Problem 1

(a) Let $D(\omega)$ be the desired filter response

$$D(\omega) = \begin{cases} 1, & |\omega| \le \frac{\pi}{3} \\ 0, & \text{else} \end{cases}$$

$$G_d(\omega) = D(\omega)\, e^{-j \frac{N-1}{2} \omega} = \begin{cases} e^{-j \frac{15}{2} \omega}, & |\omega| \le \frac{\pi}{3} \\ 0, & \text{else} \end{cases}$$

$$g[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} G_d(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} e^{-j\frac{15}{2}\omega}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{3} \operatorname{sinc}\left(\frac{\pi}{3}\left(n - \frac{15}{2}\right)\right)$$

$h[n] = g[n]\, w[n]$, where $w[n]$ is the window

$$w[n] = \begin{cases} 1, & 0 \le n \le 15 \\ 0, & \text{else} \end{cases}$$

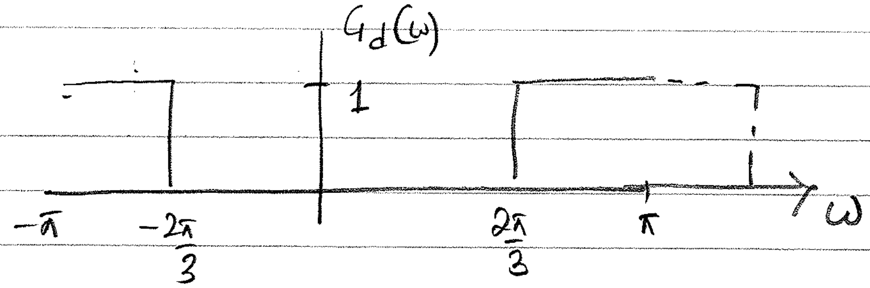
$$h[n] = \begin{cases} \frac{1}{3} \operatorname{sinc}\left(\frac{\pi}{3}\left(n - \frac{15}{2}\right)\right), & 0 \le n \le 15 \\ 0, & \text{else} \end{cases}$$

(b) The Hamming window is

$$w[n] = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right), & 0 \le n \le N-1 \\ 0, & \text{else} \end{cases}$$

$$h[n] = \begin{cases} \frac{1}{3} \operatorname{sinc}\left(\frac{\pi}{3}\left(n - \frac{15}{2}\right)\right)\left(0.54 - 0.46 \cos\left(\frac{2\pi n}{15}\right)\right), & 0 \le n \le 15 \\ 0, & \text{else} \end{cases}$$

(2) a)



$$G_d(\omega) = \begin{cases} 1 & |\omega| > \frac{2\pi}{3} \\ 0 & \text{else} \end{cases}$$

$\therefore$ Type 4 filter :

$$\Rightarrow \quad D(\omega) = \begin{cases} e^{j\left(\frac{\pi}{2} - \frac{31}{2}\omega\right)} & \frac{2\pi}{3} \leq \omega < \pi \\ 0 & |\omega| < 2\pi/3 \\ e^{-j\left(-\frac{\pi}{2} - \frac{31}{2}\right)} & -\pi < \omega \leq -\frac{2\pi}{3} \end{cases}$$

$$\Rightarrow \quad D(\omega) = \begin{cases} j\,e^{-j\,31/2} & \frac{2\pi}{3} \leq \omega < \pi \\ 0 & \text{else} \\ -j\,e^{-j\,31/2} & -\pi < \omega \leq -\frac{2\pi}{3} \end{cases}$$

We can write,

$$d[n] = \frac{1}{2\pi} \int_{2\pi/3}^{4\pi/3} D(\omega)\, e^{j\omega n}\, d\omega$$

Note that $D(\omega) = j e^{-j\frac{31}{2}\omega}$ in $\pi < \omega < \frac{4\pi}{3}$

$$\therefore \quad d[n] = \frac{1}{2\pi}\int_{\frac{2\pi}{3}}^{4\pi/3} j e^{-j\frac{31}{2}\omega} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\, j\, \frac{e^{j(n-\frac{31}{2})\omega}}{j\left(n-\frac{31}{2}\right)}\Bigg|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}$$

$$= \frac{1}{2\pi\left(n-\frac{31}{2}\right)}\left[e^{j\frac{4\pi}{3}\left(n-\frac{31}{2}\right)} - e^{+j\frac{2\pi}{3}\left(n-\frac{31}{2}\right)}\right]$$

$$= \frac{1}{2\pi\left(n-\frac{31}{2}\right)}\, e^{j\pi\left(n-\frac{31}{2}\right)}\left[e^{j\frac{\pi}{3}\left(n-\frac{31}{2}\right)} - e^{-j\frac{\pi}{3}\left(n-\frac{31}{2}\right)}\right]$$

$$= \frac{1}{2\pi\left(n-\frac{31}{2}\right)}\, e^{j\pi\left(n-\frac{31}{2}\right)}\cdot 2j\, \sin\left[\frac{\pi}{3}\left(n-\frac{31}{2}\right)\right]$$

$$= \frac{1}{\pi\left(n-\frac{31}{2}\right)}\, e^{j\pi n}\cdot e^{-j\pi\frac{31}{2}}\cdot e^{j\pi/2}\, \sin\left[\frac{\pi}{3}\left(n-\frac{31}{2}\right)\right]$$

$$\Rightarrow \quad d[n] = -(-1)^n \frac{1}{3}\, \mathrm{Sinc}\left[\frac{\pi}{3}\left(n-\frac{31}{2}\right)\right]$$

① Truncation : $h[n] = -(-1)^n \frac{1}{3}\, \mathrm{Sinc}\left[\frac{\pi}{3}\left(n-\frac{31}{2}\right)\right] \quad 0 \le n \le 31$

② Hamming : $h[n] = -(-1)^n \frac{1}{3}\, \mathrm{Sinc}\left[\frac{\pi}{3}\left(n-\frac{31}{2}\right)\right]\left[0.54 - 0.46\cos\frac{2\pi n}{31}\right]$

$0 \le n \le 31$

Problem 2

(b) Let $D_{HP}(\omega)$ be the desired high-pass filter.

$$D_{HP}(\omega) = \begin{cases} 1, & |\omega| \ge \frac{2\pi}{3} \\ 0, & \text{else} \end{cases}$$

Then, let $D_{LP}(\omega)$ be defined as follows

$$D_{LP}(\omega) = D_{HP}(\omega - \pi) = \begin{cases} 1, & |\omega| \le \frac{\pi}{3} \\ 0, & \text{else} \end{cases}$$

$$G_{dLP}(\omega) = D_{LP}(\omega)\, e^{-j\frac{N-1}{2}\omega} = \begin{cases} e^{-j\frac{31}{2}\omega}, & |\omega| \le \frac{\pi}{3} \\ 0, & \text{else} \end{cases}$$

$$g_{LP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G_{dLP}(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} e^{-j\frac{31}{2}\omega}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{3}\,\mathrm{sinc}\left(\frac{\pi}{3}\left(n - \frac{31}{2}\right)\right)$$

Use the frequency shift property to find $g_{HP}[n]$

$$g_{HP}[n] = g_{LP}[n]\, e^{j\pi n} = (-1)^n \frac{1}{3}\,\mathrm{sinc}\left(\frac{\pi}{3}\left(n - \frac{31}{2}\right)\right)$$

For a rectangular window:

$$h[n] = \begin{cases} (-1)^n \frac{1}{3}\,\mathrm{sinc}\left(\frac{\pi}{3}\left(n - \frac{31}{2}\right)\right), & 0 \le n \le 31 \\ 0, & \text{else} \end{cases}$$

For a Hamming window:

$$h[n] = \begin{cases} (-1)^n \frac{1}{3}\,\mathrm{sinc}\left(\frac{\pi}{3}\left(n - \frac{31}{2}\right)\right)\left(0.54 - 0.46\cos\left(\frac{2\pi n}{31}\right)\right), & 0 \le n \le 31 \\ 0, & \text{else} \end{cases}$$

Problem 3

(a) $$d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |w|\, e^{jwn}\, dw$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} -w\, e^{jwn}\, dw + \frac{1}{2\pi}\int_{0}^{\pi} w\, e^{jwn}\, dw$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} w\, e^{-jwn}\, dw + \frac{1}{2\pi}\int_{0}^{\pi} w\, e^{jwn}\, dw$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} 2w\cos(wn)\, dw$$

$$= \begin{cases} \frac{\pi}{2}, & n = 0 \\ \frac{(-1)^n - 1}{\pi n^2}, & n \neq 0 \end{cases}$$

(b) The filter has odd length, so we can shift the desired impulse response by $\frac{N-1}{2} = 3$ and truncate to create a causal, symmetric impulse response. Then multiply by Hamming window.

$$g[n] = \begin{cases} \frac{\pi}{2}, & n = 3 \\ \frac{(-1)^{n-3} - 1}{\pi(n-3)^2}, & 0 \le n \le 6 \text{ and } n \neq 3 \end{cases}$$

$$w_H[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{6}\right), & 0 \le n \le 6 \\ 0, & \text{else} \end{cases}$$

$$h[n] = \begin{cases} \frac{\pi}{2}, & n = 3 \\ \frac{(-1)^{n-3} - 1}{\pi(n-3)^2}\left(0.54 - 0.46\cos\left(\frac{\pi n}{3}\right)\right), & 0 \le n \le 6 \text{ and } n \neq 3 \end{cases}$$

(c) The filter has odd length and even symmetry, so it is a Type I filter.

Problem 4

1. Since the filter has an even number of coefficients, the coefficients need to be antisymmetric in order to realize a high-pass filter. Therefore, the filter is type-4 GLP.

2. One can verify that $H_d(\omega)$ has the same expression for $\frac{3\pi}{4} \leq \omega \leq \pi$ and $\pi \leq \omega \leq \frac{5\pi}{4}$ using the anti-symmetry and the 2π shift in the real part of the frequency response. Therefore,

$$H_d(\omega) = \begin{cases} e^{j(\frac{\pi}{2} - \frac{99}{2}\omega)}, & \frac{3\pi}{4} \leq \omega \leq \frac{5\pi}{4} \\ 0, & \text{otherwise} \end{cases}$$

In order to use the frequency sampling method, the inverse DFT of $H[m]$ is needed,

$$H[m] = \begin{cases} e^{j(\frac{\pi}{2} - \frac{99}{2}\frac{2\pi}{100}m)}, & 38 \leq m \leq 62 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned}
h[n] &= \frac{1}{100} \sum_{m=38}^{62} e^{j(\frac{\pi}{2} - \frac{99}{2}\frac{2\pi}{100}m)} e^{j\frac{2\pi}{100}mn} \\
&= \frac{e^{j\frac{\pi}{2}}}{100} \sum_{m=38}^{62} e^{j\frac{2\pi}{100}(n-\frac{99}{2})m} \\
&= \frac{e^{j\frac{\pi}{2}}}{100} \sum_{k=0}^{24} e^{j\frac{2\pi}{100}(n-\frac{99}{2})(k+38)} \\
&= \frac{e^{j\frac{\pi}{2}}}{100} e^{j\frac{2\pi}{100}(n-\frac{99}{2})38} \sum_{k=0}^{24} e^{j\frac{2\pi}{100}(n-\frac{99}{2})k} \\
&= \frac{e^{j\frac{\pi}{2}}}{100} e^{j\frac{2\pi}{100}(n-\frac{99}{2})38} \cdot \frac{1 - e^{j\frac{2\pi}{100}(n-\frac{99}{2})25}}{1 - e^{j\frac{2\pi}{100}(n-\frac{99}{2})}} \\
&= \frac{e^{j\frac{\pi}{2}}}{100} e^{j\frac{2\pi}{100}(n-\frac{99}{2})38} \cdot \frac{e^{j\frac{\pi}{100}(n-\frac{99}{2})25}}{e^{j\frac{\pi}{100}(n-\frac{99}{2})}} \cdot \frac{e^{-j\frac{\pi}{100}(n-\frac{99}{2})25} - e^{j\frac{\pi}{100}(n-\frac{99}{2})25}}{e^{-j\frac{\pi}{100}(n-\frac{99}{2})} - e^{j\frac{\pi}{100}(n-\frac{99}{2})}} \\
&= \frac{e^{j\frac{\pi}{2}}}{100} e^{j\frac{\pi}{100}(n-\frac{99}{2})(76+25-1)} \frac{sin(25\frac{\pi}{100}(n-\frac{99}{2}))}{sin(\frac{\pi}{100}(n-\frac{99}{2}))} \\
&= \frac{e^{j\frac{\pi}{2}}}{100} e^{j\pi(n-\frac{99}{2})} \frac{sin(25\frac{\pi}{100}(n-\frac{99}{2}))}{sin(\frac{\pi}{100}(n-\frac{99}{2}))} \\
&= -\frac{(-1)^n}{100} \frac{sin(25\frac{\pi}{100}(n-\frac{99}{2}))}{sin(\frac{\pi}{100}(n-\frac{99}{2}))}
\end{aligned}$$

Problem 5

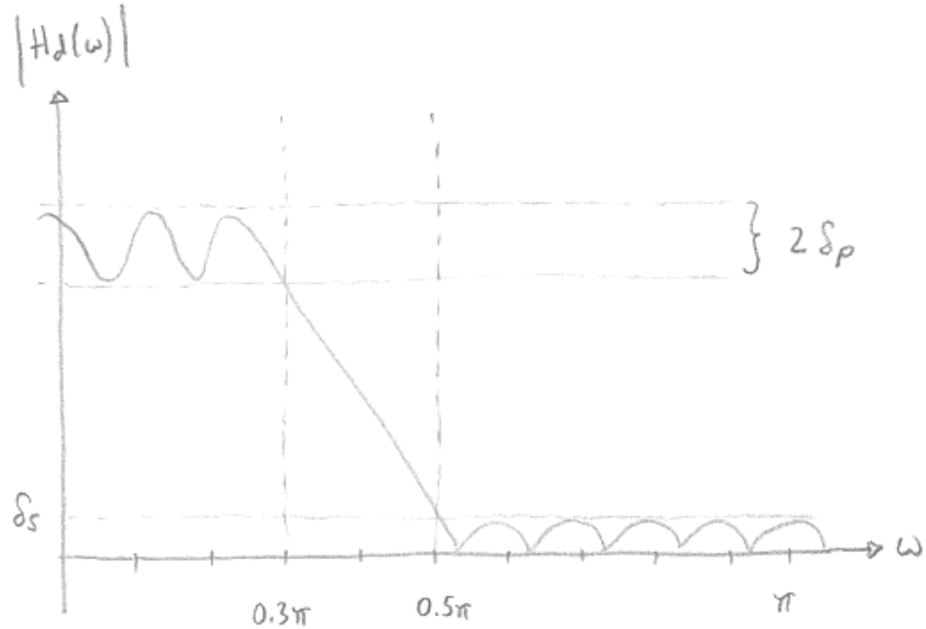
We have $\delta_p W_p = \delta_s W_s$, which gives $\delta_p = \frac{9}{4}\delta_s$. From the course notes we have

$$N \simeq \frac{-10\log_{10}(\delta_p \delta_s) - 13}{2.324\,(\omega_s - \omega_p)}$$

$$18 \simeq \frac{-10\log_{10}\left(\frac{9}{4}\delta_s^2\right) - 13}{2.324\,(0.2\pi)}$$

This gives

$\delta_s = 0.007239$ and $\delta_p = 0.01629$



The ripples are equal in magnitude throughout the passband and equal throughout the stopband with $\delta_p W_p = \delta_s W_s$.

Problem 6

Filter 1 is the equiripple filter. It has ripples of constant magnitude in the passband and stopband. The filter was designed with $W_s > W_p$.

Filter 2 is the rectangular window. It has large ripples of uneven magnitude caused by the Gibbs phenomenon.