube must be equal, i.e., $\Phi_1 = \Phi_2 = ... = \Phi_6$, therefore,

$$\Phi_i = \frac{1}{6}\times\frac{10^{-6}}{4\epsilon_o} = \frac{10^{-6}}{24\epsilon_o} \text{ V}\cdot\text{m.}$$

2. According to Example 5 in lecture 3 online, this problem can be solved by using symmetry. First, we consider Q_1. By Gauss' Law, $\oint_S \mathbf{D}_1\cdot d\mathbf{S} = Q_1$ holds for any S enclosing Q_1. Due to symmetry, half of the flux should go to the $-\widehat{x}$ direction across the yz plane (while the other half should go to the $+\widehat{x}$ direction), i.e., $\int_{x=0}\mathbf{D}_1\cdot(-\widehat{x})\,dydz = Q_1/2$. Then, we consider Q_2. Due to symmetry again, half of the total flux should go to the $+\widehat{x}$ direction across the yz plane, i.e., $\int_{x=0}\mathbf{D}_2\cdot(+\widehat{x})\,dydz = Q_2/2$. In summary, the total flux across the yz plane by both Q_1 and Q_2 in the $+\hat{x}$ direction is $\int_{x=0}(\mathbf{D}_1+\mathbf{D}_2)\cdot\widehat{x}dydz = (-Q_1+Q_2)/2$. Based on the given locations of the two charges, we have

$$\frac{Q_2 - Q_1}{2} = 3. \tag{1}$$

The flux through the $y = 1$ plane in the $+\hat{y}$ direction can be derived in a similar procedure, yielding

$$\frac{Q_2 + Q_1}{2} = 1. \tag{2}$$

Solving equations (1) and (2) we obtain $Q_1 = -2\,\text{C}$, and $Q_2 = 4\,\text{C}$.

3.

(a) There are two forces exerted on the particle: gravity and electric force. Let us assume the charged sheet is placed at $z = 0$. According to Example 1 in Lecture 3, the electric field induced by the planar sheet is

$$\mathbf{E} = \hat{z}\frac{\rho_s}{2\epsilon_0} \quad \text{for } z > 0.$$

Thus, the electric force can be written as

$$\mathbf{F} = \mathbf{E}Q = \hat{z}\frac{\rho_s Q}{2\epsilon_0}.$$

For the particle to levitate, the magnitude of the gravitational force (with acceleration $\mathbf{g} = -\hat{z}\ 9.8\ \mathrm{m/s^2}$) needs to be the same as the electric force magnitude, but in the opposite direction, such that

$$m\mathbf{g} + \mathbf{E}Q = 0.$$

As a result,

$$\begin{aligned}
\rho_s &= mg\frac{2\epsilon_0}{Q} = mg\frac{2}{Q}\frac{1}{\mu_0 c^2} \\
&= 36\pi \times 10^{-3} \times 9.8 \times \frac{2}{2 \times 10^{-6}} \times \frac{1}{4\pi \times 10^{-7} \times 9 \times 10^{16}} \\
&= 9.8\ \left(\mu\mathrm{C}/m^2\right).
\end{aligned}$$

where c denotes the speed of light in free space.

(b) In this case, the charge density of the planar sheet is reduced by half compared to that in part (a), while the gravitational force remains the same. Therefore, the total acceleration is:

$$a(t) = \frac{\mathbf{g}}{2} = -\frac{g}{2}\hat{z}.$$

Further, under constant acceleration, we find the particle velocity v and distance from the sheet d as:

$$v(t) = v_0 + \int_0^t a(t)\,dt = -\frac{gt}{2}\hat{z},$$

$$d(t) = d_0 + \int_0^t v(t)\,dt = d_0 - \frac{gt^2}{4}.$$

where $v_0 = 0$ and $d_0 = 9.8$ (m). We notice that $d(t) = 0$ when $t = 2$ (sec),which means the particle will hit the charged sheet after 2 seconds.

4. Similar to the example 4 in Lecture 3, the individual electric fields induced by these 3 slabs can be written as shifted-in-x versions of the result obtained in the example 2 of lecture 3:

$$\mathbf{E}_1 = \begin{cases} -\hat{x}\frac{\rho_1}{\epsilon_0}\frac{W}{2} = -\hat{x}\frac{5\rho_0}{2\epsilon_0}W & \text{for } x < -W \\ \hat{x}\frac{\rho_1}{\epsilon_0}\left(x + \frac{1}{2}W\right) = \hat{x}\frac{\rho_0}{\epsilon_0}\left(5x + \frac{5}{2}W\right) & \text{for } -W < x < 0 \\ \hat{x}\frac{\rho_1}{\epsilon_0}\frac{W}{2} = \hat{x}\frac{5\rho_0}{2\epsilon_0}W & \text{for } x > 0 \end{cases}$$

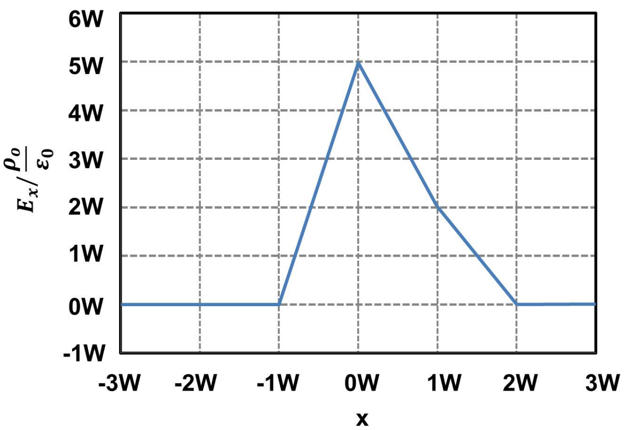
$$\mathbf{E}_2 = \begin{cases} -\hat{x}\frac{\rho_2}{\epsilon_0}\frac{W}{2} = \hat{x}\frac{3\rho_0}{2\epsilon_0}W & \text{for } x < 0 \\ \hat{x}\frac{\rho_2}{\epsilon_0}\left(x - \frac{1}{2}W\right) = -\hat{x}\frac{\rho_0}{\epsilon_0}\left(3x - \frac{3}{2}W\right) & \text{for } 0 < x < W \\ \hat{x}\frac{\rho_2}{\epsilon_0}\frac{W}{2} = -\hat{x}\frac{3\rho_0}{2\epsilon_0}W & \text{for } x > W \end{cases}$$

$$\mathbf{E}_3 = \begin{cases} -\hat{x}\frac{\rho_3}{\epsilon_0}\frac{W}{2} = \hat{x}\frac{\rho_0}{\epsilon_0}W & \text{for } x < W \\ \hat{x}\frac{\rho_3}{\epsilon_0}\left(x - \frac{3}{2}W\right) = -\hat{x}\frac{\rho_0}{\epsilon_0}(2x - 3W) & \text{for } W < x < 2W \\ \hat{x}\frac{\rho_3}{\epsilon_0}\frac{W}{2} = -\hat{x}\frac{\rho_0}{\epsilon_0}W & \text{for } x > 2W \end{cases}$$

Adding the fields we obtain the total field

$$\mathbf{E} = \hat{x}E_x = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 = \begin{cases} 0 & \text{for } x < -W \\ \hat{x}\frac{\rho_0}{\epsilon_0}(5x + 5W) & \text{for } -W < x < 0 \\ \hat{x}\frac{\rho_0}{\epsilon_0}(-3x + 5W) & \text{for } 0 < x < W \\ \hat{x}\frac{\rho_0}{\epsilon_0}(-2x + 4W) & \text{for } W < x < 2W \\ 0 & \text{for } x > 2W \end{cases}$$

which is plotted below.



5. Using the differential form of the Gauss' law, i.e., $\rho = \nabla \cdot \mathbf{D}$, let us determine the static charge $\rho(x, y, z)$ that generates the following displacement fields $\mathbf{D}$.

   (a) Given

$$\mathbf{D} = 4\text{sgn}(z)\,\hat{z}\,\frac{\text{C}}{\text{m}^2},$$

   we find that

$$\rho = \nabla \cdot \mathbf{D} = \frac{\partial}{\partial z}\,(4\text{sgn}(z)) = 8\delta(z)\,\frac{\text{C}}{\text{m}^3},$$

   where $\delta(z)$ is the Dirac delta function and $\frac{\partial}{\partial z}(\text{sgn}(z)) = 2\delta(z)$. Note that the displacement field generated by an infinite surface charge ρ_s placed at the xy-plane is

$$\mathbf{D} = \begin{cases} \frac{\rho_s}{2}\,\hat{z} & z > 0 \\ -\frac{\rho_s}{2}\,\hat{z} & z < 0 \end{cases} = \frac{\rho_s}{2}\text{sgn}(z)\,\hat{z}.$$

   Thus, an infinite surface charge density $\rho_s = 8\,\frac{\text{C}}{\text{m}^2}$ can be represented as a volumetric charge density $\rho = 8\delta(z)\,\frac{\text{C}}{\text{m}^3}$.

   (b) Given

$$\mathbf{D} = -2\text{sgn}(x+3)\,\hat{x}\,\frac{\text{C}}{\text{m}^2},$$

   we find that

$$\rho = \nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}\,(-2\text{sgn}(x+3)) = -4\delta(x+3)\,\frac{\text{C}}{\text{m}^3},$$

   Thus, an infinite surface charge density $\rho_s = -4\,\frac{\text{C}}{\text{m}^2}$ at $x = -3\text{m}$ can be represented as a volumetric charge density $\rho = -4\delta(x+3)\,\frac{\text{C}}{\text{m}^3}$.

   (c) In the case of

$$\mathbf{D} = \begin{cases} 5\,\hat{y}, & y < -5\,\text{m}, \\ -y\,\hat{y}, & |y| < 5\,\text{m}, \\ -5\,\hat{y}, & y > 5\,\text{m}, \end{cases}$$

   we find that

$$\rho = \nabla \cdot \mathbf{D} = \frac{\partial D_y}{\partial y} = \begin{cases} 0, & y < -5\,\text{m}, \\ -1\,\text{C}/\text{m}^3, & |y| < 5\,\text{m}, \\ 0, & y > 5\,\text{m}. \end{cases}$$

   where $\mathbf{D}$ is generated by a slab of charge extending from $-5 < y < 5$ m.

(d) In the case of

$$\mathbf{D} = \begin{cases} 2x - 4\,\hat{x}, & 0 < x < 2\,\mathrm{m}, \\ -x - 4\,\hat{x}, & -4 < x < 0\,\mathrm{m}, \\ 0, & otherwise, \end{cases}$$

we find that

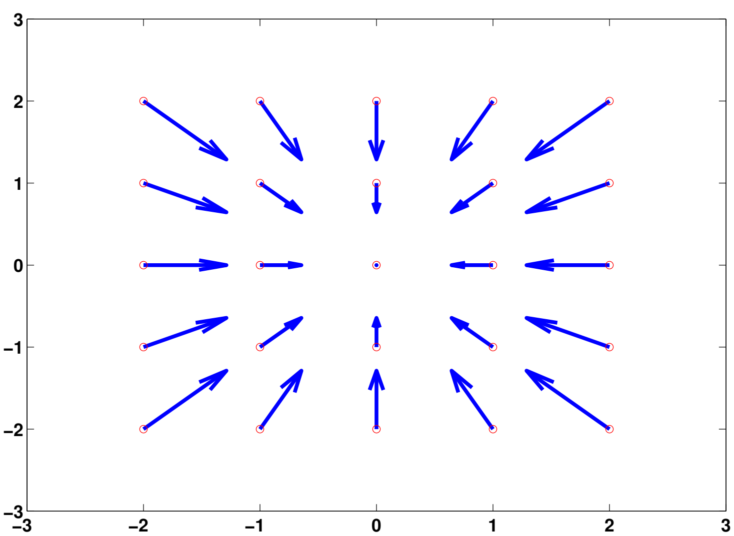
$$\rho = \nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} = \begin{cases} 2, & 0 < x < 2\,\mathrm{m}, \\ -1\,\mathrm{C/m^3}, & -4 < x < 0\,\mathrm{m}, \\ 0, & otherwise, \end{cases}$$

This displacment field is thus generated by two slabs of charge having different widths and charge densities but with overall charge neutrality.

6. Curl and divergence

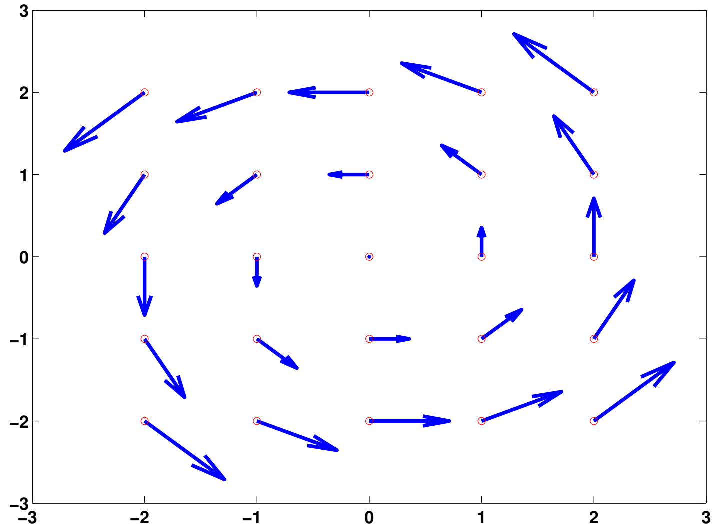
(a) $\nabla \times \mathbf{F} = \nabla \times (-x\widehat{x} - y\widehat{y}) = \begin{vmatrix} \widehat{x} & \widehat{y} & \widehat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & -y & 0 \end{vmatrix} = 0$

$\nabla \cdot \mathbf{F} = \left(\widehat{x}\frac{\partial}{\partial x} + \widehat{y}\frac{\partial}{\partial y} + \widehat{z}\frac{\partial}{\partial z}\right) \cdot (-x\widehat{x} - y\widehat{y}) = -2$



(b) $\nabla \times \mathbf{F} = \nabla \times (-y\widehat{x} + x\widehat{y}) = \begin{vmatrix} \widehat{x} & \widehat{y} & \widehat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\widehat{z}$

$\nabla \cdot \mathbf{F} = \left(\widehat{x}\frac{\partial}{\partial x} + \widehat{y}\frac{\partial}{\partial y} + \widehat{z}\frac{\partial}{\partial z}\right) \cdot (-y\widehat{x} + x\widehat{y}) = 0$



(c)

i. In (b), $\nabla \times \mathbf{F} = 2\hat{z} \neq 0$, and the field strength $|\mathbf{F}|$ is $\sqrt{x^2+y^2}$. To analyze how the field strength varies we take the gradient of $|\mathbf{F}|$, $\nabla|\mathbf{F}| = (x\hat{x} + y\hat{y})/\sqrt{x^2+y^2}$. We recognize that $\nabla|\mathbf{F}| \perp \mathbf{F}$, since $(\nabla|\mathbf{F}|) \cdot \mathbf{F} = 0$. Thus, the field strength varies **across** the direction of the field.

ii. In (a), $\nabla \cdot \mathbf{F} = \mathbf{2} \neq 0$, the field strength $|\mathbf{F}|$ is $\sqrt{x^2+y^2}$. Thus, the variation of $|\mathbf{F}|$ is $\nabla|\mathbf{F}| = (x\hat{x} + y\hat{y})/\sqrt{x^2+y^2} = -\mathbf{F}/\sqrt{x^2+y^2}$. Thus, the field strength varies **along** the direction of the field.

7.

(a) Coulomb's field of a charge Q stationed at the origin of a right-handed Cartesian coordinate system is

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_o}\frac{\hat{r}}{r^2} = \frac{Q}{4\pi\epsilon_o}\frac{(x,y,z)}{r^3}$$

where $r = (x^2+y^2+z^2)^{1/2}$. Taking its curl,

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{x} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{z},$$

we find that

$$\begin{aligned}\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= \frac{Q}{4\pi\epsilon_o}\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right) - \frac{Q}{4\pi\epsilon_o}\frac{\partial}{\partial z}\left(\frac{y}{r^3}\right) \\ &= -\frac{Q}{4\pi\epsilon_o}\left(\frac{3z}{r^4}\frac{\partial r}{\partial y} - \frac{3y}{r^4}\frac{\partial r}{\partial z}\right) = -\frac{Q}{4\pi\epsilon_o}\left(\frac{3zy}{r^5} - \frac{3yz}{r^5}\right) = 0,\end{aligned}$$

and also that

$$\begin{aligned}\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= \frac{Q}{4\pi\epsilon_o}\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) - \frac{Q}{4\pi\epsilon_o}\frac{\partial}{\partial x}\left(\frac{z}{r^3}\right) = 0, \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= \frac{Q}{4\pi\epsilon_o}\frac{\partial}{\partial x}\left(\frac{y}{r^3}\right) - \frac{Q}{4\pi\epsilon_o}\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right) = 0.\end{aligned}$$

Therefore, the curl of the electrostatic field generated by a point charge is zero,

$$\nabla \times \mathbf{E} = \mathbf{0}.$$

(b)

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_o} \\
&= \frac{Q}{4\pi\epsilon_o}\left[\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\right] \\
&= \frac{Q}{4\pi\epsilon_o}\left(\frac{1}{r^3} - \frac{3x^2}{r^5} + \frac{1}{r^3} - \frac{3y^2}{r^5} + \frac{1}{r^3} - \frac{3z^2}{r^5}\right) \\
&= \frac{Q}{4\pi\epsilon_o}\left(\frac{3}{r^3} - \frac{3(x^2+y^2+z^2)}{r^5}\right) \\
&= 0
\end{aligned}$$

8.

(a)

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos(y) & \sin(x) & 0 \end{vmatrix} = \hat{z}[\cos(x) + sin(y)],$$

$$\nabla \times (\nabla \times \mathbf{E}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \cos(x) + sin(y) \end{vmatrix} = \hat{\mathrm{x}}\cos(y) + \hat{y} sin(x).$$

(b) The differential form of Gauss' law is:

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon_0 \mathbf{E}) = \rho.$$

Therefore,

$$\rho = \nabla \cdot (\epsilon_0 \mathbf{E}) = \epsilon_0 (\nabla \cdot \mathbf{E}) = \epsilon_0 \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot [\hat{y}\sin(x) + \hat{x}\cos(y)] = 0\, C/\mathrm{m}^3.$$