1.

a) For conservation of total energy,

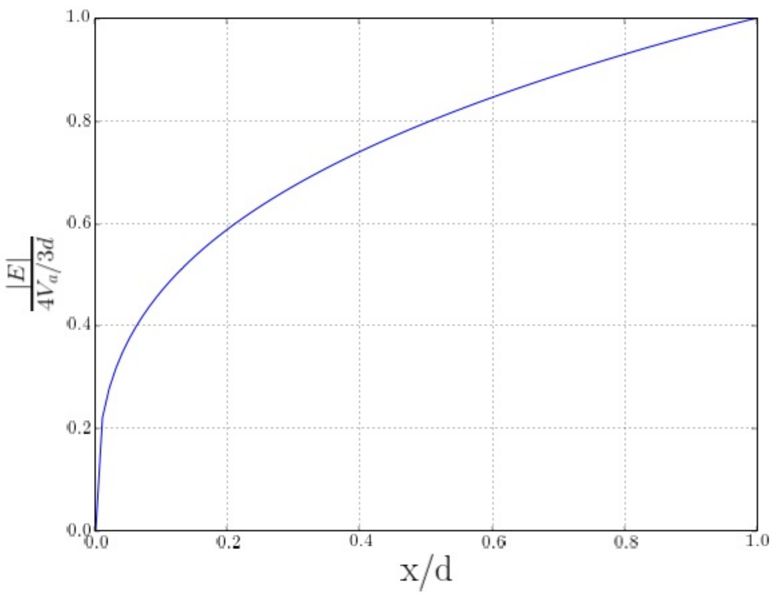
$$\frac{1}{2}m_e v_e^2 = e\,V(x)$$

Rearranging the above equation, we get,

$$v_e = \sqrt{\frac{2eV(x)}{m_e}} = \sqrt{\frac{2eV_a(x/d)^{\frac{4}{3}}}{m_e}} = 8.39\times10^5\,(\frac{x}{d})^{\frac{2}{3}}$$

b) For electric field,

$$\mathbf{E} = -\nabla V = -\frac{4V_a}{3}\,(\frac{x}{d^4})^{\frac{1}{3}}\,\hat{x}$$

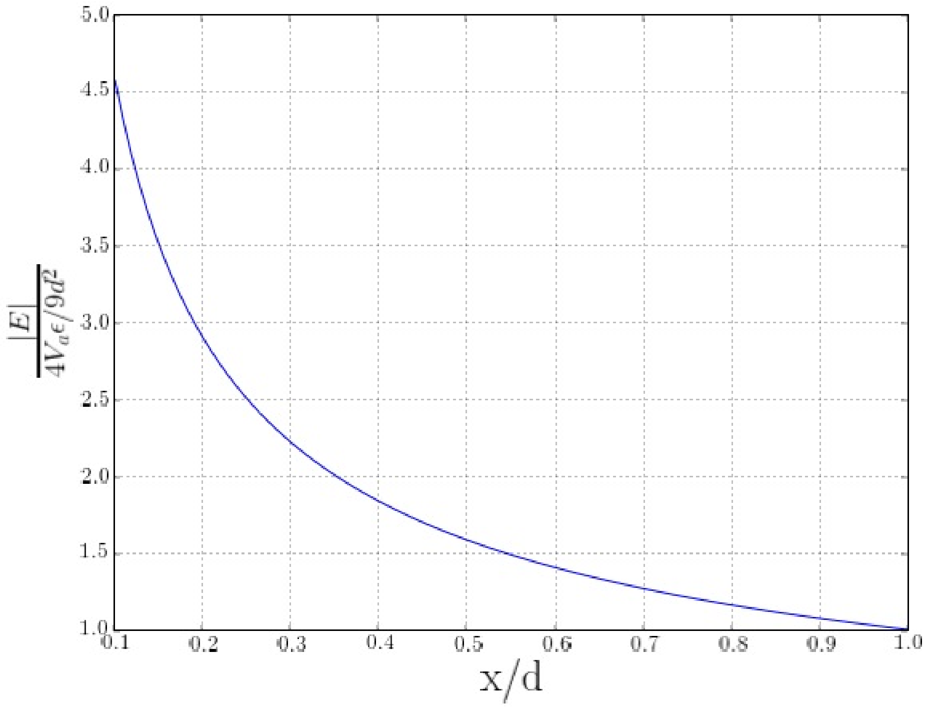


c) Using differential form of Gauss' Law,

$$\rho = \nabla\cdot\mathbf{D} = -\frac{4\epsilon_o V_a}{9d^{\frac{4}{3}}}\,(x)^{-\frac{2}{3}}$$

Alternatively, Poisson's Equation can be used,

$$\rho = -\epsilon_o\nabla^2 V = -\frac{4\epsilon_o V_a}{9d^{\frac{4}{3}}}\,(x)^{-\frac{2}{3}}$$



d) The normal component of the displacement field $\mathbf{D}$ changes by an amount equal to the surface charge density at the anode.

$$\begin{aligned} \rho_s &= \mathbf{D}.\hat{n}\,|_{x=d} = D_x\,|_{x=d}) \\ &= \frac{8\epsilon_o}{3d}\,\frac{C}{m^2} \end{aligned}$$

2. In electrostatics, we generate a curl-free vector field $\mathbf{E}(x, y, z)$ if we take the gradient of a scalar function $V(x, y, z)$. Therefore, $\nabla \times \mathbf{E} = \nabla \times (-\nabla V) = 0$. Alternatively, we can go through the calculation,

$$\mathbf{E} = -\nabla V = -\nabla(z^2 - yz) = z\hat{y} + (y - 2z)\hat{z}$$

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & z & y - 2z \end{vmatrix} = 0$$

3. Starting with the left-handside of $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$, we write

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{A}) &= \nabla \times \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + z & 2x + y & 4 \end{vmatrix} \\ &= \nabla \times (\hat{y} + 2\hat{z}) = 0. \end{aligned}$$

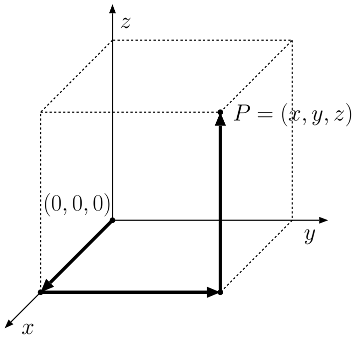
Solving the right-handside of the equation, we obtain

$$\begin{aligned}\nabla\left(\nabla\cdot\mathbf{A}\right)-\nabla^{2}\mathbf{A} &= \nabla\left(\nabla\cdot\left((x+z)\hat{x}+(2x+y)\hat{y}+4\hat{y}\right)\right)-\nabla^{2}\left((x+z)\hat{x}+(2x+y)\hat{y}+4\hat{y}\right)\\ &= \nabla\left(2\right)-\mathbf{0}=\mathbf{0}.\end{aligned}$$

Consequently, we can see that $\nabla\times\left(\nabla\times\mathbf{A}\right)=\nabla\left(\nabla\cdot\mathbf{A}\right)-\nabla^{2}\mathbf{A}$ is verified since both sides have the same solutions.

4.

a) Since the given field satisfies$\nabla\times\mathbf{E}=0$, it is said to be an electrostatic field which can be indicated as $\mathbf{E}=-\nabla V$. Thus, the electrostatic potential V at any point $P=(x,y,z)$ can be calculated by performing a vector line integral by using the path shown in the below figure.



Therefore, we can write

$$\begin{aligned}V(P)-V(0) &= -\int_{0}^{P}\mathbf{E}\cdot d\mathbf{l}\\ &= -\int_{0}^{x}E_x(x,0,0)dx-\int_{0}^{y}E_y(x,y,0)dy-\int_{0}^{z}E_z(x,y,z)dz\\ &= \frac{x^2}{2}-\frac{4}{\pi}(1-\cos(\frac{\pi y}{4}))-\frac{2z^3}{3}\,\mathrm{V}.\end{aligned}$$

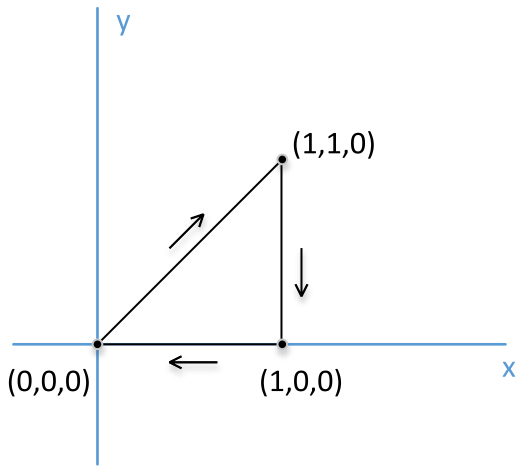
Given that $V(0)=0\,\mathrm{V}$, the electrostatic potential at $P=(1,2,3)$ is $V(1,2,3)=-\frac{4}{\pi}-\frac{35}{2}\,\mathrm{V}$.

b) Using differential form of Gauss' Law,

$$\rho=\nabla\cdot\mathbf{D}=\epsilon_o(4z+\frac{\pi}{4}\cos(\frac{\pi y}{4})-1)$$

At (0,0,0), $\rho=(\frac{\pi}{4}-1)\epsilon_o$ and at (1,2,3), $\rho=11\epsilon_o$.

5. The triangular path defined in the problem is sketched in the below figure.



Referring to the hint given in the problem, we can write

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \int_{l_1} \mathbf{E}(x,x,0) \cdot d\mathbf{l}_1 + \int_{l_2} \mathbf{E}(1,y,0) \cdot d\mathbf{l}_2 + \int_{l_3} \mathbf{E}(x,0,0) \cdot d\mathbf{l}_3.$$

Evaluating this equation for the field $\mathbf{E}$ with the $-$ sign, we obtain

$$\oint_C \mathbf{E_1} \cdot d\mathbf{l} = \int_0^1 (\hat{x}x - \hat{y}x) \cdot (\hat{x} + \hat{y})\, dx - \int_0^1 (\hat{x}y - \hat{y}) \cdot (\hat{y}) dy - \int_0^1 (-\hat{y}x) \cdot (\hat{x})\, dx = 1\,\mathrm{V}.$$

Likewise, we calculate

$$\oint_C \mathbf{E_2} \cdot d\mathbf{l} = \int_0^1 (\hat{x}x + \hat{y}x) \cdot (\hat{x} + \hat{y})\, dx - \int_0^1 (\hat{x}y + \hat{y}) \cdot (\hat{y}) dy - \int_0^1 (\hat{y}x) \cdot (\hat{x})\, dx = 0\,\mathrm{V}$$

for the field $\mathbf{E}$ with the $+$ sign. The solution can be double checked by calculating the curls of the electric fields,

$$\nabla \times \mathbf{E_1} = -\,2\hat{z} \quad \nabla \times \mathbf{E_2} = 0$$

As expected, fields with zero circulation are curl free.

6.

a) One of the boundary conditions state that at any surface S, the normal component of the displacement field $\mathbf{D}$ can change by an amount equal to the surface charge density. Applying this boundary condition to the interface at $y = 6\,\mathrm{m}$, the surface charge density is calculated as

$$\begin{aligned} \rho_s &= \hat{y} \cdot (\mathbf{D}\,|_{y=6^+} - \mathbf{D}\,|_{y=6^-}\,) \\ &= 2\epsilon_o - 4\epsilon_o = -2\epsilon_o\,\frac{\mathrm{C}}{\mathrm{m}^2}. \end{aligned}$$

b) Another boundary condition also states that the tangential component of electric field $\mathbf{E}$ needs to be continuous along the surface S. Therefore, we can write

$$E_x \big|_{y=6^+} = E_x \big|_{y=6^-} \ ,$$

Given that the space is vacuum, we can verify that $D_x|_{y=6^+} = D_x|_{y=6^-} = 2\epsilon_o \, \frac{\mathrm{C}}{\mathrm{m}^2}$. Consequently, the displacement field $\mathbf{D}$ for the region $y > 6\,\mathrm{m}$ can be expressed as

$$\mathbf{D} = 2\epsilon_o\hat{x} + 2\epsilon_o\hat{y} \, \frac{\mathrm{C}}{\mathrm{m}^2}.$$

c) Looking at the charge distribution, one can see that there is a charge density of $6\epsilon_o\,\mathrm{C/m}^2$ along the interface at $y = 0$. Thus, applying boundary conditions, we can write

$$\rho_s \quad = \quad \hat{y} \cdot (\mathbf{D} \big|_{y=0^+} - \mathbf{D} \big|_{y=0^-}),$$

from which we obtain $D_y|_{y=0^-} = 4\epsilon_o - 6\epsilon_o \, \frac{\mathrm{C}}{\mathrm{m}^2}$. Given that the space is vacuum, we can also verify that $D_x|_{y=0^+} = D_x|_{y=0^-} = 2\epsilon_o \, \frac{\mathrm{C}}{\mathrm{m}^2}$. Consequently, the displacement field $\mathbf{D}$ for the region $y < 0$ can be expressed as

$$\mathbf{D} = 2\epsilon_o\hat{x} - 2\epsilon_o\hat{y} \, \frac{\mathrm{C}}{\mathrm{m}^2}.$$

d) The displacement field, $\mathbf{D}$, is defined to be equal to $\epsilon_{medium}\mathbf{E}$. Since the region is in free space $\epsilon_{medium} = \epsilon_0$, the electric field in the region where y $<$ 0 m is

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_\mathbf{0}} = \frac{2\epsilon_o\hat{x} - 2\epsilon_o\hat{y}}{\epsilon_0} = 2\hat{x} - 2\hat{y} \, \frac{V}{m}$$

The electric field in the region where 0$<$ y $<$ 6 m is

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_\mathbf{0}} = \frac{2\epsilon_o\hat{x} + 4\epsilon_o\hat{y}}{\epsilon_0} = 2\hat{x} + 4\hat{y} \, \frac{V}{m}$$

The electric field in the region where y $>$ 6 m is

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_\mathbf{0}} = \frac{2\epsilon_o\hat{x} + 2\epsilon_o\hat{y}}{\epsilon_0} = 2\hat{x} + 2\hat{y} \, \frac{V}{m}$$

e) The voltage drop from a point P (y = 0) to ground O (y = 6) can be calculated as follows,

$$V = \int_p^O \mathbf{E} \cdot d\mathbf{l} = \int_0^6 (2\hat{x} + 4\hat{y}).d\hat{y} = \int_0^6 4\,dz = 24\,V$$

f) If the medium in the region 0 $<$ y $<$ 6 m was replaced with a dielectric with permittivity $\epsilon_{medium} = 4\epsilon_0$, it will only affect equations that deal with the electric field in this middle region since $\mathbf{E} = \frac{\mathbf{D}}{4\epsilon_\mathbf{0}}$ now (only part a will remain unchanged). Parts b and c all are solved knowing that tangential electric field boundary conditions are continuious, and thus there will be a change in the displacement field here. Part d will change as it asks to compute the electric field in this region. Part e will also change since the voltage drop is desired in this region and it is calculated by integrating along the electric field.

7.

a) In vacuum, the displacement vector is $\mathbf{D} = \epsilon_o \mathbf{E} + \mathbf{P}$. Thus, the displacement field between the plates is

$$\mathbf{D} = 4\epsilon_o \hat{z} \, \frac{\mathrm{C}}{\mathrm{m}^2},$$

from which we obtain the polarization field as $\mathbf{P} = 0 \, \frac{\mathrm{C}}{\mathrm{m}^2}$.

b) Since $\mathbf{D} = \mathbf{0}$ for $z < 0$, i.e. $\mathbf{D}\,|_{z=0^-}$ the surface charge on the plate at $z = 0$ is

$$\rho_s|_{z=0} \quad = \quad \hat{z} \cdot (\mathbf{D}\,|_{z=0^+} - \mathbf{D}\,|_{z=0^-}) = 4\epsilon_o \, \frac{\mathrm{C}}{\mathrm{m}^2}.$$

c) If the gap is filled with a dielectric of permittivity $\epsilon = 87\epsilon_o$ without changing the surface charge density then the displacement field will remain the same, i.e.,

$$\mathbf{D} = 4\epsilon_o \, \hat{z} \, \frac{\mathrm{C}}{\mathrm{m}^2}.$$

But, the electric field is now

$$\mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = \frac{4}{87} \, \hat{z} \, \frac{\mathrm{V}}{\mathrm{m}}.$$

Consequently, the polarization field becomes

$$\mathbf{P} = \mathbf{D} - \epsilon_o \mathbf{E} = \frac{344}{87}\epsilon_o \, \hat{z} \, \frac{\mathrm{C}}{\mathrm{m}^2}.$$

d) If the medium in the gap has a finite conductivity, then it will also have $\mathbf{E} = 0$ in "steady-state".Thus, $\mathbf{D} \to 0$ and $\mathbf{P} \to 0$. Because, the mobile free charges within the medium in the gap will be pushed and pulled to pile up at the surfaces until the surface charge density generates a secondary field that cancels out the fields wtihin the medium. In this particular case, the salt water shorts out the original field between the plates.

8. The solution of the problem will be given region by region.

- The region defined by $r \leq a$ is occupied by a conductor with $\sigma = 10^5\,\mathrm{S/m}$, therefore, we can directly write

$$\mathbf{D} = \mathbf{0} \, \frac{\mathrm{C}}{\mathrm{m}^2}, \qquad \mathbf{E} = \mathbf{0} \, \frac{\mathrm{V}}{\mathrm{m}}, \qquad \mathbf{P} = \mathbf{0} \, \frac{\mathrm{C}}{\mathrm{m}^2}.$$

for this particular region. In steady-state, charges can accumulate only on the surface of conducting materials. Since this material holds a net charge per unit length $Q = 4 \, \frac{C}{m}$, the surface charge density at radius r = a is

$$\rho_s|_{r=a} = \frac{Q}{Circumference} = \frac{4}{2\pi a} = \frac{2}{\pi a} \, \frac{\mathrm{C}}{\mathrm{m}^2}.$$

- The region defined by $a < r < b$ is occupied by a dielectric with $\epsilon = 4\epsilon_o$. Applying Gauss's law $\oint \mathbf{D} \cdot d\mathbf{S} = \int_V \rho dV = Q_{enc}$ and considering $\mathbf{D} = D_r \hat{r}$, we find

$$D.(2\pi r L) = Q_{enc} = 4 \times L$$

$$\mathbf{D} = \frac{4}{2\pi r}\hat{r} = \frac{2}{\pi r}\hat{r} \, \frac{\mathrm{C}}{\mathrm{m}^2}$$

for $a < r < b$. Hence, the electric field $\mathbf{E}$ is

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{1}{2\epsilon_o \pi r}\hat{r}\,\frac{V}{\mathrm{m}},$$

from which we can obtain the polarization $\mathbf{P}$ as

$$\mathbf{P} = \mathbf{D} - \epsilon_o \mathbf{E} = (\frac{1}{\pi r} - \frac{1}{2\pi r})\hat{r} = \frac{3}{2\pi r}\hat{r}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

- As the region defined by $b \leq r \leq c$ has the same material properties as region $r \leq a$, we can write

$$\mathbf{D} = \mathbf{0}\,\frac{\mathrm{C}}{\mathrm{m}^2}, \qquad \mathbf{E} = \mathbf{0}\,\frac{\mathrm{V}}{\mathrm{m}}, \qquad \mathbf{P} = \mathbf{0}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

  Then, the surface charge density at $r = b$ is given by

$$\begin{aligned} \rho_s|_{r=b} &= \hat{r}.(\mathbf{D}\,|_{r=b^+} - \mathbf{D}\,|_{r=b^-})\,|_{r=b} \\ &= \hat{r} \cdot \left(-\frac{2}{\pi r}\hat{r}\right)\Big|_{r=b} = -\frac{2}{\pi b}\,\frac{\mathrm{C}}{\mathrm{m}^2}. \end{aligned}$$

- The region defined by $c \leq r \leq d$ is occupied by a dielectric with $\epsilon = 2\epsilon_o$. Since the net charge per unit length is $4 - 2 = 2\,\frac{C}{m}$, the surface charge density at radius r = c is

$$\rho_s|_{r=c} = \frac{Q}{\text{Circumference}} = \frac{2}{2\pi c} = \frac{1}{\pi c}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

  Applying Gauss's law $\oint \mathbf{D} \cdot d\mathbf{S} = \int_V \rho dV = Q$ and considering $\mathbf{D} = D_r \hat{r}$, we find

$$\mathbf{D} = \frac{2}{2\pi r}\hat{r} = \frac{1}{\pi r}\hat{r}\,\frac{\mathrm{C}}{\mathrm{m}^2}$$

  for $c < r < d$. Hence, the electric field $\mathbf{E}$ is

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{1}{2\epsilon_o \pi r}\hat{r}\,\frac{V}{\mathrm{m}},$$

  from which we can obtain the polarization $\mathbf{P}$ as

$$\mathbf{P} = \mathbf{D} - \epsilon_o \mathbf{E} = (\frac{1}{\pi r} - \frac{1}{2\pi r})\hat{r} = \frac{1}{2\pi r}\hat{r}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

- Region defined by r > d is free space. Applying Gauss' law and noting that the total charge per unit length enclosed is $2\,\frac{C}{m}$, we get $\oint \mathbf{D} \cdot d\mathbf{S} = Q = 2 \times L\,\frac{C}{m}$ where $d\mathbf{S} = \hat{r}2\pi r$. Therefore, we can write

$$\mathbf{D} = \frac{1}{\pi r}\hat{r}\,\frac{\mathrm{C}}{\mathrm{m}^2}, \qquad \mathbf{E} = \frac{1}{\pi \epsilon_0 r}\hat{r}\,\frac{\mathrm{V}}{\mathrm{m}}, \qquad \mathbf{P}_4 = \mathbf{0}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

  Consequently, the surface charge density at $r = d$ is given by

$$\begin{aligned} \rho_s|_{r=d} &= \hat{r}.(\mathbf{D}\,|_{r=d^+} - \mathbf{D}\,|_{r=d^-})\,|_{r=d} \\ &= 0\,\frac{\mathrm{C}}{\mathrm{m}^2}. \end{aligned}$$

9.

a) The total potential at a point P can be given as follows,

$$V = \frac{Q}{4\epsilon_0 \pi r_+} - \frac{Q}{4\epsilon_0 \pi r_-}$$

where $r_{\pm}^2 = r^2 + a^2 \pm 2ra\,cos(\theta)$. Using binomial approximation $(1+p)^n \approx 1 + np$ and assuming r >> a, $\frac{1}{r}$ can be approximated as follows,

$$\begin{aligned} \frac{1}{r_{\pm}} &= \frac{1}{r}[1 + (\frac{a}{r})^2 \pm 2\,(\frac{a}{r})\,cos(\theta)]^{-\frac{1}{2}} \\ &= \frac{1}{r}[1 - \frac{1}{2}(\frac{a}{r})^2 \mp (\frac{a}{r})\,cos(\theta) + ...] \end{aligned}$$

Taking the first three terms and substituting back into the potential,

$$\begin{aligned} V &\approx \frac{Q}{4\epsilon_0 \pi r}[1 - \frac{1}{2}(\frac{a}{r})^2 - (\frac{a}{r})\,cos(\theta) - 1 + \frac{1}{2}(\frac{a}{r})^2 - (\frac{a}{r})\,cos(\theta)] \\ &= \frac{Q}{4\epsilon_0 \pi r}[-2(\frac{a}{r})\,cos(\theta)] = \frac{-2Qa\,cos(\theta)}{4\epsilon_0 \pi r^2} \end{aligned}$$

b) The electric field lines(red) and the equipotential planes(blue) can be graphed as follows,