# PDEs:

Partial Differential Equations are differential equations satisfied by a function (or functions) of more than one variable.

## Some standard PDE s

i) Laplaces Equation

Encountered in heat and mass transfer, fluid mechanics elasticity, electrostatics.

The Laplacian of a function $\phi$ is written as

$$\nabla^2 \phi$$

and defined in vector calculus notation as

$$\nabla^2 \phi = \nabla \cdot \nabla \phi \qquad \text{" div grad"}$$

In 3D cartesian coordinates

$$\left( \frac{\partial}{\partial x} \underline{i} + \frac{\partial}{\partial y} \underline{j} + \frac{\partial}{\partial z} \underline{k} \right) \cdot \left( \frac{\partial \phi}{\partial x} \underline{i} + \frac{\partial \phi}{\partial y} \underline{j} + \frac{\partial \phi}{\partial z} \underline{k} \right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Laplaces Equation $\boxed{\nabla^2 \phi = 0}$

A function $\phi$ is called harmonic if it satisfies Laplaces equation.

(ii) Poisson's Equation

$$\nabla^2 \phi = \rho(x, y, z)$$

Application: In electrostatics the potential $\phi$ satisfies Poisson equation.

(iii) Heat (or diffusion) equation

$$\frac{\partial \phi}{\partial t} = \nu \nabla^2 \phi, \qquad \nu \text{ constant}$$

Application: Temperature change in a solid.

(iv) Wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi$$

Application: Acoustics, $c$ - sound speed.

(v) Helmholtz Equation

$$\nabla^2 \phi = -k^2 \phi \qquad k\text{ - constant.}$$

(vi) Nonlinear Example: Navier-Stokes Equation

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u} + \frac{1}{\rho}\nabla P = \nu \nabla^2 \underline{u}$$

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u}) = 0$$

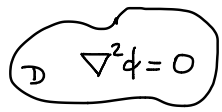
4 coupled nonlinear PDEs

$\underline{u}$ – velocity
$P$ – pressure
$\rho$ – density
$\nu$ – viscosity

# Laplaces Equation

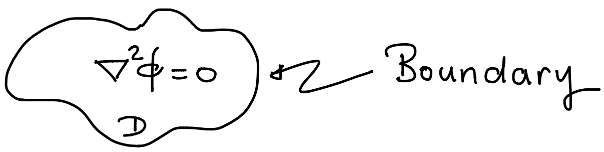
## Maximum/Minimum Principle

Let $\phi$ be a harmonic function in a 3d region $D$. Then $\phi$ has no max. or min. inside $D$




## Boundary Value Problems

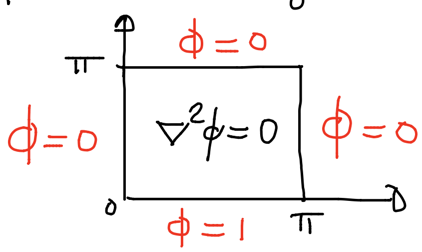
We want to find the solution of Laplaces Equation in some 2d or 3d region $D$ subject to some boundary conditions.




Dirichlet Boundary Conditions involve specifying $\phi$ on the boundary of $D$. (Think of steady temperature in a room with the temperature on the walls known.)

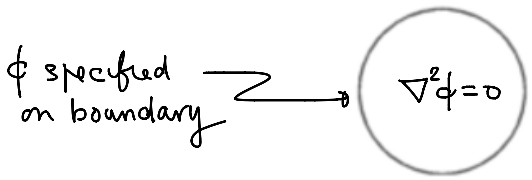
Examples

(i) Solve Laplace's Equation inside square region $0 < x < \pi$, $0 < y < \pi$ with $\phi = 0$ on 3 edges and $\phi = 1$ on lower edge.

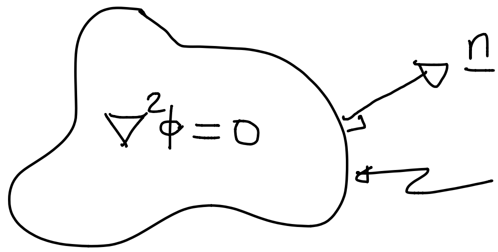


Max/Min principle — inside square $0 < \phi < 1$

(ii) Laplaces Equation in a disc.




Neumann Boundary Conditions involve specifying directional derivative of $\phi$ in direction of unit normal to boundary




$$D_{\underline{n}}\phi = \nabla\phi\cdot\underline{n} = g(\underline{x})$$

Think of temperature in a room with perfect insulation $D_{\underline{n}}\phi = 0$ on the walls. Heat walls $D_{\underline{n}}\phi = g(\underline{x})$.

## Uniqueness Theorem

If there is a solution to Laplaces equation subject to DBC/NBC then it is unique.

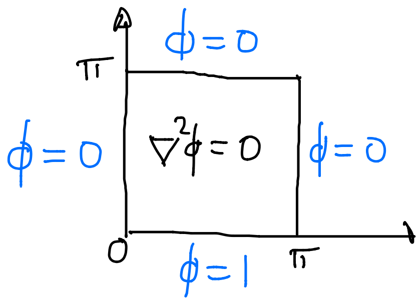
## Separation of Variables

A technique for solving linear pdes. Idea is to generate a large set of solutions (actually an infinite nº). None of these on their own satisfy the boundary conditions <u>but</u> taking a linear combination of these solutions we can find a solution which also satisfies the boundary conditions.

<u>Example</u> Solve Laplaces equation in the square

$$0 \le x \le \pi, \quad 0 \le y \le \pi$$

subject to the constant boundary conditions shown.




$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$$

Try $\phi(x,y) = X(x)\,Y(y)$ Separable form

Function of $x$ Function of $y$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \Rightarrow X''Y + XY'' = 0$$

Divide by $\phi = XY$

$$\frac{X''Y}{XY}+\frac{XY''}{XY}=0$$

$$\Rightarrow \quad \frac{X''}{X}+\frac{Y''}{Y}=0$$

or

$$\underbrace{\frac{X''}{X}}_{\text{independent of } y} = \underbrace{-\frac{Y''}{Y}}_{\text{independent of } x}$$

$$\therefore \quad \underbrace{\frac{X''}{X}}_{\text{independent of } y} = \underbrace{-\frac{Y''}{Y}}_{\text{independent of } x} = \underbrace{\text{constant}}_{\text{independent of } x \text{ and } y}$$

Three cases constant $\begin{matrix} > 0 \\ = 0 \\ < 0 \end{matrix}$

A. constant $= k^2 > 0$.

$$\frac{X''}{X} = -\frac{Y''}{Y} = k^2$$

OR

$$X'' - k^2 X = 0$$

$$Y'' + k^2 Y = 0$$

Independent solutions

$$X(x) = e^{kx}, \quad e^{-kx}$$

$$Y(y) = \cos ky, \quad \sin ky$$

Then we have solutions of form

$$\phi(x,y) = \begin{cases} e^{kx} \cos ky \\ e^{kx} \sin ky \\ e^{-kx} \cos ky \\ e^{-kx} \sin ky \end{cases}$$

B. constant $= 0$

$$X''(x) = Y''(y) = 0$$

Independent solutions

$$X(x) = x, 1 \qquad \left( X_{general} = c_1 x + c_2 1 \right)$$

$$Y(y) = y, 1$$

$$\phi(x,y) = \begin{cases} 1 \\ y \\ x \\ xy \end{cases}$$

C. Constant $= -k^2 < 0$.

$$\frac{X''}{X} = \frac{-Y''}{Y} = -k^2$$

OR

$$X'' + k^2 X = 0$$
$$Y'' - k^2 Y = 0$$

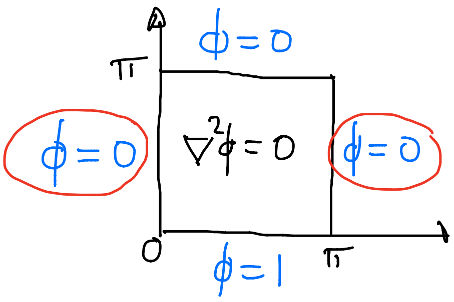
Independent solutions

$X(x) = \cos kx, \ \sin kx$

$Y(y) = e^{ky}, \ e^{-ky}$

$$\phi(x,y) = \begin{cases} \cos kx \, e^{ky} \\ \cos kx \, e^{-ky} \\ \sin kx \, e^{ky} \\ \sin kx \, e^{-ky} \end{cases}$$

None of the solutions satisfy all the boundary conditions.




Look for solutions that at least satisfy left and right boundary conditions of $\phi = 0$.

$$\phi(0,y) = \phi(\pi, y) = 0$$

constant $= -k^2$

$$\phi(x,y) = \begin{cases} \cos kx \, e^{ky} \\ \cos kx \, e^{-ky} \\ \sin kx \, e^{ky} \\ \sin kx \, e^{-ky} \end{cases}$$

Consider the solutions (from case C)

$$\phi(x,y) = e^{ky} \sin kx, \quad e^{-ky} \sin kx$$

If we choose $k$ integer then $\phi(0,y) = \phi(\pi, y) = 0$

'Useful' solutions $\phi = e^{ny} \sin nx, \; e^{-ny} \sin nx, \; n = 1, 2, 3 ..$

None of these solutions on their own satisfy top and bottom boundary conditions. Try a linear combination.

$$\phi(x,y) = \sum_{n=1}^{\infty} b_n e^{ny} \sin nx + \sum_{n=1}^{\infty} C_n e^{-ny} \sin nx$$

Can we satisfy the two remaining boundary conditions?

**Top**

$$\phi(x,\pi) = \sum_{n=1}^{\infty} \sin nx \left(e^{n\pi} b_n + e^{-n\pi} c_n\right) = 0, \quad 0 \le x \le \pi$$

Only way this can be true is that

$$e^{n\pi} b_n + e^{-n\pi} c_n = 0 \qquad \forall n$$

**Bottom**

$$\phi(x,0) = \sum_{n=1}^{\infty} b_n \sin nx + \sum_{n=1}^{\infty} c_n \sin nx = 1$$

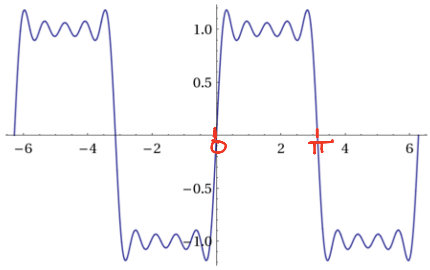
i.e. $$\sum_{n=1}^{\infty} (b_n + c_n) \sin nx = 1, \quad 0 \le x \le \pi$$

We have seen this before !!

Square wave Fourier series

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases} \qquad f(x+2\pi) = f(x)$$

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right) = \frac{4}{\pi} \sum_{\substack{n \\ \text{odd}}} \frac{1}{n} \sin nx$$




i.e. on $0 < x < \pi$ we can represent $f(x) = 1$ as a sum of sines.

We have that

$$b_n + c_n = \begin{cases} \frac{4}{\pi n}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

and (from top)

$$e^{n\pi} b_n + e^{-n\pi} c_n = 0 \quad \forall n$$

OR $$c_n = -e^{2n\pi} b_n$$

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For $n$ even $b_n = c_n = 0$

For $n$ odd

$$b_n - e^{2n\pi} b_n = \frac{4}{n\pi}$$

or $$b_n = \frac{4}{n\pi(1-e^{2n\pi})}$$

Then $$c_n = -e^{2n\pi} b_n = \frac{-4e^{2n\pi}}{n\pi(1-e^{2n\pi})}$$

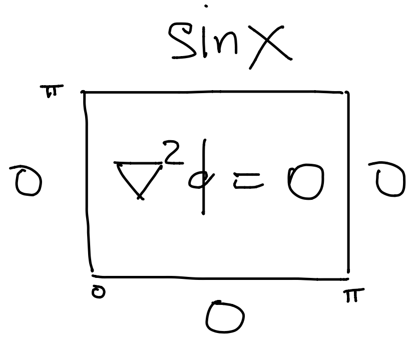
Now we need to substitute $b_n, c_n$ into our expression for $\phi$

$$\phi(x,y) = \sum_{n=1}^{\infty} b_n e^{ny} \sin nx + \sum_{n=1}^{\infty} c_n e^{-ny} \sin nx$$

ie

$$\phi(x,y) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin nx \left( \frac{e^{ny}}{1-e^{2n\pi}} + \frac{e^{-ny}}{1-e^{-2n\pi}} \right)$$

Example (Simpler)




Solve Laplaces equation with b.c

$$\phi(0,y) = 0$$
$$\phi(\pi,y) = 0$$
$$\phi(x,0) = 0$$
$$\phi(x,\pi) = \sin x$$

Start with same solution set as before

$$\phi(x,y) = \sin nx \, e^{ny}, \quad \sin nx \, e^{-ny}$$

Consider $\displaystyle \phi(x,y) = \sum_{n=1}^{\infty} \sin nx \left( b_n e^{ny} + c_n e^{-ny} \right)$

Satisfies left and right b.c s

Impose upper and lower b.c s

upper $$\phi(x,\pi) = \sum_{n=1}^{\infty} \sin nx \left(b_n e^{n\pi} + c_n e^{-n\pi}\right) = \sin x$$

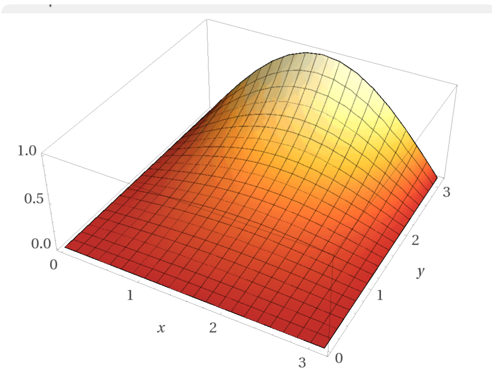
$$\phi(x,0) = \sum_{n=1}^{\infty} \sin nx \left(b_n + c_n\right) = 0$$

$$\Rightarrow \left.\begin{array}{ll} b_n e^{n\pi} + c_n e^{-n\pi} = 0 & \forall\, n \neq 1 \\ b_n + c_n = 0 & \forall\, n \end{array}\right\} \Rightarrow b_n = c_n = 0 \quad n \neq 1$$

$b_1, c_1$

$$\left.\begin{array}{l} b_1 e^{\pi} + c_1 e^{-\pi} = 1 \\ b_1 + c_1 = 0 \end{array}\right\} b_1 = \frac{1}{e^{\pi} - e^{-\pi}}, \quad c_1 = -b_1 = \frac{-1}{e^{\pi} - e^{-\pi}}$$

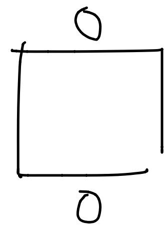
$$\phi(x,y) = \frac{\sin x}{e^{\pi} - e^{-\pi}} \left(e^{y} - e^{-y}\right)$$




$$= \frac{\sin x \, \sinh y}{\sinh \pi}$$

using $\sinh A = \dfrac{e^{A} - e^{-A}}{2}$

The examples all had zero left and right b.cs.
How would you solve problems with




Apply separation of variables as before but now use functions

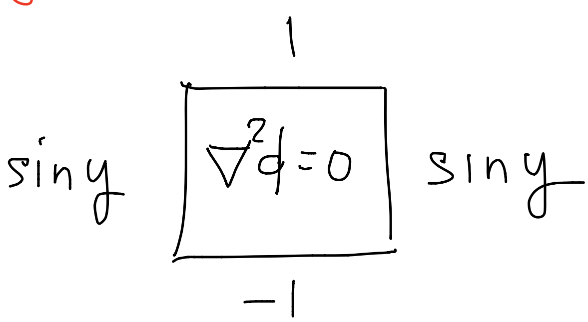
$$e^{nx}\sin ny, \quad e^{-nx}\sin ny$$

Taking a general linear combination

$$\phi(x,y) = \sum_{n=1}^{\infty} \sin ny \left(b_n e^{nx} + c_n e^{-nx}\right)$$

$b_n, c_n$ determined by satisfying left and right b.cs.

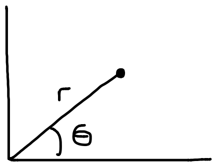
How would you solve



$$\begin{array}{ccc} & 0 & \\ \sin y & \boxed{\nabla^2 \phi_1 = 0} & \sin y \\ & 0 & \end{array} \quad + \quad \begin{array}{ccc} & 1 & \\ 0 & \boxed{\nabla^2 \phi_2 = 0} & 0 \\ & -1 & \end{array}$$

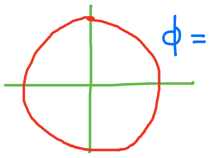
$$\phi(x,y) = \phi_1(x,y) + \phi_2(x,y)$$

# Laplaces Equation in a disc — Polar Coordinates.




$x = r\cos\theta$
$y = r\sin\theta$

$r \geqslant 0, \quad 0 < \theta \leq 2\pi$




## Separation of Variables

Try $\quad \phi(r,\theta) = R(r)\,A(\theta)$

Need 2d Laplacian in polars

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

Without ~~proof~~.

$$\therefore \nabla^2\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)R(r)\,A(\theta) \quad = 0$$

$$= R''(r)\,A(\theta) + \frac{1}{r}R'(r)\,A(\theta) + \frac{1}{r^2}R(r)\,A''(\theta) = 0$$

Divide through by $R(r)\,A(\theta)$

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{A''(\theta)}{A(\theta)} = 0$$

Not separated

Multiply by $r^2$

$$r^2 \frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \frac{A''(\theta)}{A(\theta)} = 0$$

OR

$$\underbrace{r^2 \frac{R''}{R} + r\frac{R'}{R}}_{\text{independent of } r} = \underbrace{-\frac{A''}{A}}_{\text{Independent of } \theta} = \underbrace{\text{constant}}_{\text{Independent of } r,\ \theta}$$

$\theta$ $\qquad$ $r$

Three cases $\qquad$ Constant

$> 0 \qquad = k^2$

$= 0$

$< 0 \qquad = -k^2$

## Case 1

constant $= k^2$

Equation for $A(\theta)$

$$-\frac{A''}{A} = k^2 \qquad \text{ie} \qquad A'' + k^2 A = 0$$

Solutions $\quad A(\theta): \quad \cos k\theta, \ \sin k\theta$

Require $k = n, \quad n = 1, 2, 3 \ldots.$

so that $A(\theta + 2\pi) = A(\theta)$ (Solution single valued)

$\therefore \quad A(\theta): \quad \cos n\theta, \ \sin n\theta$

## Equation for R(r)

$$r^2 \frac{R''}{R} + r\frac{R'}{R} = n^2$$

OR $\quad r^2 R'' + rR' - n^2 R = 0$

This has solutions $R(r) = r^n, \ r^{-n}$

Check $\quad R = r^n$
$R' = n r^{n-1}$
$R'' = n(n-1) r^{n-2}$

$$\begin{aligned} r^2R'' + rR' - n^2R &= r^2 n(n-1) r^{n-2} + rnr^{n-1} - n^2 r^n \\ &= (n^2-n) r^n + n r^n - n^2 r^n \\ &= 0 \quad \checkmark \end{aligned}$$

Putting solutions together

$$\phi(r,\theta) = R(r)\,A(\theta) = \begin{cases} r^n \cos n\theta \\ r^n \sin n\theta \\ r^{-n} \cos n\theta \\ r^{-n} \sin n\theta \end{cases}$$

Case 2 constant $= 0$

A equation $\quad -\frac{A''}{A} = 0 \quad$ OR $\quad A'' = 0$

Solutions $\quad A(\theta): \quad \theta, 1$

not single valued ✗

R equation

$$\frac{r^2 R''}{R} + \frac{rR'}{R} = 0$$

OR $\quad r^2 R'' + rR' = 0$

Solutions $\quad R: \quad 1, \ln r$

$\underline{\ln r}$?

$$r^2\left(-\frac{1}{r^2}\right) + r\left(\frac{1}{r}\right) = 0 \checkmark$$

Putting them together $\quad \phi(r,\theta) = R(r)\,A(\theta) = \begin{cases} \log r \\ 1 \end{cases}$

Case 3 constant $= -k^2$

$$\frac{-A''(\theta)}{A(\theta)} = -k^2$$

OR $\quad A''(\theta) = k^2 A(\theta)$

Solutions $\quad A(\theta) = e^{k\theta}, e^{-k\theta} \qquad$ but $A(\theta+2\pi) \neq A(\theta)$

not single valued

Discard this case.

Possible solutions

$r^n \cos n\theta$

$r^n \sin n\theta$

$r^{-n} \cos n\theta$

$r^{-n} \sin n\theta$

$\log r$

$1$

For Laplace equation in a disc we can discard solutions singular at origin $(r=0)$

Take a linear combination of non-singular solutions

$$1, \quad r^n \cos n\theta, \quad r^n \sin n\theta$$

$$\phi(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta\right)$$

Determine $a_n$ s and $b_n$ s from boundary conditions.

Examples (missing).

In 3d can use spherical polar coordinates.

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial}{\partial\theta} + \cot\theta\right)\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Separation of variables

$$u(r,\theta,\phi) = R(r)\,A(\theta)\,B(\phi)$$