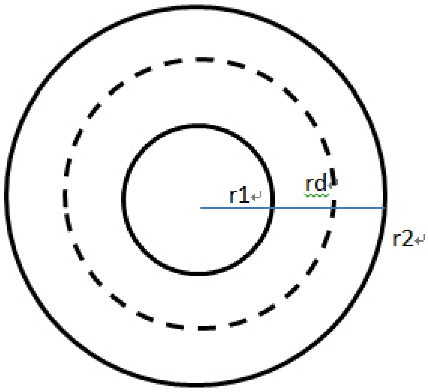
1. The problem can be sketched as shown in the following figure.



a) Refering to Gauss's Law, the electric field in the two dielectric shells are found:

$$\mathbf{E}(r) = -\nabla V = \begin{cases} -A_1\frac{1}{r}\hat{r} & , \quad r_1 < r < r_d \\ -A_2\frac{1}{r}\hat{r} & , \quad r_d < r < r_2 \end{cases}$$

from which we find the potential as

$$V(r) = \begin{cases} A_1 ln(\frac{r}{r_1}) + B_1 & , \quad r_1 < r < r_d \\ A_2 ln(\frac{r}{r_2}) + B_2 & , \quad r_d < r < r_2 \end{cases}$$

Notice our expression satisfies Laplace's equation in cylindrical coordinates:

$$\frac{\partial^2 V(r)}{\partial r^2} + \frac{1}{r}\frac{\partial V(r)}{\partial r} = 0$$

If we force the boundary conditions at $r_1$and r_2, then clearly

$$V(r) = \begin{cases} A_1 ln(\frac{r}{r_1}) & , \quad r_1 < r < r_d \\ A_2 ln(\frac{r}{r_2}) + V_p & , \quad r_d < r < r_2 \end{cases}$$

And

$$A_1 ln(\frac{r_d}{r_1}) = A_2 ln(\frac{r_d}{r_2}) + V_p$$

for the potential at $r = r_d$. Applying the other boundary condition stating that there must be no change between the normal components of the displacement vector $\mathbf{D}$ within the two dielectrics due to the fact that there are no mobile free carriers along it, we can write

$$\begin{aligned} \hat{n}\cdot(\mathbf{D}_1 - \mathbf{D}_2) &= (-\hat{r})\cdot(\epsilon_1(-A_1\hat{r}) - \epsilon_2(-A_2\hat{r})) \\ 0 &= \epsilon_1 A_1 - \epsilon_2 A_2. \end{aligned}$$

Using the last three equations, we find

$$\begin{aligned} A_1 &= \frac{\epsilon_2 V_p}{-ln(\frac{r_d}{r_2})\epsilon_1 + ln(\frac{r_d}{r_1})\epsilon_2} \\ A_2 &= \frac{\epsilon_1 V_p}{-ln(\frac{r_d}{r_2})\epsilon_1 + ln(\frac{r_d}{r_1})\epsilon_2} \end{aligned}$$

from which the electric potential can be written as

$$V(r) = \begin{cases} \frac{\epsilon_2 V_p}{-ln(\frac{r_d}{r_2})\epsilon_1 + ln(\frac{r_d}{r_1})\epsilon_2} ln(\frac{r}{r_1}), & r_1 < r < r_d, \\ \frac{\epsilon_1 V_p}{-ln(\frac{r_d}{r_2})\epsilon_1 + ln(\frac{r_d}{r_1})\epsilon_2} ln(\frac{r}{r_2}) + V_p, & r_d < r < r_2. \end{cases}$$

b) Referring to $\mathbf{E} = -\nabla V$, the electric field inside the dielectrics is given by

$$\mathbf{E}(r) = \begin{cases} \frac{\epsilon_2 V_p}{ln(\frac{r_d}{r_2})\epsilon_1 - ln(\frac{r_d}{r_1})\epsilon_2} \frac{1}{r}\,\hat{r}, & r_1 < r < r_d, \\ \frac{\epsilon_1 V_p}{ln(\frac{r_d}{r_2})\epsilon_1 - ln(\frac{r_d}{r_1})\epsilon_2} \frac{1}{r}\,\hat{r}, & r_d < r < r_2. \end{cases}$$

For $r_1 < r < r_d$, we have

$$V_p = \frac{ln(\frac{r_d}{r_2})\epsilon_1 - ln(\frac{r_d}{r_1})\epsilon_2}{\epsilon_2} E_r r$$

Given that $r_1 = r_d/2 = r_2/4 = 1\,\mathrm{m}$, $\epsilon_1 = 3\epsilon_o$, $\epsilon_2 = \epsilon_o$ and $\mathbf{E}(r_1 < r < r_d) = -5/r\,\hat{r}$, we find that

$$V_p = -\frac{ln(\frac{1}{2}) \times 3\epsilon_o - ln(2)\epsilon_o}{\epsilon_o}(-5) = 13.86\,V$$

c) Given that $r_1 = r_d/2 = r_2/4 = 1\,\mathrm{m}$,$\epsilon_1 = 3\epsilon_o$, $\epsilon_2 = \epsilon_o$ and $V_p = 13.86\,\mathrm{V}$, we have

$$\mathbf{E}(r) = \begin{cases} -5\frac{1}{r}\,\hat{r}\,\frac{\mathrm{V}}{\mathrm{m}}, & r_1 < r < r_d, \\ -15\frac{1}{r}\,\hat{r}\,\frac{\mathrm{V}}{\mathrm{m}}, & r_d < r < r_2. \end{cases}$$

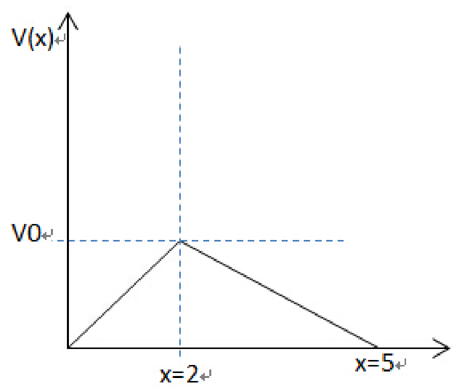
The surface charge density at $r = r_1$ is given by

$$\begin{aligned} \rho_s|_{r=r_1} &= \mathbf{D} \cdot \hat{n}|_{r=r_1} \\ &= D_r^+(r_1) - D_r^-(r_1), \end{aligned}$$

where $D_r^- = 0$. Therefore, we can write

$$\rho_s(z_o) = D_r^+(r_1) = \epsilon_1 E_r(r_1) = -15\epsilon_o\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

2. The plates at $x = 0$ and $x = 5\,\mathrm{m}$ are both grounded and have equipotentials, i.e. $V = 0$. Referring to the hint, we can show the potential change by the distance as in the below figure,



where the lines are drawn straight since both media are homogeneous. Hence, applying the Laplace's equation, the potential in each slab will be given by

$$V(x) = \begin{cases} \frac{1}{2}V_0 x & , 0 < x < 2\,\mathrm{m} \\ -\frac{V_0}{3}x + \frac{5}{3}V_0 & , 2\,\mathrm{m} < x < 5\,\mathrm{m}. \end{cases}$$

From $\mathbf{E} = -\nabla V$, we find

$$\mathbf{E} = -\nabla V = \begin{cases} -\frac{1}{2}V_0\hat{x}\,\frac{\mathrm{V}}{\mathrm{m}} & , 0 < x < 2\,\mathrm{m} \\ \frac{V_0}{3}\hat{x}\,\frac{\mathrm{V}}{\mathrm{m}} & , 2\,\mathrm{m} < x < 5\,\mathrm{m}. \end{cases}$$

Making use of the boundary condition for the interface at $x = 2\,\mathrm{m}$, we write

$$\begin{aligned} \hat{x}\cdot\left(\mathbf{D}^+_{x=2} - \mathbf{D}^-_{x=2}\right) &= -4\epsilon_o\,\frac{\mathrm{C}}{\mathrm{m}^2}, \\ 2\epsilon_o\frac{V_o}{3} - \left(-\frac{1}{2}\epsilon_o V_o\right) &= -4\epsilon_o\,\frac{\mathrm{C}}{\mathrm{m}^2} \end{aligned}$$

from which we find $V_o = -\frac{24}{7}\,\mathrm{V}$. Now, we can apply the same boundary condition to the interfaces at $x = 0$ and $x = 5\,\mathrm{m}$, respectively. Due to the fact that $\mathbf{D} = 0$ for the exterior region, we write

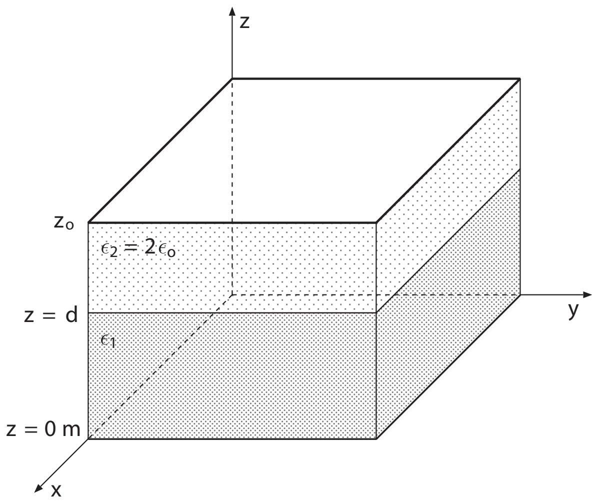
$$D_{x<2} = \rho_{x=0} \Rightarrow -\frac{1}{2}\epsilon_o V_o = \rho_{x=0}$$

from which we find the surface charge density at $x = 0$ as $\rho_{x=0} = -\frac{12}{7}\epsilon_o\,\frac{\mathrm{C}}{\mathrm{m}^2}$. Likewise,

$$-D_{x>2} = \rho_{x=5} \Rightarrow -\frac{1}{3}\epsilon_o V_o = \rho_{x=5}$$

from which we find the surface charge density at $x = 5\,\mathrm{m}$ as $\rho_{x=5} = \frac{8}{7}\epsilon_o\,\frac{\mathrm{C}}{\mathrm{m}^2}$.

3. The geometry of the problem is shown in the following figure.



a) The normal components of the displacement field in both of the dielectric slabs should satisfy the Maxwell's boundary condition

$$\hat{z}\cdot\left(\mathbf{D}^+_{z=d} - \mathbf{D}^-_{z=d}\right) = 0$$

at the interface at $z = d$. For $z < d$, the displacement field is given by

$$D_z^- = \epsilon_1 \cdot E_z^- = \frac{\epsilon_1 \epsilon_2}{-2\epsilon_1 + \epsilon_2} \frac{\mathrm{C}}{\mathrm{m}^2},$$

whereas it is given by

$$D_z^+ = \epsilon_2 \cdot E_z^+ = \frac{\epsilon_1 \epsilon_2}{-2\epsilon_1 + \epsilon_2} \frac{\mathrm{C}}{\mathrm{m}^2}$$

for $z > d$. Looking at two equations, one can say that the field given in the problem satisfies Maxwell's boundary condition regarding $\mathbf{D}$ at the boundary between the two dielectric slabs.

b) The electrostatic potential $V(z)$ for $0 < z < z_o$ is given by $V(z) - V_o = -\int_0^z E_z dz$. Thus, we write

$$V(z) = \begin{cases} -\frac{\epsilon_2}{-2\epsilon_1+\epsilon_2} z \, \mathrm{V}, & 0 < z < d, \\ -\frac{\epsilon_1}{-2\epsilon_1+\epsilon_2} z + \frac{\epsilon_1}{-2\epsilon_1+\epsilon_2} z_o + V_o \, \mathrm{V}, & d < z < z_o. \end{cases} \tag{1}$$

c) Applying the boundary condition at $z = 0$, we have $\hat{z} \cdot \left(\mathbf{D}_{z=0}^+ - \mathbf{D}_{z=0}^-\right) = \rho_s$. Since $\mathbf{D} = 0$ in the exterior region, we have $D_z^+ = \rho_s$ for the interface at $z = 0$. Therefore we write –

$$-4\epsilon_o = \frac{\epsilon_1 \epsilon_2}{-2\epsilon_1 + \epsilon_2}$$

from which we obtain $\epsilon_1 = \frac{12}{5}\epsilon_o$ after replacing ϵ_2 by $3\epsilon_o$.

d) Given that $V_o = 3\,\mathrm{V}$ and using (1), we write

$$V(d) = -\frac{\epsilon_2}{-2\epsilon_1 + \epsilon_2} d = \frac{\epsilon_1}{-2\epsilon_1 + \epsilon_2}(z_o - d) + V_o,$$

from which we obtain $z_o = 1\,\mathrm{m}$. Therefore, the thickness of region is $z_o - d = 1 - 5 = -4\,\mathrm{m}$, which is physically impossible, and we did not mean to have this when we designed the homework. We apologize for any confusion caused by this problem. You will get full credit for an answer of -4m.

e) Laplace's equation

$$\nabla^2 V = 0$$

results from the assumption that the permittivity is constant in space. In our case, however, there are two dielectrics in the region $0 < z < z_o$, which implies that the medium is not homogeneous within this region. Thus, $V(z)$ does not satisfy Laplace's equation at all points in the region (in fact, in this particular case, the equation is not satisfied only at $z = d$).

f) Knowing that $C = \frac{Q}{V}$ where $Q = \rho_s A$ and $V = |E_{1z}|\, d + |E_{2z}|\, (z_o - d)$, we have

$$C = \frac{Q}{V} = \frac{\rho_s A}{|E_{1z}|\, d + |E_{2z}|\, (z_o - d)}.$$

Using Maxwell's boundary conditions, we find that $\rho_s = \epsilon_2 |E_{2z}| = \epsilon_1 |E_{1z}|$. Then, the capacitance can be re-expressed as

$$C = \frac{A}{d/\epsilon_1 + (z_o - d)/\epsilon_2} = \frac{\epsilon_1 \epsilon_2 A}{\epsilon_1 z_o + (\epsilon_2 - \epsilon_1) d}.$$

Note: The same result can be obtained by combining the capacitances of each dielectric in series. Given that

$$C_1 = \epsilon_1 \frac{A}{d} \qquad \text{and} \qquad C_2 = \epsilon_2 \frac{A}{z_o - d},$$

we obtain

$$C = \left(C_1^{-1} + C_2^{-1}\right)^{-1} = \left(\frac{d}{\epsilon_1 A} + \frac{z_o - d}{\epsilon_2 A}\right)^{-1} = \left(\frac{\epsilon_1 z_o + (\epsilon_2 - \epsilon_1)\, d}{\epsilon_1 \epsilon_2 A}\right)^{-1} = \frac{\epsilon_1 \epsilon_2 A}{\epsilon_1 z_o + (\epsilon_2 - \epsilon_1)\, d}.$$

4.

a) Using the formula given in the problem and Lecture 10, the resistance R of this copper wire is calculated as

$$R = \frac{d}{A\sigma} \approx 0.567\,\Omega.$$

b) The voltage drop across the wire is given by $V = E \cdot d = R \cdot I$, where I is the current flowing along the wire. Assuming $I = 1.4\,\mathrm{A}$, the electric field within the wire is

$$E = \frac{R \cdot I}{d} \approx 6.35 \times 10^{-3}\,\frac{\mathrm{V}}{\mathrm{m}}.$$

c) Referring to the hint given in the problem, the current density may be expressed as $\mathbf{J} = \sigma \mathbf{E} = N_e q_e \mathbf{v}_e$ where $q_e = -1.6 \times 10^{-19}\,\mathrm{C}$. Then, the mean speed of an electron is given by

$$|\mathbf{v}_e| = \frac{\sigma\,|\mathbf{E}|}{N_e\,|q_e|} \approx 2.72 \times 10^{-5}\,\frac{\mathrm{m}}{\mathrm{s}}.$$

d) The time it would take an electron to travel from one end of the wire to the other is simply

$$t = \frac{d}{|\mathbf{v}_e|} \approx 4.6 \times 10^{6}\,\mathrm{s}.$$

The explanation is that in this case we are tracking the individual electron movement. Each individual electron has a randomized thermal motion (under no electric field). Upon application of an E field, a small drift velocity is superposed on to this randomized thermal motion. Each electron is accelerated by electric field, but scatters quickly off of impurities and other electrons and vibrations in the material atomic structure. The frequent scattering limits the average drift velocity of the electron. However, the large number of electrons in a conducting material allows for large currents to be carried by the wire.
This is to say when current flows in the wire, the electrons at one end do not need to move to the other end of the wire. Current conduction is the collective movement of all electrons inside the material. As long as the electrons on the whole are drifting towards a definite direction, we will have currents.

5.

a) Applying the integral form of Gauss's law, i.e. $\epsilon \oint \mathbf{E} \cdot d\mathbf{S} = Q_{\mathrm{V}}$ to a sphere of radius r (where $a < r < b$), we get

$$4\pi r^2 E_r = \frac{Q}{\epsilon},$$

from which we find

$$\mathbf{E} = E_r \hat{r} = \frac{Q}{4\pi\epsilon r^2}\hat{r}.$$

Taking the path integral of the electric field $\mathbf{E}$ from a to b, i.e. $V = \int_a^b \mathbf{E} \cdot d\mathbf{l}$,

$$\begin{aligned} V_{ab} &= V_a - V_b = -\int_b^a \mathbf{E} \cdot \hat{r} dr \\ &= \int_a^b \mathbf{E} \cdot \hat{r} dr = \int_a^b \frac{Q}{4\pi \epsilon r^2} dr, \end{aligned}$$

we find the potential drop V from the inner to outer shell as

$$V = \frac{Q}{4\pi\epsilon}(\frac{b-a}{ab}).$$

Since $Q = CV$, we obtain

$$\frac{1}{C} = \frac{1}{4\pi\epsilon}(\frac{b-a}{ab}),$$

implying

$$C = 4\pi\epsilon(\frac{ab}{b-a}).$$

b) Referring to the result we have found in part (a), the capacitance C is given by

$$C = 12\pi\epsilon_o \frac{a}{(1-\frac{a}{b})}.$$

Given that $b \to \infty$ and $a = 1\,\mathrm{m}$, we find

$$C = 12\pi\epsilon_o \approx 333.8\,\mathrm{pF}.$$

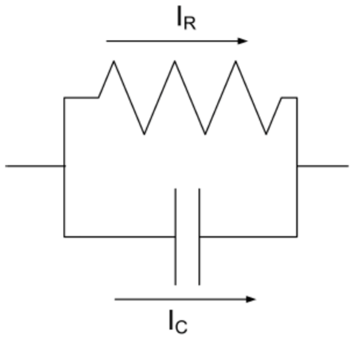
c) Using the formula given in the problem, the conductance G is given by

$$\begin{aligned} G &= \frac{\sigma}{\epsilon} C \\ &= \frac{12\pi\epsilon_o}{3\epsilon_o} \times 1 \times 10^{-6} \\ &= 4\pi \times 10^{-6} \\ &\approx 12.57\,\mu\mathrm{S}. \end{aligned}$$

To find the relation rigorously,

$$\begin{aligned} dR &= d(\tfrac{d}{A\sigma}) \quad \frac{dr}{4\pi\sigma r^2} \\ R &= \int_a^b \frac{r^{-2}}{4\pi\sigma} dr \\ &= \frac{1}{4\pi\sigma}\frac{b-a}{ab} \\ &= \frac{\epsilon}{\sigma C} \\ &= \frac{1}{G} \end{aligned}$$

d) The RC equivalent circuit that represent the physical system described above is a parallel R and C circuit as shown in the below figure.



Here, resistance $R = \frac{1}{G}$ is in parallel with capacitance C. The two terminals represent the surface of the spherical conducting shell and a point at infinity. Because the voltage across both elements is the same (equal to the voltage between the shell and infinity), they must be in parallel. In the given situation, no source is connected across the terminals, only an initial voltage is present, caused by the initial charge on the shell. The charge on the capacitor is $Q = CV$ and the equation describing the current flow can be rewritten in terms of the charge on the shell as follows. We start applying KCL on the node on the left,

$$I_R + I_C = 0,$$

$$GV + C\frac{\partial V}{\partial t} = 0,$$

$$\frac{\partial Q}{\partial t} + \frac{GQ}{C} = 0.$$

e) Initial condition for the differential equation in part (d) is that $Q(t) = 1\,\mathrm{C}$ on the inner shell at $t = 0$. To obtain an expression for $Q(t)$ in $t > 0$, we will refer to the equation we have found in part (d),

$$\frac{\partial Q}{\partial t} + \frac{GQ}{C} = 0,$$

which yields a general solution

$$Q(t) = a_1 e^{-\frac{G}{C}t} + a_2.$$

The initial condition can be written as $Q(0) = 1\,\mathrm{C}$ and at $t \to \infty$ we expect no charge $Q(\infty) = 0$. These conditions lead to $a_2 = 0$ and $a_1 = 1\,\mathrm{C}$. Therefore, the final expression will be

$$\therefore Q(t) = e^{-(\frac{G}{C}t)}\,\mathrm{C}.$$

6.

a) In the equivalent circuit representation of this system, the overall capacitance is a combination of two distinct capacitors. The capacitances C_1 (top capacitor) and C_2 (bottom capacitor) are characterized by

$$C_1 = \epsilon_1 \frac{W(L-h)}{d} \qquad \text{and} \qquad C_2 = \epsilon_2 \frac{Wh}{d},$$

respectively, where $\epsilon_1 = \epsilon_o$, $\epsilon_2 = 2\epsilon_o$, $W = 5\,\text{cm}$, $L = 30\,\text{cm}$ and $d = 1\,\text{mm}$.
Given that the capacitors are held at a potential difference $V = 2\,\text{V}$, they are connected in a parallel configuration since the voltage drop is constant across the conducting plates. Therefore, the total capacitance is written as

$$\begin{aligned} C &= C_1 + C_2 = \epsilon_o \frac{W(L-h)}{d} + 2\epsilon_o \frac{Wh}{d} \\ &= \epsilon_o \frac{W}{d}(L+h) \\ &= \epsilon_o \frac{5 \times 10^{-2}}{1 \times 10^{-3}}(0.3+h) \\ &= 22\epsilon_o\,\text{F}, \end{aligned}$$

from which we obtain $h = 0.14\,\text{m}$.

b) Assuming that the fringing effect at the boundary is negligible, the amount of charge one capacitor can hold is given by

$$Q = \rho A = CV,$$

where ρ is the surface charge density on the interior conducting plates.
Therefore, we have

$$\rho_1 = \frac{C_1 V}{A_1} = \frac{\epsilon_o}{d} V = 2 \times 10^3 \epsilon_o \,\frac{\text{C}}{\text{m}^2},$$

$$\rho_2 = \frac{C_2 V}{A_2} = \frac{2\epsilon_o}{d} V = 4 \times 10^3 \epsilon_o \,\frac{\text{C}}{\text{m}^2},$$

for the conducting plates connected to the positive terminal of the voltage source. The negative terminal has equal but opposite charge densities.

c) Based on the permittivity ($\epsilon = 2\epsilon_o$), the fluid in the tank is likely to be gasoline. Source: http://www.engineeringtoolbox.com/liquid-dielectric-constants-d_1263.html