1. Consider two infinite, plane parallel, perfectly conducting plates at $z = 0$ and $z = z_o > 0$, which hold equal and opposite surface charge densities and are kept at potentials $V = 0$ and $V = V_p > 0$, respectively. The region between the plates is filled with two slabs of perfect dielectric materials having permittivities ϵ_1 for $0 < z < d$ (region 1) and ϵ_2 for $d < z < z_o$ (region 2).

   a) Find the general solution for the electric potentials, $V = V(z)$, (in terms of V_p, d, z_0, ϵ_1, and ϵ_2) in the two regions by solving Laplace's equation piecewise and enforcing the continuity of $V(z)$ at $z = d$. **Hint**: you will also need to use the fact that there is no surface charge accumulation at a boundary between two perfect dielectrics.

   b) Given that $z_o = 4d = 2$ m, $\epsilon_1 = 3\epsilon_o$, $\epsilon_2 = \epsilon_o$, and $\mathbf{E}(0 < z < d) = -5\hat{z}$ V/m, what is the electrostatic potential V_p on the conductor plate at $z = z_0$?

   c) Given the parameters in part (b) above, what is the surface charge density ρ_s on the plate at $z = z_o$?

2. Consider two conducting plates positioned on $z = 0$ and $z = 3$ m surfaces. The plates are grounded and both have zero potential. In between the plates, on $z = 1$ m surface, there is a uniform and static surface charge of $6\,\mathrm{C/m^2}$. The permittivity of the region $z < 1$ m is ϵ_o, whereas it is $2\epsilon_o$ in the region $z > 1$ m. Determine the surface charge densities at $z = 0$ and $z = 3$ m.

   Hint: Let V_o denote the electrostatic potential at $z = 1$. Express the electric and displacement fields above and below $z = 1$ in terms of V_o and use boundary condition equations on $z = 0$, 1, and 3 m surfaces to relate V_o to pertinent surface charge densities.

3. The region between two infinite, plane parallel, perfectly conducting plates at $z = 0$ and $z = z_0$ m is filled with two slabs of perfect dielectric materials having constant electric permittivities ϵ_1 for $0 < z < d$ m (region 1) and ϵ_2 for $d < z < z_0$ (region 2), where $d = 4$ m. The bottom plate is held at constant potential V_0, while the top plate is grounded, such that $V(z_0) = 0$. The electrostatic field between the plates is known to be

$$\mathbf{E}(z) = \begin{cases} \frac{4\epsilon_2}{\epsilon_1+8\epsilon_2}\hat{z}\,\frac{\mathrm{V}}{\mathrm{m}}, & 0 < z < d \\ \frac{4\epsilon_1}{\epsilon_1+8\epsilon_2}\hat{z}\,\frac{\mathrm{V}}{\mathrm{m}}, & d < z < z_0 \end{cases}$$

   a) Verify that the above field satisfies Maxwell's boundary condition regarding $\mathbf{D}$ at the boundary between the two dielectric slabs.

   b) Write the expression for the electrostatic potential $V(z)$ throughout both regions (i.e., for $0 < z < z_0$ m) in terms of ϵ_1, ϵ_2, z_0, V_0, and d.

   c) Determine ϵ_1 if $\epsilon_2 = 2\epsilon_0$ and the surface charge density on the bottom plate (at $z = 0$ m) is $\rho_s = 4\epsilon_0\ \mathrm{C/m^2}$.

   d) What is the thickness of region 2 in meters if $V_0 = 3$ V?

   e) Does $V(z)$ determined in part (b) satisfy Laplace's equation in the region $0 < z < z_0$ m? Explain your answer.

   f) What would be the capacitance C of the structure described above if the parallel plates at $z = 0$ and $z = z_0$ m were constrained to have finite areas $A = W^2$ (where $W \gg 1$ m) facing one another? In this calculation ignore the fringing fields, and express C as a function of ϵ_1, ϵ_2, d, z_0 and A.

4. Copper is a highly conducting metal with a **conductivity** of $\sigma = 5.8 \times 10^7$ S/m and a free-electron density of $N_e = 8.45 \times 10^{28}$ m^{-3}.

   a) Determine the resistance R of a copper wire of radius $r = 1.4$ mm and length $d = 180$ m using

   $$R = \frac{1}{G} = \frac{d}{A\sigma}$$

   from Lecture 10. A denotes the cross-sectional are of the wire.

   b) What would be the magnitude of electric field $\mathbf{E}$ within the wire of part (a) if the wire were conducting a 1 A current? You may assume a uniform current distribution across the wire cross section.

   c) With the electric field determined from part (b), what would be the mean speed $|\mathbf{v}|$ of an electron in the wire? **Hint:** first deduce the current density $\mathbf{J} = \sigma\mathbf{E}$ in the wire, and then use the fact that $\mathbf{J} = N_e q\mathbf{v}$ in the copper wire, with $q = -e = -1.6 \times 10^{-19}$ C denoting the charge of an electron.

   d) How long would it take the electron in part (c) to drift from one end of the wire to the other? **Hint:** you may be surprised by the result of your calculation.

5. Consider a pair of metallic spherical shells of radii a and $b > a$ with their centers coinciding with the origin of the reference coordinate system. The medium between the shells has permittivity $\epsilon = 2\epsilon_o$ and conductivity $\sigma = 2 \times 10^{-6}$ S/m.

   a) Use the integral form of Gauss's law to relate the static field between the shells to charge Q residing on the (outer) surface of the inner shell with radius a and integrating the field from radius a to b obtain the relation $Q = CV$ where V is the potential drop from the inner to outer shell and

   $$C = 4\pi\epsilon\frac{ab}{b-a}$$

   b) Taking the limit of capacitance C as $b \to \infty$, obtain the capacitance (in pF) of a metallic shell of radius $a = 1$ m embedded in an infinite region of permittivity $\epsilon = 2\epsilon_o$ and conductivity $\sigma = 2 \times 10^{-6}$ S/m.

   c) What is the conductance (in μS) of the metallic shell configuration described in (b). **Hint:** make use of $G = \frac{\sigma}{\epsilon}C$ relation from class notes.

   d) Develop an RC equivalent circuit to represent the physical system described above. Use this equivalent circuit to derive a differential equation that governs the temporal variation of the charge on the spherical shell. **Hint:** in this circuit the lossy capacitor (see Lect 10, page 4) conducts no external current (since external circui is an effective open in this case — there is no path for the return current).

   e) Assume $Q(t) = 1$ C on the shell with radius a at $t = 0$. The radial electric field produced by $Q(t)$ will drive a radial current to discharge the shell. Obtain an expression for $Q(t)$ in $t > 0$ that describes the discarge of the metallic shell by solving the differential equation obtained in (d).