University of Illinois **Spring 2014**

ECE 310: Problem Set 2: Problems and Solutions
Fourier transform (FT), Discrete-time Fourier transform (DTFT)

Due: Wednesday February 5 at 6 p.m.
Reading: *310 Course Notes* Ch 2.1-2.5

1. **[Convolution]**
The convolution of two CT functions corresponds to the product of their CTFTs in the frequency domain. Namely:

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \longleftrightarrow X(\Omega)Y(\Omega),$$

where Ω is the frequency variable, and $X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$ denotes the Fourier transform of $x(t)$. In this problem we explore this relation in more detail with step functions as the input, i.e., $x(t) = u(t)$ and $y(t) = u(t)$.

   (a) Find the convolution of two step functions.
   Solution:

$$\begin{aligned}&u(t) * u(t)\\ =&\int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau\\ =&\begin{cases}0 & \text{if } t<0\\ \int_0^t 1\ d\tau = t & \text{if } t \geq 0\end{cases}\\ =&tu(t)\end{aligned}$$

   (b) Verify your answer by performing this operation in the transform domain.
   Solution: $X(\Omega) = Y(\Omega) = \mathcal{F}[u(t)] = \frac{1}{j\Omega} + \pi\delta(\Omega)$. The convolution $1(t) * 1(t)$ in the time domain does not exist. $\delta^2(\Omega)$ is not defined. $x(t) * y(t) \Longleftrightarrow X(\Omega)Y(\Omega) = (\frac{1}{j\Omega} + \pi\delta(\Omega))^2$. Therefore, $X(\Omega)Y(\Omega)$ is not defined. This verification cannot be done.
   However, we can verify this result by using Laplace transform. $\mathcal{L}[u(t)] = \frac{1}{s}$. Laplace transform of the convolution

$$\mathcal{L}[u(t) * u(t)] = \mathcal{L}[u(t)]\mathcal{L}[u(t)] = \frac{1}{s^2}.$$

   Laplace transform of the RHS:

$$\mathcal{L}[tu(t)] = -\frac{d\mathcal{L}[u(t)]}{ds} = -\frac{d(1/s)}{ds} = \frac{1}{s^2}.$$

   Therefore, $u(t) * u(t) = tu(t)$.

2. **[DTFT]**
Compute the DTFT of the following sequences. Plot the magnitude and phase of parts (a), (c) and (e). $\delta[n] = 1$ for $n = 0$ and $u[n] = 1$ for $n \geq 0$, and zero otherwise.

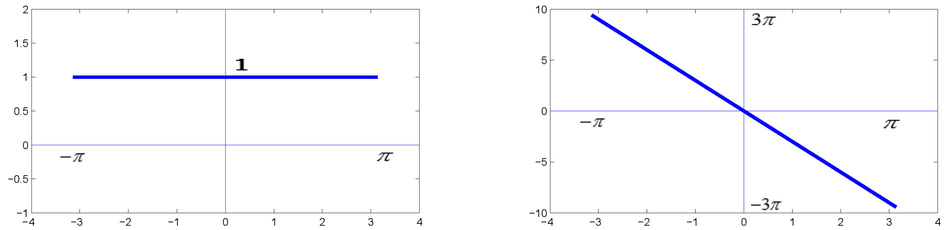


Figure 1: Magnitude and phase of 2(a)

(a) $x[n] = \delta[n-3]$
Solution: $X_d(\omega) = e^{-j\omega 3}$. Magnitude $|X_d(\omega)| = 1$. Phase $\arg\{X_d(\omega)\} = -3\omega$.

(b) $x[n] = 0.9^n u[n] + \delta[n-5]$
Solution: $X_d(\omega) = e^{-j\omega 5} + \sum_{n=0}^{\infty} 0.9^n e^{-j\omega n} = e^{-j\omega 5} + \frac{1}{1-0.9e^{-j\omega}}$

(c) $x[n] = u[n]/2^n$
Solution: $X_d(\omega) = \sum_{n=0}^{\infty} \frac{1}{2^n} e^{-j\omega n} = \frac{1}{1-e^{-j\omega}/2}$.
Magnitude $|X_d(\omega)| = \frac{1}{\sqrt{(1-\cos(\omega)/2)^2+(\sin(\omega)/2)^2}} = \frac{1}{\sqrt{5/4-\cos(\omega)}}$.
Phase $\arg\{X_d(\omega)\} = -\arctan\left(\frac{\sin(\omega)/2}{1-\cos(\omega)/2}\right)$.

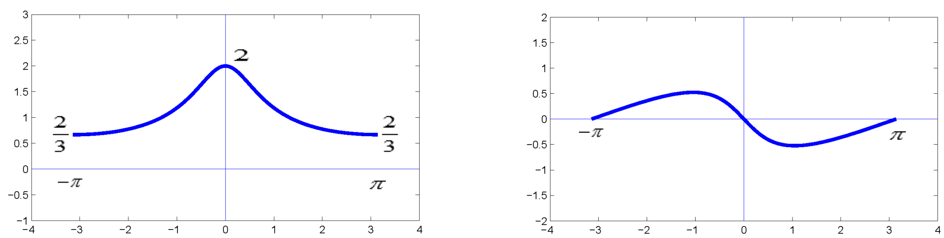


Figure 2: Magnitude and phase of 2(c)

(d) $x[n] = 2^n u[-n-1]$
Solution: $X_d(\omega) = \sum_{n=-\infty}^{-1} 2^n e^{-j\omega n} = \sum_{n=1}^{\infty} (1/2)^n e^{j\omega n} = \frac{e^{j\omega}/2}{1-e^{j\omega}/2}$.

(e) $x[n] = \frac{\sin[n\pi/2]}{\pi n} \frac{\sin[n\pi/4]}{\pi n}$
Solution: Use the DTFT pair in table 2.4 (notes page 54):

$$\frac{\sin(\omega_0 n)}{\pi n} \leftrightarrow \begin{cases} 1 & |\omega| \le \omega_0 \\ 0 & \omega_0 < |\omega| \le \pi \end{cases}.$$

Let $x_1[n] = \frac{\sin[n\pi/2]}{\pi n}$, $x_2[n] = \frac{\sin[n\pi/4]}{\pi n}$. $\arg\{X_{d1}(\omega)\} = \arg\{X_{d2}(\omega)\} = 0$. Both are real functions of ω. Magnitudes are shown in figure 3.

$x[n] = x_1[n]x_2[n] \Leftrightarrow X_d(\omega) = \frac{1}{2\pi}X_{d1}(\omega) * X_{d2}(\omega)$.

$$X_d(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_{d1}(\gamma)X_{d2}(\omega-\gamma)d\gamma$$
$$= \begin{cases} \frac{1}{2\pi}(\omega + \frac{3}{4}\pi) & -\frac{3}{4}\pi \le \omega \le -\frac{\pi}{4} \\ \frac{1}{4} & |\omega| < \frac{\pi}{4} \\ \frac{1}{2\pi}(-\omega + \frac{3}{4}\pi) & \frac{\pi}{4} \le \omega \le \frac{3}{4}\pi \\ 0 & \frac{3}{4}\pi < |\omega| \le \pi \end{cases}$$

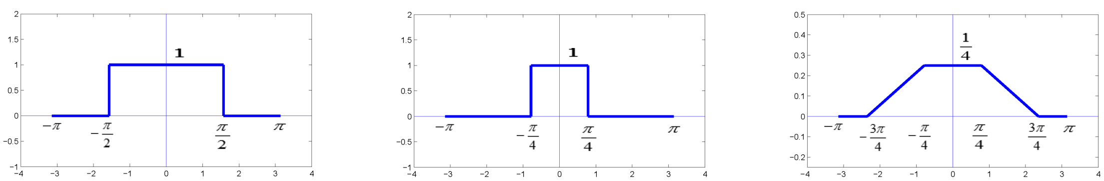


Figure 3: Magnitudes of $X_{d1}(\omega)$, $X_{d2}(\omega)$ and $X_d(\omega)$ in 2(e)

(f) $x[n] = n/2^{|n|}$
Solution: Let $y_1[n] = (1/2)^n u[n]$, $y_2[n] = 2^n u[-n-1]$, then $y[n] = (1/2)^{|n|} = y_1[n] + y_2[n]$. From part (c) and (d), we know that $Y_d(\omega) = Y_1(\omega) + Y_2(\omega) = \frac{1}{1-e^{-j\omega}/2} + \frac{e^{j\omega}/2}{1-e^{j\omega}/2}$. $x[n] = ny[n] \Leftrightarrow X_d(\omega) = j\frac{dY_d(\omega)}{d\omega}$. Hence

$$X_d(\omega) = j\frac{dY_1(\omega)}{d\omega} + j\frac{dY_2(\omega)}{d\omega}$$
$$= \frac{e^{-j\omega}/2}{(1-e^{-j\omega}/2)^2} - \frac{e^{j\omega}/2}{(1-e^{j\omega}/2)^2}.$$

(g) $x[n] = (-1)^n$
Solution: $X_d(\omega) = \sum_{n=-\infty}^{\infty}(-1)^n e^{-j\omega n} = \sum_{n=-\infty}^{\infty} e^{-j(\omega-\pi)n} = 2\pi\delta(\omega - \pi)$.

3. **[Inverse DTFT]**
Find $x[n]$ for each DTFT, $X_d(\omega)$ given below:

(a) $X_d(\omega) = e^{j\omega 3}$
Solution: $x[n] = \delta[n+3]$.

(b) $X_d(\omega) = \cos^2(\omega)$
Solution: $X_d(\omega) = \cos^2(\omega) = \frac{1}{2}(1+\cos(2\omega)) = \frac{1}{2} + \frac{1}{4}e^{-j2\omega} + \frac{1}{4}e^{j2\omega}$. Hence $x[n] = \frac{1}{2}\delta[n] + \frac{1}{4}\delta[n-2] + \frac{1}{4}\delta[n+2]$.

4. **[Forward and Inverse Transforms]**
Suppose $X_d(\omega) = \begin{cases} 1, & |\omega| \le \frac{\pi}{6} \\ 0, & \frac{\pi}{6} < |\omega| \le \pi \end{cases}$

(a) Sketch $X_d(\omega)$
Solution:

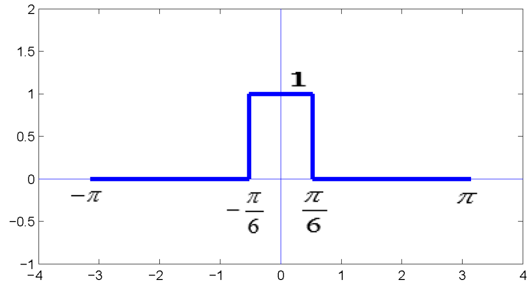


Figure 4: $X_d(\omega)$

(b) Find $x[n]$

Solution: Again, we can use the DTFT pair in the notes. Or

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi/6}^{\pi/6} e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\frac{1}{jn}(e^{j\pi n/6} - e^{-j\pi n/6}) \\ &= \frac{\sin(\pi n/6)}{\pi n}. \end{aligned}$$

(c) If $y[n] = e^{j\frac{\pi}{3}n}x[n]$, sketch $Y_d(\omega)$

Solution: Modulation in the time domain corresponds to shift in the frequency domain. (2.4.2.2 in Table 2.3, page 54.) $Y_d(\omega) = X_d(\omega - \frac{\pi}{3})$.

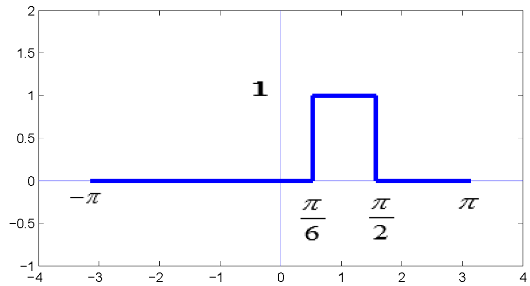


Figure 5: $Y_d(\omega)$

(d) If $y[n] = \cos(\frac{\pi}{3}n)x[n]$, sketch $Y_d(\omega)$

Solution: $y[n] = \cos(\pi n/3)x[n] = \frac{1}{2}e^{-j\pi n/3}x[n] + \frac{1}{2}e^{j\pi n/3}x[n]$. $Y_d(\omega) = \frac{1}{2}X_d(\omega + \frac{\pi}{3}) + \frac{1}{2}X_d(\omega - \frac{\pi}{3})$.

5. **[Magnitude and Phase of DTFT]**

Let $x[n] = \begin{cases} \frac{1}{N}, & 0 \le n \le N-1 \\ 0, & else \end{cases}$

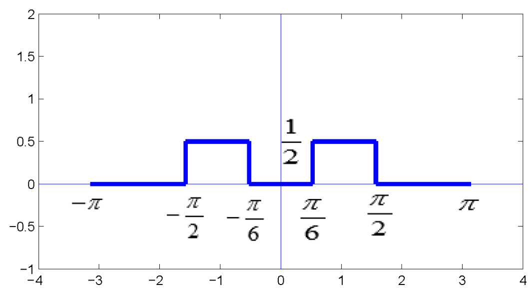


Figure 6: $Y_d(\omega)$

(a) Find closed-form expressions (no sums) for $|X_d(\omega)|$, and $\angle X_d(\omega)$
Solution:

$$\begin{aligned} X_d(\omega) &= \sum_{n=0}^{N-1} \frac{1}{N} e^{-j\omega n} \\ &= \begin{cases} \frac{1}{N}\frac{1-e^{-j\omega N}}{1-e^{-j\omega}} & \text{if } \omega \neq 2\pi m,\ m \in \mathbb{Z} \\ 1 & \text{if } \omega = 2\pi m,\ m \in \mathbb{Z} \end{cases} \end{aligned}$$

Note that $\frac{1-e^{-j\omega N}}{1-e^{-j\omega}} = \frac{e^{j\omega N/2}-e^{-j\omega N/2}}{e^{j\omega/2}-e^{-j\omega/2}} \frac{e^{-j\omega N/2}}{e^{-j\omega/2}} = \frac{\sin(\omega N/2)}{\sin(\omega/2)} e^{-j\omega(N-1)/2}$. Magnitude

$$|X_d(\omega)| = \begin{cases} \left|\frac{\sin(\omega N/2)}{N\sin(\omega/2)}\right| & \text{if } \omega \neq 2\pi m,\ m \in \mathbb{Z} \\ 1 & \text{if } \omega = 2\pi m,\ m \in \mathbb{Z} \end{cases}$$

Phase

$$\angle X_d(\omega) = \begin{cases} -\frac{\omega}{2}(N-1) & \text{if } \omega \neq 2\pi m,\ m \in \mathbb{Z}, \text{and } \operatorname{sgn}(\sin(\omega N/2)) = \operatorname{sgn}(\sin(\omega/2)) \\ -\frac{\omega}{2}(N-1)+\pi & \text{if } \omega \neq 2\pi m,\ m \in \mathbb{Z}, \text{and } \operatorname{sgn}(\sin(\omega N/2)) = -\operatorname{sgn}(\sin(\omega/2)) \\ 0 & \text{if } \omega = 2\pi m,\ m \in \mathbb{Z} \end{cases}$$

(b) For $N = 6$, plot$|X_d(\omega)|$. How will the shape of $|X_d(\omega)|$ change if N increases?
Solution: There will be more ripples if N increases. See figure 7.

(c) For $N = 6$, plot$\angle X_d(\omega)$. How will $\angle X_d(\omega)$ change as N increases?
Solution: Slope is the same as the shift in the time domain. Slope will be steeper if N increases. See figure 7. Note that there is a phase "jump" of π where the magnitude is 0.

6. **[DTFT of Complex Signals]**
Let $X_d(\omega)$ denote the DTFT of the complex valued signal $x[n]$, where the real and imaginary parts of x[n] are given below. Perform the following calculatioins without explicitly evaluating $X_d(\omega)$.

(a) Evaluate $X_d(0)$.
Solution: $X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$. When $\omega = 0$, $X_d(0) = \sum_{n=-4}^{1} x[n]e^{-j0n} = \sum_{n=-4}^{1} x[n] = (-3+1-2+2-1+3) + j(1+2+3+3+2+1) = 12j$.

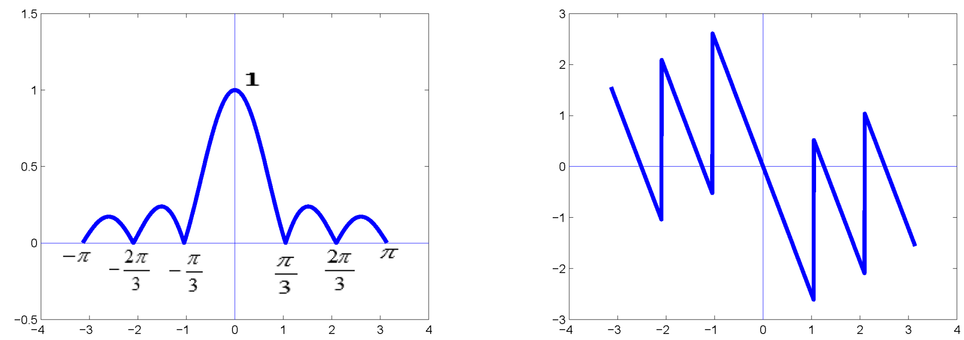
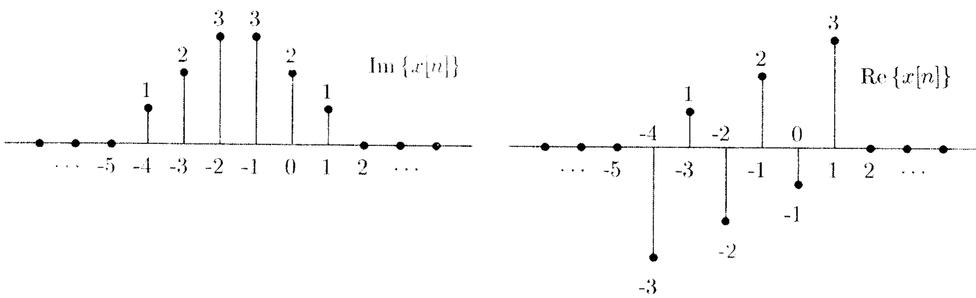


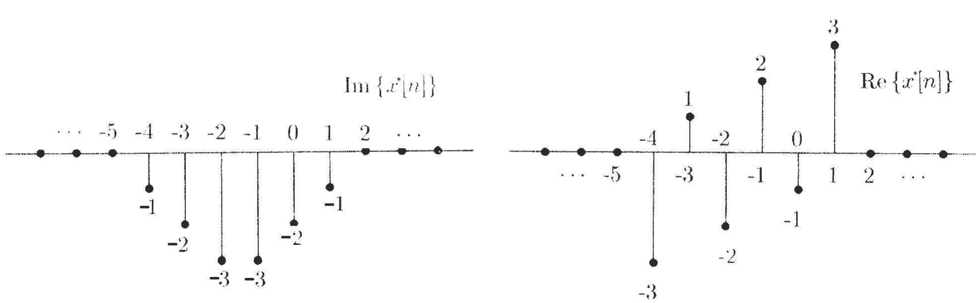
Figure 7: Magnitude and phase in problem 5



(b) Evaluate $X_d(\pi)$.
Solution: $X_d(\pi) = \sum_{n=-4}^{1} x[n]e^{-j\pi n} = \sum_{n=-4}^{1}(-1)^n x[n] = -12.$

(c) Evaluate $\int_{-\pi}^{\pi} X_d(\omega)d\omega$.
Solution: $\int_{-\pi}^{\pi} X_d(\omega)d\omega = 2\pi\frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{j\omega 0}d\omega = 2\pi x[0] = -2\pi + j4\pi.$

(d) Determine and sketch the signal whose DTFT is $X_d^*(-\omega)$.
Solution: $X_d(\omega) = \sum_n x[n]e^{-j\omega n}$. Hence $X_d^*(\omega) = \sum_n x^*[n]e^{j\omega n}$.
$X_d^*(-\omega) = \sum_n x^*[n]e^{-j\omega n}$ is the DTFT of $x^*[n]$. Sketch $x^*[n]$:



7. **[Properties of DTFT]**
Let $x[n]$ be an arbitrary signal, not necessarily real valued, with DTFT $X_d(\omega)$. Express the DTFT of the following signals in terms of $X_d(\omega)$.

(a) $x^*[n]$

Solution:

$$\begin{aligned} x^*[n] \Leftrightarrow & \sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n} \\ =& (\sum_{n=-\infty}^{\infty} x[n]e^{j\omega n})^* \\ =& X_d^*(-\omega). \end{aligned}$$

(b) $x^*[-n]$
Solution:

$$\begin{aligned} x^*[-n] \Leftrightarrow & \sum_{n=-\infty}^{\infty} x^*[-n]e^{-j\omega n} \\ =& (\sum_{n=-\infty}^{\infty} x[-n]e^{j\omega n})^* \\ =& (\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n})^* \\ =& X_d^*(\omega). \end{aligned}$$

(c) $y[n] = x[n] - x[n-2]$
Solution:

$$\begin{aligned} y[n] = x[n] - x[n-2] \Leftrightarrow & X_d(\omega) - \sum_{n=-\infty}^{\infty} x[n-2]e^{-j\omega n} \\ =& X_d(\omega) - \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega(n+2)} \\ =& X_d(\omega) - e^{-j\omega 2} \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ =& X_d(\omega)(1 - e^{-j2\omega}). \end{aligned}$$