1. A vacuum diode consists of a cathode in the $x = 0$ plane and an anode in the $x = d$ plane. The potential distribution between the plates is given by $V(x) = V_a(x/d)^{4/3}$ where $V_a = 2\,\mathrm{V}$.

   a) The electric field between the plates is

$$\mathbf{E} = -\nabla\Phi = -V_a \frac{4}{3d}\left(\frac{x}{d}\right)^{1/3} \hat{x} = -\frac{8}{3d}\left(\frac{x}{d}\right)^{1/3} \hat{x}\,\frac{\mathrm{V}}{\mathrm{m}}.$$

   Evaluating at $x = \frac{d}{4}$, we get

$$\mathbf{E}(x = \frac{d}{4}) = -\frac{2^{7/3}}{3d}\,\hat{x}\,\frac{\mathrm{V}}{\mathrm{m}}.$$

   b) The volumetric free-charge density is

$$\rho = \nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon_o \mathbf{E}) = -\frac{8\epsilon_o}{9d^2}\left(\frac{d}{x}\right)^{2/3} \frac{\mathrm{C}}{\mathrm{m}^3}.$$

   Evaluating at $x = \frac{d}{2}$, we get

$$\rho(x = \frac{d}{2}) = -\frac{2^{11/3}\epsilon_o}{9d^2}\,\frac{\mathrm{C}}{\mathrm{m}^3}.$$

   c) The surface charge density on the anode (where $\hat{n} = -\hat{x}$) is

$$\rho_S = \mathbf{D} \cdot \hat{n}|_{x=d} = -\,D_x|_{x=d} = \frac{8\epsilon_o}{3d}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

2. By definition ∇V generates a curl-free field. We can show this is true for $V = y^3 - xz$

$$\begin{aligned}
\nabla \times \mathbf{E} &= \nabla \times (-\nabla V) \\
&= \nabla \times -(-z\hat{x} + 3y^2\hat{y} - x\hat{z}) \\
&= \nabla \times \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -3y^2 & x \end{vmatrix} \\
&= 0\hat{x} + (1-1)\hat{y} + 0\hat{z} = \mathbf{0}
\end{aligned}$$

3. Given a vector field $\mathbf{A} = (z - y)\hat{y} + (z + y)\hat{z}$, let us verify that $\mathbf{A}$ satisfies the following vector identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}.$$

   Solving the left-hand side gives

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{A}) &= \nabla \times \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & z-y & z+y \end{vmatrix} \\
&= \nabla \times [(1-1)\hat{x} + 0\hat{y} + 0\hat{z}] = \mathbf{0},
\end{aligned}$$

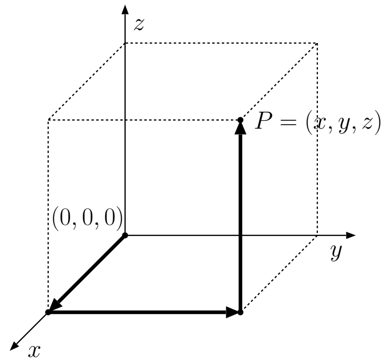
   and solving the right hand side gives

$$\begin{aligned}
\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} &= \nabla\left(\nabla \cdot ((z-y)\hat{y} + (z+y)\hat{z})\right) - \nabla^2\left((z-y)\hat{y} + (z+y)\hat{z}\right) \\
&= \nabla(-1+1) - \mathbf{0} = \mathbf{0},
\end{aligned}$$

   Since both sides are the same, the identity is verified.

4. $\mathbf{E} = 2\hat{x} + 2y\hat{y} + z\hat{z}\,\text{V/m}$ is an electrostatic field since

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2 & 2y & z \end{vmatrix} = \mathbf{0}.$$



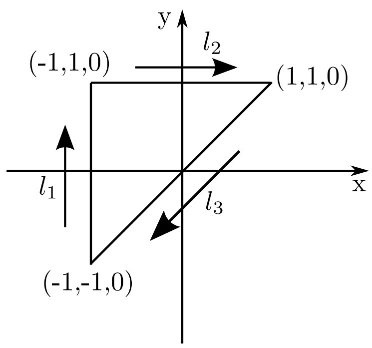
Using the path of integration shown in the above figure, we can calculate the electrostatic potential $V(x, y, z)$ at any point in space $\mathbf{P} = (x, y, z)$ as follows

$$\begin{aligned} V(\mathbf{P}) - V(\mathbf{0}) &= -\int_{\mathbf{0}}^{\mathbf{P}} \mathbf{E} \cdot d\mathbf{l} \\ &= -\int_0^x E_x(x, 0, 0)dx - \int_0^y E_y(x, y, 0)dy - \int_0^z E_z(x, y, z)dz = -(2x + y^2 + 0.5z^2)\,\text{V}. \end{aligned}$$

If $V(\mathbf{0}) = 0\,V$, the potential at $\mathbf{P} = (3, 2, 1)$ is

$$V(3, 2, 1) = -(2 \times 3 + 2^2 + 0.5 \times 1^2) = -10.5\,\text{V}.$$

5. Let us compute the circulation $\oint_C \mathbf{E} \cdot d\mathbf{l}$ over the triangular path shown below.



Given $\mathbf{E_1} = y\hat{x} + x\hat{y}\,\text{V/m}$, we find that

$$\begin{aligned} \oint_C \mathbf{E} \cdot d\mathbf{l} &= \int_{l_1} \mathbf{E}(-1, y, 0) \cdot d\mathbf{l}_1 + \int_{l_2} \mathbf{E}(x, 1, 0) \cdot d\mathbf{l}_2 + \int_{l_3} \mathbf{E}(x, x, 0) \cdot d\mathbf{l}_3 \\ &= \int_{-1}^{1} (y\hat{x} - \hat{y}) \cdot (\hat{y})dy + \int_{-1}^{1} (\hat{x} + x\hat{y}) \cdot (\hat{x})\,dx + \int_{1}^{-1} (x\hat{x} + x\hat{y}) \cdot (-\hat{x} - \hat{y})dx \\ &= \int_{-1}^{1} -1dy + \int_{-1}^{1} 1dy + \int_{1}^{-1} -2xdx + = -2 + 2 + 0 = 0\,\text{V}. \end{aligned}$$

Similarly, if $\mathbf{E_2} = y\hat{x} - x\hat{y}\,\mathrm{V/m}$, we get

$$\begin{aligned}
\oint_C \mathbf{E}\cdot d\mathbf{l} &= \int_{l_1} \mathbf{E}(-1,y,0)\cdot d\mathbf{l}_1 + \int_{l_2} \mathbf{E}(x,1,0)\cdot d\mathbf{l}_2 + \int_{l_3} \mathbf{E}(x,x,0)\cdot d\mathbf{l}_3 \\
&= \int_{-1}^{1} (y\hat{x}+\hat{y})\cdot(\hat{y})dy + \int_{-1}^{1} (\hat{x}-x\hat{y})\cdot(\hat{x})\,dx + \int_{1}^{-1} (x\hat{x}-x\hat{y})\cdot(-\hat{x}-\hat{y})dx \\
&= \int_{-1}^{1} 1dy + \int_{-1}^{1} 1dx + \int_{1}^{-1} 0dy = 2+2+0 = 4\,\mathrm{V}.
\end{aligned}$$

6. Consider a static charge density $\rho = 6\delta(z) + \rho_s\delta(z-10)\,\mathrm{C/m^3}$ in a given region, where the displacement field is $\mathbf{D} = 2\epsilon_0\hat{x} + 4\epsilon_0\hat{z}\,\mathrm{C/m^2}$ in the region $0 < z < 10\,\mathrm{m}$ and $D_z = 2\epsilon_0\,\mathrm{C/m^2}$ in the region $z > 10\,\mathrm{m}$. The volume charge density corresponds to two infinite surfaces at $z = 0\,\mathrm{m}$ and $z = 10\,\mathrm{m}$ with surface charges of $6\,\mathrm{C/m^3}$ and $\rho_s\,\mathrm{C/m^3}$, respectively.

   a) Apply boundary conditions at the interface: $\rho = \hat{n}\cdot[\mathbf{D_1} - \mathbf{D_2}]$. At $z = 10\,\mathrm{m}$, we have that the surface charge density ρ_s must be equal to the difference between the normal components of $\mathbf{D}$ on each side of the interface. Then, with $\hat{n} = \hat{z}$, we have:

$$\rho_s = D_z|_{z=10^+} - D_z|_{z=10^-} = 2\epsilon_0 - 4\epsilon_0 = -2\epsilon_0\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

   b) In the region $z > 10\,\mathrm{m}$, we know that $D_z = 2\epsilon_0\,\mathrm{C/m^2}$. In addition, since the tangential components of $\mathbf{E}$ at the interface at $z = 10\,\mathrm{m}$ must be continuous, i.e., $E_x|_{z=10^+} = E_x|_{z=10^-}$, and assuming that the charged sheets are in a vacuum, we can verify that $D_x|_{z=10^+} = D_x|_{z=10^-} = 2\epsilon_0\,\mathrm{C/m^2}$. Extending the field to the region $z > 10\,\mathrm{m}$, we find that

$$\mathbf{D} = 2\epsilon_0\hat{x} + 2\epsilon_0\hat{z}\,\frac{\mathrm{C}}{\mathrm{m}^2} \qquad \text{for } z > 10\,\mathrm{m}.$$

   c) Applying boundary conditions at the interface at $z = 0\,\mathrm{m}$, we have that $D_z|_{z=0^+} - D_z|_{z=0^-} = 6$, thus, $D_z|_{z=0^-} = 4\epsilon_0 - 6\,\mathrm{C/m^2}$. In addition, since $E_x|_{z=0^+} = E_x|_{z=0^-}$, and given that the space is a vacuum, we can verify that $D_x|_{z=0^-} = D_x|_{z=0^+} = 2\epsilon_0\,\mathrm{C/m^2}$. Extending the fields to the region $z < 0\,\mathrm{m}$, we find that

$$\mathbf{D} = 2\epsilon_0\hat{x} + (4\epsilon_0 - 6)\hat{z}\,\frac{\mathrm{C}}{\mathrm{m}^2} \qquad \text{for } z < 0\,\mathrm{m}.$$

   d) The displacement field, $\mathbf{D}$, is defined to be equal to $\epsilon_{medium}\mathbf{E}$. Since the region is in free space $\epsilon_{medium} = \epsilon_0$. The electric field in the region where $z < 0\,\mathrm{m}$ is

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \frac{2\epsilon_0\hat{x} + (4\epsilon_0 - 6)\hat{z}}{\epsilon_0} = 2\hat{x} + (4 - \frac{6}{\epsilon_0})\hat{z}\,\frac{V}{m}$$

   The electric field in the region from $z = 0\,\mathrm{m}$ to $z = 10\,\mathrm{m}$ is

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \frac{2\epsilon_0\hat{x} + 4\epsilon_0\hat{z}}{\epsilon_0} = 2\hat{x} + 4\hat{z}\,\frac{V}{m}$$

   The electric field in the region where $z > 10\,\mathrm{m}$ is

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \frac{2\epsilon_0\hat{x} + 2\epsilon_0\hat{z}}{\epsilon_0} = 2\hat{x} + 2\hat{z}\,\frac{V}{m}$$

e) We can calculate the voltage drop from $z = 0\,\text{m}$ to $z = 10\,\text{m}$ with the following equation

$$\begin{aligned}
\Delta V &= V_{10} - V_0 = -\int_0^{10} \mathbf{E} \cdot dl = -\int_0^{10} \frac{\mathbf{D}}{\epsilon_0} \cdot dl \\
&= -\int_0^{10} (2\hat{x} + 4\hat{z}) \cdot dz \\
&= -4z|_0^{10} = -40 \text{ V}
\end{aligned}$$

f) If the medium in the region $0 < z < 10\,\text{m}$ was replaced with a dielectric with permittivity $\epsilon_{medium} = 4\epsilon_0$, it will only affect equations that deal with the electric field in this middle region since $\mathbf{E} = \frac{\mathbf{D}}{4\epsilon_0}$ now (only part a will remain unchanged). Parts b and c all are solved knowing that tangential electric field boundary conditions are continuious, and thus there will be a change in the displacement field here. Part d will change as it asks to compute the electric field in this region. Part e will also change since the voltage drop is desired in this region and it is calculated by integrating along the electric field.

7. A pair of parallel conducting plates placed at $z = 0$ and $z = W$ sustains an electric field $\mathbf{E} = -3\hat{z}\,\text{V/m}$. The gap between the plates is originally occupied by a vacuum (ϵ_o, μ_o).

a) In free space $\mathbf{D} = \epsilon_o \mathbf{E}$, so the displacement field between the plates is

$$\mathbf{D} = -3\epsilon_o\,\hat{z}\,\frac{\text{C}}{\text{m}^2}.$$

By definition, $\mathbf{D} = \epsilon_o \mathbf{E} + \mathbf{P}$, thus the polarization field between the plates is

$$\mathbf{P} = \mathbf{0}\,\frac{\text{C}}{\text{m}^2}.$$

b) Since the charged plates carry equal and opposite charge densities, the fields they generate cancel each other out except between them. With $\mathbf{D} = \mathbf{0}$ for $z < 0$, we can find the surface charge density on the plate at $z = 0$ using the boundary condition equation with $\hat{n} = \hat{z}$.

$$\rho_s|_{z=0} = \hat{n} \cdot \mathbf{D}|_{z=0} = -3\epsilon_o\,\frac{\text{C}}{\text{m}^2}.$$

c) If the gap is filled with a dielectric with permittivity $\epsilon = 81\epsilon_o$ without changing the surface charge density, then the displacement field will remain the same, i.e.,

$$\mathbf{D} = -3\epsilon_o\,\hat{z}\,\frac{\text{C}}{\text{m}^2}.$$

But, the electric field is now

$$\mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = -\frac{1}{27}\,\hat{z}\,\frac{\text{V}}{\text{m}},$$

and the polarization field becomes

$$\mathbf{P} = \mathbf{D} - \epsilon_o \mathbf{E} = -\frac{80}{27}\epsilon_o\,\hat{z}\,\frac{\text{C}}{\text{m}^2}.$$

d) If the conductivity of the material between the parallel plates increases to $\sigma = 4\,\text{S/m}$, then $\mathbf{E} = \mathbf{D} = \mathbf{P} \to \mathbf{0}$ as the steady-state equilibrium is reached. This happens because in a conducting medium all equilibrium fields vanish after the rearrangement of the net charge on the bounding surface. In this particular case, the salt water shorts out the original field between the plates.

8. For the charge distribution specified in the statement of this problem, let us compute **D**, **E**, and **P** in all regions, as well as, the surface charge density on each boundary.

a) Region 1 ($r \le a$) is occupied by a conductor with $\sigma = 10^6\,\mathrm{S/m}$, therefore, we can directly write

$$\mathbf{D}_1 = \mathbf{0}\,\frac{\mathrm{C}}{\mathrm{m}^2}, \qquad \mathbf{E}_1 = \mathbf{0}\,\frac{\mathrm{V}}{\mathrm{m}}, \qquad \mathbf{P}_1 = \mathbf{0}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

In steady-state, charges can accumulate only on the surface of conducting materials. Since this material holds a net charge $Q_1 = 2\,\mathrm{C}$, the surface charge density on the sphere of radius $r = a$ is

$$\rho_S|_{r=a} = \frac{Q_1}{\mathrm{Area}} = \frac{2}{4\pi a^2} = \frac{1}{2\pi a^2}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

b) Region 2 ($a < r < b$) is occupied by a dielectric with $\epsilon_2 = 10\epsilon_o$. Applying Gauss's law $\oint \mathbf{D}\cdot d\mathbf{S} = Q_{\mathrm{enc}}$ and considering $\mathbf{D} = D_r\hat{r}$, we find

$$\mathbf{D}_2 = \frac{2}{4\pi r^2}\hat{r} = \frac{1}{2\pi r^2}\hat{r}\,\frac{\mathrm{C}}{\mathrm{m}^2} \quad \text{for } a < r < b.$$

In addition, we get

$$\mathbf{E}_2 = \frac{1}{\epsilon_2}\mathbf{D}_2 = \frac{1}{20\pi\epsilon_o r^2}\hat{r}\,\frac{\mathrm{V}}{\mathrm{m}},$$

and

$$\mathbf{P}_2 = \mathbf{D}_2 - \epsilon_o\mathbf{E}_2 = \left(\frac{1}{2\pi r^2} - \frac{1}{20\pi r^2}\right)\hat{r} = \frac{9}{20\pi r^2}\hat{r}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

c) Region 3 ($b \le r \le c$) is also occupied by a conductor and therefore

$$\mathbf{D}_3 = \mathbf{0}\,\frac{\mathrm{C}}{\mathrm{m}^2}, \qquad \mathbf{E}_3 = \mathbf{0}\,\frac{\mathrm{V}}{\mathrm{m}}, \qquad \mathbf{P}_3 = \mathbf{0}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

On the surface at $r = b$, the surface charge density is

$$\rho_S|_{r=b} = \hat{r}\cdot(\mathbf{D}_3 - \mathbf{D}_2)|_{r=b} = \hat{r}\cdot\left(-\frac{1}{2\pi r^2}\hat{r}\right)\Big|_{r=b} = -\frac{1}{2\pi b^2}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

d) Region 4 ($r > c$) is free space. Applying Gauss's law and noting that the total charge enclosed is $2 - 4 = -2\,\mathrm{C}$, we get $\oint \mathbf{D}\cdot d\mathbf{S} = \oint D_r dS = D_r 4\pi r^2 = Q_{\mathrm{enc}} = -2\,\mathrm{C}$

$$\mathbf{D}_4 = -\frac{1}{2\pi r^2}\hat{r}\,\frac{\mathrm{C}}{\mathrm{m}^2}, \qquad \mathbf{E}_4 = -\frac{1}{2\pi\epsilon_0 r^2}\hat{r}\,\frac{\mathrm{V}}{\mathrm{m}}, \qquad \mathbf{P}_4 = \mathbf{0}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

Finally, on the surface at $r = c$, the surface charge density is

$$\rho_S|_{r=c} = \hat{r}\cdot(\mathbf{D}_4 - \mathbf{D}_3)|_{r=c} = -\frac{1}{2\pi c^2}\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$