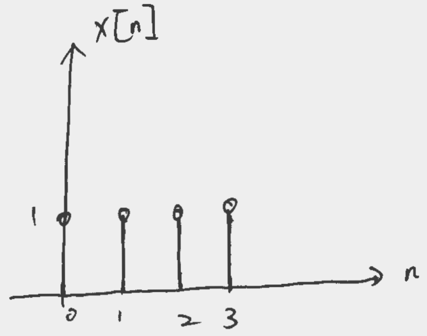
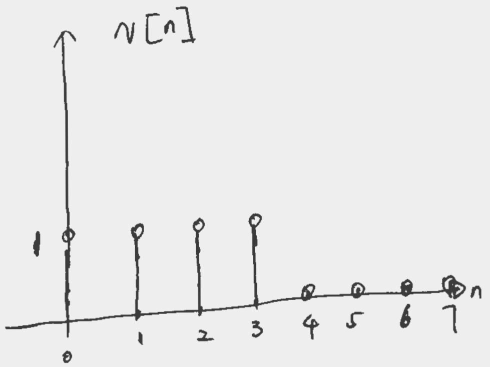
ECE 310 HW3 Soln

1. $v[n] = \begin{cases} x[n] & 0 \le n \le 3 \\ 0 & 4 \le n \le 7 \end{cases}$ $\qquad w[n] = \begin{cases} 0 & 0 \le n \le 3 \\ x[n-4] & 4 \le n \le 7 \end{cases}$

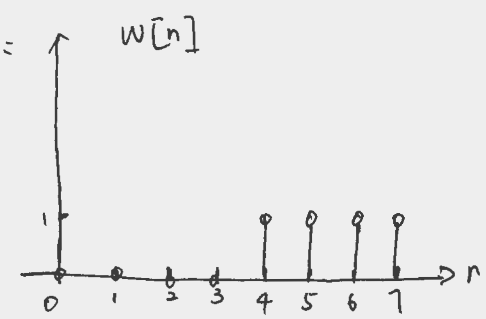
a) $x[n] =$



$v[n]$



$w[n] =$



b) $Y_d(\omega) = \sum_{n=-\infty}^{\infty} y[n] e^{-i\omega n}$

$= \sum_{n=0}^{3} x[n] e^{-i\omega n}$

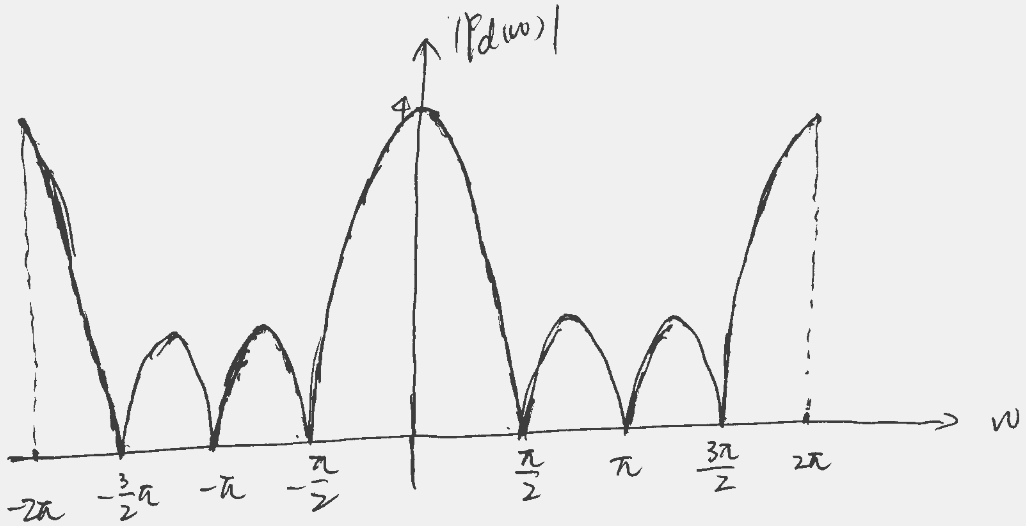
$= 1 + e^{-i\omega} + e^{-i2\omega} + e^{-i3\omega}$

$= \dfrac{1 - e^{-i4\omega}}{1 - e^{-i\omega}}$

$= \dfrac{e^{-i2\omega} \cdot (e^{i2\omega} - e^{-i2\omega})}{e^{-i\frac{\omega}{2}} (e^{i\frac{\omega}{2}} - e^{-i\frac{\omega}{2}})}$

$= e^{-i \cdot \frac{3}{2}\omega} \cdot \dfrac{\sin(2\omega)}{\sin(\frac{\omega}{2})}$

$\therefore |Y_d(\omega)| = \left| \dfrac{\sin(2\omega)}{\sin(\frac{\omega}{2})} \right|$



c) $$X[k] = \sum_{n=0}^{N-1} x[n] e^{-i\frac{2\pi kn}{N}}$$

$$= \sum_{n=0}^{3} 1 \cdot e^{\frac{-i2\pi kn}{4}}$$

$$= \frac{1 - e^{\frac{-i2\pi k}{4} \cdot 4}}{1 - e^{-i\frac{2\pi k}{4}}}$$

$$= \frac{1 - e^{-i2\pi k}}{1 - e^{-i\frac{\pi k}{2}}}$$

Shown is the derivation using the definition of the DFT. The more efficient way is to use directly the sampling property of the DFT, shown at the bottom of the page. (Full credit for either way).

$$V[k] = \sum_{n=0}^{N-1} v[n] e^{-\frac{i2\pi kn}{N}}$$

$$= \sum_{n=0}^{3} 1 \cdot e^{-\frac{i2\pi kn}{8}}$$

$$= \frac{1 - e^{\left(-\frac{i2\pi k}{8}\right) \cdot 4}}{1 - e^{-\frac{i2\pi k}{8}}}$$

$$= \frac{1 - e^{-i\pi k}}{1 - e^{-\frac{i\pi k}{4}}}$$

Same idea as for X[k] above.
Then the result is obtained in one line.

$$w[k] = \sum_{n=0}^{N-1} w[n] e^{-\frac{i2\pi kn}{N}}$$

$$= \sum_{n=4}^{7} e^{-\frac{i2\pi kn}{8}}$$

$$= e^{-i\pi k} \cdot \frac{1 - e^{-i\pi k}}{1 - e^{\frac{-i\pi k}{4}}}$$

The simpler approach is to apply the DFT shift property to v[n].

$$X[k] = \Gamma_d(\omega)\Big|_{\omega = \frac{2\pi k}{4}}$$

$$V[k] = \Gamma_d(\omega)\Big|_{\omega = \frac{2\pi k}{8}}$$

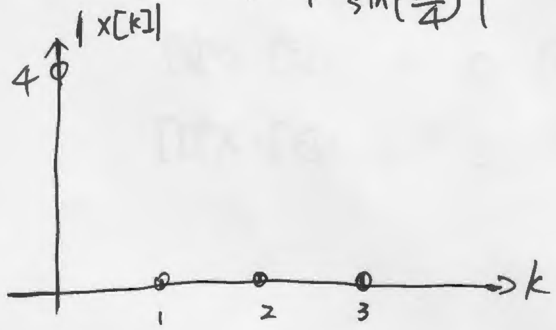
$$W[k] = V[k] \cdot e^{-i\pi k}$$

d) The magnitude of $Y_d(\omega)$ already drawn in (b)

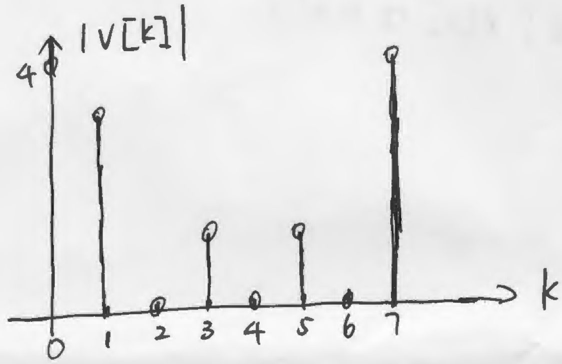
$$X[k] = \frac{1-e^{-i2\pi k}}{1-e^{-i\frac{\pi k}{2}}} = e^{-i\frac{3}{2}\cdot\frac{\pi}{2}k} \cdot \frac{\sin(2\cdot\frac{\pi k}{2})}{\sin(\frac{1}{2}\cdot\frac{\pi k}{2})}$$

$$\therefore \left| X[k] \right| = \left| \frac{\sin(\pi k)}{\sin(\frac{\pi k}{4})} \right|$$

Full credit if you just use
the sampling property as noted below

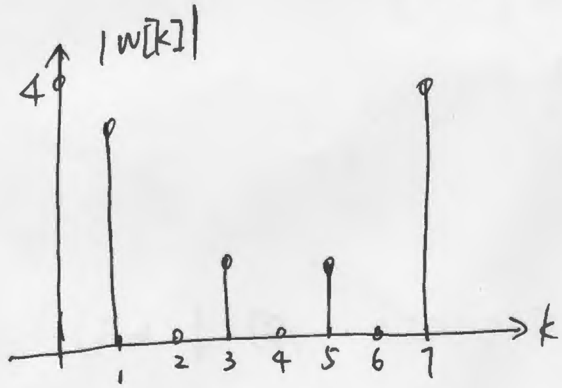


$|X[k]|$ is $|Y_d(\omega)|$ sampled with $\omega = \frac{\pi k}{2}$
with $0 \leq \omega \leq \frac{3}{2}\pi$



$|V[k]|$ is $|Y_d(\omega)|$ sampled with $\omega = \frac{\pi k}{4}$

for k=0,1, ...7



|W[k]| = |V(k]| because W[k] and V[k] only differ in phase.

2. a) Evaluate through conjugate symmetry of DFT (2.5.1.6)

$x[0] = 1 \qquad X^*[\langle\langle 4-0 \rangle\rangle_4] = X^*[0] = 1 \Rightarrow X[0] = X^*[0]$

$x[1] = e^{-j3\pi/4} \qquad X^*[\langle\langle 4-1 \rangle\rangle_4] = X^*[3] = e^{-j3\pi/4} \Rightarrow X[1] = X^*[3]$

$x[2] = 0 \qquad X^*[\langle\langle 4-2 \rangle\rangle_4] = X^*[2] = 0 \Rightarrow X[2] = X^*[2]$

$x[3] = e^{j3\pi/4} \qquad X^*[\langle\langle 4-3 \rangle\rangle_4] = X^*[1] = e^{j3\pi/4} \Rightarrow X[3] = X^*[1]$

$\therefore X[k] = X^*[\langle\langle N-k \rangle\rangle_N]$

$\therefore$ By 2.5.1.6, inverse DFT of $X[k]$ is real ($x[n]$ is real)

b) $$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot e^{\frac{j2\pi kn}{N}}$$

$$= \frac{1}{4} \sum_{k=0}^{3} X[k] \cdot e^{\frac{2\pi kn}{4}}$$

$$= \frac{1}{4} \left(1 + e^{-j3\pi/4} \cdot e^{\frac{j\pi n}{2}} + 0 \cdot e^{j\pi n} + e^{j3\pi/4} \cdot e^{j3\pi n/2}\right)$$

$$= \frac{1}{4}\left[1 + e^{j(\pi n/2 - 3\pi/4)} + e^{-j(\pi n/2 - 3\pi/4)}\right]$$

$$= \frac{1}{4}\left[1 + 2\cos(\pi n/2 - 3\pi/4)\right]$$

$x[0] = \frac{1}{4}(1-\sqrt{2}) \qquad x[1] = \frac{1}{4}(1+\sqrt{2}) \qquad x[2] = \frac{1}{4}(1+\sqrt{2}) \qquad x[3] = \frac{1}{4}(1-\sqrt{2})$

c) $Y[k] = X[k]\, e^{j2\pi k/4} \Rightarrow y[n] = x[\langle n+1 \rangle_4]$

$y[0] = x[1] = \frac{1}{4}(1+\sqrt{2}) \qquad y[1] = x[2] = \frac{1}{4}(1+\sqrt{2})$

$y[2] = x[3] = \frac{1}{4}(1-\sqrt{2}) \qquad y[3] = x[0] = \frac{1}{4}(1-\sqrt{2}$

3. $y[n] = x[n]\sin\left(\frac{2\pi J n}{N}\right) = x[n]\cdot\left(\frac{1}{2j}e^{j2\pi Jn/N} - \frac{1}{2j}e^{-j2\pi Jn/N}\right)$

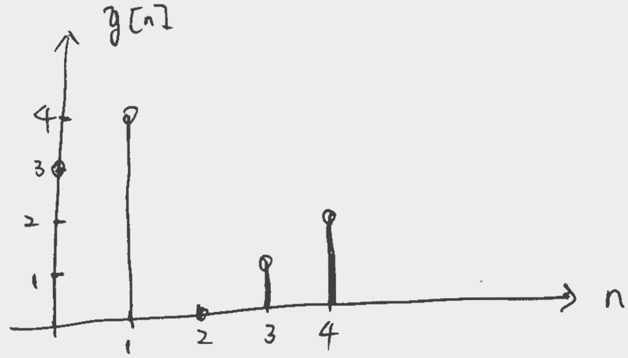
using circular frequency shift ~~pro~~ property

$Y[k] = \frac{1}{2j}X[\langle k-J\rangle_N] - \frac{1}{2j}X[\langle k+J\rangle_N]$

4. a) using the circular time shift property

$Y[k] = e^{-j(2\pi/5)\cdot 2k}X[k] \Rightarrow y[n] = x[\langle n-2\rangle_5]$

$y[n] = \{3, 4, 0, 1, 2\}$



b) $W[k] = \mathrm{Im}\{X[k]\} = \frac{1}{2}(X[k] - X^*[k])$

since $x[n]$ is real

$x[n] = x^*[n] \Rightarrow X[k] = X^*[\langle 5-k\rangle_5]$

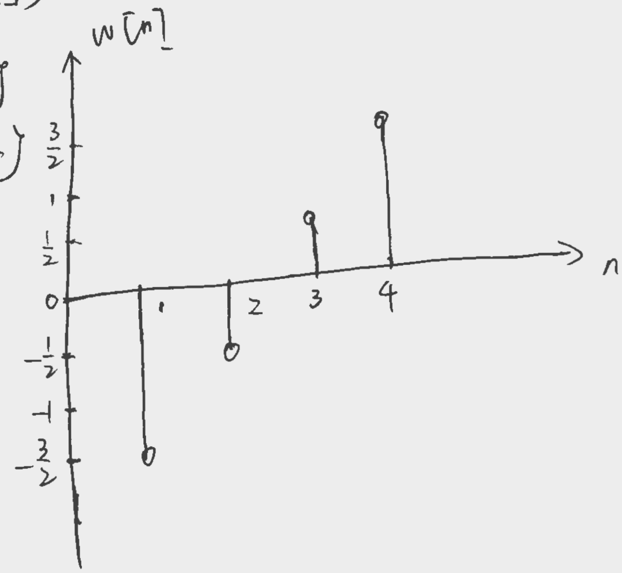
using time reversal property (2.5.1.5)

$x[n] \xleftrightarrow{DFT} X[k] \Rightarrow x[\langle -n\rangle_5] \xleftrightarrow{DFT} X[\langle 5-k\rangle_5]$

$\therefore w[n] = \frac{1}{2}(x[n] - x[\langle -n\rangle_5])$

$x[\langle -n\rangle_5] = \{0, 4, 3, 2, 1\}$

$\therefore w[n] = \{0, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$



5. $x[n] = A\cos(w_0 n) \Rightarrow v[n] = A\cos(w_0 n)\cdot(u[n]-u[n-28])$

$$V_d(w) = \sum_{n=0}^{27} \cos(0.25\pi)e^{-jwn}$$

$$= e^{-j\frac{27}{2}(w-\frac{1}{4}\pi)}\cdot\frac{\sin[14(w-\frac{1}{4}\pi)]}{2\sin[(w-\frac{1}{4}\pi)/2]} + e^{-j\frac{27}{2}(w+\frac{1}{4}\pi)}\cdot\frac{\sin[14(w+\frac{1}{4}\pi)]}{2\sin[(w+\frac{1}{4}\pi)/2]}$$

Approximation: We only consider $|V_d(w)|$ and approximate $|V_d(w)|$ as the sum of the magnitudes of the two periodic sinc function above

$$|V_d(w)| \approx \left|\frac{\sin[14(w-\frac{1}{4}\pi)]}{2\sin[(w-\frac{1}{4}\pi)/2]}\right| + \left|\frac{\sin[14(w+\frac{1}{4}\pi)]}{2\sin[(w+\frac{1}{4}\pi)/2]}\right|$$

The peaks are at $w = \pm w_0 = \{\frac{\pi}{4}, -\frac{\pi}{4}\}$ and $|V_d(\pm w_0)| \approx 14$

since $|\sin(\pi k)| = 0$ the nulls are: $w = \{\frac{1}{4}\pi + \frac{2\pi k}{28}, -\frac{1}{4}\pi + \frac{2\pi k}{28}\}$, for integer k

since $|\sin(\frac{\pi k}{2})| = 1$ for k odd, the largest sidelobe peaks:

$w = \{\frac{1}{4}\pi + \frac{3\pi}{28}, \frac{1}{4}\pi - \frac{3\pi}{28}, -\frac{1}{4}\pi + \frac{3\pi}{28}, -\frac{1}{4}\pi - \frac{3\pi}{28}\}$, with the magnitude $\frac{1}{|2\sin(\frac{3\pi}{56})|} \approx 2.985$