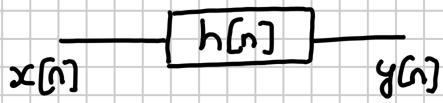
ECE 310: Lecture 24: Generalized Linear Phase

Last Class:



$$Y_d(\omega) = H_d(\omega) \cdot X_d(\omega)$$

$\Rightarrow$

$\Longleftrightarrow$ $\boxed{H_d(\omega) = K e^{-j\omega M}}$

For Causal systems, zero phase is not possible.

① **Linear Phase Response:**

non-negative in $(-\pi, \pi)$

$$H_d(\omega) = R(\omega)\, e^{j\angle H_d \omega}$$

if $\boxed{\angle H_d(\omega) = -\omega M}$, $\omega \in (-\pi, \pi)$

② **Generalized Linear Phase:** (GLP)

$$H_d(\omega) = R(\omega)\, e^{\angle H_d(\omega)}$$

$$\boxed{\angle H_d(\omega) = \alpha - \omega M}$$

③ **Group Delay:**

$$\tau_g(\omega) = \frac{-d}{d\omega} \angle H_d(\omega)$$

$$\boxed{\tau_g(\omega) = M}$$

Time delay that a signal Component of frequency ω undergoes as it passes from input to output of the system.

GLP filters have a constant group delay.

Conditions that guarantee GLP :

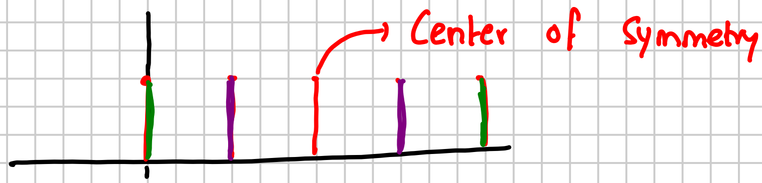
$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

① Type 1 GLP : Type I GLP is defined as a Causal System that has symmetric impulse response,

$$h(n) = h[N-1-n] \qquad 0 < n < N-1 .$$

and the filter length, N is an odd integer.

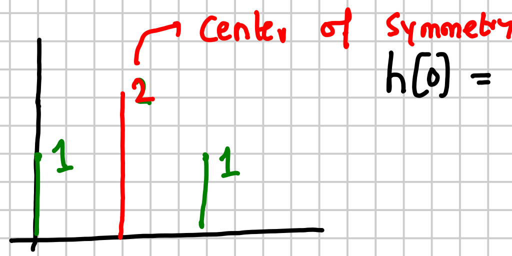
Example : ① $h(n) = \delta(n) + \delta[n-1] + \delta(n-2) + \delta(n-3] + \delta(n-4]$



$N = 5 \Rightarrow h(0) = h(4)$

$h(1) = h(3)$

② $h(n) = \delta(n) + 2\,\delta[n-1] + \delta(n-2] \Rightarrow N=3$



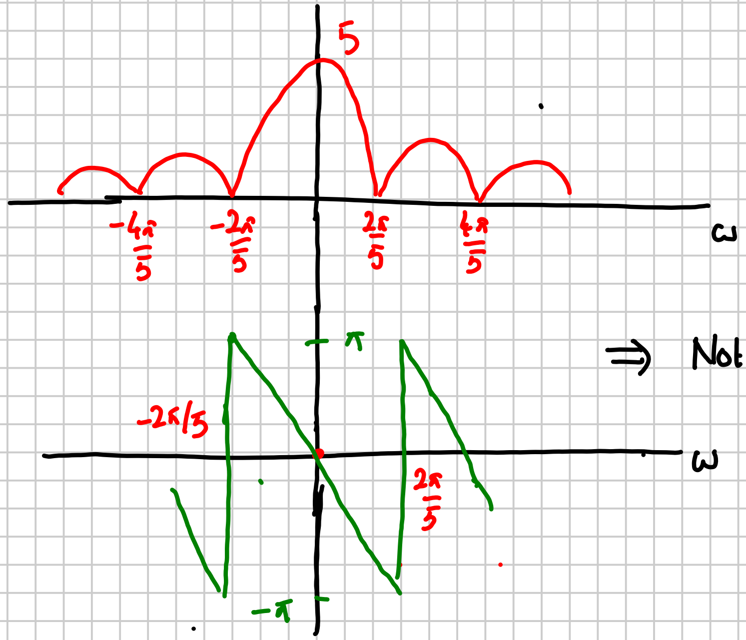
$h(0) = h(2)$

③ $h(n) = \begin{cases} 1 & 0 \leq n \leq 4 \\ 0 & \text{else} \end{cases}$

$$H_d(\omega) = \sum_{n=0}^{4} e^{-j\omega n} = \frac{1 - e^{j5\omega}}{1 - e^{j\omega}}$$

$$\Rightarrow H_d(\omega) = e^{-j5\omega/2} \frac{\sin 5\omega/2}{\sin \omega/2}$$

$\alpha = 0 \ , \ M = \frac{5}{2}$



$\Rightarrow$ Not linear phase

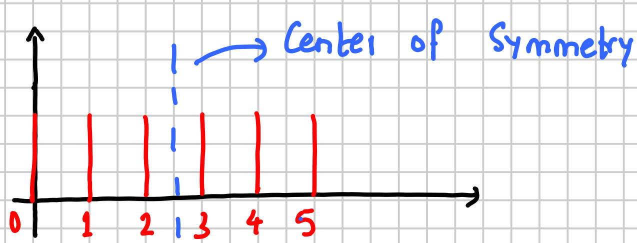
② Type II GLP: Causal System with Symmetric Impulse Response and even filter length.

$$\boxed{h(n) = h[N-1-n] \qquad \text{even } N}$$

Example:

$$h(n) = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5]$$

$N = 6$

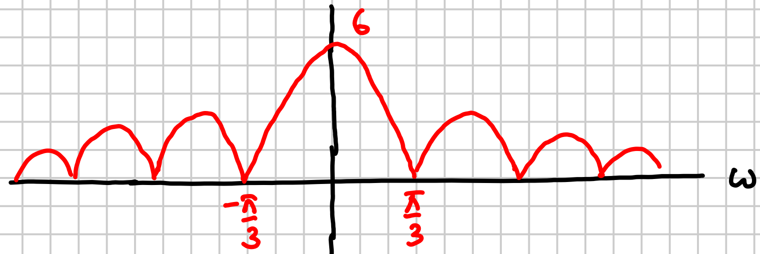


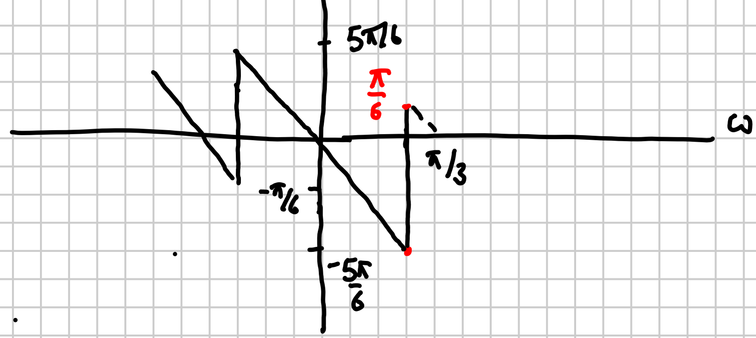
$$H_d(\omega) = \sum_{n=0}^{5} e^{-j\omega n} = \frac{1 - e^{-j6\omega}}{1 - e^{-j\omega}}$$

$$= \frac{e^{-j3\omega}\left(e^{j3\omega} - e^{-j3\omega}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}$$

$$\boxed{H_d(\omega) = e^{-j5\omega/2} \cdot \frac{\sin 3\omega}{\sin \omega/2}}$$

$\alpha = 0 \qquad M = \frac{5}{2}$





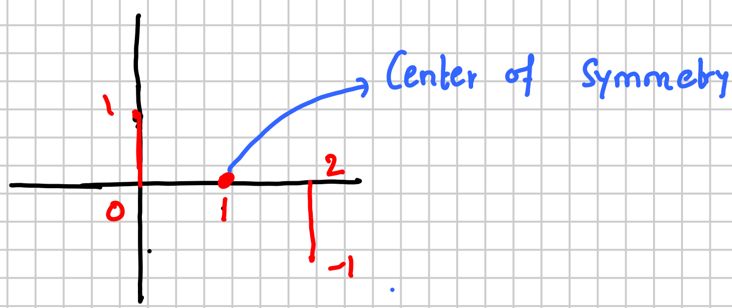
③ Type III GLP: Causal filter with anti-symmetric and odd filter length.

$$\boxed{h[n] = -h[N-1-n] \qquad N : \text{odd}}$$

Also for odd coefficient symmetry:

$$\boxed{h[M] = 0}$$

Example: $h[n] = \delta[n] - \delta[n-2] \Rightarrow N = 2$



$$H_d(\omega) = 1 - e^{-j2\omega} = e^{-j\omega}\left(e^{+j\omega} - e^{-j\omega}\right)$$

$$= e^{-j\omega} \cdot 2j\sin\omega$$

$$\Rightarrow H_d(\omega) = 2\sin\omega \, e^{-j\omega + j\pi/2}$$

$$\boxed{H_d(\omega) = 2\sin\omega \, e^{j\left(\frac{\pi}{2} - \omega\right)}}$$

$\Rightarrow \alpha = \frac{\pi}{2}$ and $M = 1$

(2) $h(n) = [1 \quad 2 \quad -1]$: Antisymmetric

But : $H_d(\omega) = 1 + 2e^{-j\omega} - 1 \cdot e^{-j2\omega}$

$$= e^{-j\omega}\left(e^{+j\omega} + 2 - e^{-j\omega}\right)$$

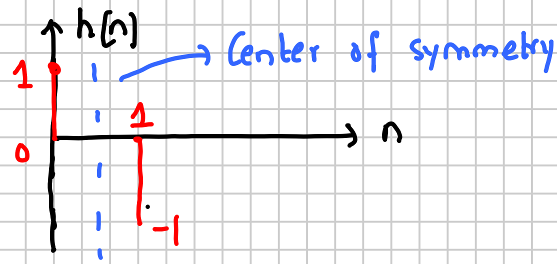
$$H_d(\omega) = 2\,e^{-j\omega}\left(1 + j\sin\omega\right)$$

Not GLP

(4) Type IV GLP : Causal filter with anti-symmetric impulse response and even filter length.

$$\boxed{h(n) = -h[N-1-n] \quad , \quad N : \text{even}}$$

$$h(n) = \delta(n) - \delta[n-1]$$



$$H_d(\omega) = 1 - e^{-j\omega}$$

$$= e^{-j\omega/2}\left[e^{j\omega/2} - e^{-j\omega/2}\right]$$

$$H_d(\omega) = e^{-j\omega/2} \cdot 2j\sin\frac{\omega}{2}$$

$$\Rightarrow H_d(\omega) = 2\sin\frac{\omega}{2}\, e^{-j\frac{\omega}{2} + j\frac{\pi}{2}}$$

$$\Rightarrow \boxed{H_d(\omega) = 2\sin\frac{\omega}{2}\, e^{j\left(\frac{\pi}{2} - \frac{\omega}{2}\right)}}$$

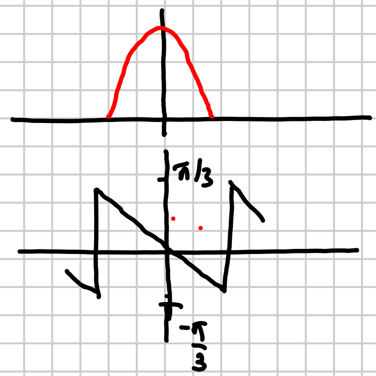
$\alpha = \frac{\pi}{2}$, $M = -\frac{1}{2}$

Examples:

① $h(n) = \{1 \quad -1 \quad 1\}$ Symmetric, odd length $\Rightarrow$ Type I

$$H_d(\omega) = 1 - e^{-j\omega} + e^{j2\omega}$$

$$= e^{-j\omega}\left[2\cos\omega - 1\right] \qquad \alpha = 0 \qquad M = 1$$



② $h(n) = \{\frac{1}{4}, -1, \frac{1}{4}\}$

$$H_d(\omega) = \frac{1}{4} - e^{-j\omega} + \frac{1}{4}e^{-j2\omega}$$

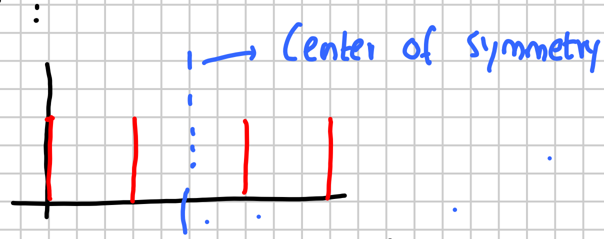
$$= e^{-j\omega}\left[\underbrace{\tfrac{1}{2}\cos\omega - 1}\right]$$

NO SIGN CHANGE IN $[-\pi, \pi]$

$\Rightarrow$ LINEAR PHASE

UNREALIZABLE FILTERS:

① Consider Type II:

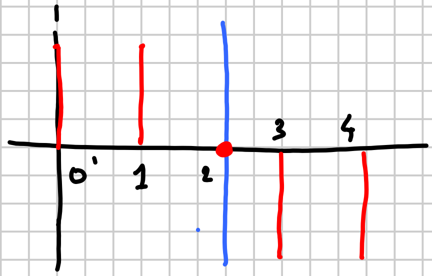


$$H(z) = h_0 + h_1 z^{-1} + h_1 z^{-2} + h_0 z^{-3}$$

$$\Rightarrow H_d(\pi) = h_0 - h_1 + h_1 - h_0 = 0$$

Type II GLP cannot be used to implement high pass / band stop filter

② Type III : Anti-Symmetric (odd-Symmetry), N = Odd



$$H(z) = h_0 + h_1 z^{-1} + 0 \cdot z^{-2} - h_1 z^{-3} - h_0 z^{-4}$$

$$\Rightarrow H_d(0) = 0$$

also $H_d(\pi) = h_0 - h_1 + h_1 - h_0 = 0$

$\Rightarrow$ Cannot realize Low / High / band stop filter

② Type IV : Anti-Symmetric (odd symmetry) & N : even

$$H(z) = h_0 + h_1 z^{-1} - h_1 z^{-2} - h_0 z^{-3}$$

$$\Rightarrow H_d(0) = h_0 + h_1 - h_1 - h_0 = 0$$

$$H_d(\pi) = 2h_0 - 2h_1$$

Cannot realize Low pass / band stop filter