(8 Pts.)

1. For each of the following questions, either circle the correct answer, or simply state the final answer in the space provided.

+2 (a) The ideal digital bandpass filter is an FIR filter. **True/ False**

+2 (b) What is the output $y[n]$ of an LSI system with frequency response $H_d(\omega)$ for the input

$x[n] = e^{jn}\cos n - 0.5$? $\boxed{\frac{1}{2} e^{j2n} H_d(2)}$

+2 (c) A signal $x_a(t)$ is such that $X_a(\Omega)$ is non-zero only between -500 Hz and 2 kHz. If $x_a(t)$ were the input to an A/D converter, what is the largest sampling period T such that there is no aliasing

at the output of the A/D? $\boxed{\frac{1}{2500}\,s = 400\,\mu s}$

+2 (d) For a real-valued impulse response, the phase response is an even function of ω. **True/ False**

(17 Pts.)

2. (a) **(10 Pts.)** Consider the system given by the following equation:

$$y[n] = 2x[n] + x[n-1] - 3x[n-2] + 11x[n-5]$$

Compute the output of the system for each of the following inputs. In each case, indicate if the input is an eigensequence of the system.

(i) $x[n] = 2\cos\left(n\pi + \frac{\pi}{4}\right)$

$h(n) = 2\delta(n) + \delta(n-1) - 3\delta(n-2) + 11\delta(n-5)$ → real valued

$H_d(\omega) = 2 + e^{-j\omega} - 3e^{-j2\omega} + 11e^{-j5\omega}$ (+1)

$y(n) = 2|H_d(\pi)| \cos\left(n\pi + \frac{\pi}{4} + \angle H_d(\pi)\right)$ → using formula (+1)

$H_d(\pi) = -13 \Rightarrow |H_d(\pi)| = 13, \ \angle H_d(\pi) = \pi$ → (+1)

$\Rightarrow y(n) = 26\cos\left(n\pi + \frac{\pi}{4} + \pi\right) = -26\cos\left(n\pi + \frac{\pi}{4}\right)$ (+1)

$= -13x(n)$ Yes, eigenSequence! (+1)

(ii) $x[n] = -3j^n$ $= -3e^{\frac{jn\pi}{2}}$

$(+1) \leftarrow y(n) = -3e^{jn\pi/2}\left(Hd(\pi/2)\right) \rightarrow$ Yes, eigenfunction $\rightarrow (+1)$

$Hd(\frac{\pi}{2}) = 2 + e^{\frac{-j\pi}{2}} - 3e^{-j\pi} + 11e^{-j5\pi/2}$

$= 5 - 12j \rightarrow (+2)$

$|Hd(\frac{\pi}{2})| = 13, \quad \angle Hd(\pi/2) = -\tan^{-1}(12/5)$

$y(n) = -39\, e^{j\left(\frac{n\pi}{2} - \tan^{-1}\left(\frac{12}{5}\right)\right)}$ $\rightarrow (+1)$

(b) **(7 Pts.)** An LSI system is described by the following equation:

$$y[n] = \frac{x[n] - x[n-1]}{2}$$

Find the frequency response and magnitude response of this system. Sketch the magnitude response. Clearly label the plot. What kind of filter is roughly implemented by this system? Why?

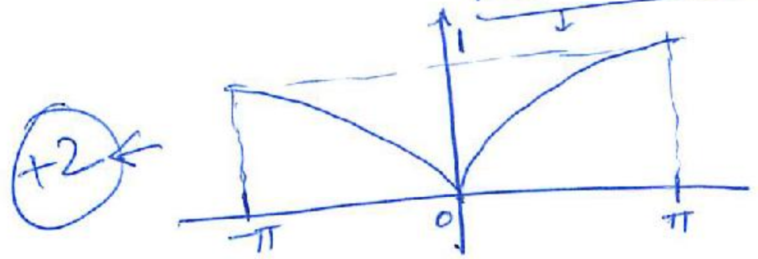
$Hd(\omega) = \frac{1 - e^{-j\omega}}{2} = \frac{e^{-\frac{j\omega}{2}}\left(e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}}\right)}{2}$

$= e^{-\frac{j\omega}{2}}\left(j \sin\frac{\omega}{2}\right)$

$= e^{j\left(\frac{\pi}{2} - \frac{\omega}{2}\right)} \sin\left(\frac{\omega}{2}\right) \rightarrow (+3)$

$|Hd(\omega)| = \left|\sin\left(\frac{\omega}{2}\right)\right| \rightarrow (+1)$

$|Hd(\omega)|$ on $[-\pi, \pi]$



$\Rightarrow$ This is an approximate High pass filter $\rightarrow (+1)$

because $Hd(0) = 0$

$Hd(\pi) = 1$

Higher frequencies are roughly passed through. & lower frequencies are attenuated out.

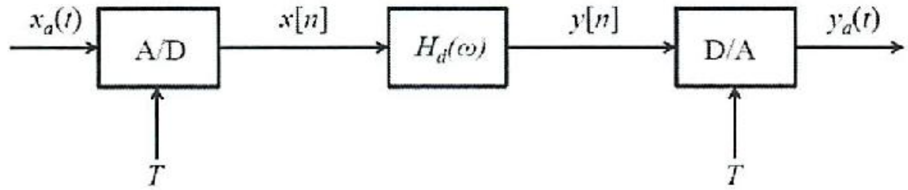


Figure 1: System for discrete-time processing of continuous time signals.

(20 Pts.)

3. Consider the system shown in Figure 1 that consists of an A/D, digital filter, and an ideal D/A. Assume that the signal $x_a(t)$ is bandlimited to 25 kHz. You are asked to implement an analog band-stop filter for this signal, that stops all frequencies between 10 kHz and 20 kHz, and passes the other frequencies. The implementation is to be done using the system in Figure 1.

   (a) **(3 Pts.)** What is the Nyquist sampling period for the input signal?

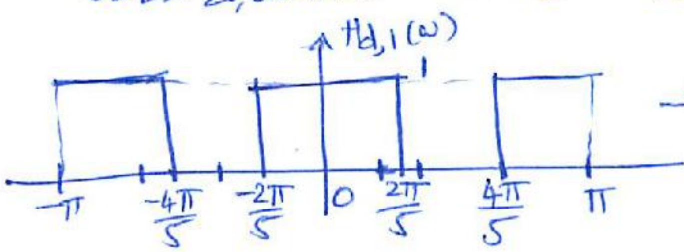
   $$T_N = \frac{\pi}{2\pi(25000)} = \frac{1}{50{,}000}\ s = 20\ \mu s \quad +3$$

   (b) **(4 Pts.)** Sketch the frequency response $H_{d,1}(\omega)$ for the necessary discrete-time filter, when the sampling is done at the Nyquist rate. Label the sketch clearly.

   $\boxed{\omega_c = \Omega_c T}$

   $\Omega_1 = 10{,}000 \times 2\pi \rightarrow \omega_1 = 2\pi/5 \rightarrow +1$

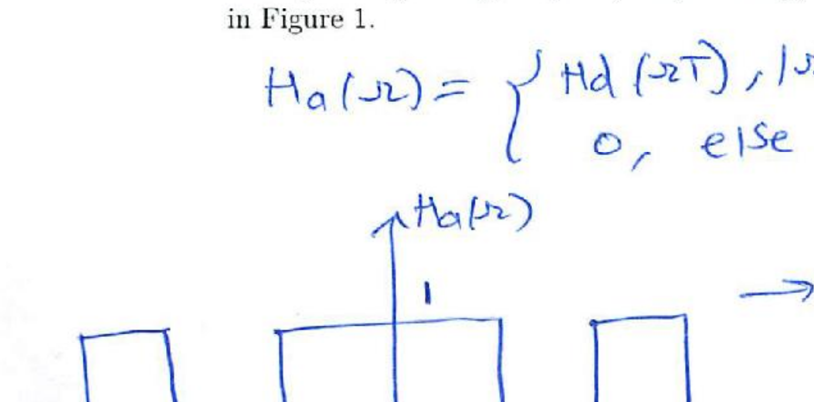
   $\Omega_2 = 20{,}000 \times 2\pi \rightarrow \omega_2 = 4\pi/5 \rightarrow +1$

   $\rightarrow +2$ for general shape

   (c) **(4 Pts.)** If the filter $H_{d,1}(\omega)$ and T in Part (b) were to be used in Figure 1, sketch the corresponding analog frequency response $H_a(\Omega)$ such that $Y_a(\Omega) = H_a(\Omega)X_a(\Omega)$ for the system in Figure 1.

   $$H_a(\Omega) = \begin{cases} H_d(\Omega T), & |\Omega| \le \frac{\pi}{T} = 50{,}000\pi \\ 0, & \text{else} \end{cases}$$

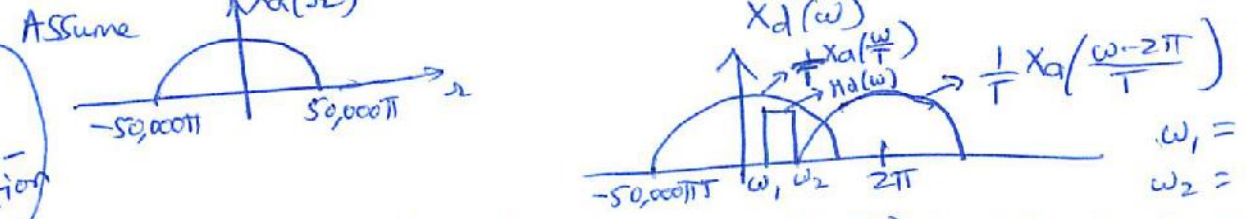
   $\rightarrow +1$ for general shape

   $\rightarrow$ for labels

Now, For Parts (d) and (e), you are asked to design an analog bandpass filter for $x_a(t)$ that passes all frequencies between 2 KHz and 12 KHz.

(d) **(5 Pts.)** Find the largest sampling period T for which the A/D, digital filter, and D/A in Figure 1 can perform the desired filtering function. Justify your answer.

Assume



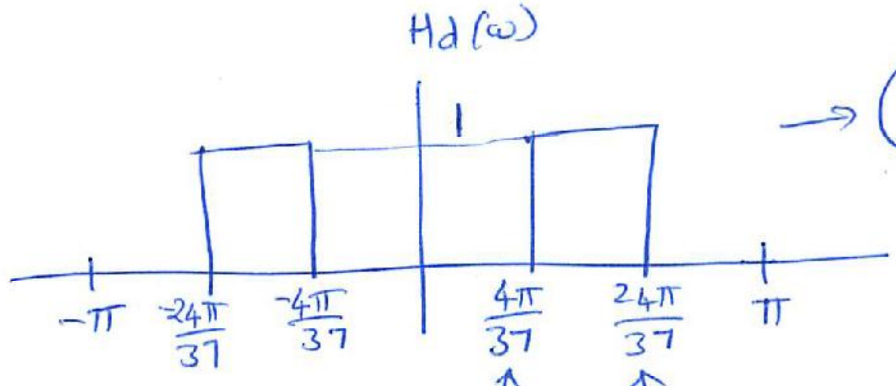
$\omega_1 = 4000\pi T$
$\omega_2 = 24000\pi T$

+3 for justification

The largest T is such that the left edge of $\frac{1}{T}X_a\left(\frac{\omega-2\pi}{T}\right)$ just touches the right edge of H_d.

$\Rightarrow \omega_2 = 2\pi - 50000\pi T \Rightarrow \boxed{T = \frac{1}{37000}\ \text{Sec}} \rightarrow +2$

(e) **(4 Pts.)** For the system using T from part (d), sketch the necessary $H_d(\omega)$. Label the sketch clearly.



$\rightarrow$ +2 for general Shape.

+1 +1 $\rightarrow$ for labels

(21 Pts.)

4. Suppose the ideal D/A in Figure 1 is replaced by a zero-order hold (ZOH), using the following interpolating function

$$g_a(t) = \begin{cases} 1 & \text{if } 0 \le t \le T \\ 0 & \text{else} \end{cases}$$

So the output of the ZOH is $y_a(t) = \sum_{n=-\infty}^{\infty} y[n] g_a(t - nT)$, where $T = \frac{1}{3}$ sec is used.

(a) **(3 Pts.)** State the mathematical relationship (equation) between the CTFT of $y_a(t)$ and the DTFT of $y[n]$ for the specific ZOH above.

$Y_a(\Omega) = Y_d(\Omega T)\, G_a(\Omega) \rightarrow +1$

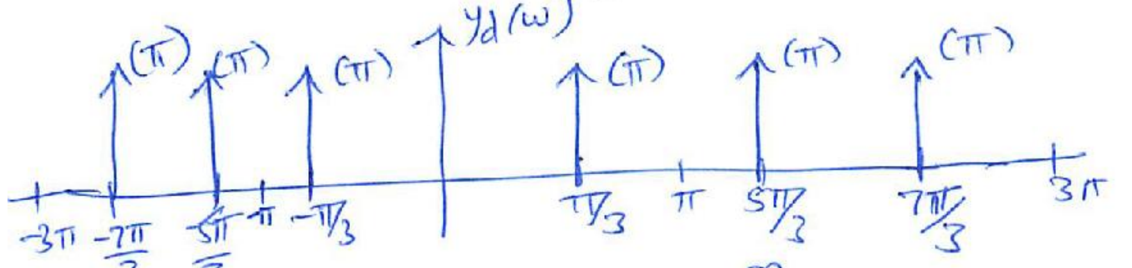
$G_a(\Omega) = T e^{-\frac{j\Omega T}{2}} \operatorname{sinc}\left(\frac{\Omega T}{2}\right) = \frac{1}{3} e^{-\frac{j\Omega}{6}} \operatorname{sinc}\left(\frac{\Omega}{6}\right) \rightarrow +1$

$\Rightarrow Y_a(\Omega) = Y_d\left(\frac{\Omega}{3}\right) \frac{1}{3} e^{-\frac{j\Omega}{6}} \operatorname{sinc}\left(\frac{\Omega}{6}\right) \rightarrow +1$

(b) **(12 Pts.)** Sketch by hand the Fourier transform of the output of the ZOH for an input $y[n] = \cos(n\pi/3)$. Do the sketch for $0 \le |\Omega| \le 9\pi$. Label the plot clearly. Determine the magnitude of the largest spurious component (in frequency domain) at the output. Show all intermediate steps for full credit.

$$Y_d(\omega) = \pi\,\delta\left(\omega - \tfrac{\pi}{3}\right) + \pi\,\delta\left(\omega + \pi/3\right), \quad |\omega| \le \pi$$

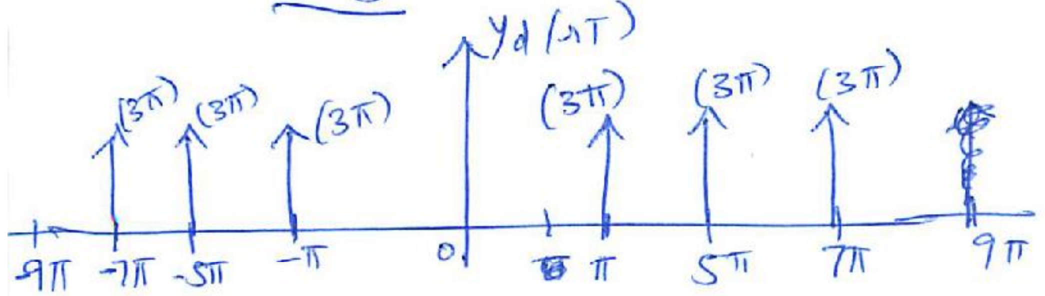
$$= \pi \sum_{k=-\infty}^{\infty} \left(\delta\left(\omega + 2k\pi - \pi/3\right) + \delta\left(\omega + 2k\pi + \pi/3\right) \right)$$



→ +3

$$Y_d(\Omega T) = \frac{\pi}{T} \sum_{k=-\infty}^{\infty} \left(\delta\left(\Omega + \frac{2k\pi}{T} - \frac{\pi}{3T}\right) + \delta\left(\Omega + \frac{2k\pi}{T} + \frac{\pi}{3T}\right) \right)$$

$T = \frac{1}{3}$



→ +3

$$Y_a(\Omega) = Y_d(\Omega T)\, G_a(\Omega)$$

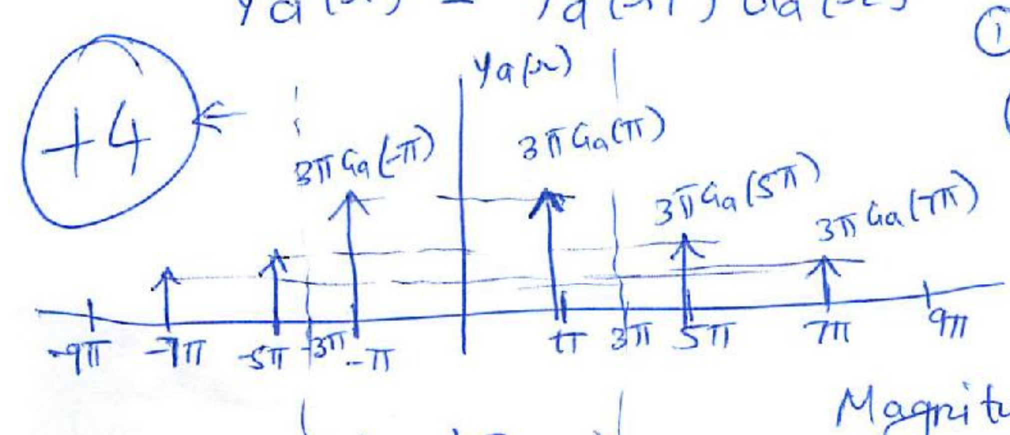
Magnitudes of the deltas

① $3\pi\,|G_a(\pi)| = 3$

② $|3\pi G_a(5\pi)| = 3/5$

③ $3\pi\,|G_a(7\pi)| = \frac{3}{7}$

+4



Magnitude of largest spurious Component = Magnitude at 5π

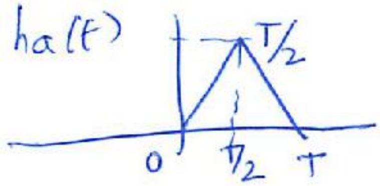
$= \boxed{3/5}$ → +2

(c) **(6 Pts.)** If the interpolating function for the ZOH is instead replaced by the following interpolating function, with $T = \frac{1}{3}$ sec

$$h_a(t) = \begin{cases} t & \text{if } 0 \le t \le 0.5T \\ T - t & \text{if } 0.5T \le t \le T \\ 0 & \text{if } t \ge T \end{cases}$$

Determine the magnitude of the largest spurious component (in frequency domain) at the output in this case. Use the same input $y[n] = \cos(n\pi/3)$ as in part (b).

Let $g_a(t)$ be the ZOH function $g_a(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{else} \end{cases}$



$h_a(t) = g_a(2t) * g_a(2t) \rightarrow$ +2

$\Rightarrow H_a(\Omega) = \left(\frac{1}{2} G_a\left(\frac{\Omega}{2}\right)\right)\left(\frac{1}{2} G_a\left(\frac{\Omega}{2}\right)\right) = \frac{1}{4} G_a^2\left(\frac{\Omega}{2}\right)$

$|H_a(\Omega)| = \frac{1}{4} \times \frac{1}{9} \operatorname{Sinc}^2\left(\frac{\Omega}{12}\right) = \frac{1}{36} \operatorname{Sinc}^2\left(\frac{\Omega}{12}\right)$ +2

The magnitude of largest Spurious Component in this case $= 3\pi\, |H_a(5\pi)|$

$= 3\pi \times \frac{1}{36} \operatorname{sinc}^2\left(\frac{5\pi}{12}\right)$

$= \frac{12}{25\pi} \sin^2(75^\circ) = \frac{12}{25\pi} \times 0.93 = \boxed{\frac{11.16}{25\pi}}$ +2

(20 Pts.)

5. Consider the causal LSI system described by the following difference equation (zero initial conditions):

$$y[n] = x[n] + \frac{1}{16} y[n-4] - x[n-4]$$

(a) **(7 Pts.)** Draw the Direct Form 2 implementation of the system. Show the system transfer function. Is the system BIBO stable? Justify.

+2 $H(z) = \dfrac{1 - z^{-4}}{1 - \frac{1}{16} z^{-4}} = \dfrac{z^4 - 1}{z^4 - 1/16}$

Poles are at $|z|^4 = \left(\frac{1}{2}\right)^4$

$\Rightarrow |z| = \frac{1}{2}$

$ROC = \{z : |z| > \frac{1}{2}\}$ for Causal

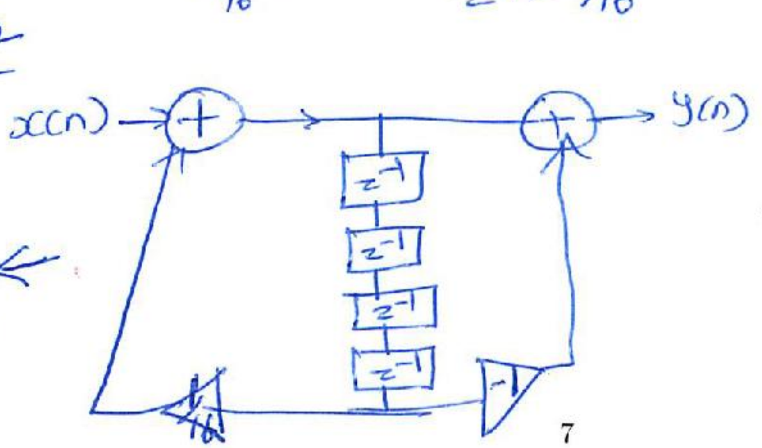
Includes unit circle

$\Rightarrow$ BIBO Stable

+1 → with explanation

Direct form 2

+4



(b) **(3 Pts.)** If a complex exponential $x[n] = e^{j\omega_0 n}$ were given as an input to the system, which frequencies won't pass through the system?

$H(z)$ has zeros where $z^4 = 1$

$z^4 = 1 = e^{j2k\pi}, k \in \mathbb{Z} \Rightarrow$ zeros are at $e^{\frac{jk\pi}{2}}$ $k = -1, 0, 1, 2$

zeros at $\{e^{-\frac{j\pi}{2}}, e^{j0}, e^{\frac{j\pi}{2}}, e^{j\pi}\} \rightarrow +1$

$\Rightarrow$ Frequencies that won't pass are $\omega_0 = \{-\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi\}$. $\rightarrow +2$

(c) **(10 Pts.)** Consider a cascade implementation of the system using second order sub-sections with real coefficients. What are the transfer functions of the sub-systems? Implement the cascade using Direct Form 1 sub-sections.

$$H(z) = \frac{z^4 - 1}{z^4 - \frac{1}{16}}$$

For roots of numerator solve $z^4 = 1 = e^{j2k\pi}, k \in \mathbb{Z}$

$\Rightarrow$ roots $\{-j, 1, j, -1\}$

For roots of denominator solve $z^4 = \frac{1}{16} = \left(\frac{1}{2}\right)^4 e^{j2k\pi}, k \in \mathbb{Z}$

$\Rightarrow$ roots are $\{-\frac{j}{2}, \frac{1}{2}, \frac{j}{2}, -\frac{1}{2}\}$

$$H(z) = \frac{(z-j)(z+j)(z-1)(z+1)}{(z-\frac{j}{2})(z+\frac{j}{2})(z-\frac{1}{2})(z+\frac{1}{2})}$$

Combine complex roots with conjugate terms.

cascade $x(n) - \boxed{H_1(z)} - \boxed{H_2(z)} - y(n)$

$\boxed{\text{We can also swap order of } H_1 \,\&\, H_2 \text{ in diagram}}$

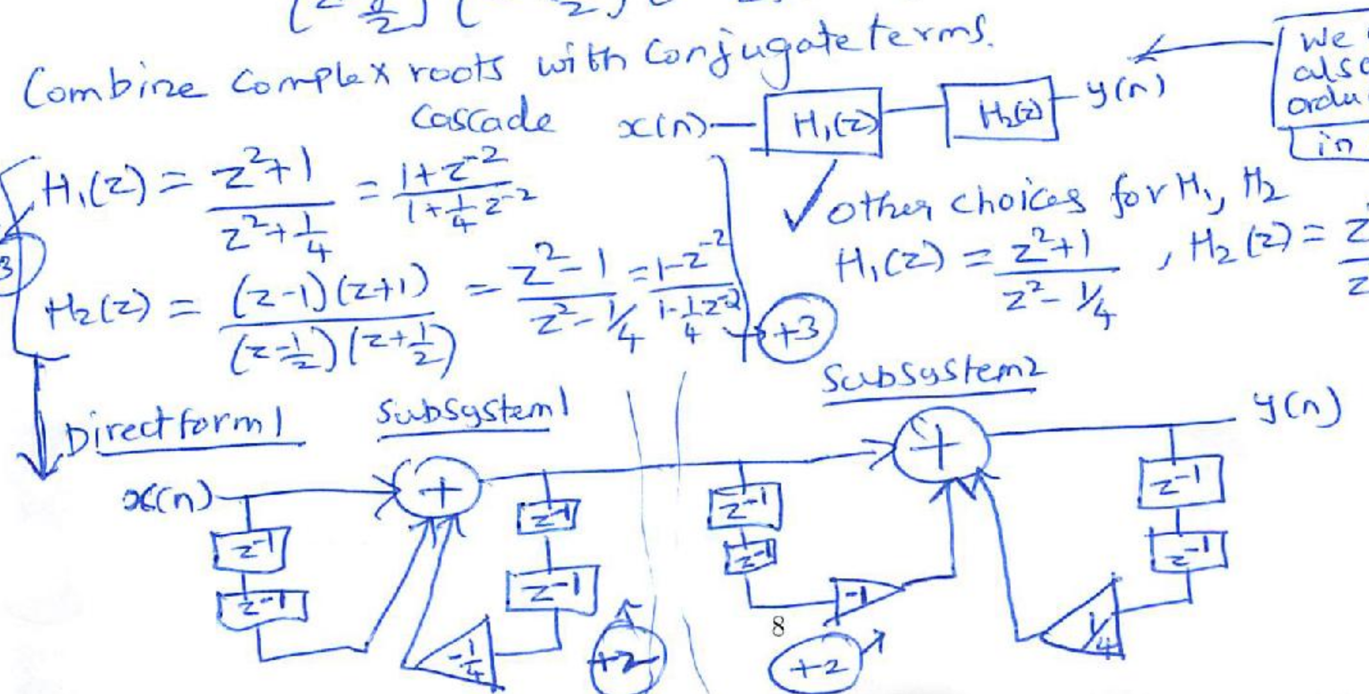
$H_1(z) = \frac{z^2+1}{z^2+\frac{1}{4}} = \frac{1+z^{-2}}{1+\frac{1}{4}z^{-2}}$

+3

$H_2(z) = \frac{(z-1)(z+1)}{(z-\frac{1}{2})(z+\frac{1}{2})} = \frac{z^2-1}{z^2-\frac{1}{4}} = \frac{1-z^{-2}}{1-\frac{1}{4}z^{-2}}$ +3

✓ other choices for H_1, H_2

$H_1(z) = \frac{z^2+1}{z^2-\frac{1}{4}}$, $H_2(z) = \frac{z^2-1}{z^2+\frac{1}{4}}$



(14 Pts.)

6. For each of the parts (i) and (ii), answer the following questions. Use (a), (b), (c), (d), (e), to denote your work for each question. State your conclusions in full sentences.

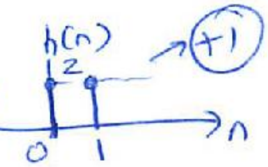
   (a) Is the system FIR or IIR? If FIR, plot $h[n]$.

   (b) If the system is FIR, is $h[n]$ symmetric, anti-symmetric, or neither?

   (c) Does $H_d(\omega)$ have Type 1 GLP, or Type II GLP, or neither? Does it have linear phase?

   (d) The frequency response of a GLP filter can be expressed as $H_d(\omega) = R(\omega)e^{j(\alpha - M\omega)}$, where $R(\omega)$ is a real function. For the systems that have GLP or linear phase, find $R(\omega)$, M, and α.

   (i) **(7 Pts.)** $\{h_n\}_{n=0}^{1} = \{2, 2\}$ (+1)



(+1)

(a) h(n) is an FIR filter (N=2)

(b) h(n) is symmetric → (+1)

(c) Hd(ω) has Type 1 GLP → (+1)

$H_d(\omega) = 2 + 2e^{-j\omega} = 2e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}\right) = e^{-j\frac{\omega}{2}}\underbrace{\left(4\cos\frac{\omega}{2}\right)}_{R(\omega)} \rightarrow$ (+1)

$R(\omega) \geq 0$ on $[-\pi, \pi] \Rightarrow$ Linear phase → (+1)

(d) $R(\omega) = 4\cos\frac{\omega}{2}$

$\alpha = 0$

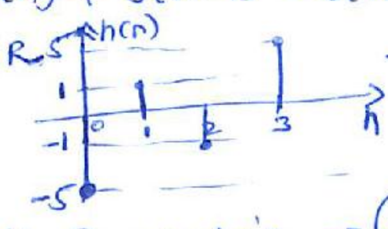
$M = \frac{1}{2} = \frac{N-1}{2}$ (+1)

   (ii) **(7 Pts.)** $y[n] = -5x[n] + x[n-1] - x[n-2] + 5x[n-3]$

$h(n) = -5\delta(n) + \delta(n-1) - \delta(n-2) + 5\delta(n-3)$

(a) h(n) is FIR → (+1)

N=4 (+1)



(b) h(n) is anti-Symmetric. → (+1)

(c) Hd(ω) has Type II GLP. → (+1)

$H_d(\omega) = -5 + e^{-j\omega} - e^{-j2\omega} + 5e^{-j3\omega} = e^{-j\frac{3\omega}{2}}\left(-5e^{j\frac{3\omega}{2}} + 5e^{-j\frac{3\omega}{2}} + e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)$

$= e^{-j\frac{3\omega}{2}}\left[-5\left(2j\sin\left(\frac{3\omega}{2}\right)\right) + 2j\sin\left(\frac{\omega}{2}\right)\right]$

(+1) ← $= e^{j\left(\frac{\pi}{2} - \frac{3\omega}{2}\right)}\underbrace{\left(2\sin\left(\frac{\omega}{2}\right) - 10\sin\left(\frac{3\omega}{2}\right)\right)}_{R(\omega)}$

R(ω) Changes Sign on $(-\pi, \pi) \Rightarrow$ Not linear phase (+1)

(d) $R(\omega) = 2\sin\left(\frac{\omega}{2}\right) - 10\sin\left(\frac{3\omega}{2}\right)$

$\alpha = \pi/2$

$M = 3/2$ → (+1)