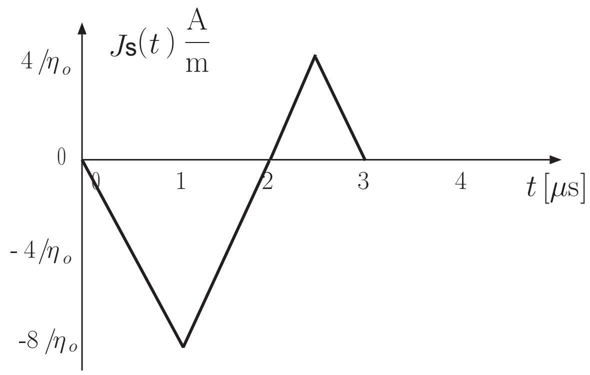
**ECE 329-Spring 2013** **Homework 8 — Solution** **Due: Tue, Mar. 12, 2013, 5PM**

1. A $z$-polarized plane TEM wave propagates in vacuum in $-\hat{y}$ direction. If the wave field varies at $y=0$ according to $E_z(0,t) = 4\Delta\left(\frac{t-1}{\tau}\right) - 2\Delta\left(\frac{t-2.5}{\tau/2}\right)$ V/m, where $\Delta\left(\frac{t}{\tau}\right)$ is the unit triangle function with width $\tau = 2\,\mu$s, we can find that

   a) The surface current density can be expressed as:

$$\mathbf{J}_s(t) = -\frac{2}{\eta_o}E_z(t)\,\hat{z} = -\frac{2}{\eta_o}[4\Delta\left(\frac{t-1\,\mu s}{2\,\mu s}\right) - 2\Delta\left(\frac{t-2.5\,\mu s}{1\,\mu s}\right)]\,\hat{z}\,\frac{\mathrm{A}}{\mathrm{m}}$$

   where $\eta_o \approx 120\pi\,\Omega$ is the intrinsic impedance of vacuum. And the plot is:




   b) The vector wave field $\vec{E}(y,t)$ is given by

$$\vec{E}(y,t) = [4\Delta(\frac{t-1+y/c}{\tau}) - 2\Delta\left(\frac{t-2.5+y/c}{\tau/2}\right)]\,\hat{z}\,\frac{\mathrm{V}}{\mathrm{m}},$$

   where $c \approx 3\times10^8\,$m/s is the wave propagation velocity in vacuum.

   c) The associated wave field $\vec{H}(y,t)$ is

$$\vec{H}(y,t) = [\frac{4}{\eta_o}\Delta(\frac{t-1+y/c}{\tau}) - \frac{2}{\eta_o}\Delta\left(\frac{t-2.5+y/c}{\tau/2}\right)]\,(-\hat{x})\,\frac{\mathrm{A}}{\mathrm{m}},$$

   d) The Poynting vector is

$$\vec{S} = \vec{E}\times\vec{H} = -\frac{16}{\eta_o}\Delta^2(\frac{t-1+y/c}{\tau}) - \frac{4}{\eta_o}\Delta^2(\frac{t-2.5+y/c}{\tau/2})\,\hat{y}\,\frac{\mathrm{W}}{\mathrm{m}^2},$$

   The poynting vector includes 2 triangle function, so there are 2 peaks, and 1 peak will arrive a point later than the other.
   So its maximum value is

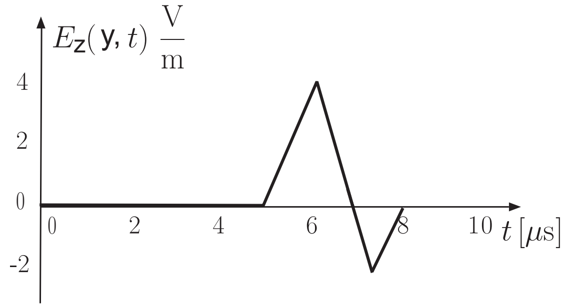
$$\max\left(\left|\vec{E}\times\vec{H}\right|\right) = \frac{16}{\eta_o} \approx \frac{2}{15\pi}\frac{\mathrm{W}}{\mathrm{m}^2}.$$

   e) The locations of the peak of $\vec{E}\times\vec{H}$ evolves according to

$$\frac{t-1+y/c}{\tau} = 0 \quad\rightarrow\quad y = (1-t)c.$$

   f) Plotting $E_z$ vs $t$ at $y = -1500\,$m.

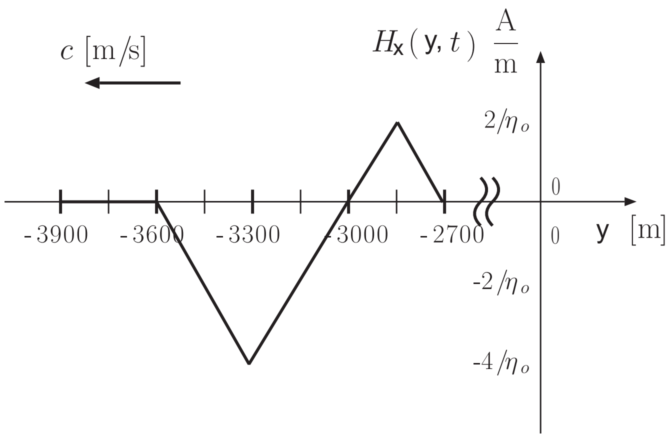
$$\begin{aligned} E_z(y,t)|_{y=-1500} &= 4\triangle\left(\frac{t-1\,\mu s-\frac{15\times10^2}{3\times10^8}\,\mathrm{s}}{2\,\mu s}\right) - 2\triangle\left(\frac{t-2.5\,\mu s-\frac{15\times10^2}{3\times10^8}\,\mathrm{s}}{1\,\mu s}\right) \\ &= 4\triangle\left(\frac{t-6\,\mu s}{2\,\mu s}\right) - 2\triangle\left(\frac{t-7.5\,\mu s}{1\,\mu s}\right)\frac{\mathrm{V}}{\mathrm{m}} \end{aligned}$$




g) Plotting $H_x$ vs $y$ at $t = 12\,\mu$s.

$$H_x(y,t)|_{t=12\,\mu\mathrm{s}} = -\frac{4}{\eta_o}\Delta(\frac{12-1\,\mu\mathrm{s}+y\cdot\frac{1}{3}\times10^{-8}}{2\,\mu\mathrm{s}}) + \frac{2}{\eta_o}\Delta(\frac{12-2.5\,\mu\mathrm{s}+y\cdot\frac{1}{3}\times10^{-8}}{1\,\mu\mathrm{s}})$$

$$= -\frac{4}{\eta_o}\Delta(\frac{3300\,\mathrm{m}+y}{600\,\mathrm{m}}) + \frac{2}{\eta_o}\Delta(\frac{2850\,\mathrm{m}+y}{300\,\mathrm{m}})\,\frac{\mathrm{A}}{\mathrm{m}}$$




2. In a homogeneous lossless dielectric with $\epsilon = \epsilon_r\epsilon_o$ and $\mu = \mu_r\mu_o$ a plane TEM wave with the following components is observed,

$$\vec{E} = \hat{x}\frac{1}{2}u(t-\frac{z}{c/2}) + \hat{y}g(t-\frac{z}{c/2})\,\frac{\mathrm{V}}{\mathrm{m}},$$

and

$$\vec{H} = \hat{x}(\frac{10z}{c} - 5t) + \hat{y}\frac{1}{120\pi}u(t-\frac{2z}{c})\,\frac{\mathrm{A}}{\mathrm{m}}.$$

a) The intrinsic impedance of the medium is

$$\eta = \left|\frac{E_x}{H_y}\right| = 60\pi\,\Omega.$$

We have used the orthogonal pair $E_x$ and $H_y$. The same relation should be valid for the orthogonal pair $E_y$ and $H_x$.

b) The propagation velocity is

$$v = \frac{c}{2} = 1.5\times10^8\,\frac{\mathrm{m}}{\mathrm{s}}.$$

c) If $\epsilon = \epsilon_r\epsilon_o$ and $\mu = \mu_r\mu_o$, then the intrinsic impedance $\eta$ can take the following form

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_r}{\epsilon_r}}\eta_o$$

while the wave propagation velocity $v$ can be written as

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{\sqrt{\epsilon_r\mu_r}}.$$

Using the results of parts (a) and (b) we find that

$$\epsilon_r = 4, \quad \mu_r = 1.$$

d) Finally, since $E_y = -\eta H_x$ (recall that the propagation direction is $\hat{z} = -\hat{y} \times \hat{x}$, hence the minus sign), we have that

$$g(t - \frac{z}{c/2}) = -60\pi \times (\frac{10z}{c} - 5t) = 300\pi \times \left(t - \frac{z}{c/2}\right),$$

then

$$g(t) = 300\pi t.$$

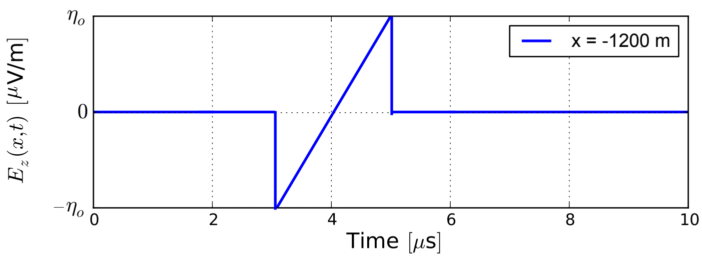
3. Consider on $x = 0$ plane, a pulse of sheet current $\vec{J}_s(t) = -\hat{z}\, 2t\, \mathrm{rect}\left(\frac{t}{\tau}\right)$ A/m, where $\tau = 2\,\mu$s. The corresponding magnetic field is: (it's direction can be verify using the right-hand-rule for Ampere's law $\oint_C \mathbf{H} \cdot d\mathbf{l} = I_C$)

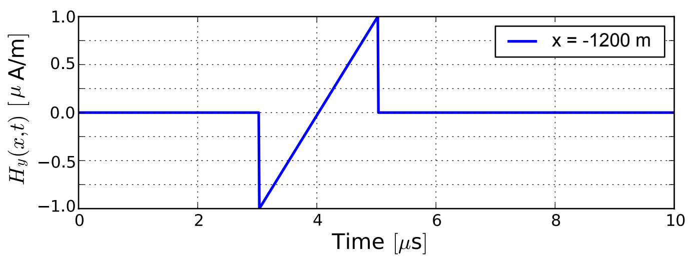
$$\begin{aligned} \vec{H}^{\pm}(x,t) &= \mp\frac{1}{2}J_{so}\left(t \mp \frac{x}{c}\right)\hat{y}\,\frac{\mathrm{A}}{\mathrm{m}} \\ &= \mp\left(t \mp \frac{x}{c}\right)\mathrm{rect}\left(\frac{t \mp \frac{x}{c}}{\tau}\right)\hat{y}\,\frac{\mathrm{A}}{\mathrm{m}} \quad \text{for } x \gtrless 0, \end{aligned}$$

(a) For $x = -1200$ m, we have that

$$E_z^-(x,t) = \eta_o\,(t - 4\mu\mathrm{s})\;\mathrm{rect}\left(\frac{t - 4\mu\mathrm{s}}{2\mu\mathrm{s}}\right)\frac{\mathrm{V}}{\mathrm{m}}$$

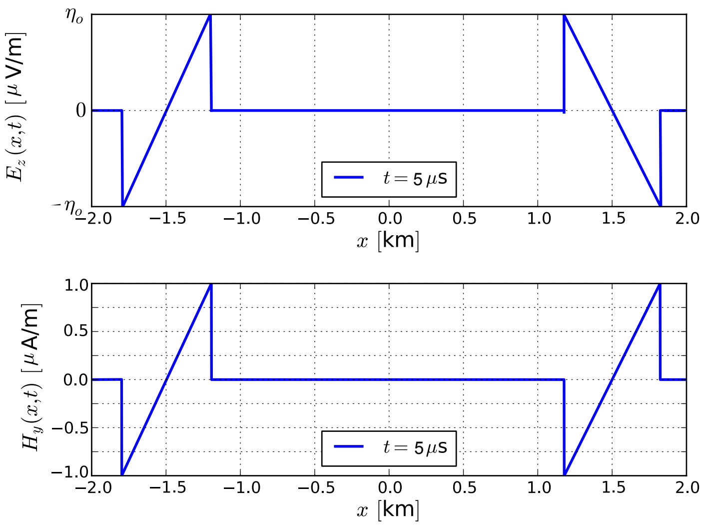
$$H_y^-(x,t) = (t - 4\mu\mathrm{s})\;\mathrm{rect}\left(\frac{t - 4\mu\mathrm{s}}{2\mu\mathrm{s}}\right)\frac{\mathrm{A}}{\mathrm{m}}.$$



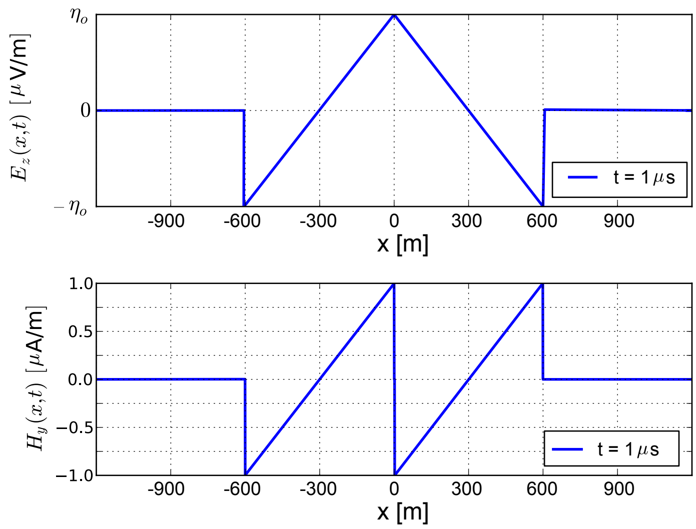
(b) For $t = 5\,\mu$s, we have that

$$
\begin{aligned}
E_z^{\pm}(x,t) &= \eta_o \left(5\mu\text{s} \mp \frac{x}{c}\right) \text{rect}\left(\frac{5\mu\text{s} \mp \frac{x}{c}}{2\mu\text{s}}\right) \frac{\text{V}}{\text{m}} \quad \text{for } x \gtrless 0 \\
&= \eta_o \left(\frac{1.5\,\text{km} \mp x}{c}\right) \text{rect}\left(\frac{1.5\,\text{km} \mp x}{0.6\,\text{km}}\right) \frac{\text{V}}{\text{m}} \quad \text{for } x \gtrless 0, \\
H_y^{\pm}(x,t) &= \mp \left(5\mu\text{s} \mp \frac{x}{c}\right) \text{rect}\left(\frac{5\mu\text{s} \mp \frac{x}{c}}{2\mu\text{s}}\right) \frac{\text{A}}{\text{m}} \quad \text{for } x \gtrless 0 \\
&= \mp \left(\frac{1.5\,\text{km} \mp x}{c}\right) \text{rect}\left(\frac{1.5\,\text{km} \mp x}{0.6\,\text{km}}\right) \frac{\text{A}}{\text{m}} \quad \text{for } x \gtrless 0.
\end{aligned}
$$




(c) For $t = 1\,\mu$s, we have that

$$
\begin{aligned}
E_z^{\pm}(x,t) &= \eta_o \left(1\mu\text{s} \mp \frac{x}{c}\right) \text{rect}\left(\frac{1\mu\text{s} \mp \frac{x}{c}}{2\mu\text{s}}\right) \frac{\text{V}}{\text{m}} \quad \text{for } x \gtrless 0 \\
&= \eta_o \left(\frac{300\,\text{m} \mp x}{c}\right) \text{rect}\left(\frac{300\,\text{m} \mp x}{600\,\text{m}}\right) \frac{\text{V}}{\text{m}} \quad \text{for } x \gtrless 0, \\
H_y^{\pm}(x,t) &= \mp \left(1\mu\text{s} \mp \frac{x}{c}\right) \text{rect}\left(\frac{1\mu\text{s} \mp \frac{x}{c}}{2\mu\text{s}}\right) \frac{\text{A}}{\text{m}} \quad \text{for } x \gtrless 0 \\
&= \mp \left(\frac{300\,\text{m} \mp x}{c}\right) \text{rect}\left(\frac{300\,\text{m} \mp x}{600\,\text{m}}\right) \frac{\text{A}}{\text{m}} \quad \text{for } x \gtrless 0.
\end{aligned}
$$




(d) The TEM wave power injected per unit volume is (see lecture 20). Recall $J_s$ exists only at $x = 0$:

$$\begin{aligned}-\vec{J}_s \cdot \vec{E} &= -\left(-\hat{z}\, 2t\,\mathrm{rect}\left(\frac{t}{\tau}\right)\right) \cdot \left(\eta_o t\,\mathrm{rect}\left(\frac{t}{\tau}\right)\,\hat{z}\right) \\ &= 2\eta_o t^2\,\mathrm{rect}^2\left(\frac{t}{\tau}\right)\,\frac{\mathrm{W}}{\mathrm{m}^2}.\end{aligned}$$

Then, the TEM wave density energy is

$$\begin{aligned}\int -\vec{J}_s \cdot \vec{E}\,dt &= \int 2\eta_o t^2\,\mathrm{rect}^2\left(\frac{t}{\tau}\right)\,dt \\ &= \int_{-\tau/2}^{\tau/2} 2\eta_o t^2\,dt \;=\; \frac{2}{3}\eta_o\,\left[t^3\right]\Big|_{-\tau/2}^{\tau/2} \\ &= \frac{1}{6}\eta_o\tau^3 = 160\pi \times 10^{-18}\,\frac{\mathrm{J}}{\mathrm{m}^2}.\end{aligned}$$

4. a) Wave: $\vec{E}_1 = -4\cos(\omega t - \beta z)\hat{y}\,\frac{\mathrm{V}}{\mathrm{m}}$

i. The electric and magnetic fields ($\vec{E}$ and $\vec{H}$) of a uniform plane wave are orthogonal to each other and to their direction of propagation, thus, it can be verified that such fields satisfy the following relation

$$\vec{E} = \eta\vec{H} \times \hat{\beta},$$

where $\hat{\beta}$ is the unit vector parallel to the propagation direction and $\eta$ is the intrinsic impedance. Using this relation and considering $\eta_o \approx 120\pi\,\Omega$ (free space), we can find the expressions for $\vec{H}$ or $\vec{E}$ that accompany the given wave fields,

$$\hat{\beta}_1 = \hat{z} \rightarrow \vec{H}_1 = \frac{4}{\eta_o}\cos(\omega t - \beta z)\hat{x}\,\frac{\mathrm{A}}{\mathrm{m}}$$

ii. The instantaneous power flow density is given by the Poynting vector $\vec{S} = \vec{E} \times \vec{H}$. Therefore, the instantaneous power that crosses some surface $A$ is given by $P = \int_A \vec{S} \cdot d\vec{A}$. In the case of uniform plane waves, this expression simplifies to

$$P = \vec{S} \cdot \hat{n}\,A,$$

where $\hat{n}$ is the vector normal to the flat area $A$. Below, we are considering $A = 1\,\mathrm{m}^2$ and $\hat{n} = \hat{z}$. For this case:

$$\vec{S}_1 = \vec{E}_1 \times \vec{H}_1 = \frac{16}{\eta_o}\cos^2(\omega t - \beta z)\hat{z}\,\frac{\mathrm{W}}{\mathrm{m}^2},$$

thus,

$$P_1 = \frac{16}{\eta_o}\cos^2(\omega t - \beta z)\,\mathrm{W}.$$

iii. We can calculate the time-average of the Poynting vector using the trig. identity: $\cos^2\theta = \frac{1}{2}\left(1 + \cos\left(2\theta\right)\right)$, and the fact that the time average of the cosine wave is zero ($\frac{1}{T}\int_T \cos(\omega t)dt = 0$):

$$\left\langle \vec{S}_1 \right\rangle = \left\langle \frac{16}{\eta_o}\cos^2(\omega t - \beta z)\hat{z}\,\frac{\mathrm{W}}{\mathrm{m}^2} \right\rangle = \frac{8}{\eta_o}\hat{z}\,\frac{\mathrm{W}}{\mathrm{m}^2}$$

Therefore, the average power that crosses some surface $A$ is given by $\langle P \rangle = \int_A \left\langle \vec{S} \right\rangle \cdot d\vec{A}$. In the case of uniform plane waves, this expression simplifies to

$$\langle P \rangle = \left\langle \vec{S} \right\rangle \cdot \hat{n}\,A,$$

where $\hat{n}$ is the vector normal to the flat area $A$. Below, we are considering $A = 1\,\mathrm{m}^2$ and $\hat{n} = \hat{z}$. For this case:

$$\langle P_1 \rangle = \frac{8}{\eta_o}\,\mathrm{W}.$$

b) Wave: $\vec{E}_2 = E_o\left(\cos(\omega t - \beta x)\hat{y} - \sin(\omega t - \beta x)\hat{z}\right)\,\frac{\mathrm{V}}{\mathrm{m}}$

i. $\hat{\beta}_2 = \hat{x} \rightarrow \vec{H}_2 = \frac{E_o}{\eta_o}\left(\cos(\omega t - \beta x)\hat{z} + \sin(\omega t - \beta x)\hat{y}\right)\,\frac{\mathrm{A}}{\mathrm{m}}$

ii. The Poynting vector is

$$\begin{aligned}
\vec{S}_2 &= \vec{E}_2 \times \vec{H}_2 \\
&= E_o\left(\cos(\omega t - \beta x)\hat{y} - \sin(\omega t - \beta x)\hat{z}\right) \times \frac{E_o}{\eta_o}\left(\cos(\omega t - \beta x)\hat{z} + \sin(\omega t - \beta x)\hat{y}\right) \\
&= \frac{E_o^2}{\eta_o}\left(\cos^2(\omega t - \beta x)\hat{z} + \sin^2(\omega t - \beta z)\hat{x}\right) \\
&= \frac{E_o^2}{\eta_o}\hat{x}\,\frac{\mathrm{W}}{\mathrm{m}^2}.
\end{aligned}$$

The wave is propagating in the $+x$ direction, therefore there is no flux of energy flowing into the $z$ direction. Therefore, the instantaneous power crossing a $1\,\mathrm{m}^2$ area in the $xy-$plane from $-z$ to $z$ is

$$P_2 = 0\,\mathrm{W},$$

iii. The time-average power crossing the $xy-$plane from $-z$ to $z$ is

$$\langle P_2 \rangle = 0\,\mathrm{W}.$$

c) Wave: $\vec{H}_3 = \cos(\omega t + \beta z + \frac{\pi}{3})\hat{x} + \sin(\omega t + \beta z - \frac{\pi}{6})\hat{y}\,\frac{\mathrm{A}}{\mathrm{m}}$

i. $\hat{\beta}_3 = -\hat{z} \rightarrow \vec{E}_3 = \eta_o\left(\cos(\omega t + \beta z + \frac{\pi}{3})\hat{y} - \sin(\omega t + \beta z - \frac{\pi}{6})\hat{x}\right)\,\frac{\mathrm{V}}{\mathrm{m}}$

ii. The Poynting vector is

$$
\begin{aligned}
\vec{S}_3 &= \vec{E}_3 \times \vec{H}_3 \\
&= \eta_o \left(\cos(\omega t + \beta z + \frac{\pi}{3})\hat{y} - \sin(\omega t + \beta z - \frac{\pi}{6})\hat{x}\right) \times \\
&\quad \left(\cos(\omega t + \beta z + \frac{\pi}{3})\hat{x} + \sin(\omega t + \beta z - \frac{\pi}{6})\hat{y}\right) \\
&= -\eta_o \left(\cos^2(\omega t + \beta z + \frac{\pi}{3})\hat{z} + \sin^2(\omega t + \beta z - \frac{\pi}{6})\hat{z}\right) \\
&= -\eta_o \left(\cos^2(\omega t + \beta z + \frac{\pi}{3})\hat{z} + \cos^2(\omega t + \beta z + \frac{\pi}{3})\hat{z}\right) \\
&= -2\eta_o \cos^2(\omega t + \beta z + \frac{\pi}{3})\hat{z}\, \frac{\mathrm{W}}{\mathrm{m}^2}.
\end{aligned}
$$

Therefore, the instantaneous power crossing a $1\,\mathrm{m}^2$ area in the $xy-$plane from $-z$ to $z$ is

$$P_3 = -2\eta_o \cos^2(\omega t + \beta z + \frac{\pi}{3})\hat{z}\, \frac{\mathrm{W}}{\mathrm{m}^2} \cdot \hat{z}\,\mathrm{m}^2 = -2\eta_o \cos^2(\omega t + \beta z + \frac{\pi}{3})\,\mathrm{W}.$$

iii. The time-average power crossing a $1\,\mathrm{m}^2$ area in the $xy-$ plane from $-z$ to $z$ is

$$\langle P_3 \rangle = -\eta_o\,\mathrm{W}.$$

d) Wave: $\vec{H}_4 = \cos(\omega t - \beta z - \frac{\pi}{2})\hat{x} + \sin(\omega t - \beta z)\hat{y}\,\frac{\mathrm{A}}{\mathrm{m}}$.

i. $\hat{\beta}_4 = \hat{z} \to \vec{E}_4 = \eta_o\left(-\cos(\omega t - \beta z - \frac{\pi}{2})\hat{y} + \sin(\omega t - \beta z)\hat{x}\right) = \eta_o \sin(\omega t - \beta z)(-\hat{y} + \hat{x})\,\frac{\mathrm{V}}{\mathrm{m}}$.

ii. The Poynting vector is

$$
\begin{aligned}
\vec{S}_4 &= \vec{E}_4 \times \vec{H}_4 \\
&= \eta_o \sin(\omega t - \beta z)(-\hat{y} + \hat{x}) \times [\cos(\omega t - \beta z - \frac{\pi}{2})\hat{x} + \sin(\omega t - \beta z)\hat{y}] \\
&= 2\eta_o \sin^2(\omega t - \beta z)\hat{z}
\end{aligned}
$$

Therefore, the instantaneous power crossing the area $A = 1\,\mathrm{m}^2$ is

$$P_4 = 2\eta_o \sin^2(\omega t - \beta z)\,\mathrm{W}.$$

iii. The time-average power crossing a $1\,\mathrm{m}^2$ area in the $xy-$ plane from $-z$ to $z$ is

$$\langle P_4 \rangle = \eta_o\,\mathrm{W}.$$

5. (a) The propagation velocity is

$$v_p = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{\sqrt{\epsilon_r \mu_r}} = \frac{c}{\sqrt{\frac{9}{4}}} = 2 \times 10^8\,\frac{\mathrm{m}}{\mathrm{s}}.$$

(b) The wave number is

$$\beta = \frac{\omega}{v_p} = \frac{6\pi \times 10^8}{2 \times 10^8} = 3\pi\,\frac{\mathrm{rad}}{\mathrm{m}}.$$

The wavelength is

$$\lambda = \frac{2\pi}{\beta} = \frac{2}{3}\,\mathrm{m}.$$

(c) The intrinsic impedance is

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_r}{\epsilon_r}}\eta_o = \sqrt{\frac{4}{9}}\eta_o = 80\pi\,\Omega.$$

(d) The electric field is

$$\begin{aligned}
\mathbf{E} &= -\eta\cos[\omega t \mp \beta(x-2)]\hat{y}\,\frac{\text{V}}{\text{m}}, \quad \text{for } x \gtrless 2, \\
\tilde{\mathbf{E}} &= -\eta e^{\mp j\beta(x-2)}\hat{y}\,\frac{\text{V}}{\text{m}}, \quad \text{for } x \gtrless 2,
\end{aligned}$$

and the magnetic field is

$$\begin{aligned}
\mathbf{H} &= \mp\cos[\omega t \mp \beta(x-2)]\hat{z}\,\frac{\text{A}}{\text{m}}, \quad \text{for } x \gtrless 2, \\
\tilde{\mathbf{H}} &= \mp e^{\mp j\beta(x-2)}\hat{z}\,\frac{\text{A}}{\text{m}}, \quad \text{for } x \gtrless 2,
\end{aligned}$$

(e) At the location of $x = 4$m, the Poynting vector is

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \eta\cos^2[\omega t \mp \beta(x-2)]\hat{x}$$

Therefore, the instantaneous power crossing the area $A = 2\,\text{m}^2$ is

$$P = 2\eta\cos^2[\omega t \mp \beta(x-2)]\,\text{W}.$$

And the time-average power crossing the $2\,\text{m}^2$ square surface is

$$\langle P\rangle = \eta\,\text{W}.$$