Signals and Systems

Stage 3: Electrical and Electronic Eng.

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Overview of Topics

Mathematical background and problem statement.

Laplace transform and solution of linear constant-coefficient ordinary differential equations.

Special case of steady-state response to sinusoidal input. Fourier transform.

Solution of linear constant-coefficient partial differential equations. Fourier series.

Discrete-time systems.

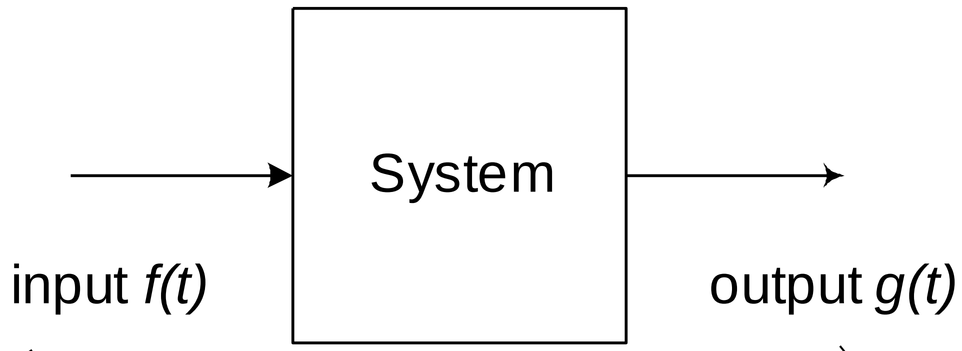
Fourier Theory

Basic Fourier Theory:

***Fourier Transform*:**

Linear vs. Nonlinear

The task of analysing a system is greatly simplified if the system is linear. If $f_n(t)$ as input produces $g_n(t)$ as output, then



$$c_1 f_1(t) + c_2 f_2(t) + \cdots \qquad\qquad c_1 g_1(t) + c_2 g_2(t) + \cdots$$

Linear vs Nonlinear

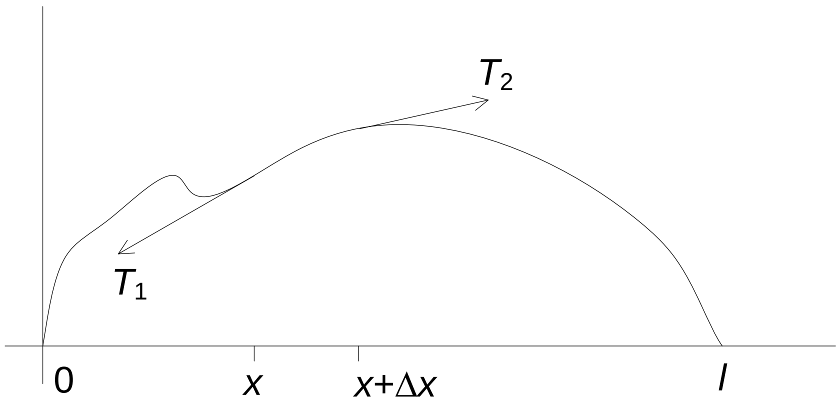
We have also noted that if the system is time-invariant as well as linear and if the input is a co-sinusoid of a certain frequency then a component of the output, namely the steady-state or long term output is (or rather can be, there is a caveat) relatively easy to find since it will also be a co-sinusoid of the exact same frequency and its amplitude and phase will be predictably related to the amplitude and phase of the input via the transfer function of the system.

Linear vs Nonlinear

Putting these two observations together we find that in principle we have a reasonably powerful idea for finding the steady-state response of an LTI system (specifically one where the transient is known to converge to zero, a so-called *stable* system) to an input which is expressible as a sum of co-sinusoids. This does raise the question of what type of input signals can be expressed as sums of co-sinusoids. As it happens the answer to this question is rather old, first arising in a question of great concern to musical theory, the vibrations of a string.

Example 3.1

The vibration of a string of length *l* and fixed at both ends:



Tension tangential to string at all points. Assume no motion in horizontal:

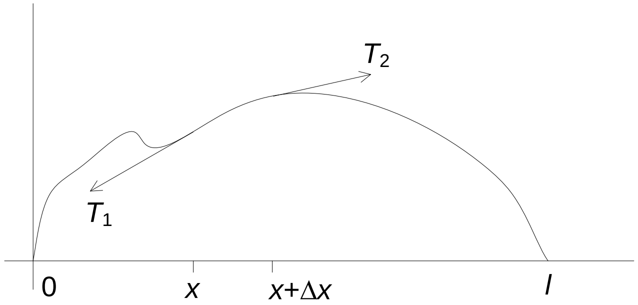
$$T_1 \cos(\theta_1) = T_2 \cos(\theta_2) = T$$

Newton's equation for motion in vertical:

$$T_2 \sin(\theta_2) - T_1 \sin(\theta_1) = \Delta m \frac{\partial^2 y}{\partial t^2} = \rho \Delta x \frac{\partial^2 y}{\partial t^2}$$

ρ is the mass per unit length. A uniform string is assumed.

Example 3.1



$$\tan(\theta_2) - \tan(\theta_1) = \frac{\rho}{T}\Delta x \frac{\partial^2 y}{\partial t^2}$$

$$\tan(\theta) = \frac{\partial y}{\partial x}$$

$$\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_{x} = \frac{\rho}{T}\Delta x \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_{x}}{\Delta x} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$

Example 3.1

The vibration of a string of length *l* and fixed at both ends is approximately described by the following equation:

$$\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$$

$$y(0,t) = y(l,t) = 0, \quad t \geq 0,$$

$$y(x,0) = f(x) \quad , \; 0 \leq x \leq l,$$

$$\frac{\partial}{\partial t} y(x,0) = 0 \quad , \; 0 \leq x \leq l$$

this is the *1D wave equation*. $y(x,t)$ is the vertical displacement of the string at position x and time t. An old idea for solving equations of this kind is to assume that the unknown $y(x,t)$ can be expressed as a product

$$y(x,t) = y_1(x) y_2(t)$$

Note

This kind of equation is called a partial differential equation (PDE). It is a linear, constant coefficient PDE. It was the Bernoullis and Euler who were responsible for much of the development of the ideas of partial differentiation. The quoted conditions are called boundary conditions and initial conditions. The problem of solving a PDE with boundary conditions is called a boundary value problem. The 1D wave equation is applicable to small, transverse vibrations of a taut, flexible string. The constant $\alpha^2 = T/\rho$, where T is the tension (assumed constant) and ρ is the mass per unit length of the string (also assumed constant).

I do not know who first thought of this *separation of variables* or *product method*. It just seems to have occurred to people generally.

Example 3.1

Substituting the hypothesised product $y(x,t) = y_1(x)y_2(t)$

$$y_1(x)\frac{d^2 y_2(t)}{dt^2} = \alpha^2 y_2(t)\frac{d^2 y_1(x)}{dx^2}$$

$$\frac{\dfrac{d^2 y_2(t)}{dt^2}}{y_2(t)} = \alpha^2 \frac{\dfrac{d^2 y_1(x)}{dx^2}}{y_1(x)}$$

As the left hand side is a function of *t* only and the right hand side a function of *x* only, for them to be equal for all *x* and *t* they must both be equal to the same constant -*k*.

Example 3.1

$$\frac{d^2 y_2(t)}{dt^2} + k y_2(t) = 0 \ , \quad \frac{d^2 y_1(x)}{dx^2} + \frac{k}{\alpha^2} y_1(x) = 0$$

Assume $k > 0$ then

$$y_2(t) = A_2 \sin\left(\sqrt{k}\, t\right) + B_2 \cos\left(\sqrt{k}\, t\right)$$

$$y_1(x) = A_1 \sin\left(\sqrt{\frac{k}{\alpha^2}}\, x\right) + B_1 \cos\left(\sqrt{\frac{k}{\alpha^2}}\, x\right)$$

It is now necessary to find the unknown coefficients A_i and B_i and to this end we recall the boundary conditions and the initial conditions.

Note

How can I justify the assumption $k > 0$? Well if $k < 0$ then the solution has takes the form

$$y_2(t) = A_2 \sinh\left(\sqrt{-k}\,t\right) + B_2 \cosh\left(\sqrt{-k}\,t\right)$$
$$y_1(x) = A_1 \sinh\left(\sqrt{\frac{-k}{\alpha^2}}\,x\right) + B_1 \cosh\left(\sqrt{\frac{-k}{\alpha^2}}\,x\right)$$

In particular the function $y_2(t)$ will not be bounded, i.e. it will grow indefinitely with time. Accordingly the purported solution to the string equation will be physically meaningless. One must be careful with mathematical equations, not all of their solutions are the right solutions.

Example 3.1

$$y_1(0)y_2(t) = y_1(l)y_2(t) = 0, \quad t \geq 0$$

implies $\qquad y_1(0) = y_1(l) = 0$

$$y_1(0) = B_1 = 0 \quad , \quad y_1(l) = A_1 \sin\left(\sqrt{\frac{k}{\alpha^2}}\, l\right) = 0$$

$$\sqrt{\frac{k}{\alpha^2}}\, l = n\pi \quad , \quad \text{i.e.} \quad k = \frac{n^2\pi^2\alpha^2}{l^2}$$

where n is any integer.

Note

Firstly we apply the boundary conditions. We find that we do not obtain a solution to the boundary value problem for just any choice of constant k. In fact the allowed values of this constant are discrete or *quantised*. This aspect of the theory was what alerted Schrödinger to the possibility that quantisation of energy might also be due to the effects of an underlying PDE.

Example 3.1

$$\frac{\partial}{\partial t} y(x,0) = y_1(x) \frac{d\, y_2(0)}{d\, t} = 0, \quad \text{for all } 0 \le x \le l$$

implies

$$\left. \frac{d\, y_2(t)}{d\, t} \right|_{t=0} = 0$$

$$\sqrt{k} A_2 = 0 \quad , \quad \text{i.e.} \quad A_2 = 0$$

Example 3.1

$$y_2(t) = B_2 \cos\left(\frac{n\pi\alpha t}{l}\right)$$

$$y_1(x) = A_1 \sin\left(\frac{n\pi x}{l}\right)$$

$$y(x,0) = A_1 \sin\left(\frac{n\pi x}{l}\right) B_2 = f(x)$$

Clearly this only holds for very special $f(x)$ and the method appears to have failed. But now we employ linearity. We regard this method as having found special solutions and consider sums of such solutions.

Example 3.1

$$y(x,t) = \sum_n y_{1n}(x) y_{2n}(t) =$$

$$\sum_n A_{1n} B_{2n} \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi\alpha t}{l}\right) = \sum_n \beta_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi\alpha t}{l}\right)$$

$$y(x,0) = \sum_n \beta_n \sin\left(\frac{n\pi x}{l}\right) = f(x)$$

The method now works if $f(x)$ can be expressed as a sum of sinusoids.

Note

$$y(x,0)=\sum_n \beta_n \sin\left(\frac{n\pi x}{l}\right)=f(x)$$

This formula would appear to be hopeless. The left-hand side is a periodic function in x being a sum of such functions. Indeed all of these functions have period $2l$. Indeed it is also odd being a sum of odd functions.
But $f(x)$ is neither periodic nor odd. In fact it is not even defined outside of the range $0 \le x \le l$. This is in fact the way out of our dilemma. The formula only has to hold for all x between 0 and l.

Example 3.1

I can define a new function which is an extension of $f(x)$ as follows: Extend the range of definition of $f(x)$ to all x as follows:

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } 0 \le x \le l \\ -f(-x) & \text{for } -l \le x \le 0 \\ \text{periodic of period } 2l & \end{cases}$$

This extended function equals $f(x)$ between 0 and l where $f(x)$ is defined. It is an *odd* extension over the range $-l$ to 0. It is explicitly periodic of period $2l$. We appear to have two definitions:

$$\tilde{f}(0) = f(0) = -f(-0) = -f(0)$$

Example 3.1

There is no problem here because of course $f(0) = 0$ is required by the boundary conditions. Likewise we appear to have two definitions:

$$\tilde{f}(l) = f(l) = \tilde{f}(l - 2l) = \tilde{f}(-l) = -f(l)$$

Again there is no problem here because of course $f(l) = 0$ is required by the boundary conditions. So we replace the previous "impossible" equation by:

$$y(x,0) = \sum_n \beta_n \sin\left(\frac{n\pi x}{l}\right) = \tilde{f}(x)$$

but require that this holds *for all x*.

Example 3.1

$$y(x,t) = \sum_n \beta_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi\alpha t}{l}\right)$$

So the vibration of the string can be expressed as a sum of special *sinusoidal vibrations* or *modes*. This is Bernoulli's principle of superposition. Of course at $t = 0$ this implies

$$y(x,0) = \sum_n \beta_n \sin\left(\frac{n\pi x}{l}\right)$$

If Bernoulli's principle of superposition gives a solution for all *initial* displacements of the string then every odd periodic signal must be expressible as a sum of sines.

Example 3.1

A consequence is that all sounds achievable by the vibration of a string, i.e. the acoustic output of all stringed instruments, can be expressed as sums of sinusoidal signals, i.e. pure tones. This begins to explain why music can be expressed in terms of notes. Notes make up music because modes make up the musical vibrations.

In terms of temporal as distinct from spatial vibrations the frequencies present are the integer multiples of $\pi\alpha/l$. We say that this is the fundamental frequency. In terms of the parameters of the problem:

$$\frac{\pi\alpha}{l} = \frac{\pi}{l}\sqrt{\frac{T}{\rho}}$$

Example 3.1

$$\frac{\pi\alpha}{l} = \frac{\pi}{l}\sqrt{\frac{T}{\rho}}$$

Hence shorter strings generate higher frequencies (which we call higher notes). Greater tension results in higher frequencies, which is why you increase the tension in a sting to tune it up. Lighter strings also generate higher frequencies which is why the strings of narrower gauge are tuned to the higher frequencies. Since many modes may be excited, many frequencies which are integer multiples of the fundamental frequency are present. If stringed instruments are to sound pleasant the apparently pure tones at frequencies which are simple multiples of one another must be concordant.

Example 3.1

In fact if I have two pure tones with the frequency of one being twice that of the other they are said to be *one octave apart*. Tones one octave apart are fully concordant. There is a tale concerning Pythagoras, which might even be true, which has him observing essentially this phenomenon, not in stringed instruments, but in beaten metal. In any event the group which he founded, the *Pythagorean Brotherhood*, was aware that mathematics plays a role in musical theory. They knew essentially of several simple frequency ratios which are concordant. This is a very important moment in human history, the first indication that mathematics may be the proper language for describing the physical phenomena of the world.

Example 3.1

Although more of a cult than a school, the Brotherhood is recognised as the first school of mathematics in known human history. Indeed the very word “mathematics” is probably due to them, meaning literally *that which is learned*. They existed as a learned society long before the first academics (the first academy was Plato’s) and long before the world’s first university (which was in Alexandria in Egypt).

Example 3.1

The solution:
$$y(x,t) = \sum_n \beta_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi\alpha t}{l}\right)$$

$$y(x,0) = \sum_n \beta_n \sin\left(\frac{n\pi x}{l}\right) = \tilde{f}(x) \quad \text{for all } x$$

is all very well, but unless I can determine those unknown coefficients β_n I have not progressed too far. As it happens Euler and Lagrange were aware of a method for determining these coefficients.

Example 3.1

We have already seen an example of an equation which was more readily solved when we reverted to complex numbers. I can rewrite as:

$$\sum_n \frac{\beta_n}{2j} e^{\frac{jn\pi x}{l}} - \sum_n \frac{\beta_n}{2j} e^{\frac{-jn\pi x}{l}} = \tilde{f}(x) \quad \text{for all } x$$

Even more generally I may consider the equation

$$\sum_{n=-\infty}^{\infty} c_n e^{\frac{jn\pi x}{l}} = \tilde{f}(x) \quad \text{for all } x$$

Example 3.1

Euler observes the following integral:

$$\int_0^{2l} e^{\frac{jn\pi x}{l}} e^{\frac{-jm\pi x}{l}} dx = \begin{cases} 0 & \text{if } n \neq m \\ 2l & \text{if } n = m \end{cases}$$

For if $n = m$:

$$\int_0^{2l} e^{\frac{jn\pi x}{l}} e^{\frac{-jn\pi x}{l}} dx = \int_0^{2l} dx = 2l$$

Example 3.1

For $n \neq m$:

$$\int_0^{2l} e^{\frac{jn\pi x}{l}} e^{\frac{-jm\pi x}{l}} dx = \int_0^{2l} e^{\frac{j(n-m)\pi x}{l}} dx = \left. \frac{e^{\frac{j(n-m)\pi x}{l}}}{\frac{j(n-m)\pi}{l}} \right|_0^{2l}$$

$$= \frac{e^{\frac{j(n-m)\pi 2l}{l}}}{\frac{j(n-m)\pi}{l}} - \frac{e^{\frac{j(n-m)\pi 0}{l}}}{\frac{j(n-m)\pi}{l}} = \frac{e^{j(n-m)2\pi} - 1}{\frac{j(n-m)\pi}{l}} = \frac{1-1}{\frac{j(n-m)\pi}{l}} = 0$$

Example 3.1

Alternatively I may write:

$$\frac{1}{2l}\int_{0}^{2l} e^{\frac{jn\pi x}{l}} e^{\frac{-jm\pi x}{l}}\,dx = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

There are very deep interpretations of this observation of Euler's and indeed a real understanding of Fourier theory is only achieved by delving into these interpretations. This is how I presented the material of this module for many years, initially with a high degree of success, and in more recent years with a high degree of failure. I have finally responded to the student feedback and given up. We will not give very much thought to these interpretations this year.

Example 3.1

Reconsider the equation in question:

$$\sum_{n=-\infty}^{\infty} c_n e^{\frac{jn\pi x}{l}} = \tilde{f}(x) \quad \text{for all } x$$

$$\frac{1}{2l}\int_0^{2l}\left(\sum_{n=-\infty}^{\infty} c_n e^{\frac{jn\pi x}{l}}\right) e^{\frac{-jm\pi x}{l}}\,dx = \frac{1}{2l}\int_0^{2l} \tilde{f}(x) e^{\frac{-jm\pi x}{l}}\,dx$$

$$= \sum_{n=-\infty}^{\infty} c_n \left(\frac{1}{2l}\int_0^{2l} e^{\frac{jn\pi x}{l}} e^{\frac{-jm\pi x}{l}}\,dx\right) = c_m$$

Example 3.1

So it is possible to find the coefficients through an integral formula:

$$\sum_{n=-\infty}^{\infty} c_n e^{\frac{jn\pi x}{l}} = \tilde{f}(x) \quad \text{for all } x$$

$$c_n = \frac{1}{2l} \int_0^{2l} \tilde{f}(x) e^{\frac{-jn\pi x}{l}} dx$$

These coefficients are called the *Fourier coefficients* of the periodic signal $\tilde{f}(x)$ whereas the sum

$$\sum_{n=-\infty}^{\infty} c_n e^{\frac{jn\pi x}{l}}$$

is called the *Fourier series*.

Fourier

Daniel Bernoulli had introduced his modes long before Joseph Fourier arrived on the scene. Euler, Lagrange and others were well aware of how to calculate the Fourier coefficients again long before Fourier. So why is the theory called after Fourier. Essentially prior to Fourier everyone felt that there was a problem with Bernoulli's way of solving equations such as the 1D wave equation. The problem was that the initial deflection $f(x)$, which could in principle be anything (subject to the requirement that $f(0) = f(l) = 0$). Yet this initial deflection would also permit the expression

$$f\left(x\right)=\sum_{n}\beta_{n}\sin\left(\frac{n\pi x}{l}\right)$$

Fourier

Instinctively it seems mathematicians recognised that on the right hand side of this summation we have very well behaved functions, the sinusoids. We can differentiate them as many times as we like. It seemed that, since the sum of two functions which we can differentiate as often as we like is another such function, this should also be true of arbitrary sums of such functions. So the mathematicians from Euler to Lagrange instinctively felt that the formula could only hold for functions $f(x)$ which were infinitely differentiable. But the method purported to work for essentially *any* function $f(x)$. It was not until the nineteenth century that we learnt that an *infinite* sum of continuous (let alone differentiable) functions need not be continuous.

Fourier

Indeed infinite sums behave very differently from finite sums when it comes to functions and their properties and Fourier theory has played a major role in teaching us this hard won lesson. Fourier, on the other hand, was a little blasé about this issue. He simply declared that he was going to assume that everything was fine and that Daniel Bernoulli's method was without flaw. On the basis of this he went on to give the first serious consideration of series of the form of the Fourier series, i.e. series of the forms:

$$\sum_n \alpha_n \cos\left(\frac{n\pi x}{l}\right) \qquad \sum_n \beta_n \sin\left(\frac{n\pi x}{l}\right) \qquad \sum_n c_n e^{\frac{jn\pi x}{l}}$$

Fourier

Such series are now called the *Fourier cosine series*, the *Fourier sine series* and the *Fourier series* or *complex Fourier series*. Actually Fourier gave rather a lot of thought to the first two, although I will say very little about them. Series in general play a very significant role in mathematics. Indeed Newton's take on the calculus was based on his belief that the relatively new (in his day) decimal system for representing numbers ought to have some counterpart for representing functions. He found this counterpart, it is what we now call the Maclaurin series and he proved to be something of a master in manipulating series of this kind. Most of his significant mathematical achievements, including the calculus, derive from this skill.

Fourier

In general series, or any other form of representation system only become valuable if we have some associated mathematical problem which is made relatively easy when using this representation. So decimal numbers or more generally place representation systems, are valuable because both addition and multiplication become simpler to achieve in these terms. The Maclaurin series is valuable because integration and differentiation become almost trivial (and rather obviously “inverses” of one another) when the functions involved are represented in this way. The problem which most naturally associates with the various Fourier series is that of solving PDEs using the method of separation of variables.

Interpretation

However, the first application of this method to the problem of the vibrating string opens up a very important possibility of *interpreting* what these series mean. I recollect the formula:

$$y(x,t) = \sum_n \beta_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi\alpha t}{l}\right)$$

In most stringed instruments the actual vibration of the string produces relatively little sound. This vibration must be amplified, this being done either by the sound box, or in electric instruments by an electronic amplifier. In this case the vibration, essentially at some particular location along the string is amplified.

Interpretation

At the point half way along the string we have:

$$y\left(\tfrac{l}{2},t\right)=\sum_n \beta_n \sin\left(\frac{n\pi l}{2l}\right)\cos\left(\frac{n\pi\alpha t}{l}\right)=\sum_n \beta_n \sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi\alpha t}{l}\right)$$

In temporal terms this vibration is expressible as a sum of cosines of radian frequencies which are integer multiples of a fundamental frequency $\pi\alpha/l$. Let us denote this fundamental frequency by ω_0. Obviously some of these cosines are zero-weighted, for if n is even then $\sin(n\pi/2)$ is zero. So in fact the vibration is expressible as a sum of cosines of radian frequencies which are *odd* integer multiples of the fundamental frequency ω_0.

Interpretation

Now a sinusoidal signal (cosine or sine) sounds like a pure tone. In the case of the string Bernoulli's *modes*, in so far as they express the temporal vibration at a particular point, are just pure tones at different frequencies, namely at frequencies which are integer multiples of one frequency, the fundamental frequency. So Bernoulli's solution which offers the vibration as a sum of modes, or special vibrations, is actually expressing the vibration as a sum of pure tones. So if we follow Fourier and accept that Bernoulli's method of superposition can actually find the general solution, then every vibration of a string fixed at both ends can be expressed as or decomposed as a sum of vibrations of a very pure kind, namely pure tones.

Interpretation

Not all instruments are stringed instruments, i.e. instruments which operate by amplifying the vibrations of a string. Some are wind instruments. These operate by setting up vibrations in a column of air. Sometimes this column of air is fixed at both ends (in a flute for example) and sometimes it is fixed at one end only (in a trumpet for example). The equations which describe these vibrations are, substantially the same as those which describe the vibrations in a string. Moreover, as we shall ultimately see, Bernoulli's modal analysis can be applied with just a little change to determine the vibrations of a string fixed at just one end. The upshot is that Bernoulli's modes work for all stringed and wind instruments.

Interpretation

We conclude that the temporal vibrations of stringed and wind instruments can be decomposed into sums of pure vibrations, i.e. pure tones. Moreover, the vibrations of a string are due to the flexible nature of the string. More general structures can also vibrate, although these vibrations may be in time and in several spatial dimensions. In fact the tendency for structures to vibrate can be a cause for some considerable concern in the design of bridges, tall buildings, vehicle chasses, *etc*. If, as Fourier did, we accept as fully general Bernoulli's method of expressing solutions as sums of special solutions (modes), then we conclude for general structures that their vibrations can also be expressed as sums of pure vibrations, i.e. pure tones.

Interpretation

If we focus on temporal vibrations, which I intend to do, then the three different series considered by Fourier are:

$$\sum_n \alpha_n \cos(n\omega_0 t) \qquad \sum_n \beta_n \sin(n\omega_0 t) \qquad \sum_n c_n e^{jn\omega_0 t}$$

these are respectively the Fourier cosine series, the Fourier sine series and the complex Fourier series also know as the Fourier series. The first two obviously offer an expression as a sum of pure tones, namely cosines and sines respectively. The latter does not so obviously achieve this, but note

$$\sum_n c_n e^{jn\omega_0 t} = \sum_n c_n \cos(n\omega_0 t) + j\sum_n c_n \sin(n\omega_0 t)$$

Interpretation

So all three of these series do indeed describe a decomposition into a sum of pure tones where the frequencies of these tones are all integer multiples of a fundamental frequency. Now let $f(t)$ be a general temporal vibration. Under what conditions can I decompose this vibration into a sum of pure vibrations whose frequencies are integer multiples of a fundamental frequency, i.e. when can I write:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

for suitable coefficients c_n and suitable fundamental frequency ω_0?

Interpretation

Well every term on the right-hand side is periodic with period (although not necessarily least period) $T = 2\pi/\omega_0$ as for all t:

$$e^{jn\omega_0(t+T)} = e^{jn\omega_0 t}e^{jn\omega_0 T} = e^{jn\omega_0 t}e^{jn2\pi} = e^{jn\omega_0 t}$$

The sum of such signals will “apparently” again be periodic of period T. Actually we are considering a possible infinite sum so the proof of this is tricky. Nevertheless, the statement is, with certain caveats, essentially correct. So it appears that a signal $f(t)$ will only permit such a decomposition if it is itself periodic of period T. Fourier’s acceptance of Bernoulli’s method amounts substantially to the assumption that this is the only condition required of $f(t)$.

Jean Baptiste Joseph Fourier

1768-1830

Conjectured that *all* periodic functions could be written as sums of sinusoids, i.e. that the sinusoids form a *basis* for the set of periodic functions of a given period.

Note

Joseph Fourier was born in Auxerre, France. He lived his life during the period of the French Revolution in which he played a significant part. He accompanied and played a senior role in the scientific and literary component of the Napoleonic expedition to Egypt. He was prefect of Grenoble during Napoleon’s reign and by all accounts proved exceptional in this role. His great contribution to knowledge is concerned with the theory of heat. He commenced work on this topic in 1804. His memoir on the topic was first read to the Paris Institute in 1807 and a committee was set up to report upon it. The committee, consisting of Lagrange, Laplace, Monge and Lacroix, was not unequivocally impressed.

Note

Fourier's conjecture, as stated, is false. Fortunately it is not very false, in the sense that the additional conditions that need to be imposed upon the signal for it to be true are relatively mild and very commonly satisfied by practical signals. One such set of conditions is known as the *Dirichlet conditions*. Of the committee that considered Fourier's manuscript, Lagrange (one of the greatest mathematicians of all time) made the most serious objection: what if the periodic signal had points of discontinuity (e.g. the square wave below). In this case Fourier would be contesting that a sum of continuous signals (sinusoids) would add up to a discontinuous signal. Fourier was contesting this and it turns out that he was correct.

Note

As noted above most mathematicians up until the mid nineteenth century felt that a sum of continuous functions should be continuous (a finite sum certainly was). It turns out that they were wrong, infinite sums just do not work the same as finite sums. It was Weierstrass who most clearly observed this fact. Nonetheless this belief prevented many mathematicians from taking the steps that Fourier dared to take, although many (most notably Euler) were aware of almost all of the mathematical ideas which Fourier employed. Fourier theory is a rare, possibly unique case, of Euler acting conservatively. His *modus operandi* was to allow his remarkable mathematical imagination to run wild.

Note

Actually the convergence properties of Fourier series are really rather subtle. Rather than looking foolish in retrospect, Lagrange, in offering his concerns for Fourier's work, appears to be what indeed he was, an incredibly gifted mathematician.

Interpretation

Difficulties notwithstanding if we accept Fourier's conjecture, then any periodic signal $f(t)$ of period T permits a decomposition of the form:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \quad , \quad \omega_0 = \frac{2\pi}{T}$$

This decomposition of the signal f is called the Fourier series of f. I have said *the* Fourier series not *a* Fourier series. This is because the coefficients c_n of the Fourier series are in fact uniquely defined. Indeed we have already seen how to find them in our work on Euler's observation concerning special integrals above. As I am now working in temporal rather than spatial terms I repeat this observation.

Interpretation

Euler observes the following integral:

$$\frac{1}{T}\int_0^T e^{jn\omega_0 t}e^{-jm\omega_0 t}dt = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases} \qquad \text{where } \omega_0 = \tfrac{2\pi}{T}$$

For if $n = m$:

$$\frac{1}{T}\int_0^T e^{jn\omega_0 t}e^{-jn\omega_0 t}dt = \frac{1}{T}\int_0^T dt = 1$$

Interpretation

For $n \neq m$:

$$\frac{1}{T}\int_0^T e^{jn\omega_0 t} e^{-jm\omega_0 t}dt = \frac{1}{T}\int_0^T e^{j(n-m)\omega_0 t}dt = \frac{1}{T}\frac{e^{j(n-m)\omega_0 t}}{j(n-m)\omega_0}\Bigg|_0^T$$

$$= \frac{e^{j(n-m)\omega_0 T}}{j(n-m)\omega_0 T} - \frac{e^0}{j(n-m)\omega_0 T} = \frac{e^{j(n-m)2\pi} - 1}{j(n-m)2\pi} = \frac{1-1}{j(n-m)2\pi} = 0$$

Interpretation

So if:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \quad , \quad \omega_0 = \frac{2\pi}{T}$$

then

$$\frac{1}{T}\int_0^T f(t) e^{-jm\omega_0 t} dt = \frac{1}{T}\int_0^T \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} e^{-jm\omega_0 t} dt$$

$$= \sum_{n=-\infty}^{\infty} c_n \frac{1}{T}\int_0^T e^{jn\omega_0 t} e^{-jm\omega_0 t} dt = c_m$$

i.e.

$$c_n = \frac{1}{T}\int_0^T f(t) e^{-jn\omega_0 t} dt$$

Fourier Series

Definition 3.1: Given any periodic signal f of period T the *Fourier coefficients* of f are the associated numbers:

$$c_n = \frac{1}{T}\int_0^T f(t)\exp(-jn\omega_0 t)dt$$

Definition 3.2: Given any periodic signal f of period T the *Fourier series* of f equals the summation:

$$\sum_{n=-\infty}^{\infty} c_n \exp(jn\omega_0 t)$$

where c_n are the Fourier coefficients of f and where $\omega_0 = 2\pi/T$.

Note

In essence Fourier's conjecture is that every periodic signal is equal to its own Fourier series. As noted above, this statement is false in general. However it is sufficiently close to the truth that Fourier theory survives this failure of what appears to be its foundational conjecture and proves to be a very powerful and useful theory nonetheless.

Interpretation

Whether a signal is equal to its Fourier series or not, the Fourier series still tells us what pure tones comprise at least a part of the signal. We interpret the Fourier series as informing us of which pure tones are present in the signal and how strongly. A large magnitude Fourier coefficient implies that the corresponding pure tone is strongly present. A low magnitude or zero magnitude Fourier coefficient that the corresponding pure tone is either largely or entirely absent. The Fourier series is describing the *frequency content* of the signal.

Interpretation

We have come to favour this interpretation of the Fourier coefficients and the Fourier series so completely that in fact we have reversed the process. The Fourier series and the Fourier coefficients are how we *define* the frequency content of a periodic signal.

Interpretation

To put this interpretation/definition to the test consider the special case of the zeroth Fourier coefficient. The associated pure tone is the signal:

$$\exp\left(j0\omega_0 t\right)=1$$

Although this is a pure tone it is a zero frequency tone, i.e. a constant. From the formula we have:

$$c_0 = \frac{1}{T}\int_0^T f\left(t\right)\exp\left(-j0\omega_0 t\right)dt = \frac{1}{T}\int_0^T f\left(t\right)dt$$

Interpretation

$$c_0 = \frac{1}{T}\int_0^T f(t)dt$$

You may perhaps have met this integral before in the case of a periodic signal of period *T*. The number equal to this integral is called the *average value* or just the *average* of the signal. The interpretation of the Fourier series is that c_0 measures the degree to which a zero frequency pure tone or constant is present in the signal. In electronic engineering we usually do not talk about zero frequency, we say *DC* instead.

Interpretation

We also call a constant signal a *DC signal*. So the degree to which DC is present in a periodic signal is measured, apparently, by the average value of the signal and indeed this makes some sense since the average value is the constant value about which the signal fluctuates.

I should note that the phrase frequency content is not universally used. Commonly the term *power spectrum* or *spectrum* is used instead as if it were synonymous. It is not, it is more general, so I will stick to the phrase frequency content.

Note

Many textbooks and many practitioners do not much care for the complex Fourier series because it involves complex functions although in fact the actual signal $f(t)$ will of course be real. There is an alternative form of the series which is valid in this case.

Fourier Coefficients: Properties

If f is periodic of period T and real then

$$c_{-n} = \overline{c_n} \text{ for all } n.$$

Proof:

$$c_{-n} = \frac{1}{T}\int_0^T f(t)\exp(jn\omega_0 t)\,dt$$

$$= \overline{\frac{1}{T}\int_0^T f(t)\exp(-jn\omega_0 t)\,dt}$$

$$= \overline{c_n}$$

▲

Trigonometric Fourier Series

Lemma 3-1: If signal *f* is periodic of period *T* and real then its Fourier series may be expressed as follows:

$$c_0 + \sum_{n=1}^{\infty}\left(\alpha_n \cos\left(n\omega_0 t\right) + \beta_n \sin\left(n\omega_0 t\right)\right)$$

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$\alpha_n = 2\,\mathrm{Re}(c_n) = \frac{2}{T}\int_0^T f(t)\cos\left(n\omega_0 t\right)dt$$

$$\beta_n = -2\,\mathrm{Im}(c_n) = \frac{2}{T}\int_0^T f(t)\sin\left(n\omega_0 t\right)dt$$

Note

This series is more satisfactory since it expresses a real periodic signal (or real vibration) in terms of real pure tones, namely cosines and sines. The price paid is that we have three separate formulae for calculating the coefficients appearing and, moreover, these formulae involve integrations which include trigonometric functions. The goal in integration is to get the integration into the form of integrating exponentials, because that is easy. In this respect this reformulation involves a huge retrograde step.

Proof

$$\sum_{n=-\infty}^{-1} c_n \exp(jn\omega_0 t) + c_0 + \sum_{n=1}^{\infty} c_n \exp(jn\omega_0 t)$$

$$= c_0 + \sum_{n=1}^{\infty} |c_n| \left(\exp\left(j\left(n\omega_0 t + \theta_n \right)\right) + \exp\left(- j\left(n\omega_0 t + \theta_n \right)\right)\right)$$

where $c_n = |c_n| \exp(j\theta_n)$.

$$= c_0 + \sum_{n=1}^{\infty} 2|c_n| \cos\left(n\omega_0 t + \theta_n \right)$$

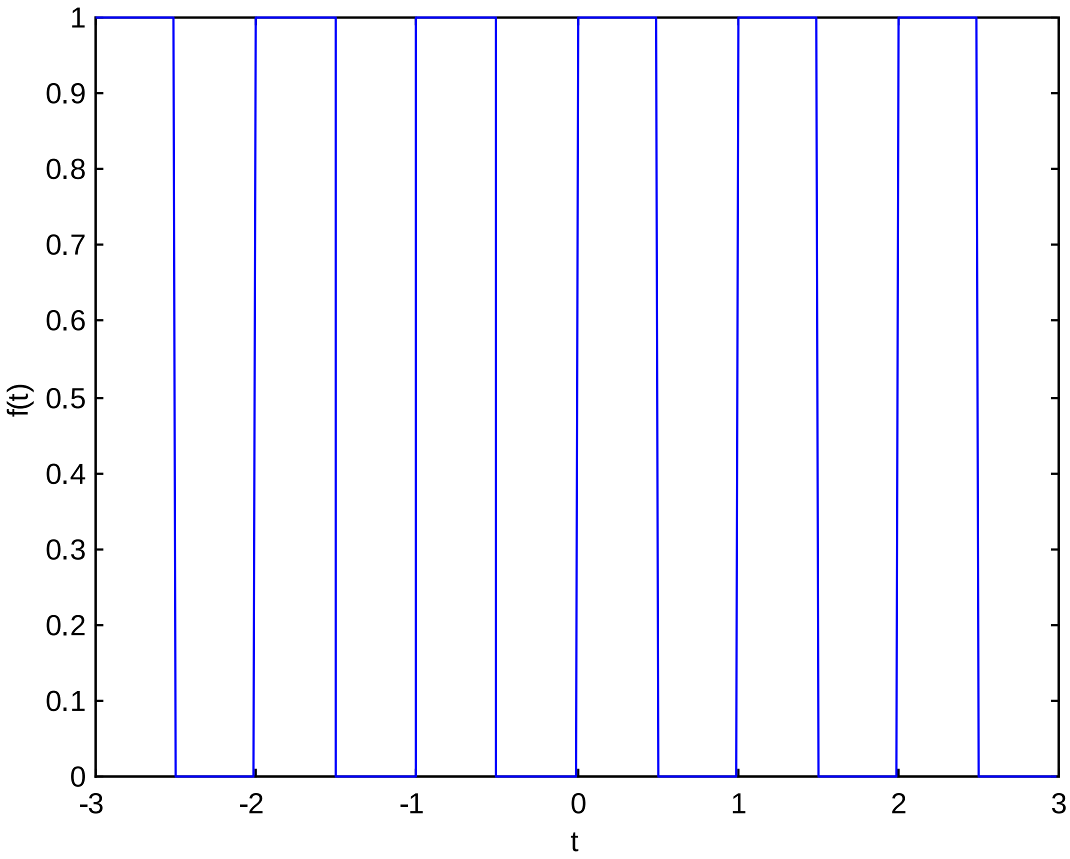
$$= c_0 + \sum_{n=1}^{\infty} \left(2|c_n| \cos(\theta_n) \right) \cos(n\omega_0 t) + \left(-2|c_n| \sin(\theta_n) \right) \sin(n\omega_0 t)$$

▲

Example 3.2:

$$f(t)=\begin{cases}1 & \text{if} \quad 0\le t\le 0.5\\ 0 & \text{if} \quad 0.5<t<1\end{cases}$$

$f(t)$ periodic of period 1 sec.



Example 3.2:

$$f(t) = \begin{cases} 1 & \text{if} \quad 0 \le t \le 0.5 \\ 0 & \text{if} \quad 0.5 < t < 1 \end{cases}$$

$$c_n = \int_0^1 f(t) e^{-jn\omega_0 t} dt = \int_0^{0.5} e^{-jn\omega_0 t} dt \quad , \quad \omega_0 = \tfrac{2\pi}{1} = 2\pi \ \text{rad/sec}$$

$$c_n = \left. \frac{e^{-jn2\pi t}}{-jn2\pi} \right|_0^{0.5} = \frac{e^{-jn\pi} - 1}{-jn2\pi} = \frac{(-1)^n - 1}{-jn2\pi} \quad \text{if} \ \ n \ne 0.$$

$$c_n = \begin{cases} 0 & \text{if} \ \ n \ne 0, \ \text{even} \\ \frac{-j}{n\pi} & \text{if} \ \ n \ \text{odd} \end{cases}$$

Example 3.2:

$$f(t)=\begin{cases}1 & \text{if} \quad 0\le t\le 0.5\\ 0 & \text{if} \quad 0.5<t<1\end{cases}$$

$$c_0=\int_0^{0.5} dt=\tfrac{1}{2}$$

$$\alpha_n=2\,\mathrm{Re}\left(c_n\right)=0 \quad \text{for all } n\ge 1$$

$$\beta_n=-2\,\mathrm{Im}\left(c_n\right)=\begin{cases}0 & \text{if } n\ge 1 \ \text{ even}\\ \frac{2}{n\pi} & \text{if } n\ge 1 \ \text{ odd}\end{cases}$$

$$c_0+\sum_{n=1}^{\infty}\left(\alpha_n\cos\left(n\omega_0 t\right)+\beta_n\sin\left(n\omega_0 t\right)\right)=\tfrac{1}{2}+\sum_{m=0}^{\infty}\frac{2}{\left(2m+1\right)\ \pi}\sin\left(\left(2m+1\right)2\pi t\right)$$

Example 3.2:

$$f(t) = \begin{cases} 1 & \text{if} \quad 0 \le t \le 0.5 \\ 0 & \text{if} \quad 0.5 < t < 1 \end{cases}$$

The trigonometric Fourier series is:

$$\tfrac{1}{2} + \tfrac{2}{\pi}\sin(2\pi t) + \tfrac{2}{3\pi}\sin(6\pi t) + \tfrac{2}{5\pi}\sin(10\pi t) + \cdots$$

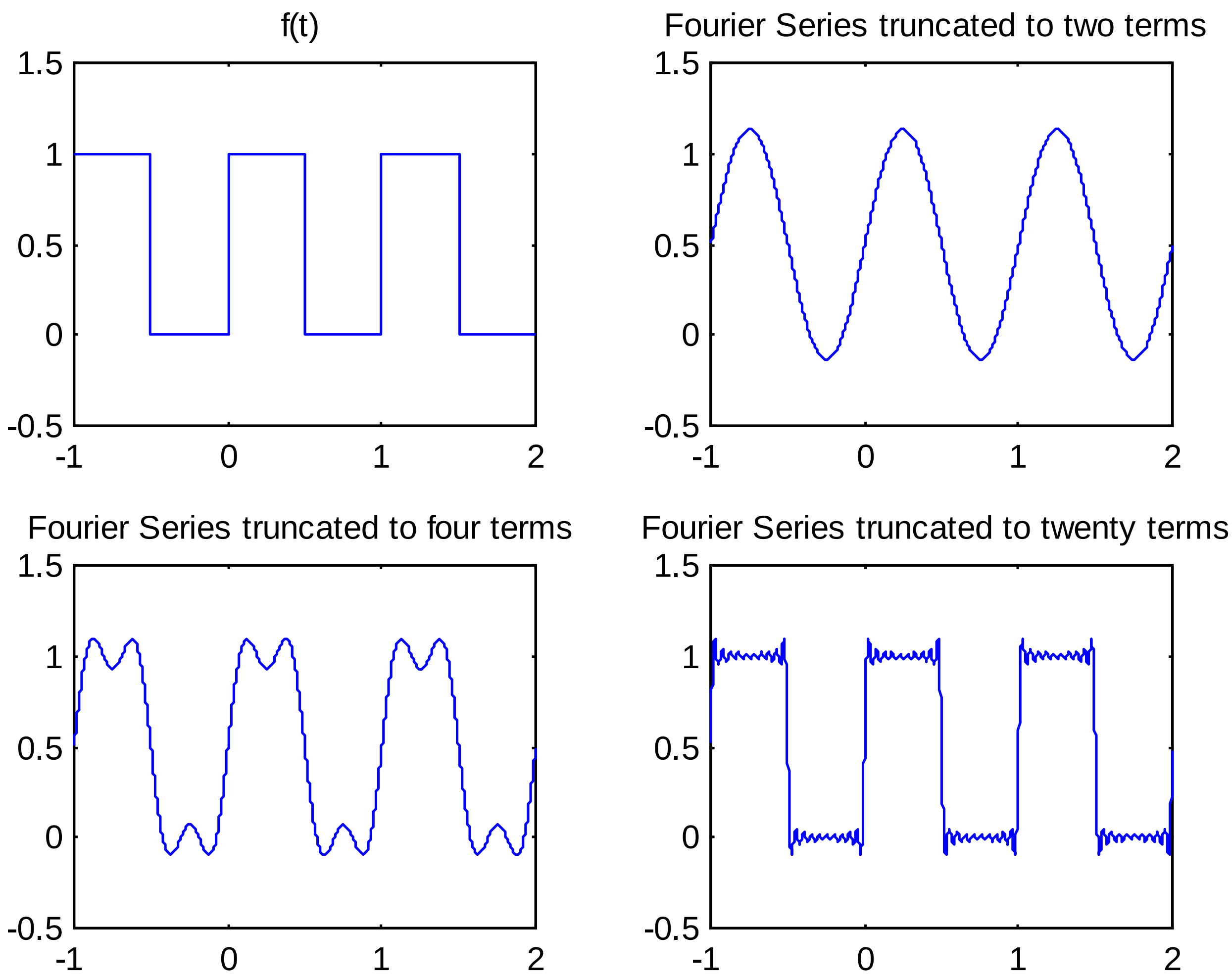
The DC component is ½, which makes sense. The average of the signal is clearly equal to ½. The fundamental frequency is 2π rad/sec. Only odd integer multiples of this frequency appear.

Example 3.2:

$$f(t) = \begin{cases} 1 & \text{if} \quad 0 \le t \le 0.5 \\ 0 & \text{if} \quad 0.5 < t < 1 \end{cases}$$

Obviously I cannot work with an infinite series. In general I will have to truncate. It is in doing this that I bring some trouble down upon myself. My intuition would be that if I retain say one million terms then the discrepancy between the truncated series and the signal to which it converges (assuming that it does converge, which it certainly does in this case) should become negligible. This intuition proves sound, but not in the way I expect.

Gibbs' Phenomenon



The peak difference between signal and truncated series is approximately 0.09 regardless of how large one chooses N .

Note

The peak error does not decrease as the number of terms N retained is increased. This phenomenon is called the *Gibbs' Phenomenon* (although it had been noted and explained by Wilbraham 60 years before Gibbs). It is due to the particular kind of approximation which the Fourier series yields and the associated convergence properties of that series.

Note

If I have a series approximation to a periodic signal $f(t)$ of period T where that approximation has the form:

$$\sum_{n=-N}^{N} d_n e^{jn\omega_0 t} \quad , \quad \omega_0 = \tfrac{2\pi}{T}$$

then it is possible to find the choice of the coefficients d_n which produces the minimum *mean square error*, i.e. for which we have a minimal value of:

$$\frac{1}{T}\int_0^T \left| f(t) - \sum_{n=-N}^{N} d_n e^{jn\omega_0 t} \right|^2 dt$$

Note

Indeed this “best” choice is to take the coefficients d_n to equal the Fourier coefficients c_n. So, as an approximation, the truncated Fourier series is “best” in the sense that it minimises the mean square error. It does not minimise the peak error. What is actually happening above is that whereas the peak error is essentially unchanging, errors of comparable size are occurring over an ever narrowing range of times. For the abundance of times the error is much smaller.

Note

Intimately related to the fact that the Fourier series minimises the mean square error is the fact that the series does not converge to the signal $f(t)$, but rather to a certain related signal which is usually very similar to it. Actually the Fourier series converges to $f(t)$ for those times t such that f is continuous at t. For those times t for which f is discontinuous the Fourier series converges to *the average of the left and right hand limits*, i.e.

$$\lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{jn\omega_0 t} = \frac{f\left(t^-\right)+f\left(t^+\right)}{2}$$

Note

For the square wave above this implies that the series converges to ½ at the points of discontinuity 0, 0.5, … It is this slightly unusual form of the convergence which is the cause of the Gibbs' phenomenon. It underscores the nontrivial nature of Lagrange's objections to Fourier's conjecture. Far from retrospectively looking foolish for having denounced one of the world's great mathematical theories, Lagrange emerges as prescient. It would take generations for a full resolution of the convergence problem to emerge and many would argue that the last word has not yet been written on this topic.

Interpretation

$$\underbrace{c_0}_{\text{DC Component (Average Value)}} + \underbrace{\alpha_1 \cos(\omega_0 t) + \beta_1 \sin(\omega_0 t)}_{\text{Fundamental}} + \sum_{n=2}^{\infty} \underbrace{\left(\alpha_n \cos(n\omega_0 t) + \beta_n \sin(n\omega_0 t)\right)}_{n\text{th harmonic}}$$

The term “harmonic” is employed because the frequency of these terms, being an integer multiple of the fundamental frequency, these tones will be in harmony with the fundamental tone.

Numerical Fourier Series

$$c_n = \frac{1}{T}\int_0^T f(t)\exp(-jn\omega_0 t)dt$$

Divide the integration range [0,T] into N equal sub-intervals.

$$c_n = \frac{1}{T}\sum_{k=0}^{N-1}\int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} f(t)\exp(-jn\omega_0 t)dt$$

Numerical Fourier Series

For large N and assuming $n \neq 0$ Cauchy's first mean value theorem for integration gives

$$c_n \cong \frac{1}{T} \sum_{k=0}^{N-1} f\left(\frac{kT}{N}\right) \int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} \exp(-jn\omega_0 t)\,dt$$

$$= \left(\frac{1 - \exp(-j2\pi n / N)}{j2\pi n}\right) \sum_{k=0}^{N-1} f\left(\frac{kT}{N}\right) \exp\left(\frac{-j2\pi nk}{N}\right)$$

Numerical Fourier Series

On the other hand for large N and assuming $n=0$

$$c_0 \cong \frac{1}{T}\sum_{k=0}^{N-1} f\left(\frac{kT}{N}\right)\int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} dt$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} f\left(\frac{kT}{N}\right)$$

Numerical Fourier Series

Let

$$F_n = \sum_{k=0}^{N-1} f\left(\frac{kT}{N}\right) \exp\left(\frac{-j2\pi nk}{N}\right)$$

$$W_n = \begin{cases} \left(\dfrac{1-\exp(-j2\pi n/N)}{j2\pi n/N}\right) & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases}$$

$$c_n \cong \frac{W_n F_n}{N} \qquad \text{for } n = 0,1,\ldots,N-1$$

Note

This method yields approximate values for the first N Fourier coefficients.

We focus on calculating c_n for non-negative n only, since the coefficients with negative indices can then be obtained by the Hermitian symmetry property (assuming f is real).

The approximate formula does not employ an expression for f, but rather the values attained by f at the specific times 0, T/N, $2T/N$, …,$(N\text{-}1)T/N$. These values are called *uniformly-spaced samples* of the signal f.

If n is small in comparison with N then W_n is approximately one. In consequence one almost always employs the cruder approximation $c_n \simeq F_n/N$ with the proviso that N be large (as above) and that n be small in comparison with N. In other words the approximation is not acceptable for all $n = 0,1, \ldots, N\text{-}1$.

Numerical Fourier Series

There exists a very efficient algorithm called the *Fast Fourier Transform* (FFT) for calculating the numbers F_n for each $n = 0,1, \ldots, N$-1.

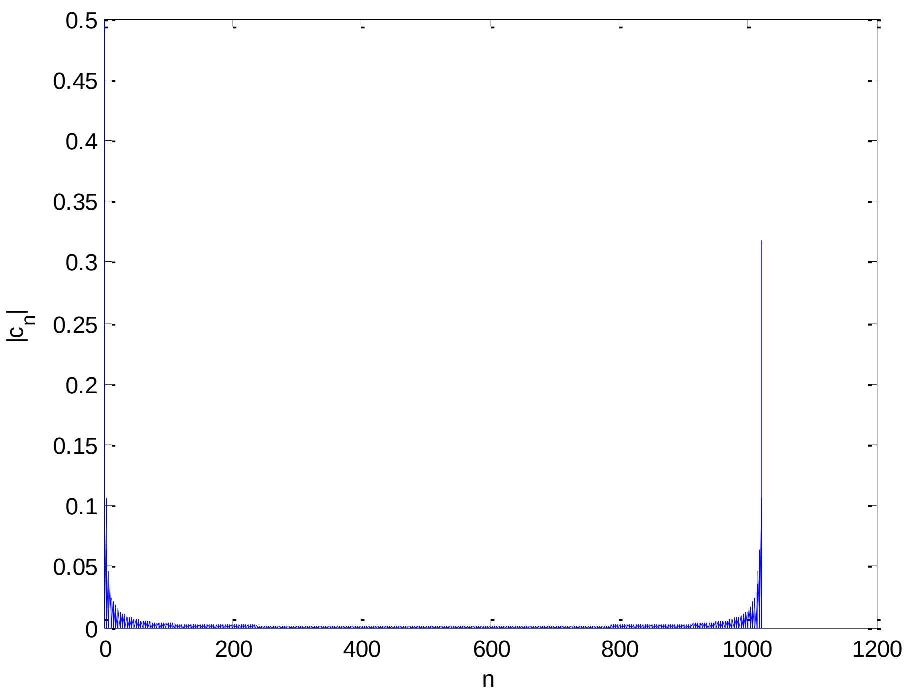
It is intrinsic that F_n is periodic in n with period N. Since the Fourier coefficients are *not* periodic in general (indeed they are essentially never periodic) this is indicating that there is a problem with the approximation.

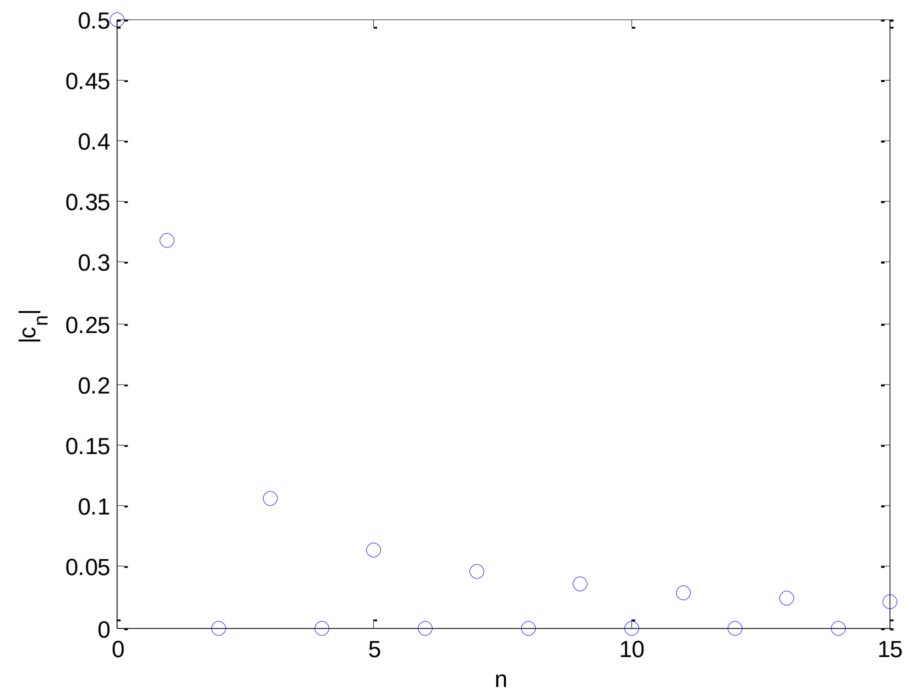
We have assumed that the approximation $c_n \cong W_n F_n / N$ (or F_n / N) is good provided N is large. Whereas this proviso is certainly necessary for good approximation it is not sufficient. The periodicity of F_n is hinting at this. A detailed analysis indicates that regardless of the size of N the approximation cannot be relied upon for indices n greater than $N/2$. In consequence the approximation is only employed at most for Fourier coefficients c_n with $n = 0,1, \ldots, N/2 - 1$ and in practice should only be used for far fewer than this.

The **fft** command in Matlab will approximately evaluate the Fourier coefficients of a signal whose samples are stored in the vector *f provided the first element of f is the sample of the signal at time* $t = 0$.

Example 3.2

$$f(t)=\begin{cases}1 & \text{if} \quad 0\le t\le 0.5\\ 0 & \text{if} \quad 0.5<t<1\end{cases}$$





Code

```
>>  N = 1024   set number of samples, selecting a power of 2

>>  t = [0:1/N:1-(1/N)];        create vector of N sample times uniformly distributed between 0 and 1

>>  f = [ones(1,N/2) zeros(1,N/2)];    create vector of associated samples

>>  FF = fft(f)/N;        find approximate Fourier coefficients

>>  nvec = [0:N-1];     create vector of coefficient indices

>>  plot(nvec,abs(FF))    plot magnitude of coefficients vs indices

>>  xlabel('n')

>>  ylabel('|c_n|')
```

Note

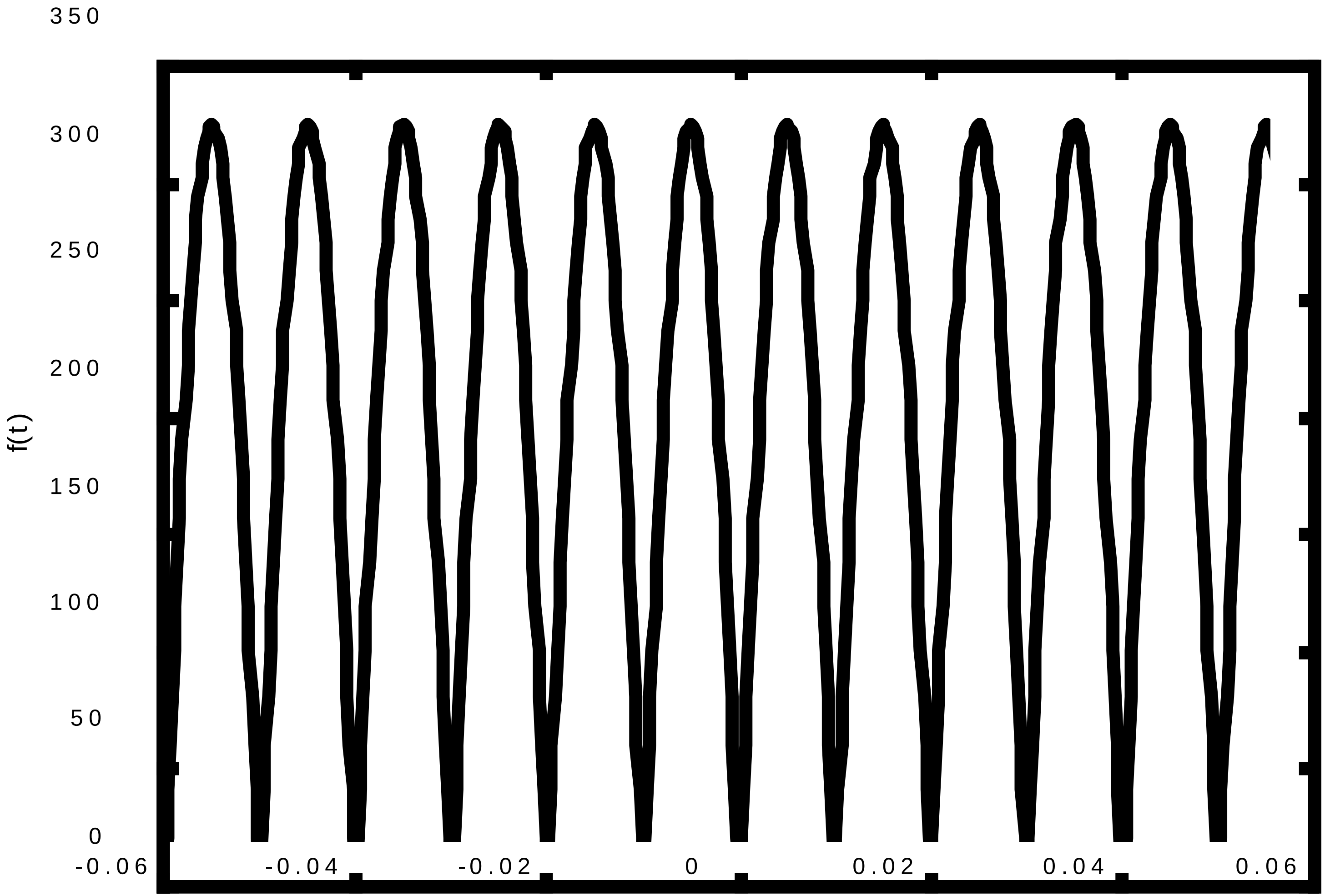
Note the symmetry about $N/2 = 512$ in subplot on left (which shows calculated F_n for all $n = 0,1,\ldots, N$-1). This is due to periodicity and hermitian symmetry.

```
>> plot(nvec(1:16),abs(FF(1:16)), 'o')    plot magnitude of first 16 coefficients vs indices. Do not allow Matlab plotting to join points with line segments. Plot o at each data point.

>> xlabel('n')

>> ylabel('|c_n|')
```

Plot on right zooms in on the first 16 values of the modulus of c_n . Note that approximation appears to be good.

Example 3.3:

$$f(t) = \left| 230\sqrt{2}\sin\left(100\pi t\right) \right| \text{ if } 0 \le t \le 0.02$$

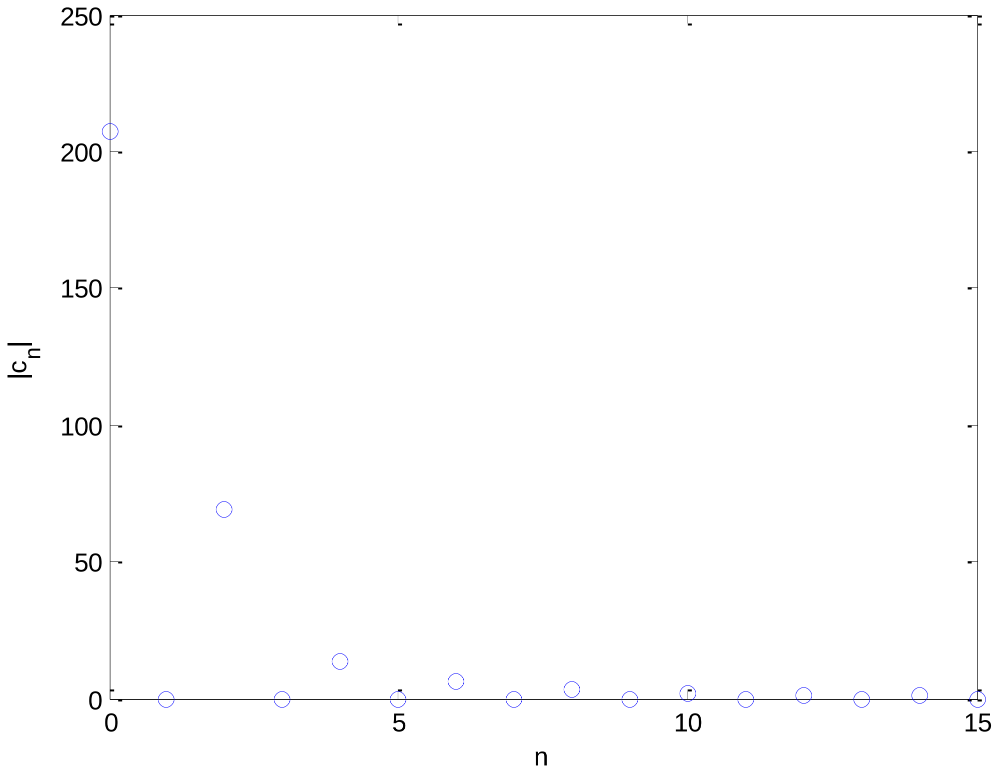


Code

```
>>  N = 1024   set number of samples, selecting a power of 2

>>  t = (0.02/N)*[0:N-1];       create vector of N sample times uniformly distributed between 0 and 0.02

>>  f = abs(230*sqrt(2)*sin(100*pi*t));       create vector of associated samples

>>  FF = fft(f)/N;       find approximate Fourier coefficients

>>  plot([0:15],abs(FF(1:16)), 'o')    plot magnitude of first sixteen coefficients vs indices

>>  xlabel('n')

>>  ylabel('|c_n|')
```

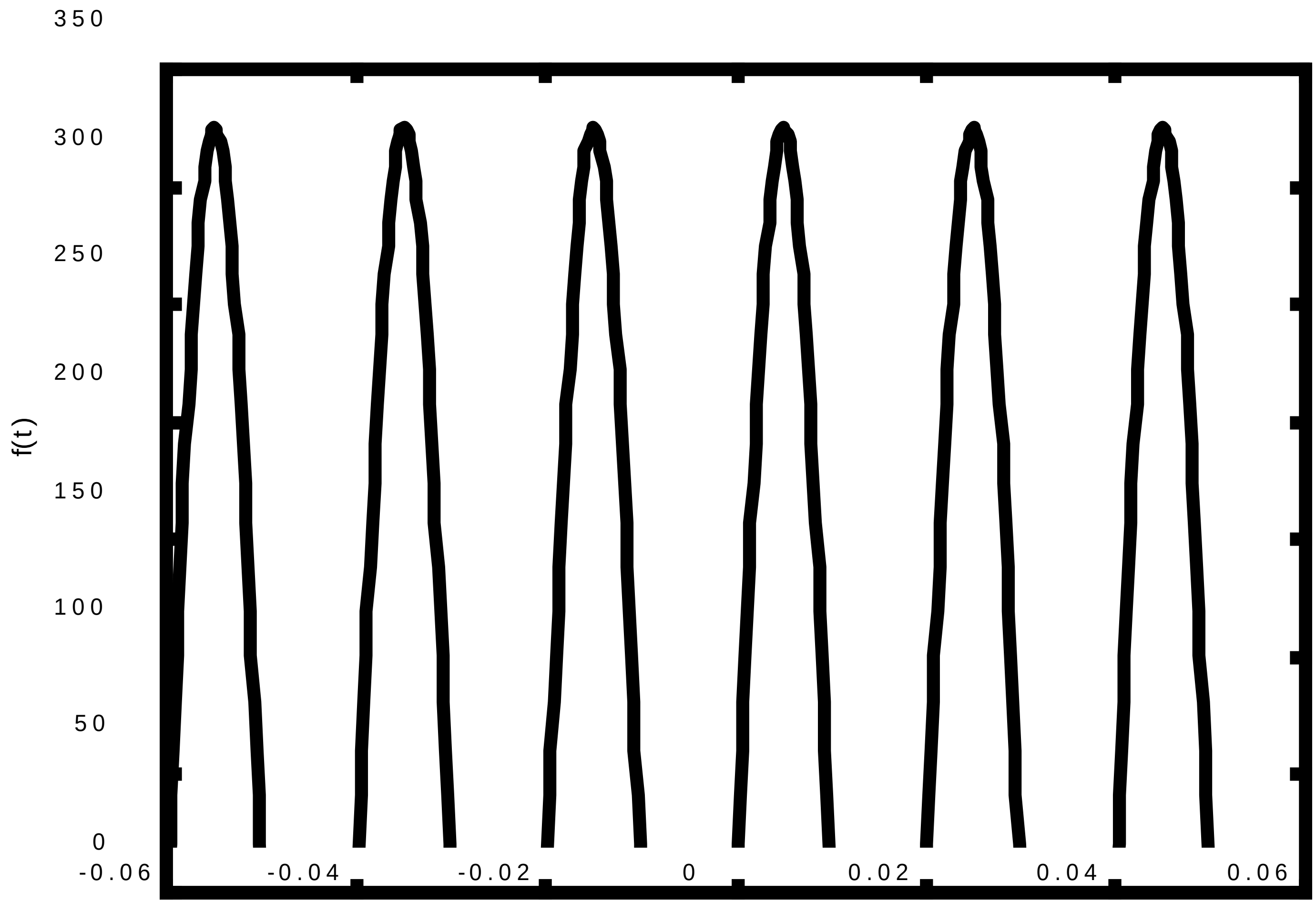
Example 3.3:

$$f(t) = \left| 230\sqrt{2}\sin(100\pi t) \right| \text{ if } 0 \le t \le 0.02$$



Example 3.4:

$$f(t)=\begin{cases}230\sqrt{2}\sin(100\pi t) & \text{if } 0\le t\le 0.01\\ 0 & \text{if } 0.01<t<0.02\end{cases}$$

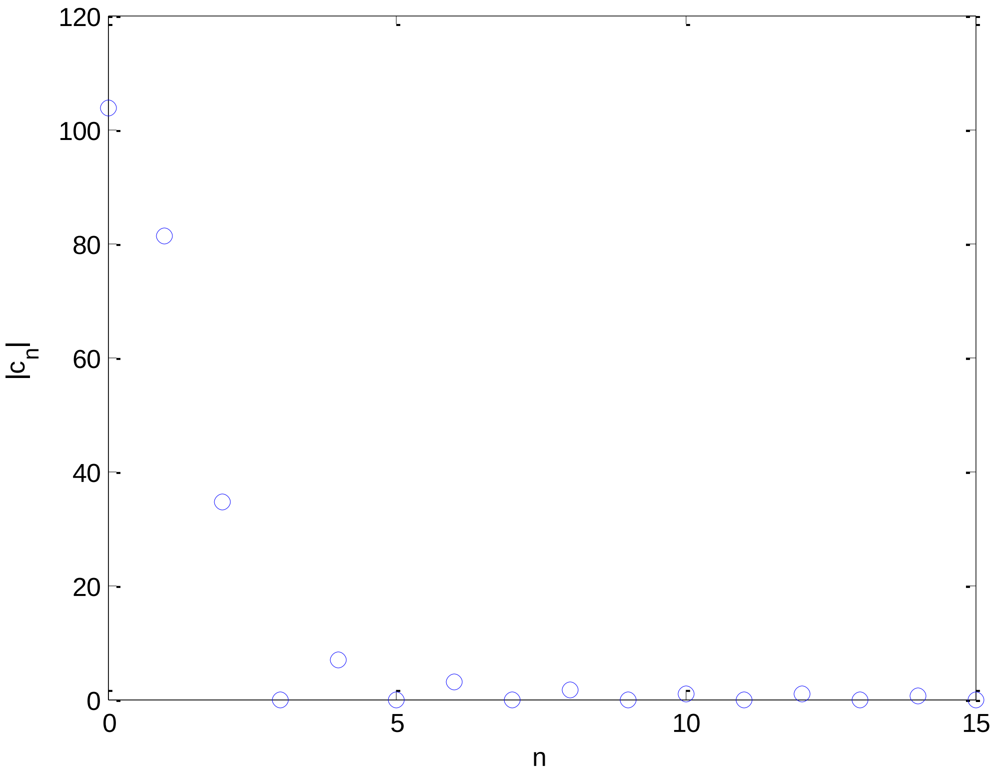


Code

```
>> N = 1024   set number of samples, selecting a power of 2

>> t = (0.02/N)*[0:N-1];      create vector of N sample times uniformly distributed between 0 and 0.02

>> f1 = abs(230*sqrt(2)*sin(100*pi*t(1:N/2)));      create vector of first half of associated samples

>> f2 = zeros(1:N/2);  create vector of second half of associated samples

>> f = [f1 f2];            combine samples

>> FF = fft(f)/N;        find approximate Fourier coefficients

>> plot([0:15],abs(FF(1:16)), 'o')    plot magnitude of first sixteen coefficients vs indices

>> xlabel('n'), ylabel('|c_n|')
```

Example 3.4:

$$f(t) = \begin{cases} 230\sqrt{2}\sin(100\pi t) & \text{if } 0 \le t \le 0.01 \\ 0 & \text{if } 0.01 < t < 0.02 \end{cases}$$



Note

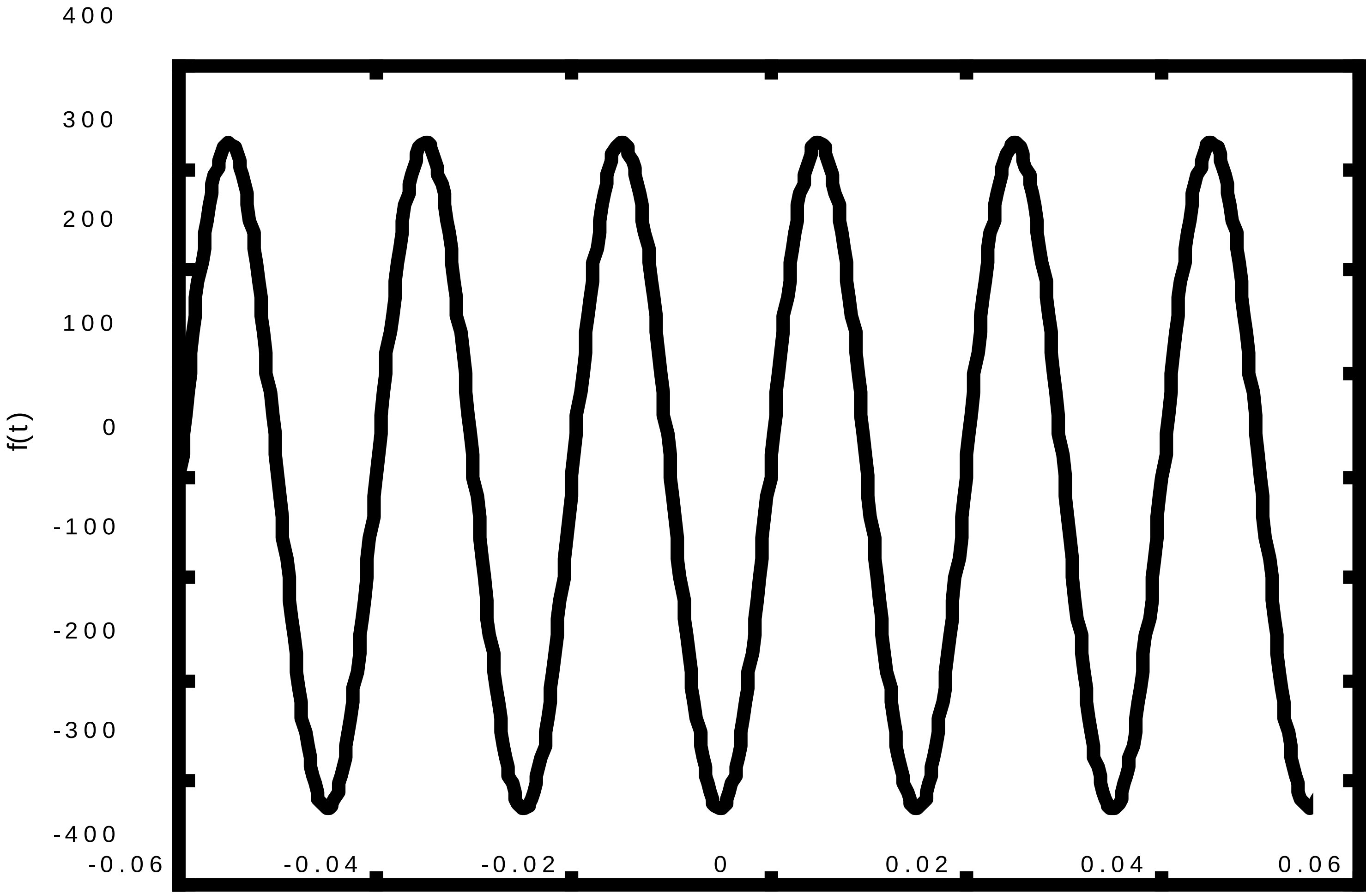
The signal of example 3.3 is a 50 Hz, 230 V rms sinusoid which has been *full-wave rectified*. The signal of example 3.4 is a 50 Hz, 230 V rms sinusoid which has been *half-wave rectified* . Rectification comprises the opening salvo of a power supply whose purpose is to convert the AC input voltage to a DC output voltage. The first phase of this process is to convert the signal from one with zero DC value to one with non-zero DC value. The next phase *may be* said to comprise extracting this DC value and eliminating the fluctuations, i.e. filtering the signal such that the DC component is passed but the higher frequency components are blocked. I say “may be”, the actual process is somewhat trickier than this.

Note

If I interpret the rectification process this way which is the better option, to full-wave rectify or to half-wave rectify. Intuition suggests the former surely. It seems mad to block half the signal (and therefore half the energy) and the half-wave rectified sinusoid surely has more extreme fluctuations. Mathematics agrees. The DC component of the full-wave rectified sinusoid is twice that of the half wave rectified sinusoid. The lowest frequency harmonic which appears in the full-wave rectified sinusoid is at 100Hz (being the second harmonic, the fundamental being absent). The lowest frequency harmonic which appears in the half wave-rectified sinusoid is at half this value, namely 50 Hz. An electrical filter is cheaper and more effective if it does not have to *cut off* too quickly.

Example 3.5:

$$f(t) = 230\sqrt{2}\sin(100\pi t)$$



Note

This is a rather good example to consider. Actually I can state what the trigonometric Fourier series of this function is by inspection. But students, attempting to perform the necessary analytic work to calculate the series usually make a particular error and end up concluding that all of the Fourier coefficients are zero, thereby declaring that the trigonometric Fourier series is zero. In fairness a good number subsequently have the sense that something has gone wrong, even if they cannot discern what. I propose to determine the series both analytically and numerically, although as I say it is in fact possible at a glance to already declare what the answer is.

Example 3.5:

$$f(t) = 230\sqrt{2}\sin(100\pi t)$$

$$T = 0.02 \ \text{sec}, \quad \omega_0 = 100\pi \ \text{rad/sec}$$

$$c_n = \frac{1}{0.02}\int_0^{0.02} f(t)e^{-jn100\pi t}dt = (50)\left(230\sqrt{2}\right)\int_0^{0.02}\sin(100\pi t)e^{-jn100\pi t}dt$$

$$c_n = (50)\left(230\sqrt{2}\right)\int_0^{0.02}\left(\frac{e^{j100\pi t} - e^{-j100\pi t}}{2j}\right)e^{-jn100\pi t}dt$$

$$c_n = \frac{(50)\left(230\sqrt{2}\right)}{2j}\left(\int_0^{0.02} e^{j100\pi(n-1)t}dt - \int_0^{0.02} e^{-j100\pi(n+1)t}dt\right)$$

Example 3.5:

$$c_n = \frac{(50)\left(230\sqrt{2}\right)}{2j}\left(\left.\frac{e^{j100\pi(n-1)t}}{j100\pi(n-1)}\right|_0^{0.02} - \left.\frac{e^{-j100\pi(n+1)t}}{-j100\pi(n+1)}\right|_0^{0.02}\right)$$

$$c_n = \frac{(50)\left(230\sqrt{2}\right)}{2j}\left(\left(\frac{e^{j2\pi(n-1)}-1}{j100\pi(n-1)}\right)-\left(\frac{e^{-j2\pi(n+1)}-1}{-j100\pi(n+1)}\right)\right)$$

$$c_n = \frac{(50)\left(230\sqrt{2}\right)}{2j}\left(\left(\frac{0}{j100\pi(n-1)}\right)-\left(\frac{0}{-j100\pi(n+1)}\right)\right)=0$$

Note

This is the calculation which students tend to first perform, if they succeed at all. It seems that all of the Fourier coefficients are zero and I must deduce that the Fourier series itself is zero. Since the Fourier series converges to $f(t)$ in this case (since f is continuous) I appear to have shown that the signal equals zero, although of course it does not. The error in the argument is overlooking that we divide by n-1 and by n+1. Accordingly the argument as presented allows me to deduce certainly that $c_n = 0$ for all n not equal to +/- 1 and in fact this is true. The argument is invalid for and therefore says nothing about c_1 and c_{-1}. As the second of these is the conjugate of the first we may focus on repairing the argument for the case $n = 1$.

Example 3.5:

Looking back over the previous argument I find that no error was made up to the point:

$$c_n = \frac{(50)\left(230\sqrt{2}\right)}{2j}\left(\int_0^{0.02} e^{j100\pi(n-1)t}\,dt - \int_0^{0.02} e^{-j100\pi(n+1)t}\,dt\right)$$

I explicitly set $n = 1$ in this formula.

$$c_1 = \frac{(50)\left(230\sqrt{2}\right)}{2j}\left(\int_0^{0.02} dt - \int_0^{0.02} e^{-j200\pi t}\,dt\right)$$

Example 3.5:

$$c_1 = \frac{(50)\left(230\sqrt{2}\right)}{2j}\left(0.02 - \left.\frac{e^{-j200\pi t}}{-j200\pi}\right|_0^{0.02}\right) =$$

$$\frac{(50)\left(230\sqrt{2}\right)}{2j}\left(0.02 - \frac{\left(e^{-j4\pi} - 1\right)}{-j200\pi}\right) = \frac{(50)\left(230\sqrt{2}\right)}{2j}(0.02)$$

$$= \frac{\left(230\sqrt{2}\right)}{2j} = -j\frac{\left(230\sqrt{2}\right)}{2}$$

Example 3.5:

$$c_0 = 0$$

$$\alpha_n = 2\,\mathrm{Re}(c_n) = 0 \quad \text{for all } n \geq 1$$

$$\beta_n = -2\,\mathrm{Im}(c_n) = 0 \quad \text{for all } n \geq 2$$

$$\beta_1 = -2\,\mathrm{Im}(c_1) = 230\sqrt{2}$$

$$c_0 + \sum_{n=1}^{\infty}\left(\alpha_n \cos(n\omega_0 t) + \beta_n \sin(n\omega_0 t)\right) = 230\sqrt{2}\sin(100\pi t)$$

Note

So the trigonometric Fourier series of the signal

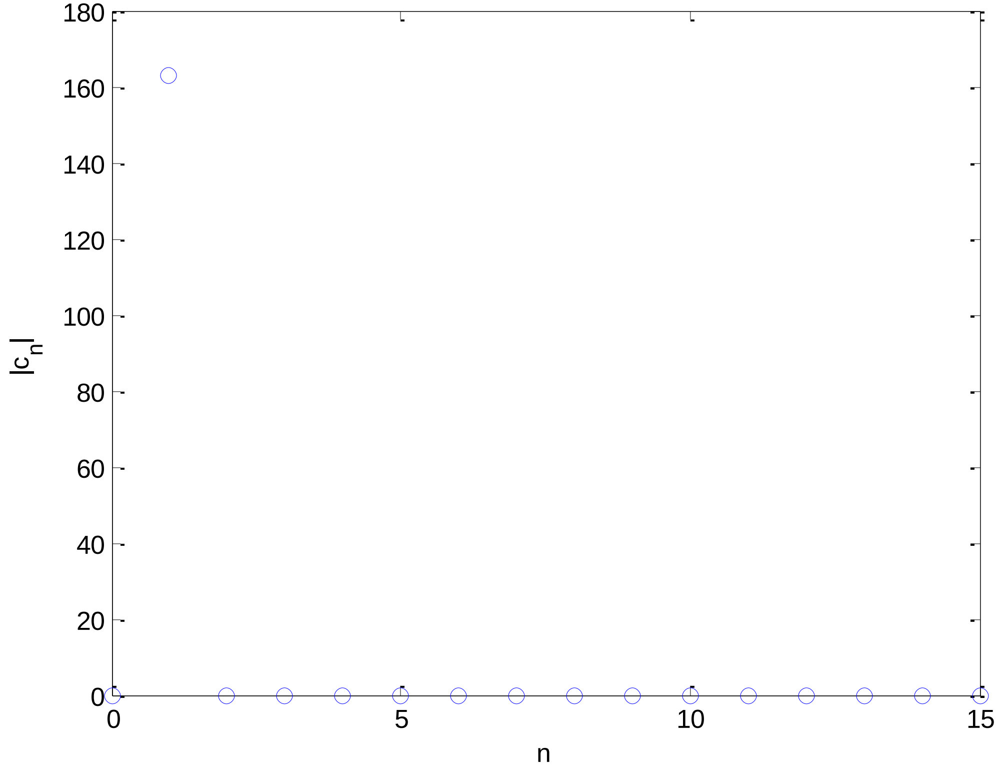
$$230\sqrt{2}\,\sin(100\pi t)$$

is equal to the signal itself. That makes sense. The trigonometric Fourier series is an expression for a signal as a sum of sinusoids whose frequencies are all integer multiples of a fundamental frequency. Moreover the Fourier coefficients and along with them the series are unique. If the signal already is a sum of sinusoids whose frequencies are integer multiples of a fundamental frequency (or is simply equal to one such sinusoid as in this case) then it is already expressed as a trigonometric Fourier series. So actually we could have just observed this at the outset.

Code

```
>>  N = 1024   set number of samples, selecting a power of 2

>>  t = (0.02/N)*[0:N-1];      create vector of N sample times uniformly distributed between 0 and 0.02

>>  f = 230*sqrt(2)*sin(100*pi*t);     create vector of associated samples

>>  FF = fft(f)/N;       find approximate Fourier coefficients

>>  plot([0:15],abs(FF(1:16)), 'o')    plot magnitude of first sixteen coefficients vs indices

>>  xlabel('n'), ylabel('|c_n|')
```

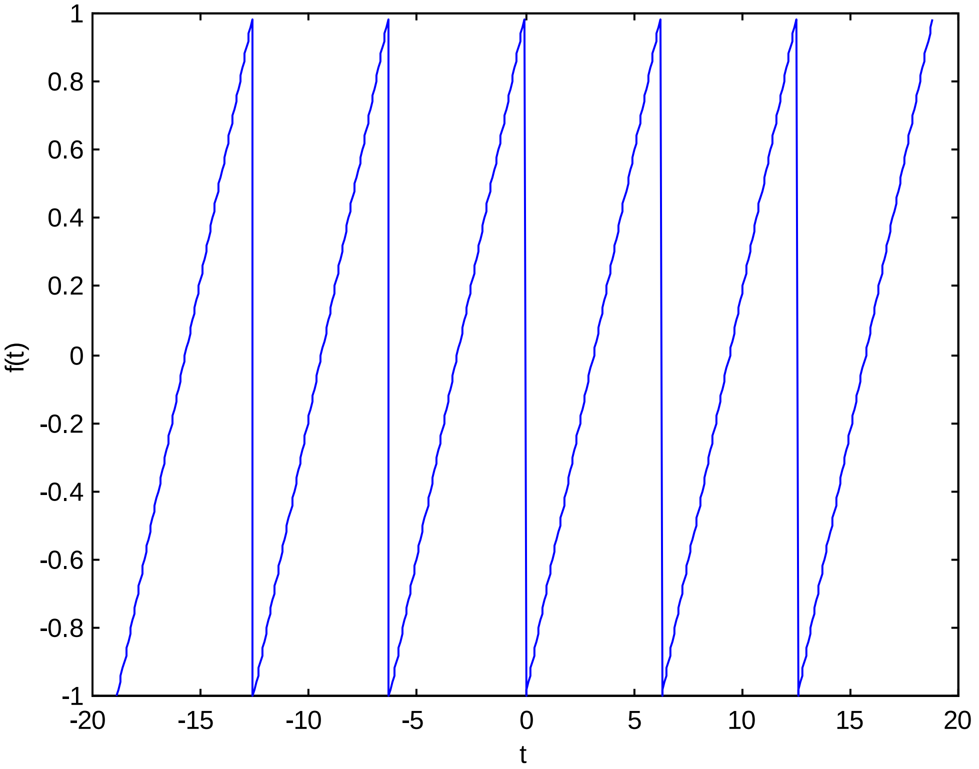
Example 3.5:



The numerical method confirms that the only frequency present in the signal is 50Hz, which makes sense as the signal is a pure 50Hz tone. Note $\frac{230\sqrt{2}}{2} = 162.6346$

Example 3.6:

$$f(t) = \frac{t}{\pi} \quad \text{if} \quad -\pi \le t \le \pi$$



Example 3.6:

$$f(t) = \frac{t}{\pi} \quad , \quad -\pi \le t < \pi$$

$$T = 2\pi \text{ sec}, \quad \omega_0 = 1 \text{ rad/sec}$$

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(t) e^{-jnt} dt = \frac{1}{2\pi}\int_0^{\pi} \frac{t}{\pi} e^{-jnt} dt + \frac{1}{2\pi}\int_{\pi}^{2\pi} f(t) e^{-jnt} dt$$

A problem arises. I do not have an explicit formula for $f(t)$ valid in the range π to 2π. I must use the fact that the signal is periodic. In integration terminology I make a substitution.

$$c_n = \frac{1}{2\pi}\int_0^{\pi} \frac{t}{\pi} e^{-jnt} dt + \frac{1}{2\pi}\int_{-\pi}^{0} f(t+2\pi) e^{-jn(t+2\pi)} dt$$

Example 3.6:

$$c_n = \frac{1}{2\pi}\int_0^{\pi} \frac{t}{\pi} e^{-jnt}\,dt + \frac{1}{2\pi}\int_{-\pi}^{0} f(t) e^{-jnt} e^{-jn2\pi}\,dt$$

$$c_n = \frac{1}{2\pi}\int_0^{\pi} \frac{t}{\pi} e^{-jnt}\,dt + \frac{1}{2\pi}\int_{-\pi}^{0} \frac{t}{\pi} e^{-jnt}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{t}{\pi} e^{-jnt}\,dt$$

Now the obvious way to approach the integral is integration by parts:

$$c_n = \frac{1}{2\pi}\left(\frac{t}{\pi}\right)\left(\frac{e^{-jnt}}{-jn}\right)\Bigg|_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-jnt}}{-jn\pi}\,dt \quad \text{if} \quad n \neq 0$$

Note it was useful to pretend to forget the division by zero above, but I do not intend to do that again.

Example 3.6:

$$c_n = \frac{1}{2\pi}\left(\frac{e^{-jn\pi}}{-jn}\right) - \frac{1}{2\pi}\left(\frac{e^{jn\pi}}{jn}\right) + \frac{1}{2jn\pi^2}\int_{-\pi}^{\pi} e^{-jnt}dt \quad \text{if} \quad n \neq 0$$

$$c_n = \frac{(-1)^n}{2\pi}\left(\frac{1}{-jn} - \frac{1}{jn}\right) + \frac{1}{2jn\pi^2}\left(\left.\frac{e^{-jnt}}{-jn}\right|_{-\pi}^{\pi}\right) \quad \text{if} \quad n \neq 0$$

$$c_n = \frac{(-1)^n}{2\pi}\left(\frac{2j}{n}\right) + \frac{1}{2jn\pi^2}\left(\frac{e^{-jn\pi} - e^{jn\pi}}{-jn}\right) = \frac{j(-1)^n}{\pi n} \quad \text{if} \quad n \neq 0$$

Example 3.6:

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{t}{\pi}dt = \left.\frac{t^2}{4\pi^2}\right|_{-\pi}^{\pi} = 0$$

$$\alpha_n = 2\,\mathrm{Re}(c_n) = 0 \quad \text{for} \quad n \ge 1$$

$$\beta_n = -2\,\mathrm{Im}(c_n) = \frac{2(-1)^{n+1}}{\pi n} \quad \text{for} \quad n \ge 1$$

$$c_0 + \sum_{n=1}^{\infty}\left(\alpha_n \cos(n\omega_0 t) + \beta_n \sin(n\omega_0 t)\right) =$$

$$\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{\pi n}\sin(nt) = \frac{2}{\pi}\sin(t) - \frac{2}{2\pi}\sin(2t) + \frac{2}{3\pi}\sin(3t) - \cdots$$

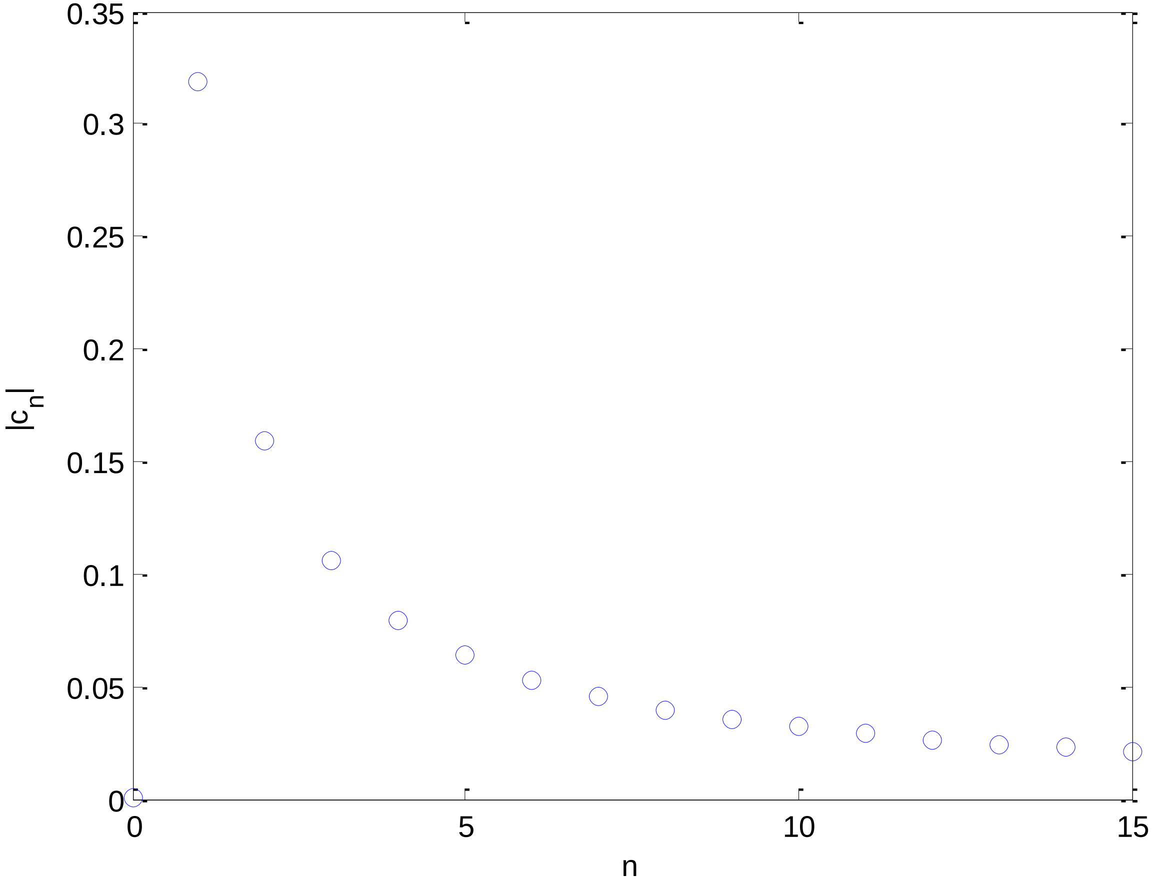
Note

The example serves to illustrate a problem which may arise in the numerical solution. Matlab requires that the sample values be presented to the **fft** command in the form of a vector. It assumes that the first sample appearing as the first element of this vector is the sample value at the time $t = 0$. Since out formula is valid over a range starting rather at the time $t = -\pi$ we have to work a little harder to ensure that Matlab's **fft** command is given the input it requires.

Code


```
>> N = 1024    set number of samples, selecting a power of 2

>> T = 2*pi    set period

>> t = [-T/2:T/N:(T/2)-(T/N)];    create vector of N sample times uniformly distributed between -π and π

>> f = t/pi;    create vector of associated samples

>> f1 = f(1:length(f)/2)    get first half of samples

>> f2 = f((length(f)/2)+1:length(f))    get second half of samples

>> fadj = [f2 f1]    adjust sample order so that first sample is for t = 0

>> FF = fft(fadj)/N;    find approximate Fourier coefficients

>> plot([0:15],abs(FF(1:16)), 'o')    plot magnitude of first sixteen coefficients vs indices
```

Example 3.6:



The numerical result is consistent with the analytical result.

Note

Now I started this section with a rather different observation, namely that if the input to a linear system is a sum of inputs then the output is the corresponding sum of outputs. We have seen that if the input is a sinusoid then the steady-state output can be relatively easily found once one knows the transfer function of the system. The opening question of this section was whether I can leverage these facts to obtain a method for finding the steady-state response of linear, time-invariant (stable) systems to inputs which are more general than sinusoids. The answer, simply, is yes I can.

Fourier Analysis

Let a real, linear, time-invariant, stable system have transfer function $H(s)$. Note that by "stable" I mean that the poles of $H(s)$ have negative real part. By "real" I mean that $H(s)$ is a real rational polynomial. Suppose the input $f(t)$ to the system is periodic of period T. Suppose moreover that this input signal satisfies some set of conditions which ensure that it's Fourier series converges to the signal itself. I might note that one particularly well-known set of conditions of this kind is due to Dirichlet. Very well:

$$f(t) = c_0 + \sum_{n=1}^{\infty} \alpha_n \cos(n\omega_0 t) + \beta_n \sin(n\omega_0 t) \quad , \quad \omega_0 = \tfrac{2\pi}{T}$$

Fourier Analysis

By linearity the response (and in particular the steady state response) is the sum of the responses to each of the terms in this summation. But each of these terms has the form:

$$A\cos\left(n\omega_0 t + \phi\right)$$

since:

$$c_0 = c_0 \cos\left(0\omega_0 t + 0\right)$$
$$\alpha_n \cos\left(n\omega_0 t\right) = \alpha_n \cos\left(n\omega_0 t + 0\right)$$
$$\beta_n \sin\left(n\omega_0 t\right) = \beta_n \cos\left(n\omega_0 t - \tfrac{\pi}{2}\right)$$

Fourier Analysis

So the steady-state response is:

$$c_0 H(j0) + \sum_{n=1}^{\infty} \alpha_n \left| H(jn\omega_0) \right| \cos\left(n\omega_0 t + Arg\left(H(jn\omega_0)\right)\right)$$

$$+ \sum_{n=1}^{\infty} \beta_n \left| H(jn\omega_0) \right| \sin\left(n\omega_0 t + Arg\left(H(jn\omega_0)\right)\right)$$

A few points need to be made to justify this assertion. For now however we see that it offers a formula for the general steady-state response of a general real, LTI, stable system to a fairly general periodic input signal. Granted you may not much like the look of the formula, at least at first, but surely it is better than having no formula.

Fourier Analysis

Now for some of those points to be made. Given the input component:

$$c_0 = c_0 \cos\left(0\omega_0 t + 0\right)$$

the associated steady-state output component will be:

$$c_0 \left|H\left(j0\right)\right| \cos\left(0\omega_0 t + Arg\left(H\left(j0\right)\right)\right) =$$

$$c_0 \left|H\left(j0\right)\right| \cos\left(Arg\left(H\left(j0\right)\right)\right)$$

But the system is real. Accordingly $H(j0) = H(0)$ is real. If it is non-negative then

$$c_0 \left|H\left(j0\right)\right| \cos\left(Arg\left(H\left(j0\right)\right)\right) = c_0 H\left(j0\right) \cos\left(0\right) = c_0 H\left(j0\right)$$

If it is negative then

$$c_0 \left|H\left(j0\right)\right| \cos\left(Arg\left(H\left(j0\right)\right)\right) = c_0\left(-H\left(j0\right)\right) \cos\left(\pi\right) = c_0 H\left(j0\right)$$

Fourier Analysis

Given the input component:

$$\beta_n \sin\left(n\omega_0 t\right) = \beta_n \cos\left(n\omega_0 t - \tfrac{\pi}{2}\right)$$

the associated steady-state output component will be:

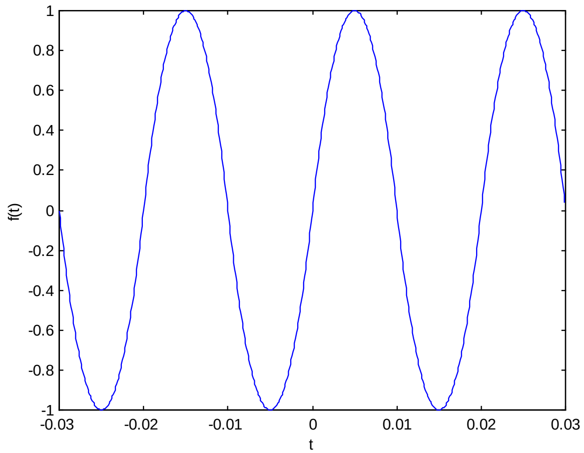
$$\beta_n \left|H\left(jn\omega_0\right)\right| \cos\left(n\omega_0 t - \tfrac{\pi}{2} + Arg\left(H\left(jn\omega_0\right)\right)\right) =$$
$$\beta_n \left|H\left(jn\omega_0\right)\right| \sin\left(n\omega_0 t + Arg\left(H\left(jn\omega_0\right)\right)\right)$$

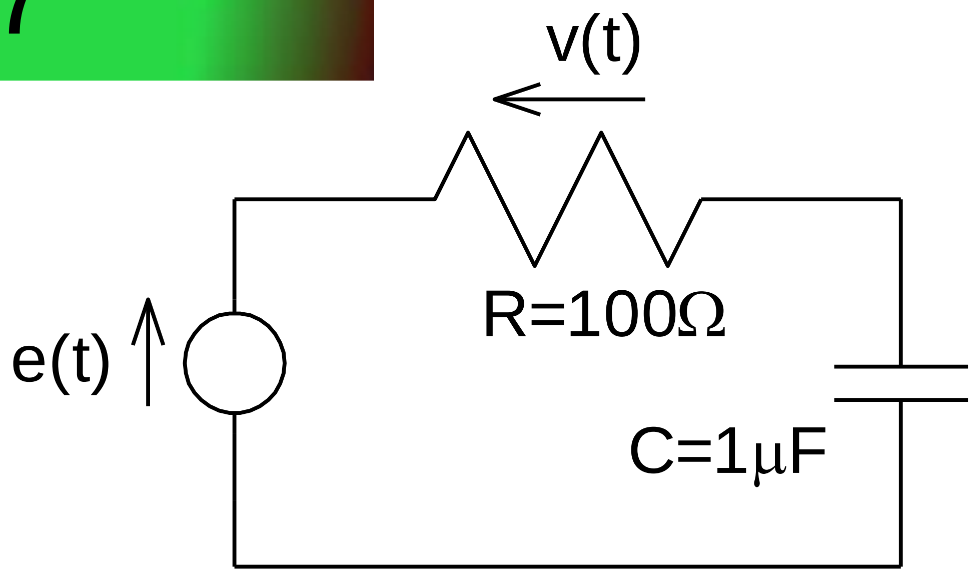
We see that the iconic cosine in implies cosine out holds *verbatim* when I replace cosine by sine.

Note

What are, in effect, special cases of Fourier analysis when applied to electrical circuits go by the name of *phasor analysis*. The analysis method is mainly due to Steinmetz. The presentation, at least in its original form, was rather graphical. One drew arrows representing complex numbers in the complex/Argand plane rather than employing the basic algebra of complex numbers. This reflected, probably, the generally lower level of mathematical education of engineers at the time. Like the mathematicians before them, engineers fought rather fiercely against having to engage with complex numbers.

Example 3.7





$$e(t)=\begin{cases}(200)^2 t(0.01-t) & \text{for } 0\le t\le 0.01\\ (200)^2 t(0.01+t) & \text{for } -0.01\le t\le 0\end{cases}$$

$$e(t)=\frac{32}{\pi^3}\sum_{\substack{n=1\\ n\ \text{odd}}}^{\infty}\frac{\sin(100n\pi t)}{n^3}$$

Note

We have here a signal which clearly is *not* a sinusoid (as we see from the formula) but looks like a sinusoid when we plot it. The reason for this is that the fundamental is

$$\frac{32}{\pi^3}\sin(100\pi t)$$

whereas the harmonics are

$$\frac{32}{27\pi^3}\sin(300\pi t)+\frac{32}{125\pi^3}\sin(500\pi t)+\cdots$$

Note

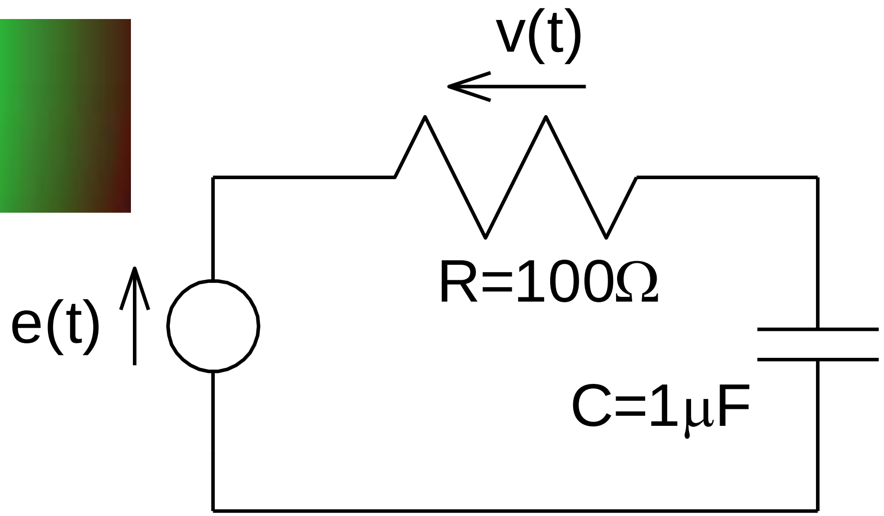
There is a measure called *total harmonic distortion* which compares essentially the "size" of all of the harmonics with that of the fundamental. Unfortunately there is an inconsistency in the definition of size used. One version takes power as the measure of size. Given this definition the total harmonic distortion of the signal above is (B_3 = 1/42 is a *Bernoulli number*):

$$\frac{\displaystyle\sum_{\substack{n=3 \\ n \text{ odd}}}^{\infty} \left(\frac{32}{\pi^3 n^3}\right)^2}{\left(\dfrac{32}{\pi^3}\right)^2} = \sum_{\substack{n=3 \\ n \text{ odd}}}^{\infty} \frac{1}{n^6} = \frac{\left(2^6 - 1\right)\pi^6 B_3}{2(6!)} - 1 = 0.0014$$

Note

Almost all of the power of the signal is in the fundamental. Less that 0.15% of it is in the harmonics. This is why the signal looks and indeed sounds like a sinusoid. Essentially we only see and hear the fundamental frequency. There is a slight discrepancy. The amplitude of the fundamental is $32/\pi^3 = 1.032$, whereas the amplitude of the signal which we see is 1.

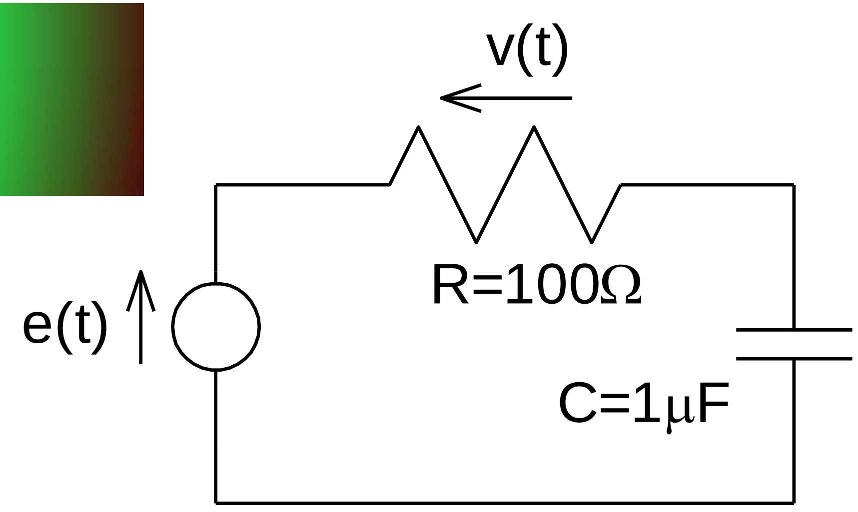
Example 3.7



$$e(t) = v(t) + \frac{1}{RC}\int v(t)dt\,, \qquad E(j\omega) = V(j\omega) + \frac{1}{RC}\frac{V(j\omega)}{j\omega}$$

$$H(j\omega) = \frac{j\omega 10^{-4}}{1 + j\omega 10^{-4}}$$

Example 3.7



$$v(t) = \frac{32}{\pi^3} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{(n\pi/100)}{\sqrt{1+(n\pi/100)^2}} \frac{\sin(100n\pi t + \theta_n)}{n^3}$$

$$\theta_n = Arg(H(j100n\pi)) = \frac{\pi}{2} - \tan^{-1}\left(\frac{n\pi}{100}\right)$$

Code

>> Np = [1e-4 0], Dp = [1e-4 1], Gp = tf(Np,Dp) *create system variable representing the system. Note use of scientific e notation.*

>> T = 1/50, N = 1024 *period of input signal, number of samples per half period*

>> t1 = [0:T/(2*N);(T/2) – (T/(2*N))]; *create vector of N times uniformly distributed between 0 and T/2.*

>> f1 = (200^2)*t1.*((T/2) – t1) ; *vector of associated samples*

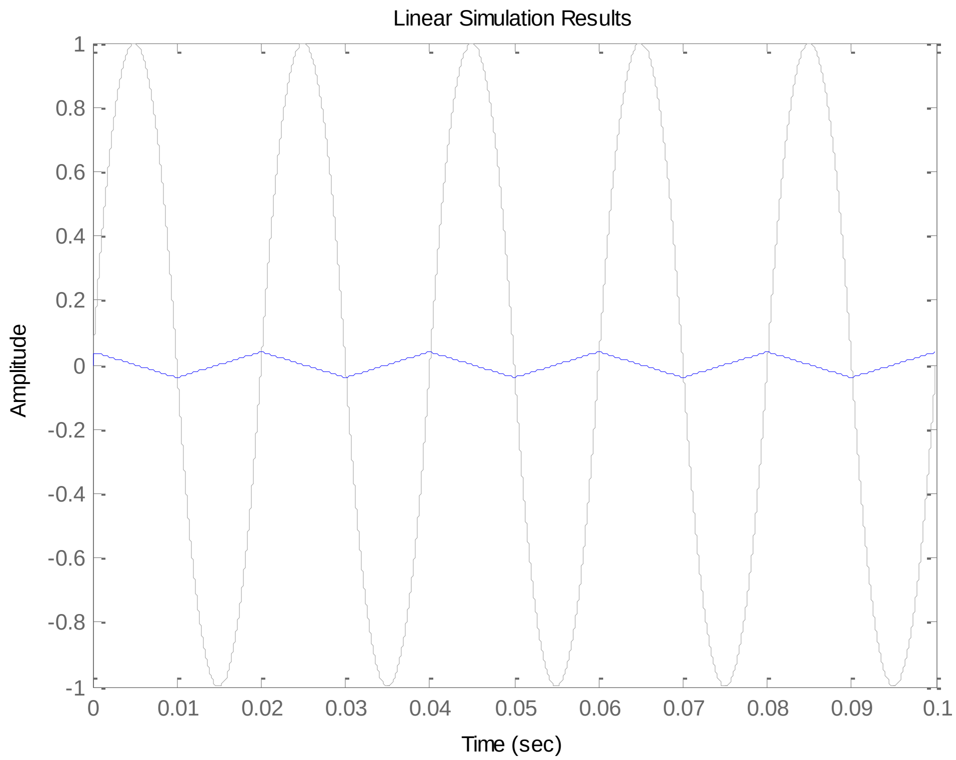
>> t2 =(T/2) + t1; *create vector of N times uniformly distributed between T/2 and T*

>> f2 = (200^2)*(t2 – T).*((T/2) + (t2 – T)) ; *vector of associated samples. Note subtraction of one period to bring arguments into valid range of given formula.*

Code

```
>>  t = [t1 t2 t1+T t2+T t1+(2*T) t2+(2*T) t1+(3*T) t2+(3*T)
t1+(4*T) t2+(4*T)];    times for first five cycles of input signal.

>>  f = [f1 f2 f1 f2 f1 f2 f1 f2 f1 f2];  associated samples. There are
much better ways of doing this periodic signal extension but this will
do.

>>  lsim(Gp,f,t);        find the response of the system to first five cycles
of the input signal.
```

Note



Although the input looks sinusoidal the output certainly does not. In fact it looks like a triangular wave.

Code

```
>> Np = [1e-4 0], Dp = [1e-4 1], Gp = tf(Np,Dp)     create system variable representing the system

>> nvec = [1:2:11] ;  create vector of first odd integers

>> Gpvec = polyval(Np,i*100*pi*nvec)./polyval(Dp,i*100*pi*nvec);
        create vector of values of Gp(jnω0) for values of n in nvec

>> alpha = (32/pi^3)./(nvec.^3);       create vector of coefficients of Fourier series of input signal

>> angle(Gpvec)    find output phases for first terms

>> alphaout = abs(Gpvec).*alpha  find output amplitudes for first few terms
```

Note

We have here a signal which does not look like a sinusoid when we plot it. The reason for this is that the fundamental is

$$\frac{32}{\pi^3}(0.0314)\sin(100\pi t + 1.5394)$$

whereas the harmonics are

$$\frac{32}{\pi^3}(0.0035)\sin(300\pi t + 1.4768) + \frac{32}{\pi^3}(0.0012)\sin(500\pi t + 1.415) + \cdots$$

Note

The total harmonic distortion is

$$\frac{0.0035^2 + 0.0012^2 + 0.0006^2 + 0.0004^2 + \cdots}{0.0314^2}$$

which is slightly more than 0.0144. This means that a little over 1.4% of the power of the signal is in the harmonics. The total harmonic distortion is much higher so we can see it and hear it.