Signals and Systems

Stage 3: Electrical & Electronic Eng.

Dr Paul Curran

Room 145, Engineering.

paul.curran@ucd.ie

+353-1-7161846

Models

In the introductory notes we considered, briefly, some background mathematical theory concerning the value of and methods for solving linear, constant coefficient ordinary differential equations. Although these equations provide a very powerful model paradigm for systems operating close to either equilibrium or some other desired operating point, it transpires that there are some other useful paradigms. Although ancient in heritage one of these alternative model paradigms really found its feet with the introduction of probability theory by Fermat and Pascal in response to a problem posed by Antoine Gombaud, Chevalier de Méré. The *problem of points* asks how two players in a game of chance should divide the pot in the event that the game is interrupted before its conclusion.

Models: Example 5

Let a bag contain r red marbles and g green marbles. All marbles are either red or green. We perform the experiment of drawing a marble noting its colour and then replacing it. We repeat this experiment n times. For a given k lying between 0 and n what is the probability of drawing a red marble exactly k times in these n experiments (or trials)?

The clever idea is to denote this unknown probability $P(n,k)$. Now, on the nth trial/experiment there are two possible mechanisms by which we end up having drawn exactly k red marbles.

Models: Example 5

Mechanism I: we draw a red marble on the nth iteration of the experiment having drawn k-1 red marbles in the previous n-1 iterations.

Mechanism II: we draw a green marble on the nth iteration of the experiment having drawn k red marbles in the previous n-1 iterations.

The probability that mechanism I will yield k red marbles is the probability of drawing a red marble in the nth iteration given that we have already drawn k-1 red marbles in the previous n-1 iterations.

Models: Example 5

Using the basic theory of conditional probabilities, let A denote the event where we draw a red marble on the *n*th iteration and let B denote the event that we draw *k*-1 red marbles in the first *n*-1 iterations.

$$p(A) = p(A \mid B)p(B)$$

$$p(A) = \left(\frac{r}{r+g}\right)P(n-1, k-1)$$

Models: Example 5

The probability that mechanism II will yield *k* red marbles is the probability of drawing a green marble in the *n*th iteration given that we have already drawn *k* red marbles in the previous *n*-1 iterations.

Using the basic theory of conditional probabilities, let A denote the event where we draw a green marble on the *n*th iteration and let B denote the event that we draw *k* red marbles in the first *n*-1 iterations.

$$p(A) = p(A \mid B)p(B) \qquad p(A) = \left(\frac{g}{r+g}\right)P(n-1,k)$$

Models: Example 5

As the mechanisms describe mutually exclusive events we find that the probability of k red marbles drawn in n trials is

$$P(n,k)=\left(\frac{r}{r+g}\right)P(n-1,k-1)+\left(\frac{g}{r+g}\right)P(n-1,k)$$

We obtain a *difference equation*. This type of equation in fact is sometimes called a *partial difference equation*. It is also sometimes called a *recurrence*.

Models

It is actually rather common for problems in either combinatorics or probability theory to be “solved” in this way, i.e. by finding a difference equation of which they are a solution. So it is not too surprising that a very good method for solving equations of this kind was first introduced in the field of probability theory by one of the greatest early contributors to that field, de Moivre.

Models: Example 6

The previous example is not in fact an example of the kind of difference equation which we will be considering. A more relevant example would be the simple equation governing radioactive decay. If the half-life of a radioactive material is τ then, if $M(t)$ denotes the mass remaining at time t:

$$M\left(t+\tau\right)=\tfrac{1}{2}M\left(t\right)$$

This is not a difference equation, but it can be related to one and the means to do this is the general mechanism by which such equations arise in practice.

Models: Example 6

$$M(t+\tau) = \tfrac{1}{2} M(t)$$

The critical idea is to *sample*. We ask for an equation which will tell us, not what the remaining mass will be at all times, but rather what will it be at particular times only. In this case the obvious choice of "particular times" is the integer multiples of the half life. In other words we note that, setting $t = n\tau$ we obtain:

$$M(n\tau+\tau) = \tfrac{1}{2} M(n\tau)$$

Let $M_n = M(n\tau)$ then:

$$M_{n+1} = \tfrac{1}{2} M_n$$

Models: Example 6

$$M_{n+1} = \tfrac{1}{2} M_n$$

So we obtain a rather simple difference equation. If the initial mass is M_0 then it is very easy to solve this equation:

$$M_n = \left(\tfrac{1}{2}\right)^n M_0$$

The radioactive material decays away such that after ten half lives only 1 part in 1024 remains.

Models: Example 7

The two examples which we have considered so far are atypical of difference equations in general. A more typical example is offered by Verhulst's logistic growth law. Malthus had previously offered a rather trivial law of population growth:

$$P_{n+1} = \alpha P_n$$

where P_n is the population at sample time n. The sampling rate might be once every minute, once every day, once every year, *etc* depending upon the nature of the population (eg cells, flies, people).

Models: Example 7

Sampling is again the method by which a continuously varying quantity is prepared for description by a difference equation. Obviously Malthus' law is rather preposterous if the parameter α exceeds 1 as it now predicts a population which grows without bound. Upon reading Malthus' essay Verhulst immediately saw that a modification which recognised the finiteness of available resources was surely appropriate. Verhulst offers the following:

$$P_{n+1} = \alpha P_n \left(1 - \gamma P_n\right)$$

Models: Example 7

Verhulst's law may be modified: $\gamma P_{n+1} = \alpha \gamma P_n (1 - \gamma P_n)$

or, letting $Q_n = \gamma P_n$: $Q_{n+1} = \alpha Q_n (1 - Q_n)$

This important difference equation is called the *logistic equation*. Its behaviour as the parameter α is varied is surprisingly complicated. It is my understanding that Verhulst did not really have any better success than Malthus in terms of finding an equation which could model the population dynamics of large animals. Malthus' law in this regard was too simple and so is Verhulst's.

Models: Example 7

$$Q_{n+1} = \alpha Q_n \left(1 - Q_n\right)$$

From the perspective of this module the logistic equation proves to be far too complicated to admit any kind of simple analysis. Ultimately the reason for this is that it is not linear, unlike the equations of examples 5 and 6. It transpires that the solutions to rather simple looking nonlinear difference equations can have astonishingly complicated behaviour, behaviour so complex that we have only been aware of it at all for about one century and have only been able to analyse special cases of it for about the last 40 years. Mostly we treat these equations by simulation, i.e. we get a computer to solve them.

Models: Example 6

How might we go about solving the equation

$$M_{n+1} = \tfrac{1}{2} M_n$$

One method, similar to one of the approaches which we took in the case of differential equations, is to guess the form of the solution. Accordingly suppose I guess that the solution has the form:

$$M_n = A\alpha^n$$

for some A and some α. For this to actually be a solution I require:

$$M_{n+1} = A\alpha^{n+1} = \tfrac{1}{2} M_n = \tfrac{1}{2} A\alpha^n$$

Models: Example 6

It is clear that the proposed solution is indeed a solution if $\alpha = ½$. Of course the proposed solution satisfies the initial condition

$$M_0 = A\alpha^0 = A$$

Like differential equations, difference equations must be provided with suitable initial conditions in order for the associated mathematical problem to be complete. Given the two choices, $A = M_0$ and $\alpha = ½$ the purported solution becomes the actual solution

$$M_n = M_0\left(\tfrac{1}{2}\right)^n$$

Models: Example 6

This method of solution leaves a lot to be desired. Specifically how did I make the guess concerning the form of the solution? Although I can improve it somewhat I choose to jump straight to de Moivre's method.

$$M_{n+1} = \tfrac{1}{2} M_n$$

de Moivre solves by doing something rather strange and decidedly more complicated. He forms and equates two associated series:

$$\sum_{n=0}^{\infty} M_{n+1} z^{-n} = \sum_{n=0}^{\infty} \tfrac{1}{2} M_n z^{-n}$$

Models: Example 6

Processing the two series separately:

$$\sum_{n=0}^{\infty} M_{n+1} z^{-n} = z\left(\sum_{n=0}^{\infty} M_{n+1} z^{-(n+1)}\right) = z\left(\sum_{n=1}^{\infty} M_n z^{-n}\right) = z\left(\sum_{n=0}^{\infty} M_n z^{-n} - M_0\right)$$

$$\left(\sum_{n=0}^{\infty} \tfrac{1}{2} M_n z^{-n}\right) = \tfrac{1}{2}\left(\sum_{n=0}^{\infty} M_n z^{-n}\right)$$

Equating:

$$z\left(\sum_{n=0}^{\infty} M_n z^{-n} - M_0\right) = \tfrac{1}{2}\sum_{n=0}^{\infty} M_n z^{-n}$$

Models: Example 6

$$\left(z-\tfrac{1}{2}\right)\left(\sum_{n=0}^{\infty} M_n z^{-n}\right) = zM_0$$

$$\left(\sum_{n=0}^{\infty} M_n z^{-n}\right) = \left(\frac{z}{z-\tfrac{1}{2}}\right)M_0 = \left(\frac{1}{1-\tfrac{1}{2}z^{-1}}\right)M_0$$

My choice of the complex number z in all of this must clearly be such that the various series involved converge. It is fairly obvious that this occurs provided the modulus of z is sufficiently large. Accordingly I may assume in particular that $\frac{1}{2} z^{-1}$ has modulus less than 1.

Models: Example 6

Although somewhat in disguise the following famous result is largely included as proposition 35 of book IX of Euclid's *Elements*.

$$a\sum_{n=0}^{\infty} r^n = \left(\frac{a}{1-r}\right) \quad \text{if } \left|r\right|<1$$

Viète extends this result to complex *r*. We obtain:

$$\sum_{n=0}^{\infty} M_n z^{-n} = M_0\left(\sum_{n=0}^{\infty}\left(\tfrac{1}{2}z^{-1}\right)^n\right) = \sum_{n=0}^{\infty} M_0\left(\tfrac{1}{2}\right)^n z^{-n}$$

Models: Example 6

Of course this does not hold for one z. It holds for all z having suitably large modulus. It follows from some very deep results in complex analysis, that

$$M_n = M_0\left(\tfrac{1}{2}\right)^n$$

which of course is the solution presented above. It is somewhat clear I think that this method of solving difference equations has rather a lot in common with the Laplace transform method of solving ordinary differential equations.

de Moivre's Idea

Sequences and functions can be related through the idea of *series* and in particular through the idea of *power series*. The Maclaurin series is probably the best known example. For example, the formula

$$\exp(-t) = 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \cdots$$

relates the sequence: { 1, -1, 1/2, -1/6, ... } to the function exp(-*t*) which effectively *generates* it. We say that exp(-*t*) is the *generating function* for the sequence.

de Moivre's Idea

de Moivre was one of the early group of mathematicians who developed probability theory. Difference equations were fairly naturally arising in this field, as were certain sequences. Of particular significance was the sequence of *moments* of a random variable. The first moment of random variable X is its expected value $E(X)$. The second moment of the random variable X is the expected value of its square, $E(X^2)$. Higher moments are similarly defined. We may consider the *generating function* for this sequence of moments.

de Moivre's Idea

This function is:

$$E(1)+tE(X)+\frac{t^2}{2}E(X^2)+\frac{t^3}{6}E(X^3)+\cdots$$

The clever observation is that it is commonly easier to find this generating function (thereby finding all of the moments in one shot) than to look for the individual moments. From the basic properties of the expectation operator the moment generating function equals

$$E\left(1+tX+\frac{t^2}{2}X^2+\frac{t^3}{6}X^3+\cdots\right)=E\left(e^{tX}\right)$$

de Moivre's Idea

It is a tangent but I should note that if the probability density function (pdf) of the random variable X is $f(x)$ then

$$E\left(e^{tX}\right)=\int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

assuming the random variable to be a real random variable. Should the variable be restricted to being non-negative then

$$E\left(e^{tX}\right)=\int_{0}^{\infty} e^{tx} f(x)\,dx$$

apart from a different notation this is basically the Laplace transform of the pdf.

de Moivre’s Idea

de Moivre has the idea to apply essentially the concept of the generating function to permit a solution of linear difference equations. So, instead of looking for the solution to the difference equation, i.e. a sequence (commonly called the *orbit* in modern terminology), we look instead for the generating function for this sequence. Again in modern terminology the “generating function” is not written as a function in t but rather as a function in z^{-1} as we had above. Moreover it is expressed as a power series not as a Maclaurin series.

de Moivre's Idea

Consider the difference equation: $x_n + 2x_{n-1} = 1$

de Moivre's idea is to look, not for the sequence $\{x_n\}$, but rather, for the *generating function* of the sequence.

$$\sum_{n=0}^{\infty} x_n z^{-n} = X(z) \quad , \quad \sum_{n=0}^{\infty} \left(x_n + 2x_{n-1}\right) z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$$\sum_{n=0}^{\infty} x_n z^{-n} + 2\sum_{n=0}^{\infty} x_{n-1} z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$$X(z) + 2\left(\left(\sum_{(n-1)=0}^{\infty} x_{n-1} z^{-(n-1)-1}\right) + x_{-1}\right) = \sum_{n=0}^{\infty} z^{-n}$$

$$X(z) + 2\left(z^{-1} X(z) + x_{-1}\right) = \sum_{n=0}^{\infty} z^{-n}$$

de Moivre's Idea

Now by employing Viète's excellent formula :

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

$$\left(1+2z^{-1}\right)X(z)+2x_{-1}=\left(\frac{1}{1-z^{-1}}\right)$$

$$X(z)=\frac{\left(\frac{1}{1-z^{-1}}\right)-2x_{-1}}{\left(1+2z^{-1}\right)}$$

a solution of sorts.

de Moivre's Idea

Now employing partial fractions:

$$X(z) = \frac{1}{\left(1 - z^{-1}\right)\left(1 + 2z^{-1}\right)} - \frac{2x_{-1}}{\left(1 + 2z^{-1}\right)}$$

$$= \frac{\frac{1}{3}}{1 - z^{-1}} + \frac{\frac{2}{3}}{1 + 2z^{-1}} - \frac{2x_{-1}}{\left(1 + 2z^{-1}\right)} = \frac{\frac{1}{3}}{1 - z^{-1}} + \frac{\frac{2}{3} - 2x_{-1}}{1 + \left(\frac{1}{2}z\right)^{-1}}$$

$$= \frac{1}{3}\sum_{n=0}^{\infty} z^{-n} + \left(\frac{2}{3} - 2x_{-1}\right)\sum_{n=0}^{\infty}\left(\frac{1}{2}z\right)^{-n}$$

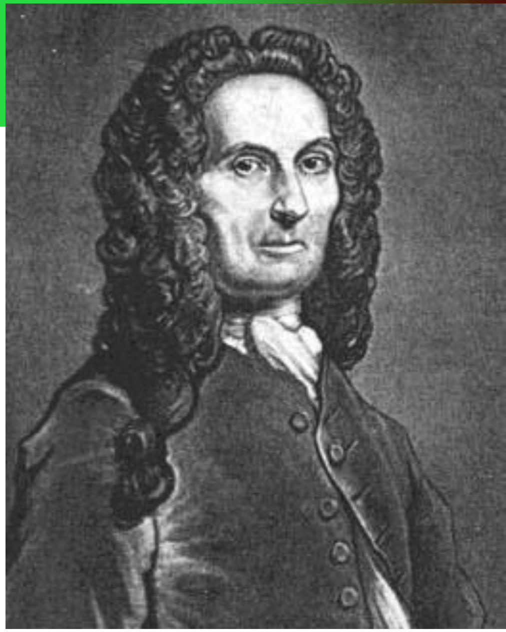
$$= \sum_{n=0}^{\infty}\left(\frac{1}{3} + \left(\frac{2}{3} - 2x_{-1}\right)2^{n}\right)z^{-n} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$x(n) = \frac{1}{3} + \left(\frac{2}{3} - 2x_{-1}\right)2^{n}$$

de Moivre’s Idea

de Moivre’s idea amounts to the discrete-time version of the Laplace transform method, i.e. the version of it which can be applied to difference equations. de Moivre’s priority is completely unrecognised. In engineering theory the method was essentially rediscovered, almost simultaneously in the USA and the Soviet Union. A variety of competing names continue to be used, but in the English language literature the rather uninspiring name *Z-transform* is now dominant. If Wikipedia articles are hit and miss, which they are, then the article on the Z-transform is a definite miss. It does not even mention Tsypkin whom most Russian language authors recognise as the modern inventor, or rather reinventor of the method.

Z-Transform



1667-1754

Given signal $f(n)$ let

$$F(z) = \sum_{n=0}^{\infty} f(n) z^{-n} \quad , \quad z \in C$$

Definition 6.1: Given signal $f(n)$ the function, $F(z)$, defined as above for all z for which the summation converges and by analytic continuation for all other z, is called the *Z-Transform* of $f(n)$, denoted $\mathcal{Z}(f)$.

Example 6.1

Definition 6.2: The discrete-time *step function* $u(n)$ is given by

$$u(n) = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n \geq 0 \end{cases}$$

$$U(z) = \frac{1}{1 - z^{-1}}$$

Z-Transform Properties: 1

If $\mathcal{Z}\left(f_1(n)\right) = F_1(z)$ and $\mathcal{Z}\left(f_2(n)\right) = F_2(z)$

then

$$\mathcal{Z}\left(\alpha_1 f_1(n) + \alpha_2 f_2(n)\right) = \alpha_1 F_1(z) + \alpha_2 F_2(z)$$

This property is called *linearity*.

Z-Transform Properties: 2

If $\mathcal{Z}\left(f\left(n\right)\right) = F\left(z\right)$ then

$$\mathcal{Z}\left(\lambda^{-n} f\left(n\right)\right) = F\left(\lambda z\right)$$

This property is called *modulation*.

Z-Transform Properties: 3

If $\mathcal{Z}\left(f(n)\right)=F(z)$ then

$$\mathcal{Z}\left(f(n-n_0)\right)=z^{-n_0}F(z)+z^{-(n_0-1)}f(-1)+z^{-(n_0-2)}f(-2)+\cdots+f(-n_0)$$

$$\mathcal{Z}\left(f(n+n_0)\right)=z^{n_0}F(z)-z^{n_0}f(0)-z^{(n_0-1)}f(-1)-\cdots-zf(n_0-1)$$

These are called the *shifting* properties. In the formulae the shift n_0 is assumed greater than or equal to 1.

Example 6.2

Signal	Z-Transform
$u(n)$	$\dfrac{1}{1-z^{-1}}$
$\lambda^{-n}u(n)$	$\dfrac{1}{1-\lambda^{-1}z^{-1}}$
$\delta(n)$	1
$\sin(\Omega_0 n)$	$\dfrac{\sin(\Omega_0)z^{-1}}{1-2\cos(\Omega_0)z^{-1}+z^{-2}}$
$\cos(\Omega_0 n)$	$\dfrac{1-\cos(\Omega_0)z^{-1}}{1-2\cos(\Omega_0)z^{-1}+z^{-2}}$

Example 6.2

Definition 6.3: The *Kronecker delta sequence*

$\delta(n)$ is given by

$$\delta(n) = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases}$$

$$\Delta(z) = 1$$

Example 6.3

Consider the following difference equation, known as the *Fibonacci equation*:

$$y(n) = y(n-1) + y(n-2), \quad y(-1) = 0 \quad , \quad y(-2) = 1$$

$$Y = z^{-1}Y + z^{-2}Y + 1$$

$$Y = \frac{1}{1 - z^{-1} - z^{-2}}$$

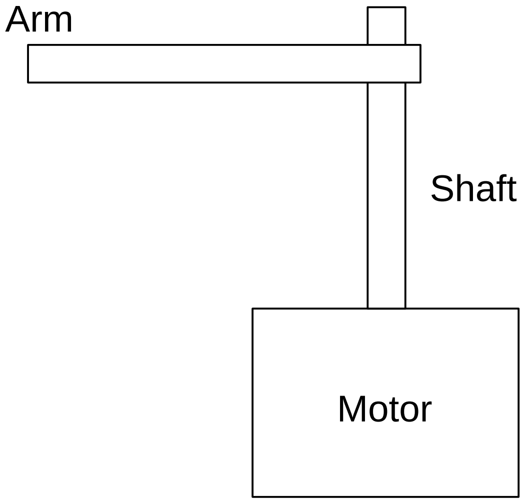
The Z-transform of the solution is obtained.

Example 6.4

A robotic arm is fixed to the shaft of a DC motor and free to rotate in a plane. I am going to consider a simpler model than I considered before.

$$T = Ki \quad , \quad \frac{d\omega}{dt} = \frac{K}{J} i$$

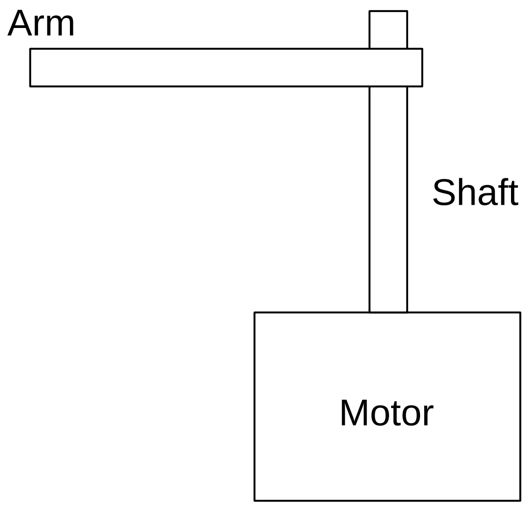
Because it can be difficult to finely adjust a current I will not bother. Instead I will operate the current control in an on/off manner, either setting it to zero (off) or to maximum (on).



Example 6.4

To obtain a finer level of control we adjust the current control setting regularly, indeed I will assume periodically. By adjusting the amount of time for which the current is on or off in a cycle we achieve the effect of a graded adjustment of current.

This type of adjustment of the control setting to achieve a masquerade of continuous adjustment is called *pulse width modulation* (PWM).



Example 6.4

Simple model of system (ignoring friction, *etc*.). Let the period of the PWM scheme be T. This period must be considerably lower than the natural periods associated with the system behaviour.

$$T = Ki \quad , \quad i(t) = \begin{cases} 0 & nT \le t < nT + (1-\rho_n)T \\ I_0 & nT + (1-\rho_n)T \le t < (n+1)T \end{cases}$$

$$\frac{d\omega}{dt} = \frac{K}{J} i \quad , \quad \omega((n+1)T) = \omega(nT) + \frac{K I_0 T \rho_n}{J}$$

Example 6.4

In the expression for the current above:

$$i(t)=\begin{cases}0 & nT \le t < nT+(1-\rho_n)T \\ I_0 & nT+(1-\rho_n)T \le t < (n+1)T\end{cases}$$

the current is off (set to zero) for the first fraction 1-ρ_n of the period and on (set to maximum) for the remaining fraction ρ_n of the period. The parameter ρ_n is called the *duty cycle*. If the duty cycle is 50% and if the period T is short (i.e. the switching is high frequency) then the system will react in approximately the same way as it would to a current set equal to half the maximum.

Example 6.4

$$\omega(nT) = \omega((n-1)T) + \frac{KI_0T}{J}\rho_{n-1}$$

Let $\omega_n = \omega(nT)$, i.e. employ the usual sampling method to convert to a difference equation.

$$\omega_n = \omega_{n-1} + \frac{KI_0T}{J}\rho_{n-1}$$

Taking Z-transforms:

$$\Omega(z) = z^{-1}\Omega(z) + \omega(-T) + \frac{KI_0T}{J}z^{-1}R(z)$$

Example 6.4

Accordingly:
$$\Omega = \left(\frac{\left({KI_0T}/{J} \right) z^{-1}}{1 - z^{-1}} \right) R + \left(\frac{\omega(-T)}{1 - z^{-1}} \right)$$

This is what things looked like in the case of differential equations when the Laplace transform was employed. The transform of the output is “something” multiplied by the transform of the input plus a second term whose numerator depends only on the initial conditions and whose denominator is the same as that of the “something”. Granted the transform in question is a new transform.

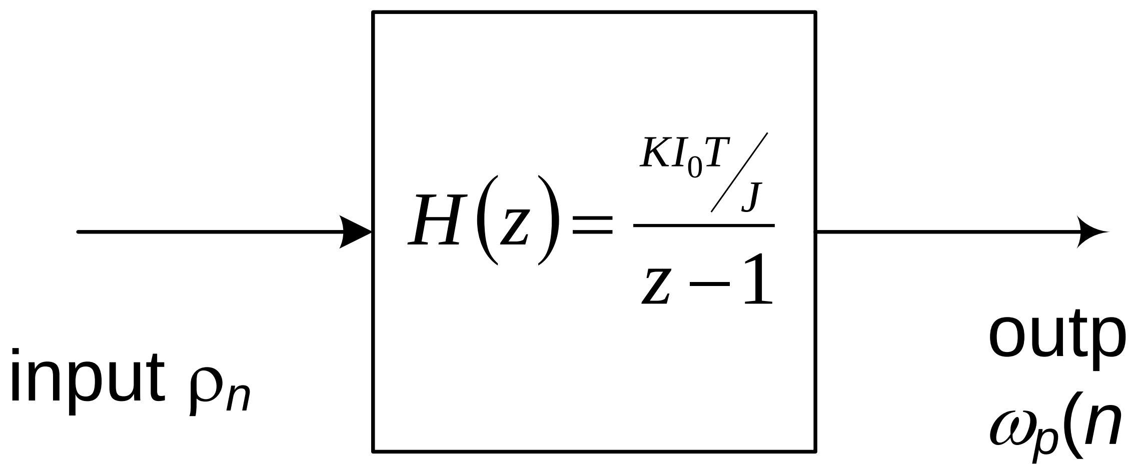
Example 6.4

The solution it seems is the sum of a particular solution which satisfies zero initial conditions (the forced response) and a solution to the unforced equation (the free response). This is true in general for linear, constant coefficient difference equations. Indeed much of the general theory of linear constant coefficient ODEs holds verbatim for linear, constant coefficient difference equations.

Example 6.4

$$\omega_{n+1} = \omega_n + \frac{KI_0T\rho_n}{J}$$

$$H(z) = \frac{\Omega_p}{R} = \frac{\left(KI_0T/J\right)z^{-1}}{1-z^{-1}}$$



z-domain model of DC motor with PWM and arm

In general: Solving linear, constant coefficient DEs

$$a_m g(n-m) + \cdots + a_1 g(n-1) + a_0 g(n)$$
$$= b_{m-1} f(n-m+1) + \cdots + b_1 f(n-1) + b_0 f(n)$$

where $f(n)$ is a specified causal function (i.e. zero for $n < 0$) and initial conditions $g(-1), g(-2), \ldots, g(-m)$ are given.

Solving linear, constant coefficient DEs

$$a_m\left(z^{-m}G(z)+z^{-(m-1)}g(-1)+z^{-(m-2)}g(-2)+\cdots-g(-m)\right)$$
$$+\cdots+a_1\left(z^{-1}G(z)+g(-1)\right)+a_0G(z)$$

$$=b_{m-1}z^{-(m-1)}F(z)+\cdots+b_1z^{-1}F(z)+b_0F(z)$$

$$\left(a_mz^{-m}+\cdots+a_1z^{-1}+a_0\right)G(z)=\left(b_{m-1}z^{-(m-1)}+\cdots+b_1z^{-1}+b_0\right)F(z)$$
$$-\left(a_mg(-m)+\cdots+a_1g(-1)\right)-\cdots-z^{-(m-1)}a_mg(-1)$$

Solving linear, constant coefficient DEs

$$N\left(z^{-1}\right)=b_{m-1}z^{-(m-1)}+\cdots+b_1z^{-1}+b_0$$

$$D\left(z^{-1}\right)=a_m z^{-m}+\cdots+a_1z^{-1}+a_0$$

$$Q\left(z^{-1}\right)=-\left(a_m g(-m)+\cdots+a_1 g(-1)\right)+\cdots+z^{-(m-1)}a_m g(-1)$$

$$D\left(z^{-1}\right)G(z)=N\left(z^{-1}\right)F(z)+Q\left(z^{-1}\right)$$

$$G(z)=\frac{N\left(z^{-1}\right)}{D\left(z^{-1}\right)}F(z)+\frac{Q\left(z^{-1}\right)}{D\left(z^{-1}\right)}=H(z)F(z)+\frac{Q\left(z^{-1}\right)}{D\left(z^{-1}\right)}$$

Note

The solution is indeed seen to be a sum of two terms. The first term (whose Z-transform is $H(z)F(z)$) depends upon the forcing term or input f. It depends not at all on the initial conditions and indeed has itself zero initial conditions. This is the forced response. The second term (whose Z-transform is $Q(z^{-1})/D(z^{-1})$) depends upon the initial conditions but not at all on the input. It satisfies the associated difference equation where the input has been set to zero, i.e. the associated *homogeneous equation*. This is the free response. It tells us how the system shakes off its initial energy.

de Moivre's Idea

We have not made sufficient progress in what has been considered above. We obtain expressions not for the sequences which comprise the solutions to difference equations but rather for their moment generating functions and to be precise for their Z-transforms. Of course we are interested in finding the actual sequences. For de Moivre's elegant idea to really deliver on its promise we must develop a technique for going backwards, i.e. given the Z-transform of a sequence we must develop a method for finding the sequence itself. We need the inverse Z-transform.

Inverse Z-Transform

Definition 6.4: Given $F(z)$ the function $f(n)$ satisfying $f(n) = 0$ for $n < 0$ and $\mathcal{Z}(f(n)) = F(z)$ is called the inverse Z-transform of F, denoted $\mathcal{Z}^{-1}(F)$.

Partial Fraction Expansion

Assume $F(z)=\dfrac{N(z^{-1})}{D(z^{-1})}$, N and D polynomials having no common zeros such that $\deg(N)\leq\deg(D)$ and such that denominator D is monic. Then there exist constants

$$A_0\,, A_{11}\,, \ldots, A_{1m_1}\,, \ldots, A_{p1}\,, \ldots, A_{pm_p}$$

such that

$F(z)=$

$$A_0+\frac{A_{11}}{\left(z^{-1}-\lambda_1\right)}+\cdots+\frac{A_{1m_1}}{\left(z^{-1}-\lambda_1\right)^{m_1}}+\cdots+\frac{A_{p1}}{\left(z^{-1}-\lambda_p\right)}+\cdots+\frac{A_{pm_p}}{\left(z^{-1}-\lambda_p\right)^{m_p}}$$

Inverse Z-Transform

$$\mathcal{Z}^{-1}(A) = A\delta(n)$$

$$\mathcal{Z}^{-1}\left(\frac{A}{\left(z^{-1}-\lambda\right)^{k}}\right) = \frac{A(-1)^{k}}{\lambda^{k}}\binom{n+k-1}{k-1}\lambda^{-n}u(n)$$

Inverse Z-Transform

Assume

$$F(z) = A_0 + \frac{A_{11}}{\left(z^{-1} - \lambda_1\right)} + \cdots + \frac{A_{1m_1}}{\left(z^{-1} - \lambda_1\right)^{m_1}} + \cdots + \frac{A_{p1}}{\left(z^{-1} - \lambda_p\right)} + \cdots + \frac{A_{pm_p}}{\left(z^{-1} - \lambda_p\right)^{m_p}}$$

then

$$f(n) = A_0 \delta(n) + \left(\frac{-A_{11}}{\lambda_1}\right) \lambda_1^{-n} u(n) + \cdots + \left(\frac{(-1)^{m_1} A_{1m_1}}{\lambda_1^{m_1}}\right) \binom{n + m_1 - 1}{m_1 - 1} \lambda_1^{-n} u(n)$$

$$+ \cdots + \left(\frac{-A_{p1}}{\lambda_p}\right) \lambda_p^{-n} u(n) + \cdots + \left(\frac{(-1)^{m_p} A_{pm_p}}{\lambda_p^{m_p}}\right) \binom{n + m_p - 1}{m_p - 1} \lambda_p^{-n} u(n)$$

is the inverse Z - transform of $F(z)$.

Example 6.5

Find $\mathcal{Z}^{-1}\left(\dfrac{z^{-1}+1}{z^{-2}+z^{-1}-6}\right)$

$$\frac{z^{-1}+1}{z^{-2}+z^{-1}-6}=\frac{z^{-1}+1}{\left(z^{-1}-2\right)\left(z^{-1}+3\right)}=A_0+\frac{A_{11}}{z^{-1}-2}+\frac{A_{21}}{z^{-1}+3}$$

$$\frac{z^{-1}+1}{z^{-2}+z^{-1}-6}=\frac{0.6}{z^{-1}-2}+\frac{0.4}{z^{-1}+3}$$

$$\mathcal{Z}^{-1}\left(\frac{z^{-1}+1}{z^{-2}+z^{-1}-6}\right)=\left(\frac{-0.6}{2}\right)2^{-n}u(n)+\left(\frac{-0.4}{-3}\right)(-3)^{-n}u(n)$$

Example 6.5

$$F(z) = \left(\frac{z^{-1} + 1}{z^{-2} + z^{-1} - 6} \right)$$

```
>>  Num = [1  1]          create numerator polynomial Num

>>  Den = [1  1  -6]      create denominator polynomial Den
```

I have said before that a polynomial in *s* is created in Matlab by simply listing the coefficients in descending order of powers of *s*. I also admitted that this does not create a polynomial in *s*. It creates a vector and it is only when I hand this vector over to a Matlab function which assumes this vector to represent a polynomial in *s* that it becomes anything other than a vector. In this last statement I lied. Matlab cannot know whether it is a polynomial in *s* or in any other variable.

Example 6.5

Employ the **residue** command as usual:

```
>> [R,P,K] = residue(Num,Den)
```

```
R =

    0.4000
    0.6000


P =

    -3
     2
```

As usual K = [ ] , however in this case we do not ignore it. This value of K informs that $A_0 = 0$.

Example 6.6

Find inverse Z - transform of

$$F(z)=\left(\frac{0.25}{\left(0.25z^{-2}-\frac{1}{\sqrt{2}}z^{-1}+1\right)\left(z^{-1}-\sqrt{2}\right)}\right)$$

$$F(z)=\frac{0.5}{z^{-1}-\sqrt{2}}-\frac{0.25}{z^{-1}-\sqrt{2}-j\sqrt{2}}-\frac{0.25}{z^{-1}-\sqrt{2}+j\sqrt{2}}$$

$$f(n)=\left(\frac{-0.5}{\sqrt{2}}\right)\left(\sqrt{2}\right)^{-n}u(n)+$$

$$\left(\frac{0.25}{\sqrt{2}(1+j)}\right)\left(\sqrt{2}(1+j)\right)^{-n}u(n)+\left(\frac{0.25}{\sqrt{2}(1-j)}\right)\left(\sqrt{2}(1-j)\right)^{-n}u(n)$$

Example 6.6

$$F(z) = \left(\frac{0.25}{\left(0.25z^{-2} - \frac{1}{\sqrt{2}} z^{-1} + 1\right)\left(z^{-1} - \sqrt{2}\right)} \right)$$

>> Num = [0.25] *create numerator polynomial Num*

>> Den = conv([0.25 –1/sqrt(2) 1],[1 –sqrt(2)]) *create denominator polynomial Den*

>> [R,P,K] = residue(Num,Den)

R =	P =	K =
-0.2500 + 0.0000i	1.4142 + 1.4142i	[]
-0.2500 - 0.0000i	1.4142 - 1.4142i	
0.5000	1.4142	

Example 6.6

$$f(n) = \left(\frac{-0.5}{\sqrt{2}}\right)\left(\sqrt{2}\right)^{-n} u(n) +$$

$$\left(\frac{0.25}{\sqrt{2}(1+j)}\right)\left(\sqrt{2}(1+j)\right)^{-n} u(n) + \left(\frac{0.25}{\sqrt{2}(1-j)}\right)\left(\sqrt{2}(1-j)\right)^{-n} u(n)$$

$$= -(0.5)\left(\frac{1}{\sqrt{2}}\right)^{n+1} u(n) + (0.25)\left(2e^{j\frac{\pi}{4}}\right)^{-(n+1)} u(n) + (0.25)\left(2e^{-j\frac{\pi}{4}}\right)^{-(n+1)} u(n)$$

$$= -(0.5)\left(\frac{1}{\sqrt{2}}\right)^{n+1} u(n) + \left(\frac{0.25}{2^{n+1}}\right)\left(e^{-j\frac{\pi}{4}(n+1)} + e^{j\frac{\pi}{4}(n+1)}\right) u(n)$$

$$= -(0.5)\left(\frac{1}{\sqrt{2}}\right)^{n+1} u(n) + \left(\frac{0.25}{2^{n}}\right)\cos\left(\frac{\pi(n+1)}{4}\right) u(n)$$

Example 6.7

Consider the Fibonacci equation:

$$y(n) = y(n-1) + y(n-2), \quad y(-1) = 0 \quad , \quad y(-2) = 1$$

$$Y = \frac{1}{1 - z^{-1} - z^{-2}} = \frac{\frac{1}{\sqrt{5}}}{z^{-1} + \left(\frac{1+\sqrt{5}}{2}\right)} - \frac{\frac{1}{\sqrt{5}}}{z^{-1} + \left(\frac{1-\sqrt{5}}{2}\right)}$$

$$y(n) = \left(\frac{-\frac{1}{\sqrt{5}}}{-\left(\frac{1+\sqrt{5}}{2}\right)}\right)\left(-\left(\frac{1+\sqrt{5}}{2}\right)\right)^{-n} u(n) + \left(\frac{\frac{1}{\sqrt{5}}}{-\left(\frac{1-\sqrt{5}}{2}\right)}\right)\left(-\left(\frac{1-\sqrt{5}}{2}\right)\right)^{-n} u(n)$$

$$= \left(\frac{-1}{\sqrt{5}}\right)\left(\frac{-1}{\left(\frac{1+\sqrt{5}}{2}\right)}\right)^{n+1} u(n) + \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)}\right)^{n+1} u(n)$$

Example 6.7

This is a fairly unlikely formula for a solution. Since the initial conditions are integers and since the difference equation just calls for us to add the previous two values it follows that the solution is a sequence of integers. With the irrational number $\sqrt{5}$ appearing all over the place it is certainly not obvious that the formula produces integers for every $n \geq 0$. Obvious or not, it does. This is essentially one of the first hints of something really deep. Its revelation would have to wait for Galois, probably the most mathematically-gifted person to ever live.

Example 6.7

$$\left|\frac{-1}{\left(\frac{1+\sqrt{5}}{2}\right)}\right|<1 \quad , \quad \left|\frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)}\right|=\frac{1+\sqrt{5}}{2}>1$$

$$\lim_{n\to\infty}\left(\frac{y(n)}{y(n-1)}\right)=\frac{1+\sqrt{5}}{2}=\Phi \qquad \text{the Golden ratio.}$$

If you care to look it up you will find a very significant literature concerning this number. Its emergence as a limiting ratio here seems to explain its ubiquity in nature and that ubiquity explains our own preference for it.

Discrete-time

Annual rainfall, quarterly earnings, weekly box office receipts, daily calorie consumption, hourly electricity consumption are all quantities regularly dealt with by scientists. These quantities have quite distinct physical natures and as such are very dissimilar. Nevertheless, they share some common properties: they usually vary with time (i.e. from year to year, day to day, *etc*.) and at a particular time (i.e. for a particular year, day, etc.) they take on a particular value (a real number).

If each of these is to be an example of a *discrete-time signal* what then is the appropriate definition of such a signal?

Discrete-time Signals

Definition 7.1: A discrete-time signal is a function of a discrete variable (this variable is commonly given the symbol n denoting index or discrete-time).

Mathematicians refer to discrete-time signals as *sequences*. Discrete-time signals most commonly arise from sampling. A second commonly used symbol for the index is k.

Example 7.1

Discrete-time signals are ubiquitous in numerical problems. Consider for example a numerical procedure (the Newton-Raphson method) for calculating the square root of 2:

$$x(n) = \tfrac{1}{2}\, x(n-1) + \frac{1}{x(n-1)} \quad , \quad x(0) = 1$$

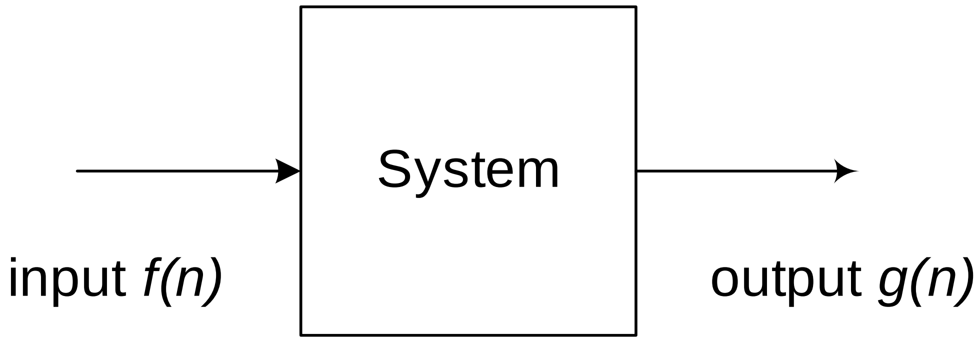
$$x(1) = \tfrac{3}{2} = 1.5 \quad , \quad x(2) = \tfrac{17}{12} = 1.4167 \quad ,$$

$$x(3) = \tfrac{577}{408} = 1.412 \quad , \quad \ldots$$

Note

I have used subscript notation previously, denoting the nth element of a sequence by x_n for example. There is a second commonly used notation however which denotes the nth element by $x(n)$. I will permit myself a certain level of carelessness and employ both notations to some extent.

Systems



eg square function on calculator.

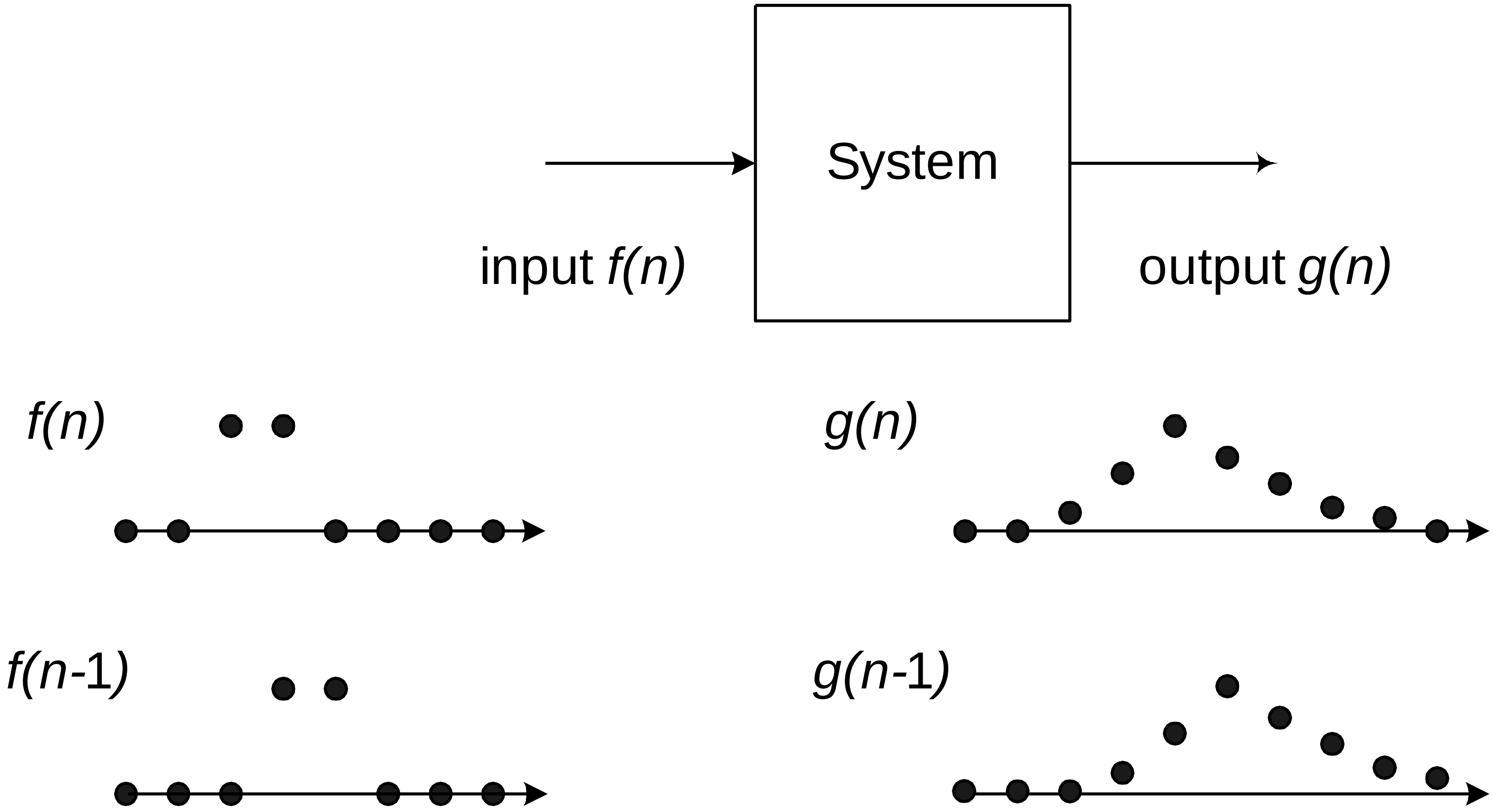
Definition 7.2: A discrete-time single-input single-output (SISO) system processes one discrete-time signal called the input (or excitation) and produces as a result a second discrete-time signal called the output (or response).

Time or Index-Invariance

Definition 7.3: A discrete-time, SISO system is time-invariant if it obeys the following rule:

If signal $f(n)$ as input produces signal $g(n)$ as output then delayed signal $f(n-m)$ as input produces delayed signal $g(n-m)$ as output for any delay m.

To illustrate:



Linearity

Definition 7.4: A discrete-time, SISO system is linear if it obeys the following rule:

If signal f_1 as input produces signal g_1 as output and signal f_2 as input produces signal g_2 as output then, given any real numbers α_1 and α_2, composite signal $\alpha_1 f_1 + \alpha_2 f_2$ as input produces composite signal $\alpha_1 g_1 + \alpha_2 g_2$ as output.

eg Multiply by 2 on calculator (input = number to be multiplied, output = twice this number)

Linearity

The particularly astute might object to this example (*sic* the multiply by 2 function on a calculator). I cannot continue to do this indefinitely. Eventually I get to a number where, if I try to multiply it by 2 it becomes too big to be stored/represented in the calculator. Actually I am happy with this example of a linear discrete-time system precisely for this reason. It reveals that in truth linear discrete-time systems do not exist (just as linear continuous-time systems do not exist). They are a useful fiction only. The multiply by 2 function might not always be linear, but it is linear provided you take care to keep the numbers sufficiently small.

Example 7.2

A temperature sensor measures the temperature at a certain location every minute. The measurement is subject to measurement noise. To provide some form of noise reduction the sensor records the average of the current and the previous temperature measurements.

$$R(n) = \tfrac{1}{2}T(n) + \tfrac{1}{2}T(n-1)$$

This may be regarded as a system with input equal to the sequence of temperature measurements and output equal to the sequence of recorded averages. It is linear and time-invariant and is an example of a *moving average*.

System Models

As noted previously in order to obtain scientific predictions of system behaviour one must have a set of mathematical equations which at least approximately describe this behaviour.

Such a set of equations is called a *model* of the system. Fundamental sciences (such as physics and chemistry) are concerned with finding such models.

Models

Consider the growth of a population of blowflies:

$$x(n) = a_1 x(n-1) - a_2 x(n-1) - bx^2(n-1)$$
$$= ax(n-1) - bx^2(n-1)$$

where $x(n)$ denotes the population on the first day of the n^{th} fortnight of the experiment. This model of course is essentially the same as that of Verhulst.

Clearly the special solution of extinction, $x(n) = 0$, is possible. This is an *equilibrium* solution.

Linear Models

For small x , i.e. close to extinction, competition is irrelevant as resources far exceed requirements so that we approximately obtain the simple growth law:

$$x(n) = ax(n-1)$$

a linear, constant coefficient, difference equation having solution:

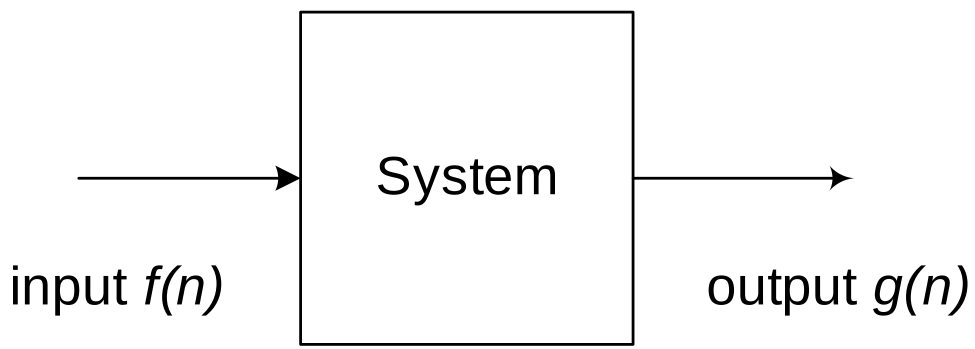
$$x(n) = a^n x(0)$$

The simple linear model predicts exponential growth. In practice exponential growth will not persist in the majority of populations due to increasing competition.

Linear vs. Nonlinear

The task of analysing a system is greatly simplified if the system is linear.

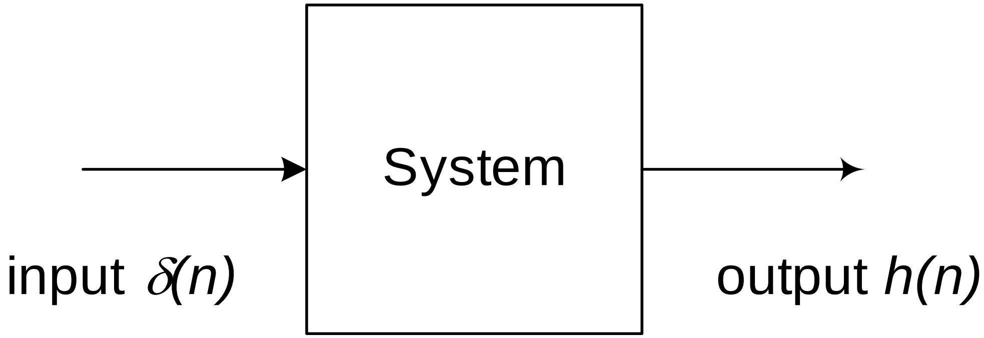
If $f_k(n)$ as input produces $g_k(n)$ as output, then



$$c_1 f_1(n) + c_2 f_2(n) + \cdots \qquad\qquad c_1 g_1(n) + c_2 g_2(n) + \cdots$$

Impulse Response

Definition 7.5: The response of a SISO, linear time-invariant (LTI) discrete-time system to an input equal to the Kronecker delta sequence $\delta(n)$ is called the *impulse response* of the system.



Example 7.2

Recall the temperature measurement with moving average

$$R(n) = \tfrac{1}{2}T(n) + \tfrac{1}{2}T(n-1)$$

Although we will develop a better method of calculation consider the impulse response by means of direct evaluation:

$$h(-1) = \tfrac{1}{2}\delta(-1) + \tfrac{1}{2}\delta(-2) = 0$$

$$h(0) = \tfrac{1}{2}\delta(0) + \tfrac{1}{2}\delta(-1) = \tfrac{1}{2}$$

$$h(1) = \tfrac{1}{2}\delta(1) + \tfrac{1}{2}\delta(0) = \tfrac{1}{2}$$

$$h(2) = \tfrac{1}{2}\delta(2) + \tfrac{1}{2}\delta(1) = 0$$

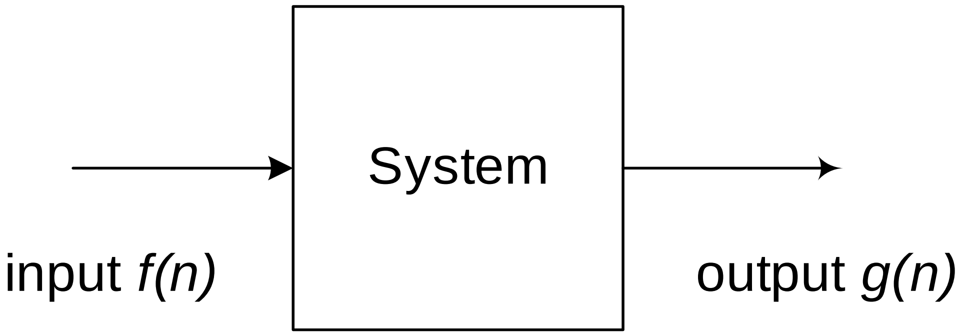
$$h(n) = 0 \quad \text{if} \quad n \neq 0,1$$

$$h(n) = \tfrac{1}{2} \quad \text{if} \quad n = 0,1$$

The Convolution Sum

Theorem 7.1: Let the impulse response of a SISO, LTI discrete-time system be $h(n)$. Then, under mild conditions, the response $g(n)$ to an excitation $f(n)$ is given by Dirichlet's formula

$$g(n) = \sum_{m=-\infty}^{\infty} f(m)h(n-m)$$



Argument

i/p $\delta(n) \Rightarrow$ o/p $h(n)$.

i/p $\delta(n-m) \Rightarrow$ o/p $h(n-m)$ (time - invariance).

i/p $f(m)\delta(n-m) \Rightarrow$ o/p $f(m)h(n-m)$ (linearity).

i/p $\sum_{m=-\infty}^{\infty} f(m)\delta(n-m) \Rightarrow$ o/p $\sum_{m=-\infty}^{\infty} f(m)h(n-m)$ (linearity).

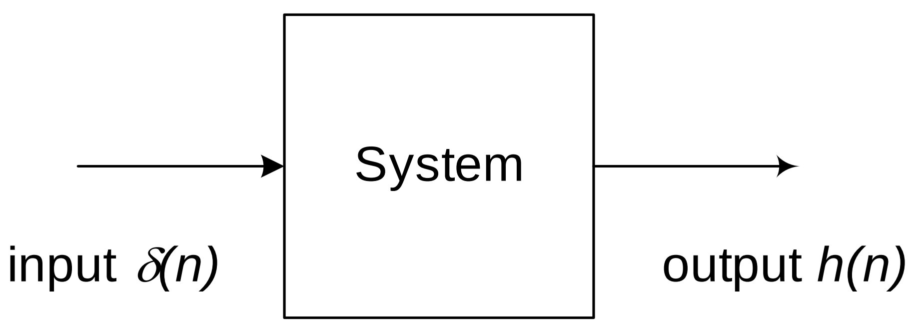
i/p $f(n) \Rightarrow$ o/p $\sum_{m=-\infty}^{\infty} f(m)h(n-m)$ (sifting).

Causal Systems

Definition 7.5: A discrete time signal $f(n)$ is causal if $f(n) = 0$ for all $n < 0$.

Definition 7.6: The response of an LTI, SISO system to a unit impulse $\delta(n)$ is called the impulse response of the system.

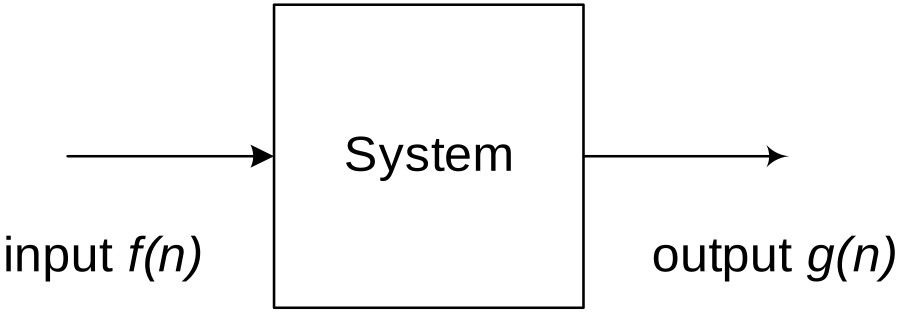
Definition 7.7: An LTI, SISO system is causal if the impulse response $h(n)$ is causal.



The Convolution Sum

Theorem 7.2: Let the impulse response of a SISO, LTI, causal system be $h(n)$. Then, under suitable conditions, the response $g(n)$ to a causal excitation $f(n)$ is given by Dirichlet's formula

$$g(n) = \sum_{m=0}^{n} f(m)h(n-m) \qquad \text{for } n \geq 0.$$



Z-Transform: Properties 4

Theorem 7.3: Convolution.

If $\mathcal{Z}\left(f_1(n)\right) = F_1(z)$ and $\mathcal{Z}\left(f_2(n)\right) = F_2(z)$ then

$$\mathcal{Z}\left(\sum_{m=0}^{n} f_1(m) f_2(n-m)\right) = F_1(z)F_2(z)$$

The Transfer Function

Theorem 7.4: Let the impulse response of a SISO, LTI, causal system be $h(n)$, let the input $f(n)$ be a causal signal then the output $g(n)$ is causal and

$$G(z) = F(z)H(z)$$

Example 7.2

Recall the temperature measurement with moving average

$$R(n) = \tfrac{1}{2}T(n) + \tfrac{1}{2}T(n-1)$$

taking the Z-transform:

$$R(z) = \tfrac{1}{2}T(z) + \tfrac{1}{2}z^{-1}T(z)$$

$$R(z) = \tfrac{1}{2}\left(1 + z^{-1}\right)T(z)$$

giving the transfer function: $\tfrac{1}{2}\left(1 + z^{-1}\right)$

Frequency Response

As in the continuous-time case we observe that:

(i) the Z-transform method comprises an effective method for solving linear, constant coefficient difference equations.

(ii) the general solution to such an equation takes the form of the sum of the free response and the forced response. The forced response is described in terms of convolution and the special form of the free response derives from the partial fraction expansion and its inverse Z-transform.

Frequency Response

A third general point may be made:

(iii) the steady-state, forced response of a linear, constant coefficient difference equation to a co-sinusoidal forcing term is a co-sinusoid of the same frequency. This point requires extensive elaboration. However we start with a simple example.

Example 7.2:

$$R(n) = \tfrac{1}{2}T(n) + \tfrac{1}{2}T(n-1)$$

$$T(n) = 25^{\circ} \quad \text{for all } n \geq 0$$

$$T(n) = 0^{\circ} \quad \text{for all } n < 0$$

$$R(0) = 12.5^{\circ}$$

$$R(n) = 25^{\circ} \quad \text{for all } n \geq 1$$

Note

Note that the solution is a sum of two components –

a transient which passes very quickly and a steady state. In this case the steady-state is a constant averaged temperature of 25^o. Of course the input is a constant temperature also of 25^o (for $n \geq 0$) in this case. So DC (i.e. zero frequency) input has produced DC output *in the long run*.

Consider a co-sinusoidal input.

Example 7.2:

$$R(n) = \tfrac{1}{2}T(n) + \tfrac{1}{2}T(n-1)$$

$$T(n) = 5\cos\left(\tfrac{n\pi}{2}\right)^{\mathrm{o}} \quad \text{for all } n \geq 0$$

$$T(n) = 0^{\mathrm{o}} \quad \text{for all } n < 0$$

$$R(0) = 2.5^{\mathrm{o}}$$

$$R(1) = 2.5\left(\cos\left(\tfrac{\pi}{2}\right) + 1\right)^{\mathrm{o}} = 2.5^{\mathrm{o}}$$

$$R(2) = 2.5\left(\cos(\pi) + \cos\left(\tfrac{\pi}{2}\right)\right)^{\mathrm{o}} = -2.5^{\mathrm{o}}$$

Example 7.2:

$$R(3) = 2.5\left(\cos\left(\tfrac{3\pi}{2}\right) + \cos(\pi)\right)^{\mathrm{o}} = -2.5^{\mathrm{o}}$$

$$R(n) = 2.5^{\mathrm{o}} \quad \text{for } n = 0, 1, 4, 5, 8, 9, 12, \ldots$$

$$R(n) = -2.5^{\mathrm{o}} \quad \text{for } n = 2, 3, 6, 7, 10, 11, \ldots$$

Example 7.2

$$R(z) = \left(\tfrac{1}{2} + \tfrac{1}{2} z^{-1}\right) T(z)$$

$$T(z) = \frac{5\left(1 - \cos\left(\frac{\pi}{2}\right) z^{-1}\right)}{1 - 2\cos\left(\frac{\pi}{2}\right) z^{-1} + z^{-2}} = \frac{5}{1 + z^{-2}}$$

$$R(z) = 2.5\left(\frac{1 + z^{-1}}{1 + z^{-2}}\right)$$

$$R(z) = \frac{1.25 - 1.25j}{z^{-1} - j} + \frac{1.25 + 1.25j}{z^{-1} + j}$$

Example 7.2

$$R(n) =$$

$$(1.25 - 1.25j)\left(\frac{-1}{j}\right) j^{-n} + (1.25 + 1.25j)\left(\frac{-1}{-j}\right)(-j)^{-n}$$

$$= (1.25 + 1.25j)(-j)^{n} + (1.25 - 1.25j)(j)^{n}$$

$$= \left(\frac{5\sqrt{2}}{4}\right) e^{\frac{j\pi}{4}} \left(e^{-\frac{j\pi}{2}}\right)^{n} + \left(\frac{5\sqrt{2}}{4}\right) e^{-\frac{j\pi}{4}} \left(e^{\frac{j\pi}{2}}\right)^{n}$$

Example 7.2

$$R(n) = \left(\frac{5}{\sqrt{2}}\right)\cos\left(\frac{n\pi}{2} - \frac{\pi}{4}\right)$$

Here no transient is present. In this case the steady-state is a co-sinusoid of frequency $\pi/2$ rad/sample. Note the units of rad/sample not rad/sec. Of course the input is a co-sinusoid of frequency $\pi/2$ rad/sample. So $\pi/2$ rad/sample input has produced $\pi/2$ rad/sample output *in the long run*.

Frequency Response

Consider the more general situation where a system is described by a linear, constant-coefficient difference equation with input equal to a co-sinusoid starting at discrete-time $n = 0$

$$f(n) = A\cos(\Omega_0 n)u(n)$$

A is the amplitude and Ω_0 is the frequency in rad/sample. It is this change in units more than anything which determines control theorists to denote the frequency of a discrete-time sinusoid by a different symbol from that of a continuous-time sinusoid.

Frequency Response

$$G(z) = \frac{N(z^{-1})}{D(z^{-1})} F(z) + \frac{Q(z^{-1})}{D(z^{-1})} = H(z)F(z) + \frac{Q(z^{-1})}{D(z^{-1})}$$

$$F(z) = A\left(\frac{1-\cos(\Omega_0)z^{-1}}{1-2\cos(\Omega_0)z^{-1} + z^{-2}}\right) = \frac{A(1-\cos(\Omega_0)z^{-1})}{\left(z^{-1}-e^{j\Omega_0}\right)\left(z^{-1}-e^{-j\Omega_0}\right)}$$

Frequency Response

$$G(z) = \frac{A_1}{z^{-1} - e^{j\Omega_0}} + \frac{\bar{A}_1}{z^{-1} - e^{-j\Omega_0}} + \frac{A_{11}}{z^{-1} - \lambda_1} + \cdots$$

where λ_1, ... denote the roots of $D(z^{-1})$, i.e. $1/\lambda_1$ denote the poles of the transfer function $H(z)$. Assume that all of these poles have modulus less than unity so that all but the first two terms of the partial fraction expansion correspond to transient terms in the solution.

Frequency Response

Now Heaviside's method gives:

$$A_1 = \frac{AN\left(e^{j\Omega_0}\right)\left(1-\cos\left(\Omega_0\right)e^{j\Omega_0}\right)}{\left(e^{j\Omega_0}-e^{-j\Omega_0}\right)\,D\left(e^{j\Omega_0}\right)} = H\left(e^{-j\Omega_0}\right)\frac{A\left(\frac{1}{2}-\frac{1}{2}e^{j2\Omega_0}\right)}{\left(2j\sin\left(\Omega_0\right)\right)}$$

$$= H\left(e^{-j\Omega_0}\right)\frac{\left(-A\right)\left(\frac{1}{2}e^{j\Omega_0}-\frac{1}{2}e^{-j\Omega_0}\right)e^{j\Omega_0}}{\left(2j\sin\left(\Omega_0\right)\right)} = H\left(e^{-j\Omega_0}\right)\frac{\left(-A\right)e^{j\Omega_0}}{2}$$

Frequency Response

$$g(n) = A_1\left(\frac{-1}{e^{j\Omega_0}}\right)e^{-j\Omega_0 n} + \overline{A}_1\left(\frac{-1}{e^{-j\Omega_0}}\right)e^{j\Omega_0 n} + \cdots$$

$$g(n) =$$

$$\left(\frac{A}{2}\right)H\left(e^{-j\Omega_0}\right)e^{j\Omega_0}e^{-j\Omega_0(n+1)} + \left(\frac{A}{2}\right)H\left(e^{j\Omega_0}\right)e^{-j\Omega_0}e^{j\Omega_0(n+1)} + \cdots$$

$$= \left(\frac{A}{2}\right)H\left(e^{-j\Omega_0}\right)e^{-j\Omega_0 n} + \left(\frac{A}{2}\right)H\left(e^{j\Omega_0}\right)e^{j\Omega_0 n} + \cdots$$

Frequency Response

$$g(n) =$$

$$\left(\frac{A}{2}\right)\left|H\left(e^{j\Omega_0}\right)\right| e^{-jArg\left(H\left(e^{j\Omega_0}\right)\right)} e^{-j\Omega_0 n} +$$

$$\left(\frac{A}{2}\right)\left|H\left(e^{j\Omega_0}\right)\right| e^{jArg\left(H\left(e^{j\Omega_0}\right)\right)} e^{j\Omega_0 n} + \cdots$$

$$g(n) = A\left|H\left(e^{j\Omega_0}\right)\right| \cos\left(\Omega_0 n + Arg\left(H\left(e^{j\Omega_0}\right)\right)\right) + \cdots$$

Frequency Response

So the steady-state response of a system with transfer function $H(z)$ to a co-sinusoidal input

$$f(n) = A\cos(\Omega_0 n)$$

is a co-sinusoidal output of the same frequency as the input:

$$A\left|H\left(e^{j\Omega_0}\right)\right|\cos\left(\Omega_0 n + Arg\left(H\left(e^{j\Omega_0}\right)\right)\right)$$

The output co-sinusoid has a different amplitude and phase.

Frequency Response

The output amplitude is the input amplitude multiplied by:

$$\left|H\left(e^{j\Omega_0}\right)\right|$$

the modulus of the transfer function evaluated at z = $\exp(j\Omega_0)$. This is called the *gain* at the frequency Ω_0.

The output phase is the input phase plus:

$$Arg\left(H\left(e^{j\Omega_0}\right)\right)$$

the phase or argument of the transfer function evaluated at $z = \exp(j\Omega_0)$.

Frequency Response

The transfer function evaluated at $z = \exp(j\Omega_0)$ is called the *frequency response* of the system. It tells us how the system reacts to a sinusoidal input. If the frequency response at a particular frequency Ω rad/sample is rather close to zero, i.e. the modulus is very small, then the system gain at that frequency is very small. Accordingly even a rather large amplitude input at this frequency will produce as a result a very low amplitude output. The system is said to *block* or *filter out* this frequency.

Frequency Response

On the other hand, if the frequency response at a particular frequency Ω rad/sample is rather large, i.e. the modulus is quite large, then the system gain at that frequency is high. Accordingly the system shows a resonance at this frequency since for a modest input signal amplitude at this frequency the output signal amplitude is higher, possibly considerably higher.

Frequency Response

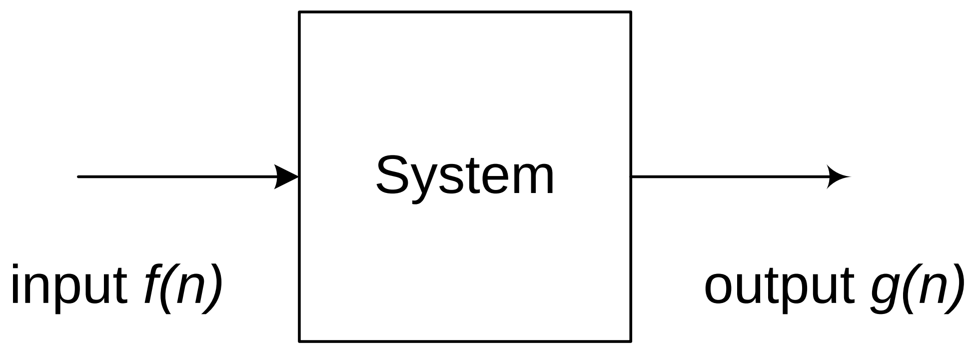
It is possible to establish this result by considering a general linear, constant-coefficient difference equation with an input which is a discrete-time co-sinusoid, by simply assuming that the associated response is also a discrete-time sinusoid of the same frequency (although having possibly different amplitude and phase) and then following your nose to determine what this output amplitude and phase must be. The calculation is a bit of a slog and does not add anything to what has just been said. Moreover it does not actually prove very much. It does not for example justify the assumption that the response is a sinusoid of the same frequency as the input.

Frequency Response

In the continuous time case these results concerning the frequency response hinted at, and were ultimately explained by Fourier theory. We may say the same here and accordingly are led to the last topic of the module: discrete-time Fourier theory.

Linear vs. Nonlinear

The task of analysing a discrete-time system is greatly simplified if the system is linear. If $f_k(n)$ as input produces $g_k(n)$ as output, the



$$c_1 f_1(n) + c_2 f_2(n) + \cdots \qquad\qquad c_1 g_1(n) + c_2 g_2(n) + \cdots$$

Linear vs Nonlinear

We have also noted that if the system is time-invariant as well as linear and if the input is a co-sinusoid of a certain frequency then a component of the output, namely the steady-state or long term output is (or rather can be, there is a caveat) relatively easy to find since it will also be a co-sinusoid of the exact same frequency and its amplitude and phase will be predictably related to the amplitude and phase of the input via the transfer function of the system.

Linear vs Nonlinear

Putting these two observations together we find that in principle we have a reasonably powerful idea for finding the steady-state response of an LTI system (specifically one where the transient is known to converge to zero, a so-called *stable* system) to an input which is expressible as a sum of co-sinusoids. This raises the question of what type of input signals can be expressed as sums of co-sinusoids. In short we wish to achieve for discrete-time signals what we achieved with the Fourier series for continuous-time signals. We want a *discrete-time Fourier series*.

Discrete-time Fourier Series

What do we mean by a discrete-time Fourier series? The Fourier series is the expression of a signal as a sum of sinusoids all of whose frequencies are integer multiples of a fundamental frequency. So the discrete-time Fourier series will be an expression of the discrete-time signal as a sum of discrete-time sinusoids all of whose frequencies are integer multiples of a fundamental frequency, i.e:

$$f(n) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 n}$$

for suitable coefficients c_k and suitable fundamental frequency Ω_0 rad/sample.

Discrete-time Fourier Series

Every term on the right-hand side is periodic with period (although not necessarily least period) $N = 2\pi/\Omega_0$ as for all k:

$$e^{jk\Omega_0(n+N)} = e^{jk\Omega_0 n} e^{jk\Omega_0 N} = e^{jk\Omega_0 n} e^{jk2\pi} = e^{jk\Omega_0 n}$$

The sum of such signals will "apparently" again be periodic of period N. So it appears that a signal $f(n)$ will only permit such a decomposition if it is itself periodic of period N. The discrete-time Fourier series can be expected to give representations for periodic discrete-time signals only. How can we find appropriate values for the coefficients?

Discrete-time Fourier Series

The purported expansion as a sum of discrete-time complex exponentials

$$f(n) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 n}$$

is highly redundant at present. Note that:

$$e^{j(k+N)\Omega_0 n} = e^{jk\Omega_0 n} e^{jN\Omega_0 n} = e^{jk\Omega_0 n} e^{j2\pi n} = e^{jk\Omega_0 n}$$

so that the complex exponentials in the summation are not distinct functions. Indeed the ($k+N$) th equals the kth. I should rewrite the sum as a sum of at least potentially distinct terms:

$$f(n) = \sum_{k=0}^{N-1} c_k e^{jk\Omega_0 n}$$

Discrete-time Fourier Series

There are greatly superior ways for finding these values but, as we did with the Fourier series, we will not consider them. Rather we will simply and rather randomly observe that for any given Ω_0 and N such that $N\Omega_0 = 2\pi$:

$$\frac{1}{N}\sum_{n=0}^{N-1}\exp\left(jk\Omega_0 n\right)\overline{\exp\left(jl\Omega_0 n\right)} = \begin{cases} 0 & \text{if } k \neq l \\ 1 & \text{if } k = l \end{cases}$$

Proof

$$\frac{1}{N}\sum_{n=0}^{N-1}\exp\left(jk\Omega_0 n\right)\overline{\exp\left(jl\Omega_0 n\right)}$$

$$\frac{1}{N}\sum_{n=0}^{N-1}\exp\left(jk\Omega_0 n\right)\exp\left(-jl\Omega_0 n\right) \qquad = \frac{1}{N}\sum_{n=0}^{N-1}\exp\left(j\left(k-l\right)\Omega_0 n\right)$$

If $k \neq l :$ $\quad = \dfrac{1}{N}\left(1+r+\cdots+r^{N-1}\right)$

where $r = \exp\left(j\left(k-l\right)\Omega_0\right)$ so that $r^N = 1,\ r \neq 1$

$$\text{i.e.} = \frac{1}{N}\left(\frac{1-r^N}{1-r}\right) = 0.$$

Proof

$$\text{If } k = l: \qquad = \frac{1}{N}\sum_{n=0}^{N-1} 1 = 1$$

▲

Discrete-time Fourier Series

If $f(n)=\sum_{m=0}^{N-1} c_m \exp(jm\Omega_0 n)$ for suitable coefficients c_m then

$$\frac{1}{N}\sum_{n=0}^{N-1} f(n)\exp(-jk\Omega_0 n)$$

$$=\frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} c_m \exp(jm\Omega_0 n)\exp(-jk\Omega_0 n)$$

$$=\sum_{m=0}^{N-1} c_m\left(\frac{1}{N}\sum_{n=0}^{N-1}\exp(jm\Omega_0 n)\exp(-jk\Omega_0 n)\right)=c_k$$

Discrete-time Fourier Series

Definition 8.1: Given any signal $f(n)$ of period N the Fourier coefficients of f are the associated numbers:

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} f(n)\exp(-jk\Omega_0 n)$$

Definition 8.2: Given any signal $f(n)$ of period N the discrete-time Fourier series of f equals the summation

$$\sum_{k=0}^{N-1} c_k \exp(jk\Omega_0 n)$$

where c_k are the Fourier coefficients of f.

Fourier Coefficients: Properties 1

If $f(n)$ is periodic of period N then $c_k = c_{k+N}$ for all k.

Proof:

$$c_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} f(n)\exp\big(j(k+N)\Omega_0 n\big)$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} f(n)\exp\big(jk\,\Omega_0 n\big)$$

$$= c_k$$

▲

Fourier Coefficients: Properties 2

If f is real then $c_{-k} = \overline{c_k}$ for all k.

Proof:

$$c_{-k} = \frac{1}{N}\sum_{n=0}^{N-1} f(n)\exp(jk\Omega_0 n)$$

$$= \overline{\frac{1}{N}\sum_{n=0}^{N-1} f(n)\exp(-jk\Omega_0 n)}$$

$$= \overline{c_k}$$

▲

Trigonometric Fourier Series

Lemma 8-1: If signal f is periodic of period N and real then its Fourier series may be expressed as follows:

$$c_0 + \sum_{k=1}^{M}\left(\alpha_k \cos\left(k\Omega_0 n\right) + \beta_k \sin\left(k\Omega_0 n\right)\right)$$

$$M = \frac{N-1}{2} \text{ if } N \text{ odd}, M = \frac{N}{2} \text{ if } N \text{ even.}$$

$$c_0 = \frac{1}{N}\sum_{n=0}^{N-1} f\left(n\right)$$

$$\alpha_k = 2\,\mathrm{Re}\left(c_k\right) = \frac{2}{N}\sum_{n=0}^{N-1} f\left(n\right)\cos\left(k\Omega_0 n\right)$$

$$\beta_k = -2\,\mathrm{Im}\left(c_k\right) = \frac{2}{N}\sum_{n=0}^{N-1} f\left(n\right)\sin\left(k\Omega_0 n\right)$$

Numerical Fourier Series

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} f(n)\exp(-jk\Omega_0 n)$$

From the discussion in the continuous-time case (with a slight switch of notation) we recall that the FFT is an algorithm for efficient calculation of the numbers:

$$F_k = \sum_{n=0}^{N-1} f\left(\frac{nT}{N}\right)\exp\left(\frac{-j2\pi kn}{N}\right) \qquad c_k = \frac{F_k}{N}$$

Interpretation

Just as in the continuous-time case we interpret the Fourier series as describing the *frequency content* of the signal.

Interpretation

To put this interpretation/definition to the test consider the special case of the zeroth Fourier coefficient. The associated pure tone is the signal:

$$\exp\left(j0\Omega_0 n\right) = 1$$

Although this is a pure tone it is a zero frequency tone, i.e. a constant. From the formula we have:

$$c_0 = \frac{1}{N}\sum_{n=0}^{N-1} f\left(n\right)\exp\left(-j0\Omega_0 n\right) = \frac{1}{N}\sum_{n=0}^{N-1} f\left(n\right)$$

Interpretation

$$c_0 = \frac{1}{N}\sum_{n=0}^{N-1} f(n)$$

The number equal to this summation is called the *average value* or just the *average* of the signal. The interpretation of the Fourier series is that c_0 measures the degree to which a zero frequency pure tone or constant is present in the signal.

Example 8.1

An LTI, SISO, causal, discrete-time system with input $x(n)$ and output $y(n)$ is governed by the recursion/difference equation:

$$y(n) - 2.501y(n-1) + 2.1157y(n-2) - 0.5917y(n-3) =$$
$$0.0013x(n-1) + 0.0024x(n-2) + 0.0001x(n-3) \qquad \text{for } n \geq 0$$

subject to the initial conditions $y(-1) = y(-2) = y(-3) = 0.$

Given an input $x(n)$ which is zero for n not equal to 0 or 1 and 0.5 for n equal to 0 or 1, find a formula for the resulting output of the system

Let the input to the system be a periodic signal of period 10. For n = 0 and 1 this signal equals 0.5. For n = 2, 3, … , 9 this signal is equal to 0. Find the resulting steady-state output of the system.

Example 8.1

The first task is to solve the difference equation for a known input. All of the initial conditions are zero which renders the task far simpler than it might otherwise be. The "obvious" approach to solving the equation is to have recourse to the methods presented in the module, namely the Z-transform.

$$Y(z)=\left(\frac{0.0013z^{-1}+0.0024z^{-2}+0.0001z^{-3}}{1-2.501z^{-1}+2.1157z^{-2}-0.5917z^{-3}}\right)X(z)$$

$$X(z)=0.5+0.5z^{-1}$$

Example 8.1

$$Y(z)=\left(\frac{0.0013z^{-1}+0.0024z^{-2}+0.0001z^{-3}}{1-2.501z^{-1}+2.1157z^{-2}-0.5917z^{-3}}\right)\left(0.5+0.5z^{-1}\right)$$

Employing the Matlab command **residue**:

$$Y(z)=\frac{-0.0472}{z^{-1}-1.5639}+\frac{0.0179-0.0114j}{z^{-1}-1.0059-0.2625j}+\frac{0.0179+0.0114j}{z^{-1}-1.0059+0.2625j}$$
$$-0.0001z^{-1}-0.0024$$

It should be noted that the third output argument of **residue** K is not null in this case reflecting the last two, slightly unusual terms in the partial fraction expansion.

Example 8.1

$$y(n)=\frac{0.0472}{1.5639^{n+1}}-\frac{0.0179-0.0114j}{(1.0059+0.2625j)^{n+1}}-\frac{0.0179+0.0114j}{(1.0059-0.2625j)^{n+1}}$$
$$-0.0001\delta(n-1)-0.0024\delta(n)$$

$$y(n)=\frac{0.0472}{1.5639^{n+1}}-\frac{0.0212e^{-j0.5671}}{(1.0396)^{n+1}e^{-j0.2553(n+1)}}-\frac{0.0212e^{j0.5671}}{(1.0396)^{n+1}e^{j0.2553(n+1)}}$$
$$-0.0001\delta(n-1)-0.0024\delta(n)$$

$$y(n)=\frac{0.0472}{1.5639^{n+1}}-\frac{0.0212e^{j(0.2553(n+1)-0.5671)}}{(1.0396)^{n+1}}-\frac{0.0212e^{-j(0.2553(n+1)-0.5671)}}{(1.0396)^{n+1}}$$
$$-0.0001\delta(n-1)-0.0024\delta(n)$$

Example 8.1

$$y(n)=\frac{0.0472}{1.5639^{n+1}}-\frac{0.0424}{(1.0396)^{n+1}}\cos(0.2553n-0.3118)$$
$$-0.0001\delta(n-1)-0.0024\delta(n)$$

Example 8.1

The second task is to find the steady-state output given a periodic input. Fourier theory offers the means to solve this problem. There are three broad parts to this solution. Firstly find the transfer function:

$$H(z) = \left(\frac{0.0013z^{-1} + 0.0024z^{-2} + 0.0001z^{-3}}{1 - 2.501z^{-1} + 2.1157z^{-2} - 0.5917z^{-3}} \right)$$

Secondly express the periodic input signal as a sum of sinusoids, i.e. find its discrete-time Fourier series.

Example 8.1

The input signal is periodic of period $N = 10$.

$$x(n) = \begin{cases} 0.5 & \text{for } n = 0, 1 \\ 0 & \text{for } n = 2, \ldots, 9 \end{cases}$$

```
>>  N = 10   Set equal to period of signal.
>>  x = [0.5 0.5 zeros(1,N-2)];
>>  XX = fft(x)/N;
```

0.1000 0.0905 - 0.0294i 0.0655 - 0.0476i 0.0345 - 0.0476i

0.0095 - 0.0294i 0 0.0095 + 0.0294i 0.0345 + 0.0476i

0.0655 + 0.0476i 0.0905 + 0.0294i

Example 8.1

The input signal is

$$x(n) = 0.1 + 0.1809\cos\left(\tfrac{\pi}{5}n\right) + 0.0588\sin\left(\tfrac{\pi}{5}n\right) +$$
$$0.1309\cos\left(\tfrac{2\pi}{5}n\right) + 0.0951\sin\left(\tfrac{2\pi}{5}n\right) + 0.0691\cos\left(\tfrac{3\pi}{5}n\right) +$$
$$0.0951\sin\left(\tfrac{3\pi}{5}n\right) + 0.0191\cos\left(\tfrac{4\pi}{5}n\right) + 0.0588\sin\left(\tfrac{4\pi}{5}n\right)$$

Thirdly we must determine $H\left(e^{j\Omega}\right)$ for Ω equal to all of these frequencies 0, $\pi/5$, $2\pi/5$, $3\pi/5$ and $4\pi/5$.

Example 8.1

```
>> Num = [0.0001 0.0024 0.0013 0];
>> Den = [-0.5917 2.1157 -2.501 1];
>> w = [0:4]*(pi/5);
>> H = polyval(Num,exp(-i*w))./polyval(Den,exp(-i*w))
```

0.1652 -0.0112 + 0.0156i -0.0002 + 0.0024i 0.0002 + 0.0007i

0.0002 + 0.0002i

Example 8.1

```
>> abs(H)
>> angle(H)
```

0.1652 0.0192 0.0024 0.0007 0.0003

0 2.1921 1.6754 1.3413 0.9062

Employing Fourier theory:

$$y(n) = 0.0165 + 0.0035\cos\left(\tfrac{\pi}{5}n + 2.1921\right) + 0.0011\sin\left(\tfrac{\pi}{5}n + 2.1921\right) + 0.00031\cos\left(\tfrac{2\pi}{5}n + 1.6754\right) + 0.00023\sin\left(\tfrac{2\pi}{5}n + 1.6754\right) + 0.000047\cos\left(\tfrac{3\pi}{5}n + 1.3413\right) + 0.00006\sin\left(\tfrac{3\pi}{5}n + 1.3413\right) + 0.000005\cos\left(\tfrac{4\pi}{5}n + 0.9062\right) + 0.000016\sin\left(\tfrac{4\pi}{5}n + 0.9062\right)$$

Note

As in the case of continuous-time signals most discrete-time signals will not be periodic. We must consider how to construct a theory of the frequency content of a discrete-time signal in the event that it is not periodic. We follow Fourier by imagining that a non-periodic signal is in fact just a limiting case of a periodic signal where the period goes to infinity.

Fourier's limit argument

Suppose the period $N = 2M+1$ is odd. Invoking periodicity and the established properties of Fourier coefficients we have:

$$f(n) = \sum_{k=-M}^{M} c_k e^{jk\left(\frac{2\pi}{2M+1}\right)n} \qquad c_k = \frac{1}{2M+1} \sum_{n=-M}^{M} f(n) e^{-jk\left(\frac{2\pi}{2M+1}\right)n}$$

$$f(n) = \sum_{k=-M}^{M} \left(\frac{1}{2M+1} \sum_{n=-M}^{M} f(n) e^{-j\frac{2\pi kn}{2M+1}} \right) e^{j\frac{2\pi kn}{2M+1}}$$

Fourier's limit argument

$$\Omega = \frac{2\pi k}{2M+1} \qquad d\Omega = \frac{2\pi}{2M+1}$$

$$f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty} f(n)e^{-j\Omega n}\right)e^{j\Omega n}\,d\Omega$$

$$-\pi \le \Omega \le \pi$$

Discrete-time Fourier Transform

Discrete-time Fourier Transform

$$F\left(e^{j\Omega}\right) = \mathcal{F}\left(f\left(n\right)\right) = \sum_{n=-\infty}^{\infty} f\left(n\right)\exp\left(-j\Omega n\right)$$

$$-\pi \leq \Omega \leq \pi.$$

Inverse Discrete-time Fourier Transform

$$f\left(n\right) = \mathcal{F}^{-1}\left(F\left(e^{j\Omega}\right)\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F\left(e^{j\Omega}\right)\exp\left(j\Omega n\right)d\Omega$$

Discrete-time Fourier Transform: Properties 1

If f is real then $F\left(e^{-j\Omega}\right) = \overline{F\left(e^{j\Omega}\right)}$

for all Ω.

Proof: $$F\left(e^{-j\Omega}\right) = \sum_{n=-\infty}^{\infty} f(n)\exp(j\Omega n)$$

$$= \overline{\sum_{n=-\infty}^{\infty} f(n)\exp(-j\Omega n)}$$

$$= \overline{F\left(e^{j\Omega}\right)}$$

▲

Discrete-time Fourier Transform: Properties 2

If $\mathcal{F}\left(f_1(n)\right)=F_1\left(e^{j\Omega}\right)$ and $\mathcal{F}\left(f_2(n)\right)=F_2\left(e^{j\Omega}\right)$

then

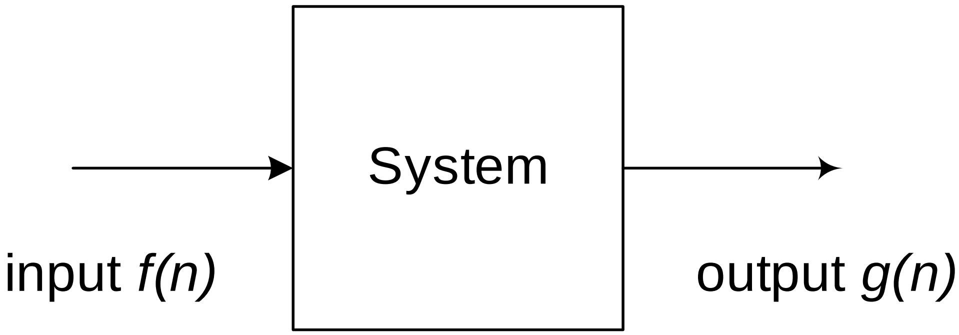
$$\mathcal{F}\left(\alpha_1 f_1(n)+\alpha_2 f_2(n)\right)=\alpha_1 F_1\left(e^{j\Omega}\right)+\alpha_2 F_2\left(e^{j\Omega}\right)$$

This property is called *linearity*.

Recall: The Convolution Sum

Theorem 8.1: Let the impulse response of a SISO, LTI discrete-time system be $h(n)$. Then, under mild conditions, the response $g(n)$ to an excitation $f(n)$ is given by Dirichlet's formula

$$g(n) = \sum_{m=-\infty}^{\infty} f(m)h(n-m)$$



Fourier Transform: Properties 3

Theorem 8.2: Convolution

If $\mathcal{F}\left(f_1(n)\right) = F_1\left(e^{j\Omega}\right)$ and $\mathcal{F}\left(f_2(n)\right) = F_2\left(e^{j\Omega}\right)$

then $\mathcal{F}\left(\sum_{m=-\infty}^{\infty} f_1(m) f_2(n-m)\right) = F_1\left(e^{j\Omega}\right) F_2\left(e^{j\Omega}\right)$

The Transfer Function

Theorem 8.3: Let the impulse response of a discrete-time SISO, LTI system be $h(n)$, let the input be $f(n)$ and let the output be $g(n)$, then, under mild conditions

$$G\left(e^{j\Omega}\right) = F\left(e^{j\Omega}\right)H\left(e^{j\Omega}\right)$$

Note

The discrete-time Fourier transform of the impulse response $h(n)$ of a discrete-time system is called the transfer function:

$$H\left(e^{j\Omega}\right)$$

I appear to have made a bad error of notation here. I already denoted the Z-transform of the impulse response by $H(z)$ and called this the transfer function.

Note

If the system is causal, i.e. if the impulse response $h(n)$ is zero for $n < 0$ then:

$$H\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\Omega n} = \sum_{n=0}^{\infty} h(n)\left(e^{j\Omega}\right)^{-n}$$

$$= \sum_{n=0}^{\infty} h(n) z^{-n}\Bigg|_{z=e^{j\Omega}} = H(z)\Big|_{z=e^{j\Omega}}$$

In fact the notation is actually well-chosen.

Example 8.2

Recall the temperature measurement with moving average

$$R(n) = \tfrac{1}{2}T(n) + \tfrac{1}{2}T(n-1)$$

$$h(n) = 0 \quad \text{if} \quad n \neq 0,1$$
$$h(n) = \tfrac{1}{2} \quad \text{if} \quad n = 0,1$$

$$H\left(e^{j\Omega}\right) = \tfrac{1}{2} + \tfrac{1}{2}e^{-j\Omega}$$

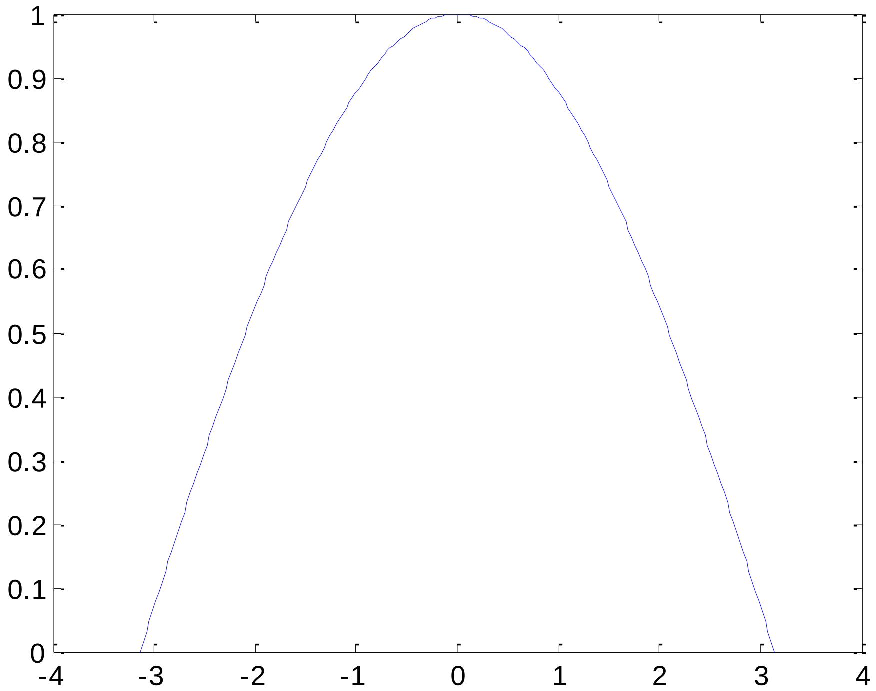
$$R\left(e^{j\Omega}\right) = \left(\tfrac{1}{2} + \tfrac{1}{2}e^{-j\Omega}\right)T\left(e^{j\Omega}\right)$$

Example 8.2

$$R\left(e^{j\Omega}\right)=\left(\tfrac{1}{2}+\tfrac{1}{2}e^{-j\Omega}\right)T\left(e^{j\Omega}\right)$$

$$H\left(e^{j\Omega}\right)=\tfrac{1}{2}+\tfrac{1}{2}e^{-j\Omega}$$

Plot of $|H(e^{j\Omega})|$ vs. Ω, $-\pi < \Omega < \pi$



Low frequencies are passed, high frequencies are blocked.

Example 8.2

Alternatively

$$R(n) = \tfrac{1}{2}T(n) + \tfrac{1}{2}T(n-1)$$

$$R(z) = \tfrac{1}{2}T(z) + \tfrac{1}{2}z^{-1}T(z) = \left(\tfrac{1}{2} + \tfrac{1}{2}z^{-1}\right)T(z)$$

$$H(z) = \frac{1}{2} + \frac{1}{2}z^{-1}$$

$$H\left(e^{j\Omega}\right) = \frac{1}{2} + \frac{1}{2}e^{-j\Omega}$$

Note

The Z-transform amounts to a more elementary method for determining the transfer function of a system given that the system is described by difference equations. Accordingly the relationship which has been established relating the Z-transform to the discrete-time Fourier transform yields a convenient method for determining the transfer function relative to the discrete-time Fourier transform for causal systems described by difference equations.

Note

As in the case of the Z-transform, the value of the discrete-time Fourier transform is that it yields a system theory, i.e. a way to solve difference equations, but also it has physical meaning, so it yields a signal theory. Like its continuous time counterpart the discrete-time Fourier transform not only describes in an elegant manner (namely the transfer function) how a system behaves, it also explains what the system is through the physical interpretation of the discrete-time Fourier transform as the frequency content.

Note

There is a curious difference between the discrete-time Fourier transform and the continuous-time Fourier transform. As I have defined it the discrete-time Fourier transform is only defined for frequencies Ω between $-\pi$ and π rad/sample. I could employ the same definition to define the discrete-time Fourier transform for frequencies outside of this range but I would find it to be periodic of period 2π, i.e. it would just repeat outside this range. This is because the pure tones themselves are periodic with respect to Ω.

Note

In the same manner if I describe the action of a system by describing its transfer function relative to the discrete-time Fourier transform then I find again that this transfer function is periodic of period 2π, so that how it behaves between $-\pi$ and π tells me how it behaves throughout. For this module I will not employ the periodic versions and will accept that this is how things turn out, discrete-time Fourier transforms can be effectively only defined by restricting the frequency to the finite range $-\pi$ to π. When you take a module which covers some sampling theory however you will find that there is somewhat more to this story.

Note

As in the case of continuous-time signals we now have two methods which purport to describe the frequency content of discrete-time periodic signals, the discrete-time Fourier series and the discrete-time Fourier transform. If the two methods are correct then they should agree with one another.

Fourier Transform of Periodic Signals

$$\mathcal{F}^{-1}\left(2\pi\delta\left(\Omega-\Omega_0\right)\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}2\pi\delta\left(\Omega-\Omega_0\right)\exp\left(j\Omega n\right)d\Omega$$

$$=\int_{-\pi}^{\pi}\delta\left(\Omega-\Omega_0\right)\exp\left(j\Omega n\right)d\Omega=\exp\left(j\Omega_0 n\right)$$

Fourier Transform of Periodic Signals

$$\mathcal{F}\left(\exp\left(j\Omega_0 n\right)\right) = 2\pi\delta\left(\Omega - \Omega_0\right)$$

$$\mathcal{F}\left(A\sin\left(\Omega_0 n\right)\right) = \frac{A\pi}{j}\left(\delta\left(\Omega - \Omega_0\right) - \delta\left(\Omega + \Omega_0\right)\right)$$

$$\mathcal{F}\left(A\cos\left(\Omega_0 n\right)\right) = A\pi\left(\delta\left(\Omega - \Omega_0\right) + \delta\left(\Omega + \Omega_0\right)\right)$$

Note

It is possible to employ the discrete-time Fourier transform to acquire a much simpler proof of the result given above concerning the steady-state response of a discrete-time LTI system to a sinusoidal input. In fact we will prove an even more general result than the previous.

Sinusoidal Steady State Response

Theorem 8.4 : If an LTI, SISO, discrete-time system has transfer function $H\left(e^{j\Omega}\right)$ and input $f(n)=\exp\left(j\Omega_0 n\right)$ then under certain conditions the steady-state output of the system is

$$g(n)=H\left(e^{j\Omega_0}\right)\exp\left(j\Omega_0 n\right)$$

Sinusoidal Steady State Response

$$g(n) = \mathcal{F}^{-1}\left(G\left(e^{j\Omega}\right)\right) =$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j\Omega}\right) 2\pi\delta(\Omega - \Omega_0)\exp(j\Omega n)\,d\Omega$$

$$= H\left(e^{j\Omega_0}\right)\exp(j\Omega_0 n)$$

Sinusoidal Steady State Response

Theorem 8.4*: If a real, LTI, SISO, discrete-time system has transfer function

$$H\left(e^{j\Omega}\right)=\left|H\left(e^{j\Omega}\right)\right|\exp\left(j\theta(\Omega)\right)$$

and input $f(n)=A\sin\left(\Omega_0 n+\Phi\right)$ then under certain conditions the steady-state output of the system is

$$g(n)=A\left|H\left(e^{j\Omega_0}\right)\right|\sin\left(\Omega_0 n+\Phi+\theta(\Omega_0)\right)$$

Sinusoidal Steady State Response

$$G\left(e^{j\Omega}\right)=H\left(e^{j\Omega}\right)\left(\frac{A\pi}{j}e^{j\phi}\delta\left(\Omega-\Omega_0\right)-\frac{A\pi}{j}e^{-j\phi}\delta\left(\Omega+\Omega_0\right)\right)$$

$$g(n)=$$

$$H\left(e^{j\Omega_0}\right)\frac{A}{2j}e^{j\phi}\exp\left(j\Omega_0 n\right)-H\left(e^{-j\Omega_0}\right)\frac{A}{2j}e^{-j\phi}\exp\left(-j\Omega_0 n\right)$$

$$=\left|H\left(e^{j\Omega_0}\right)\right|\frac{A}{2j}e^{j\left(\Omega_0 n+\phi+\theta\left(\Omega_0\right)\right)}-\left|H\left(e^{j\Omega_0}\right)\right|\frac{A}{2j}e^{-j\left(\Omega_0 n+\phi+\theta\left(\Omega_0\right)\right)}$$

$$=\left|H\left(e^{j\Omega_0}\right)\right|A\sin\left(\Omega_0 n+\phi+\theta\left(\Omega_0\right)\right)$$