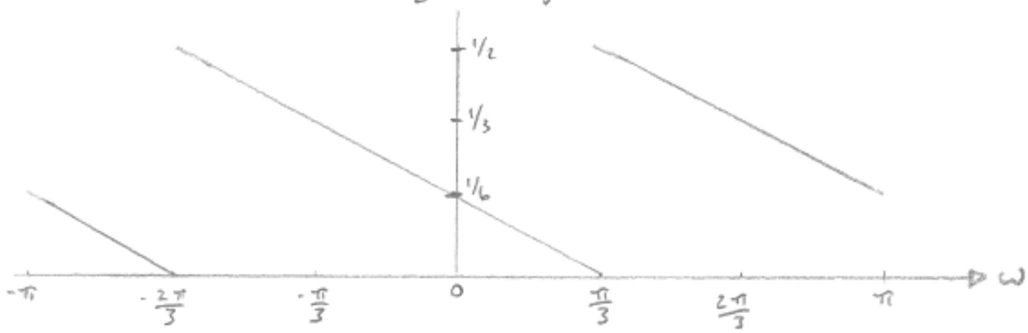
Problem 1

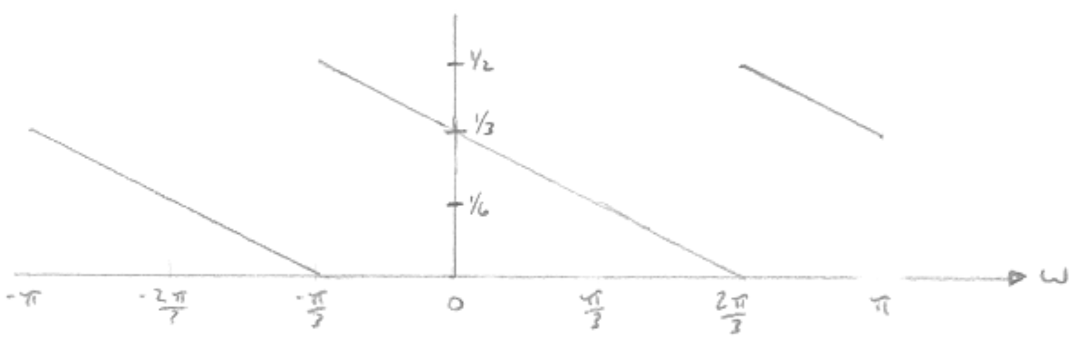
$y[n] = x[n] \cos(\pi n/3)$

Using the modulation property, $Y_d(\omega) = \frac{1}{2}\left[X_d(\omega - \frac{\pi}{3}) + X_d(\omega + \frac{\pi}{3})\right]$

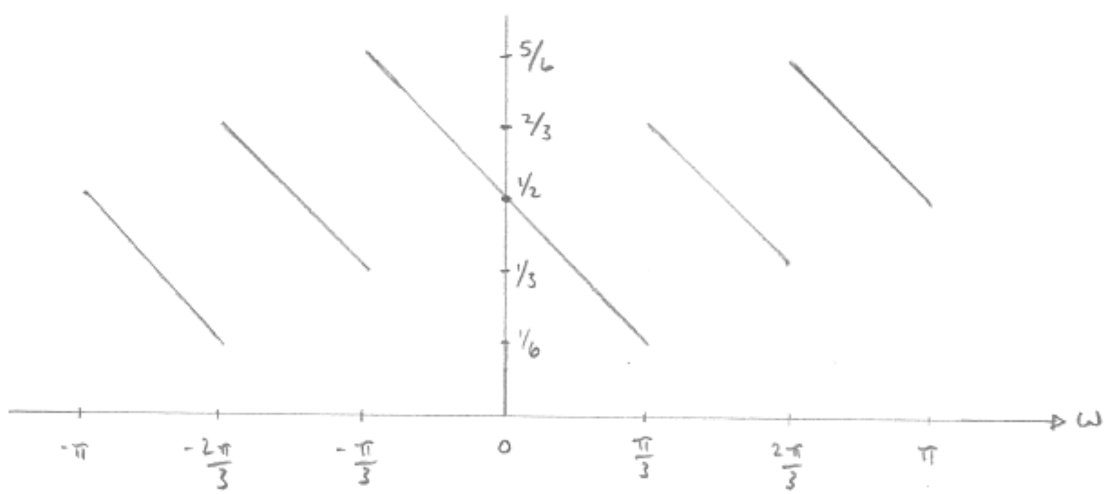
$\frac{1}{2} X_d(\omega - \frac{\pi}{3})$



$\frac{1}{2} X_d(\omega + \frac{\pi}{3})$



$Y_d(\omega)$



Problem 2

(a) $x[n] = \delta[n+3] + \delta[n-3]$

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

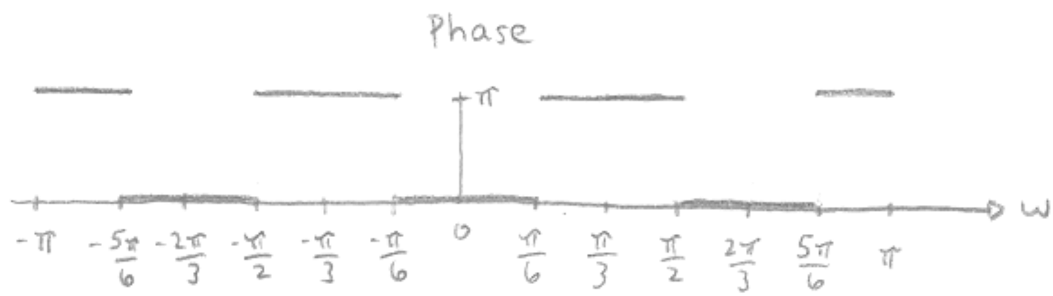
$$= e^{j3\omega} + e^{-j3\omega}$$

$$= 2\cos(3\omega)$$

$$|X_d(\omega)| = 2\,|\cos(3\omega)|$$

$$X_d(\omega) = \begin{cases} 2\,|\cos(3\omega)|\, e^{j0}, & \cos(3\omega) > 0 \\ 2\,|\cos(3\omega)|\, e^{j\pi}, & \cos(3\omega) < 0 \end{cases}$$

$$\measuredangle X_d(\omega) = \begin{cases} 0, & \cos(3\omega) > 0 \\ \pi, & \cos(3\omega) < 0 \end{cases}$$



Problem 2

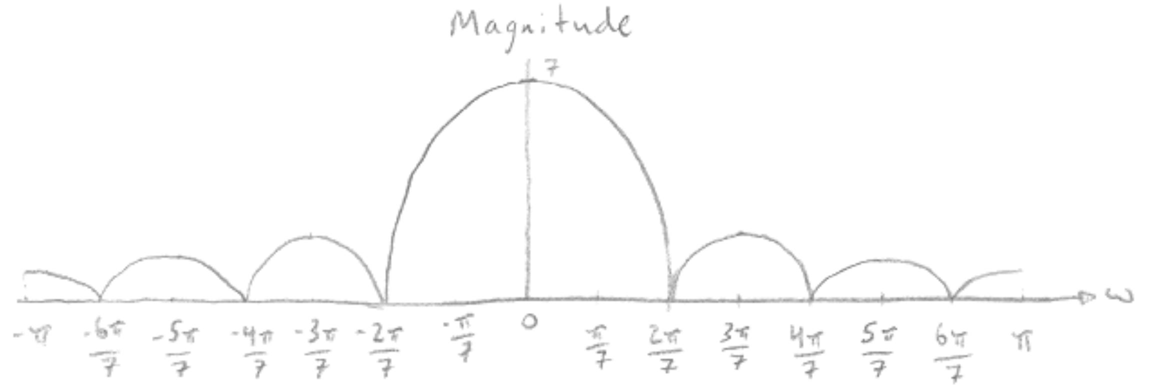
(b) $x[n] = u[n] - u[n-7]$

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=0}^{6} (e^{-j\omega})^n = \frac{1 - e^{-j7\omega}}{1 - e^{-j\omega}}$$

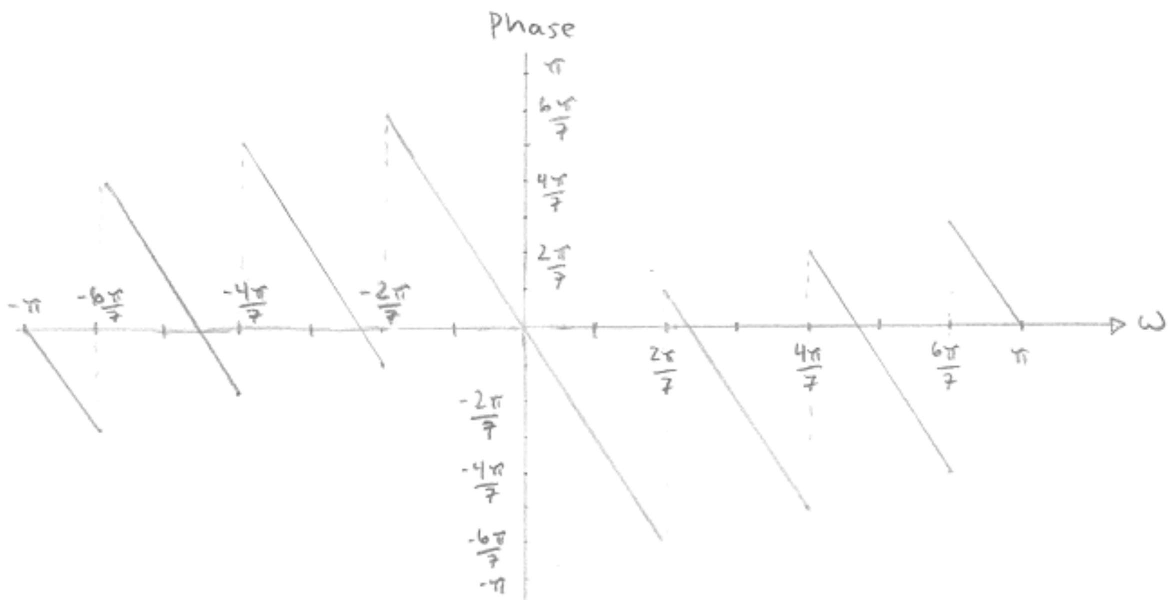
To find magnitude and phase, factor as follows:

$$X_d(\omega) = \frac{e^{-j\frac{7}{2}\omega}\left(e^{j\frac{7}{2}\omega} - e^{-j\frac{7}{2}\omega}\right)}{e^{-j\frac{1}{2}\omega}\left(e^{j\frac{1}{2}\omega} - e^{-j\frac{1}{2}\omega}\right)} = e^{-j3\omega}\frac{\sin(\frac{7}{2}\omega)}{\sin(\frac{1}{2}\omega)}$$

$$|X_d(\omega)| = \left|\frac{\sin(\frac{7}{2}\omega)}{\sin(\frac{1}{2}\omega)}\right| \qquad X_d(0) = \sum_{n=0}^{6} 1 = 7$$



$$\measuredangle X_d(\omega) = \begin{cases} -3\omega, & \sin(\frac{7}{2}\omega)/\sin(\frac{1}{2}\omega) > 0 \\ \pi - 3\omega, & \sin(\frac{7}{2}\omega)/\sin(\frac{1}{2}\omega) < 0 \end{cases}$$



Problem 2

(c) $x[n] = \left(\frac{1}{4}\right)^n u[n]$

$$X_d(\omega) = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\frac{e^{-j\omega}}{4}\right)^n = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$= \frac{4}{4-e^{-j\omega}}$$

(d) $x[n] = \left(\frac{1}{4}\right)^n u[n+4] = \left(\frac{1}{4}\right)^{-4} \left(\frac{1}{4}\right)^{n+4} u[n+4]$

Using time shift and linearity properties and part (c):

$$X_d(\omega) = (4^4)\left(\frac{4}{4-e^{-j\omega}}\right) e^{j4\omega}$$

(e) $x[n] = \left(\frac{1}{4}\right)^n e^{j\pi n/3} u[n]$

Using Frequency shift property and part (c):

$$X_d(\omega) = \frac{4}{4-e^{-j(\omega-\frac{\pi}{3})}}$$

(f) $x[n] = \left(\frac{1}{4}\right)^n \cos\left(\frac{\pi n}{3}\right) u[n-5]$

Let $z[n] = \left(\frac{1}{4}\right)^n u[n-5] = \left(\frac{1}{4}\right)^5 \left(\frac{1}{4}\right)^{n-5} u[n-5]$

Using time shift, linearity, and part (c):

$$Z_d(\omega) = \left(\frac{1}{4}\right)^5 \left(\frac{4}{4-e^{-j\omega}}\right) e^{-j5\omega}$$

Then, $x[n] = z[n] \cos\left(\frac{\pi n}{3}\right)$

Using modulation property:

$$X_d(\omega) = \frac{1}{2} Z_d\left(\omega+\frac{\pi}{3}\right) + \frac{1}{2} Z_d\left(\omega-\frac{\pi}{3}\right)$$

$$Z_d(\omega) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)^4 \left(\frac{e^{-j5(\omega+\frac{\pi}{3})}}{4-e^{-j(\omega+\frac{\pi}{3})}} + \frac{e^{-j5(\omega-\frac{\pi}{3})}}{4-e^{-j(\omega-\frac{\pi}{3})}}\right)$$

Problem 2

(h) $x[n] = n\left(\frac{1}{4}\right)^n u[n-3] = \left(\frac{1}{4}\right)^3 n \left(\frac{1}{4}\right)^{n-3} u[n-3]$

$$= \left(\frac{1}{4}\right)^3 (n-3)\left(\frac{1}{4}\right)^{n-3} u[n-3] + \left(\frac{1}{4}\right)^3 (3)\left(\frac{1}{4}\right)^{n-3} u[n-3]$$

Use the following DTFT pairs and the time shift property:

$$a^n u[n] \xleftrightarrow{\text{DTFT}} \frac{1}{1-ae^{-j\omega}}, \quad \text{if } |a|<1$$

$$n a^n u[n] \xleftrightarrow{\text{DTFT}} \frac{ae^{-j\omega}}{(1-ae^{-j\omega})^2}, \quad \text{if } |a|<1$$

$$X_a(\omega) = \left(\frac{1}{4}\right)^3 \frac{\frac{1}{4}e^{-j\omega}\cdot e^{-j3\omega}}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2} + 3\left(\frac{1}{4}\right)^3 \frac{1\cdot e^{-j3\omega}}{1-\frac{1}{4}e^{-j\omega}}$$

$$= \frac{\left(\frac{1}{4}\right)^2 e^{-j4\omega}}{(4-e^{-j\omega})^2} + \frac{3\left(\frac{1}{4}\right)^2 e^{-j3\omega}}{4-e^{-j\omega}}$$

Problem 3

$x[n]$ is real-valued if and only if its DTFT $X_d(\omega)$ is Hermitian symmetric, ie, $X_d(\omega) = X_d^*(-\omega)$.

(a) $X_d(\omega) = \sin^2(\omega) - j\sin(2\omega)$

$$X_d^*(-\omega) = \left(\sin^2(-\omega) - j\sin(-2\omega)\right)^*$$
$$= \sin^2(\omega) - j\sin(2\omega)$$
$$= X_d(\omega)$$, therefore $x[n]$ is real-valued.

(b) $X_d(\omega) = e^{j\cos\omega} - e^{j\sin\omega}$

$$X_d^*(-\omega) = \left(e^{j\cos(-\omega)} - e^{j\sin(-\omega)}\right)^*$$
$$= \left(e^{j\cos\omega} - e^{-j\sin\omega}\right)^*$$
$$= e^{-j\cos\omega} - e^{j\sin\omega}$$
$$\neq X_d(\omega)$$, therefore $x[n]$ is not real-valued.

(c) $X_d(\omega) = e^{\cos\omega} - e^{j\sin\omega}$

$$X_d^*(-\omega) = \left(e^{\cos(-\omega)} - e^{j\sin(-\omega)}\right)^*$$
$$= \left(e^{\cos\omega} - e^{-j\sin\omega}\right)^*$$
$$= e^{\cos\omega} - e^{j\sin\omega}$$
$$= X_d(\omega)$$, therefore $x[n]$ is real-valued.

Problem 4

(a) $y[n] = x^*[n]$

$$Y_d(\omega) = \sum_{n=-\infty}^{\infty} x^*[n] e^{-j\omega n}$$

$$= \left[\sum_{n=-\infty}^{\infty} x[n] e^{j\omega n} \right]^*$$

$$= X_d^*(-\omega)$$

(b) $y[n] = x^*[-n+4]$

Let $z[n] = x^*[n]$ and $w[n] = z[-n]$, then $y[n] = w[n-4]$

From 4 (a) above, $Z_d(\omega) = X_d^*(-\omega)$

Using time reversal property, $W_d(\omega) = Z_d(-\omega)$

Using time shift property, $Y_d(\omega) = W_d(\omega) e^{-j4\omega}$

Combining all of the above:

$W_d(\omega) = X_d^*(\omega)$ and $Y_d(\omega) = X_d^*(\omega) e^{-j4\omega}$

(c) $y[n] = x[n-2] \cos(\pi n/5)$

Let $z[n] = x[n-2]$, then $y[n] = z[n] \cos(\pi n/5)$

Using time shift property, $Z_d(\omega) = X_d(\omega) e^{-j2\omega}$

Using the modulation property, $Y_d(\omega) = \frac{1}{2}\left[Z_d(\omega - \frac{\pi}{5}) + Z_d(\omega + \frac{\pi}{5}) \right]$

Combining the above:

$$Y_d(\omega) = \frac{1}{2}\left[X_d(\omega - \tfrac{\pi}{5}) e^{-j2(\omega - \frac{\pi}{5})} + X_d(\omega + \tfrac{\pi}{5}) e^{-j2(\omega + \frac{\pi}{5})} \right]$$

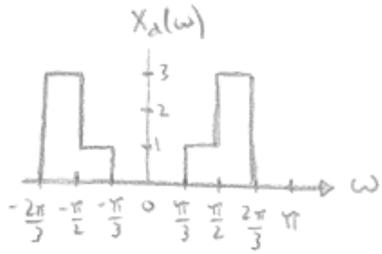
Problem 5

(a) $X_d(\omega) = 1 + 3e^{-j2\omega} - j3\sin(3\omega) = 1 + 3e^{-j2\omega} - j3\left(\dfrac{e^{j3\omega} - e^{-j3\omega}}{2j}\right)$

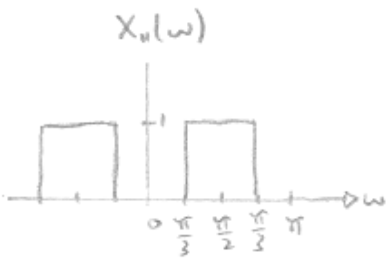
$= 1 + 3e^{-j2\omega} - \frac{3}{2}e^{j3\omega} + \frac{3}{2}e^{-j3\omega}$

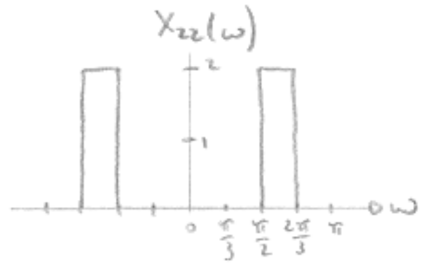
$x[n] = \delta[n] + 3\delta[n-2] - \frac{3}{2}\delta[n+3] + \frac{3}{2}\delta[n-3]$

(b) Assuming $x[n]$ is real, $X_d(\omega) = X_d^*(-\omega)$. Therefore,

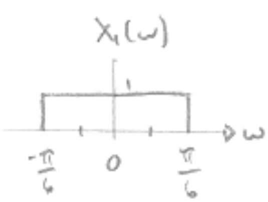


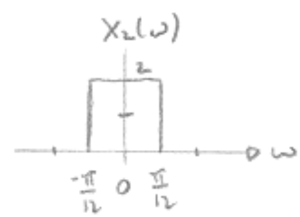
Next, divide $X_d(\omega)$ into two parts, $X_d(\omega) = X_{11}(\omega) + X_{22}(\omega)$





Then, $X_{11}(\omega)$ and $X_{22}(\omega)$ can be given in terms of $X_1(\omega)$ and $X_2(\omega)$ below.





$X_d(\omega) = X_{11}(\omega) + X_{22}(\omega) = X_1(\omega - \frac{\pi}{2}) + X_1(\omega + \frac{\pi}{2}) + X_2(\omega - \frac{7\pi}{12}) + X_2(\omega + \frac{7\pi}{12})$

Using the modulation property:

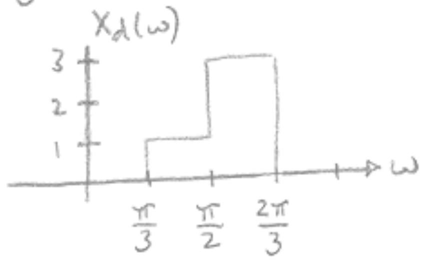
$x[n] = 2x_1[n]\cos(n\frac{\pi}{2}) + 2x_2[n]\cos(n\frac{7\pi}{12})$

$\frac{1}{2\pi}\int_{-a}^{a} e^{j\omega n}\,d\omega = \frac{1}{2\pi jn}\left(e^{jan} - e^{-jan}\right) = \frac{\sin(na)}{\pi n} = \frac{a}{\pi}\,\text{sinc}(na)$

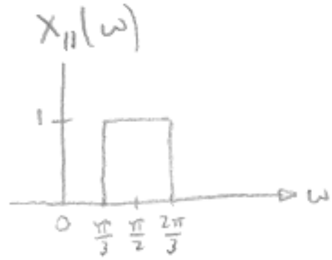
$x[n] = \frac{1}{3}\,\text{sinc}\left(\frac{\pi n}{6}\right)\cos\left(\frac{\pi n}{2}\right) + \frac{1}{3}\,\text{sinc}\left(\frac{\pi n}{12}\right)\cos\left(\frac{7\pi n}{12}\right)$

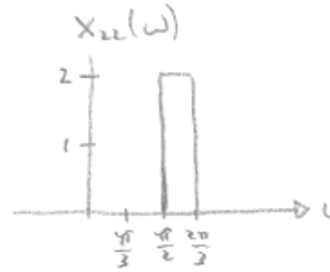
Problem 5

(b) Assuming $x[n]$ is complex,

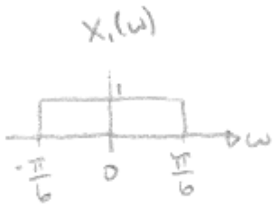


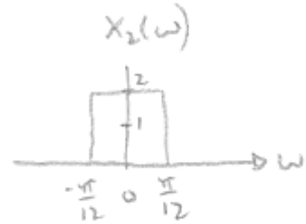
Divide $X_d(\omega)$ into two parts, $X_d(\omega) = X_{11}(\omega) + X_{22}(\omega)$





$X_{11}(\omega)$ and $X_{22}(\omega)$ are frequency shifted versions of $X_1(\omega)$ and $X_2(\omega)$ given below.





$$X_d(\omega) = X_{11}(\omega) + X_{22}(\omega) = X_1\left(\omega - \frac{\pi}{2}\right) + X_2\left(\omega - \frac{7\pi}{12}\right)$$

Using the frequency shifting property:

$$x[n] = x_1[n]\, e^{j\frac{\pi}{2}n} + x_2[n]\, e^{j\frac{7\pi}{12}n}$$

$$x[n] = \frac{1}{6}\,\text{sinc}\left(\frac{\pi}{6}n\right) e^{j\frac{\pi}{2}n} + \frac{1}{6}\,\text{sinc}\left(\frac{\pi}{12}n\right) e^{j\frac{7\pi}{12}n}$$

Problem 6

(a) $X_d(0) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\cdot 0\cdot \omega} = \sum_{n=-\infty}^{\infty} x[n]$

$= (3 - 1 + 2 - 2 + 1 - 3) + j(1 + 2 + 3 + 3 + 2 + 1)$

$= j\,12$

(b) $X_d(\pi) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\pi n} = \sum_{n=-4}^{1} x[n]\,(-1)^n$

$= \sum_{n \text{ even}} x[n] - \sum_{n \text{ odd}} x[n]$

$= [(3 + 2 + 1) - (-1 - 2 - 3)] + j[(1 + 3 + 2) - (2 + 3 + 1)]$

$= 0$

(c) Using the inverse DTFT equation:

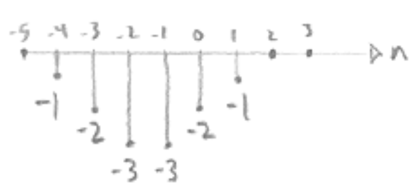
$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(\omega)\, e^{j\omega n}\, d\omega$

Let $n = 0$ and solve for the desired integral

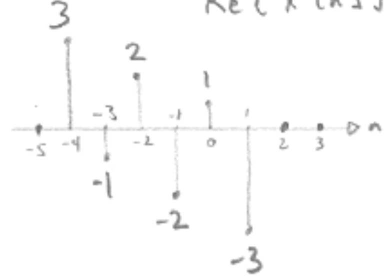
$\int_{-\pi}^{\pi} X_d(\omega)\, d\omega = 2\pi\, x[0] = 2\pi(1 + 2j) = \pi(2 + 4j)$

(d) $X_d^*(-\omega) = \sum_{n=-\infty}^{\infty} x^*[n]\, e^{-j\omega n}$, the desired signal is $x^*[n]$

$Im\{x^*[n]\}$



$Re\{x^*[n]\}$



Problem 7

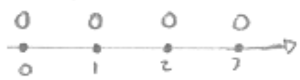
(a) $x[n] = \delta[n-4], \quad 0 \le n \le 3$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j 2\pi k \frac{n}{N}}, \quad 0 \le k \le N-1$$

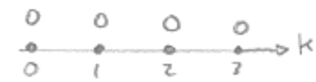
$$= \sum_{n=0}^{3} \delta[n-4]\, e^{-j 2\pi k \frac{n}{4}}, \quad 0 \le k \le 3$$

$$= 0, \quad 0 \le k \le 3$$

$|X[k]|$



$\measuredangle X[k]$



(b) $x[n] = \begin{cases} 3, & 0 \le n \le 3 \\ 0, & 4 \le n \le 7 \end{cases}$

$$X[k] = \sum_{n=0}^{3} 3\, e^{-j 2\pi k \frac{n}{8}} + \sum_{n=4}^{7} 0 \cdot e^{-j 2\pi k \frac{n}{8}}$$

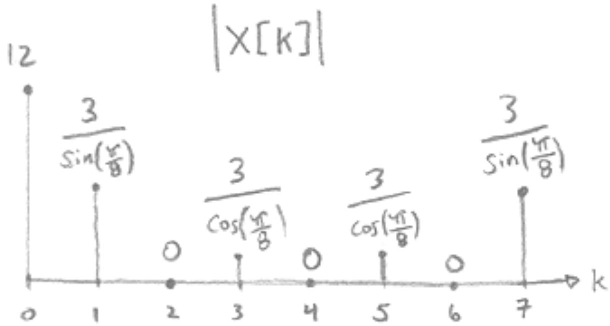
$$= 3 \left(\frac{1 - e^{-j\pi k}}{1 - e^{-j\pi \frac{k}{4}}} \right) = 3\, \frac{e^{-j\frac{\pi}{2}k}\left(e^{j\frac{\pi}{2}k} - e^{-j\frac{\pi}{2}k}\right)}{e^{-j\frac{\pi}{8}k}\left(e^{j\frac{\pi}{8}k} - e^{-j\frac{\pi}{8}k}\right)}$$

$$= 3\, e^{-j\frac{3\pi}{8}k} \left(\frac{\sin(\frac{\pi}{2}k)}{\sin(\frac{\pi}{8}k)} \right), \quad 1 \le k \le 7$$

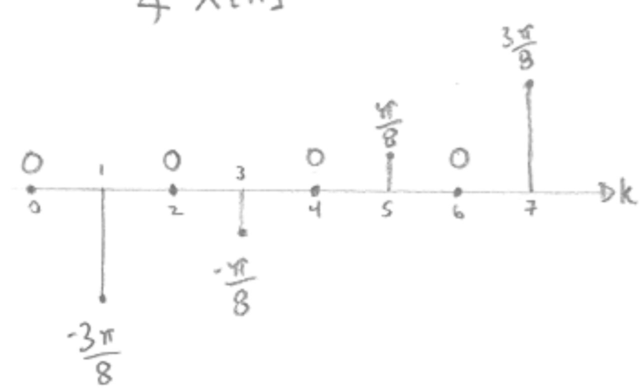
For $k = 0$, use DFT sum:

$$X[0] = \sum_{n=0}^{3} x[n] = 12$$

$|X[k]|$



$\measuredangle X[k]$



Problem 7

(c) $x[n] = \cos(\frac{\pi}{3}n) \quad , \quad 0 \le n \le 5$

$$X[k] = \sum_{n=0}^{5} \cos(\tfrac{\pi}{3}n)\, e^{-j2\pi k\frac{n}{6}}$$

$$= \sum_{n=0}^{5} \left[\frac{e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3}n}}{2}\right] e^{-j\pi\frac{k}{3}}$$

$$= \frac{1}{2}\sum_{n=0}^{5} \left(e^{j\frac{\pi}{3}(1-k)}\right)^n + \frac{1}{2}\sum_{n=0}^{5} \left(e^{-j\frac{\pi}{3}(1+k)}\right)^n$$

$$= \frac{1}{2}\left(\frac{1-e^{j2\pi(1-k)}}{1-e^{j\frac{\pi}{3}(1-k)}}\right) + \frac{1}{2}\left(\frac{1-e^{-j2\pi(1+k)}}{1-e^{-j\frac{\pi}{3}(1-k)}}\right), \quad \begin{matrix} 0 \le k \le 5 \\ k \ne 1 \end{matrix}$$

$$X[1] = \frac{1}{2}\sum_{n=0}^{5} 1 + \frac{1}{2}\sum_{n=0}^{5} \left(e^{-j\frac{2\pi}{3}}\right)^n$$

$$= 3 + \frac{1}{2}\left(\frac{1-e^{-j4\pi}}{1-e^{-j\frac{2\pi}{3}}}\right) = 3 \quad , \quad k=1$$

(d) $x[n] = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \qquad 0 \le n \le 12$

$$X[k] = \sum_{n=0}^{6} 1 \cdot e^{-j2\pi k\frac{2n}{13}}$$

$$= \sum_{n=0}^{6} \left(e^{-j\frac{4\pi k}{13}}\right)^n$$

$$= \frac{1-e^{-j\frac{28\pi k}{13}}}{1-e^{-j\frac{4\pi k}{13}}} \quad , \quad k \ne 0$$

$$X[k] = \begin{cases} 7 & , \quad k=0 \\ \dfrac{1-e^{-j\frac{28\pi k}{13}}}{1-e^{-j\frac{4\pi k}{13}}} & , \quad k \ne 0 \end{cases}$$