1. Permittivity and permeability

a) Seeing that this is an RC circuit, the time constant, τ, is equal to RC. We know that $\tau_{slab} = 1.5\tau_{air}$ and and have some slab of length l and area A. Then, we can find the time constant.

$$\begin{aligned}\tau &= RC \\ &= \rho\frac{l}{A}\epsilon\frac{A}{l} \\ &= \rho\epsilon\end{aligned}$$

Now we can compare τ with τ_0 to solve for ϵ. It is assumed that the slab has the same resistivity, ρ, as air.

$$\begin{aligned}\tau_{slab} &= 1.5\tau_{air} \\ \rho_{slab}\epsilon_{slab} &= 1.25\rho_{air}\epsilon_0 \\ \epsilon_{slab} &= 1.5\epsilon_0\end{aligned}$$

b) If instead of a slab length l the slab is now one quarter of the the length, we can still use an RC circuit but now have a capacitance of air in series with the capacitance of the slab. We know that $C_{air} = \epsilon_0\frac{A}{l}$, and therefore $C_{1/4\,air} = \frac{4}{3}\epsilon_0\frac{A}{l}$ and $C_{3/4\,slab} = 4\epsilon_r\epsilon_0\frac{A}{l}$. Let us use the formula for capacitance in series to solve this:

$$\begin{aligned}\frac{1}{C_{eq}} &= \frac{1}{C_{1/4\,air}} + \frac{1}{C_{3/4\,slab}} \\ &= \frac{1}{4C_{air}} + \frac{3}{4\epsilon_r C_{air}} \\ C_{eq} &= \frac{4\epsilon_r C_{air}}{3+\epsilon_r}\end{aligned}$$

Once again we can compare τ_{eq} with τ_{air} to solve for ϵ_r. It is assumed that the slab has the same resistance, R, as air.

$$\begin{aligned}\tau_{eq} &= 1.5\tau_{air} \\ R_{eq}C_{eq} &= 1.5R_{air}C_{air} \\ \frac{4\epsilon_r C_{air}}{3+\epsilon_r} &= 1.5C_{air} \\ \epsilon_r &= \frac{9}{5}\end{aligned}$$

Thus $\epsilon_{slab} = \epsilon_r\epsilon_0 = \frac{9}{5}\epsilon_0$

c) Seeing that this is an LR circuit, $\tau = L/R$. We know that $\tau_{rod} = 0.9995\tau_{air}$ and have some slab of length l and area A which is inside a solenoid with parameters K and N. Knowing the inductance of a cylindrical solenoid:

$$L = \frac{\mu K N^2 A}{l} \quad \rightarrow \quad \tau = \frac{\mu K N^2 A}{Rl}$$

Once again we can compare τ_{eq} with τ_{air} to solve for μ. It is assumed that the slab has the same resistance, R, as air.

$$\begin{aligned}\tau_{rod} &= 0.9995\tau_{air} \\ \frac{\mu_{rod}KN^2A}{R_{rod}l} &= 0.9995\frac{\mu_0 KN^2A}{R_{air}l} \\ \mu &= 0.9995\mu_0\end{aligned}$$

Since $\mu_{rod} < \mu_0$, this slab is dimagnetic.

2. Continuity equation.

   a) Taking the divergence of $\mathbf{J} = \left(2y^2z^2\,\hat{x} + 3z\,\hat{y} + 4z(x-x_o)^2\,\hat{z}\right)$ A/m^2 we get

$$\nabla \cdot \mathbf{J} = \frac{\partial}{\partial x}\left(2y^2z^2\right) + \frac{\partial}{\partial y}\left(3z\right) + \frac{\partial}{\partial z}\left(4z(x-x_o)^2\right) = 4(x-x_o)^2.$$

   Since $\nabla \cdot \mathbf{J}$ is time independent, we have that

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J} \quad \rightarrow \quad \rho(\mathbf{r}, t) = -\left(4(x-x_o)^2\right)t + \rho_o\,\frac{\mathrm{C}}{\mathrm{m}^3}.$$

   Evaluating $\rho(\mathbf{r}, t)$ at $\mathbf{r} = \vec{0}$ and given that $\rho_o = 0$ and $x_o = 3$, we find

$$\rho(\vec{0}, t) = -36t\,\frac{\mathrm{C}}{\mathrm{m}^3}.$$

   b) Since the units of J_x, J_y, and J_z are A/m^2, we get

$$\begin{aligned}
[J_x] = \left[2y^2z^2\right] &= \frac{\mathrm{A}}{\mathrm{m}^2} \quad \rightarrow \quad [2] = \frac{\mathrm{A}}{\mathrm{m}^6}, \\
[J_y] = [3z] &= \frac{\mathrm{A}}{\mathrm{m}^2} \quad \rightarrow \quad [3] = \frac{\mathrm{A}}{\mathrm{m}^3}, \\
[J_z] = \left[4z(x-x_o)^2\right] &= \frac{\mathrm{A}}{\mathrm{m}^2} \quad \rightarrow \quad [4] = \frac{\mathrm{A}}{\mathrm{m}^5}.
\end{aligned}$$

3. Charge density in a conductor.

   a) In a homogeneous conductor where $\mathbf{J} = \sigma\mathbf{E}$, Gauss's law $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_o}$ implies

$$\nabla \cdot \mathbf{J} = \frac{\sigma}{\epsilon_o}\rho.$$

   Then, using the continuity equation $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$, we can show that

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon_o}\rho = 0.$$

   b) This first order differential equation can be rewritten as follows,

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} = -\frac{\sigma}{\epsilon_o} \quad \rightarrow \quad \frac{\partial \ln \rho}{\partial t} = -\frac{\sigma}{\epsilon_o}.$$

   Integrating over time from 0 to t, we get

$$\int_0^t \frac{\partial \ln \rho}{\partial t}dt = -\int_0^t \frac{\sigma}{\epsilon_o}dt \quad \rightarrow \quad \ln \rho - \ln \rho_o = -\frac{\sigma}{\epsilon_o}t,$$

   and thus, the solution for the charge density is

$$\rho = \rho_o e^{-\frac{\sigma}{\epsilon_o}t},$$

   where ρ_o is the density distribution at time $t = 0$. Making $\rho_o = \sin(40z)$ C/m^3, we find that

$$\rho = \sin(40z)\,e^{-\frac{\sigma}{\epsilon_o}t}\,\frac{\mathrm{C}}{\mathrm{m}^3} \qquad \text{for } t \geq 0.$$

c) The time it takes for ρ to reduce to $0.01\sin(40z)\,\mathrm{C/m^3}$ (assuming $\sigma = 10^7\,\mathrm{S/m}$) is calculated as follows,

$$e^{-\frac{\sigma}{\epsilon_o}t} = 0.01 \quad \rightarrow \quad t = -\frac{\epsilon_o}{\sigma}\ln 0.01 = -\frac{8.8542\times 10^{-12}}{10^7}(-4.6052) = 4.08\times 10^{-18}\,\mathrm{s}.$$

d)

i. At $t = 0$ and from Gauss's law, $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_o}$, we know that there is a non-zero electric field $\mathbf{E}$ associated to the non-zero charge density distribution $\rho(\mathbf{r}, t)$. From the lecture notes, chapter 10, we know that the stored electrostatic energy per unit volume has a non-zero value: $w = \frac{1}{2}\epsilon_o \mathbf{E}\cdot\mathbf{E}$.

ii. As $t \to \infty$ without any external source, the charge density $\rho \to 0$, and so does the electric field $\mathbf{E} \to 0$. That means that the electrostatic energy per unit volume is 0.

The stored energy at $t = 0$ can be seen as the stored energy in a capacitor C. The conductor has a finite conductivity σ and will have a resistance $R \propto 1/\sigma$. From ECE210 we know that the energy stored in a capacitor in an RC circuit will completely dissipate through the resistor R in the absence of any other energy source.

4. E and H Fields

a) An electric field given by

$$\mathbf{E} = \sin(\omega t - \beta y)\hat{z}\,\frac{\mathrm{V}}{\mathrm{m}}$$

is propagating at a velocity $v = \frac{\omega}{\beta} = c$. The medium has $\mu = \mu_r\mu_o = \mu_o$, which implies $\mu_r = 1$ and thus using $\frac{1}{\sqrt{\epsilon_r\mu_r}} = 1$, we get $\epsilon_r = 1$. Using Faraday's law $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$, first we get

$$\nabla\times\mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \sin(\omega t - \beta y) \end{vmatrix} = \frac{\partial}{\partial y}\left(\sin(\omega t - \beta y)\right)\hat{x} = -\beta\cos(\omega t - \beta y)\hat{x}.$$

Now equating the above result to $-\frac{\partial\mathbf{B}}{\partial t}$, we get

$$-\beta\cos(\omega t - \beta y)\hat{x} = -\frac{\partial\mathbf{B}}{\partial t}.$$

Integrating both sides of the above equation and dividing by $\mu = \mu_0 = 1.2566\times 10^{-6}$ will give us

$$\begin{aligned}\mathbf{H} &= \tfrac{\mathbf{B}}{\mu} = \frac{\beta}{\omega\mu_o}\sin(\omega t - \beta y)\hat{x} = \frac{1}{c\mu_o}\sin(\omega t - \beta y)\hat{x} \\ &= 2.65\times 10^{-3}\sin(\omega t - \beta y)\hat{x}\,\frac{\mathrm{A}}{\mathrm{m}}.\end{aligned}$$

b) A magnetic field given by

$$\mathbf{H} = \cos(\omega t + \beta x)\hat{z}\,\frac{\mathrm{A}}{\mathrm{m}}$$

is propagating at a velocity $v = \frac{\omega}{\beta} = \frac{2}{3}c$. The medium is homogeneous with $\epsilon = \epsilon_r\epsilon_o$ and $\mu = \mu_r\mu_o$, and thus using $\frac{1}{\sqrt{\epsilon_r\mu_r}} = \frac{2}{3}$, we get $\mu_r = \frac{9}{4}\times\frac{1}{\epsilon_r} = \frac{9}{4}\times\frac{1}{2.25} = 1$ ($\therefore\ \mu = \mu_0$). Using Ampere's law $\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}$ (where $\mathbf{J} = \sigma\mathbf{E} = \vec{0}$ as $\sigma = 0$), first we get

$$\nabla\times\mathbf{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \cos(\omega t + \beta x) \end{vmatrix} = -\frac{\partial}{\partial x}\left(\cos(\omega t + \beta x)\right)\hat{y} = \beta\sin(\omega t + \beta x)\hat{y}.$$

Now equating the above result to $\frac{\partial \mathbf{D}}{\partial t}$, we get

$$\beta \sin(\omega t + \beta x)\hat{y} = \frac{\partial \mathbf{D}}{\partial t}.$$

Integrating both sides of the above equation and dividing by ϵ will give us

$$\begin{aligned}
\mathbf{E} &= -\frac{\beta}{\omega\epsilon}\cos(\omega t + \beta x)\hat{y} = -\frac{3}{2c \times 2.25\epsilon_0}\cos(\omega t + \beta x)\hat{y} \\
&= -251.15\,\cos(\omega t + \beta x)\hat{y}\,\frac{\mathrm{V}}{\mathrm{m}}.
\end{aligned}$$

5. Verifying vector identity

$$\mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H} = \nabla \cdot (\mathbf{E} \times \mathbf{H})$$

for $\mathbf{E} = 4e^{-\alpha z}\hat{y}$ and $\mathbf{H} = 2e^{-\alpha z}\hat{x}$. Solving the left-hand side of the identity gives

$$\begin{aligned}
\mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H} &= (2e^{-\alpha z}\hat{x}) \cdot \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & 4e^{-\alpha z} & 0 \end{vmatrix} - (4e^{-\alpha z}\hat{y}) \cdot \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \frac{\partial}{\partial z} \\ 2e^{-\alpha z} & 0 & 0 \end{vmatrix} \\
&= (2e^{-\alpha z}\hat{x}) \cdot (4\alpha e^{-\alpha z}\hat{x}) - (4e^{-\alpha z}\hat{y}) \cdot (-2\alpha e^{-\alpha z}\hat{y}) \\
&= 8\alpha e^{-2\alpha z} + 8\alpha e^{-2\alpha z} = 16\alpha e^{-2\alpha z}
\end{aligned}$$

and the right-hand side gives

$$\begin{aligned}
\nabla \cdot (\mathbf{E} \times \mathbf{H}) &= \nabla \cdot \left(4e^{-\alpha z}\hat{y} \times 2e^{-\alpha z}\hat{x}\right) \\
&= \nabla \cdot \left(-8e^{-2\alpha z}\hat{z}\right) \\
&= \frac{\partial}{\partial z}\left(-8e^{-2\alpha z}\right) = 16\alpha e^{-2\alpha z},
\end{aligned}$$

hence, the identity is verified.