ECE 310 : Lecture 25 : FIR Filter Design : Windowing

Last class : Generalized Linear Phase (GLP)

$$\boxed{H_d(\omega) = R(\omega)\, e^{+j(\alpha - \omega M)}}$$

① Type I :

$$\boxed{h(n) = h(N-1-n)} \qquad \boxed{N : \text{odd integer}}$$

$$\boxed{\begin{gathered} H_d(\omega) = R(\omega)\, e^{-j\omega M} \\ \alpha = 0 \qquad M = \frac{N-1}{2} \end{gathered}}$$

② Type II :

$$\boxed{h(n) = h(N-1-n)} \qquad \boxed{N : \text{even integer}}$$

$$\boxed{\begin{gathered} H_d(\omega) = R(\omega)\, e^{-j\omega M} \\ \alpha = 0 \;, \quad M = \frac{N-1}{2} \end{gathered}}$$

③ Type III :

$$\boxed{h[n] = -h(N-1-n)} \qquad \boxed{N : \text{odd integer}}$$

$$\boxed{\begin{gathered} H_d(\omega) = R(\omega)\, e^{j(\alpha - \omega M)} \\ \alpha = \frac{\pi}{2} \;, \quad M = \frac{N-1}{2} \end{gathered}}$$

ALSO NOTE : $\boxed{h(M) = 0}$ for odd coefficient symmetry

④ Type IV :

$\boxed{h(n) = h[N-1-n]}$ $\boxed{N : \text{even integer}}$

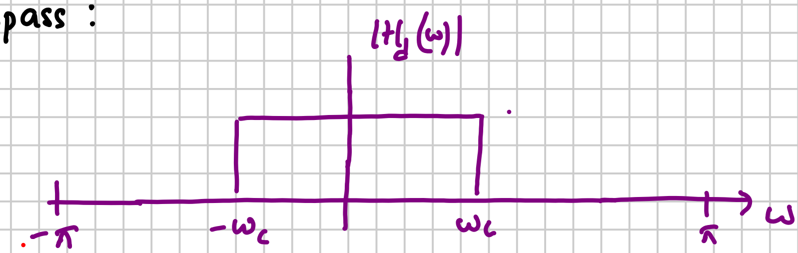
$$\boxed{\begin{array}{c} H_d(\omega) = R(\omega)\, e^{-j\omega M} \\ \alpha = 0 \qquad M = \frac{N-1}{2} \end{array}}$$

FIR FILTER DESIGN : Windowing

Filters : LSI systems that modify a certain set of frequencies relative to others.

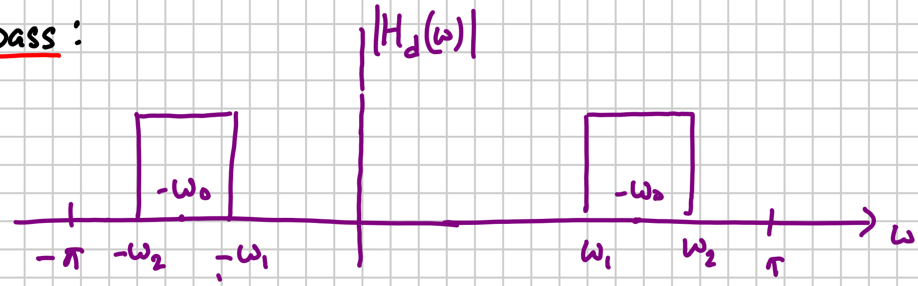
Classes of filters and ideal characteristics :

① Lowpass :

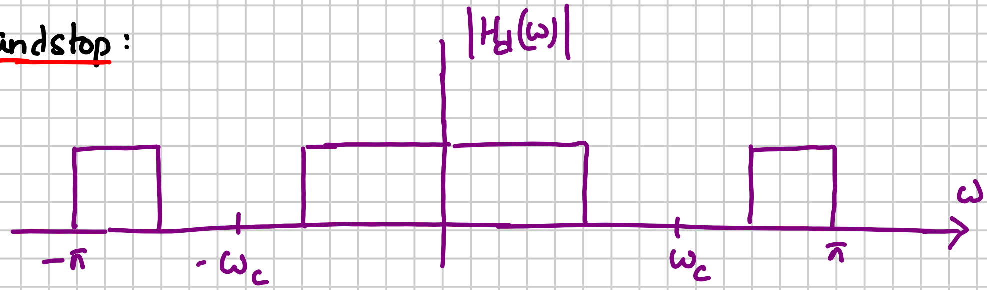


② High pass

③ Bandpass :

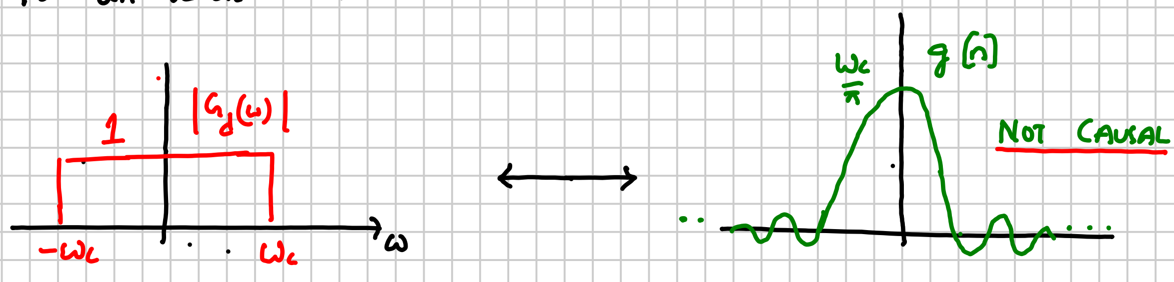


④ Bandstop:



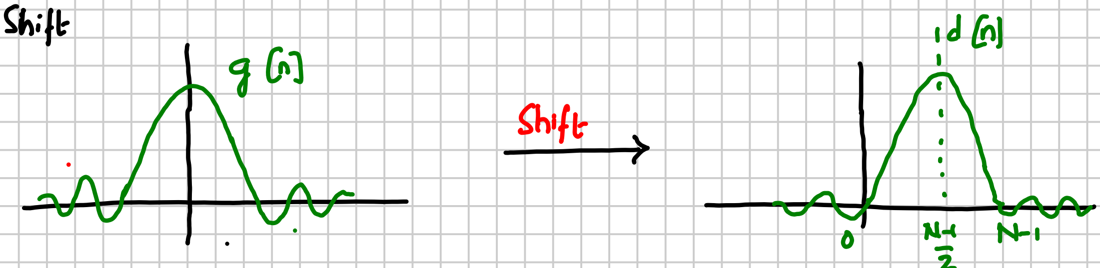
Recall: Ideal filters are not causal.

For an ideal LPF:



Filter must be made causal: ① Shift + ② Truncate

① Shift



$$\boxed{d(n) = g[n-M], \qquad M = \frac{N-1}{2}}$$

② Window:

Define a new system

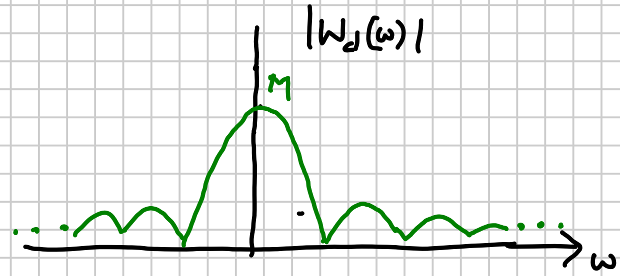
$$h(n) = \begin{cases} d[n] & 0 \leq n \leq N-1 \\ 0 & \text{else} \end{cases} \qquad ①$$

Equation ① can be seen as the product of shifted version of desired impulse response $g(n)$ and a window $w(n)$

Let $$w[n] = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow W_d(\omega) = e^{j\omega M} \frac{\sin \omega M}{\sin \frac{\omega}{2}}$$

Magnitude response of the rectangular window:



Note that the window function has piecewise linear phase

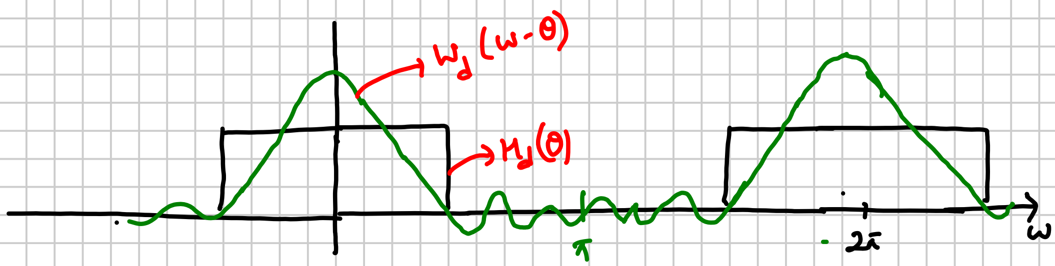
$$\angle W_d(\omega) = \begin{cases} -\omega M & \sin\left(\frac{\omega M}{2}\right) \geq 0 \\ -\omega M + \pi & \sin\left(\frac{\omega M}{2}\right) < 0 \end{cases}$$

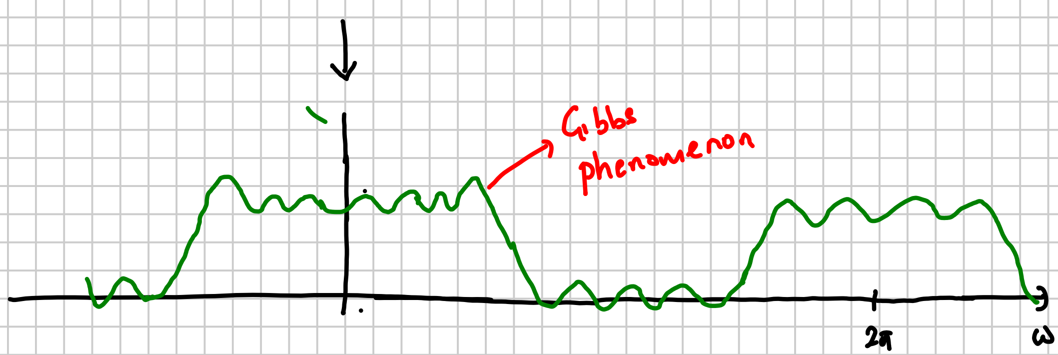
In the transfor domain:

$$\boxed{H_d(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D(\theta)\, W(\omega-\theta)\, d\theta}$$

Periodic Convolution

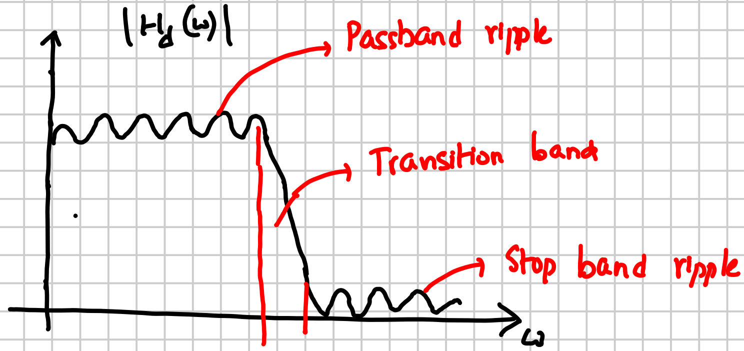
Hence we have (in frequency domain):





Gibbs Phenomenon: At sharp transitions we get tall ripples. As N increases ripples become narrower, but the heights of the ripples nearest to discontinuity remain large

Causality Implications:



Design of FIR filters using Windows:

① Get desired Response $G_d(\omega)$

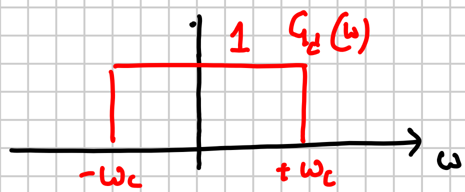
② Multiply $G_d(\omega)$ by phase term = Shifting pulse in time domain

③ Get $d(n)$ = IDTFT ($D(\omega)$) → We get shifted filter in time domain

④ Apply window.

Examples:

① Design LPF, length $N = 30$, cut-off $\omega_c = \frac{\pi}{4}$



$\boxed{N = 30}$ → even length ⇒ Type II GLP FIR

⇒ even Symmetry

$$\boxed{M = \frac{N-1}{2} = \frac{29}{2}}$$

$$D(\omega) = \begin{cases} 1\, e^{j\omega M} & |\omega| < \pi/4 \\ 0 & \text{else} \end{cases}$$

$$d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\, e^{j\omega M} \cdot e^{j\omega n}\, d\omega$$

$$\Rightarrow \boxed{d[n] = \frac{1}{4}\,\text{Sinc}\left[\frac{\pi}{4}\left(n-\frac{29}{2}\right)\right]} \rightarrow \text{Shifted Sinc}$$

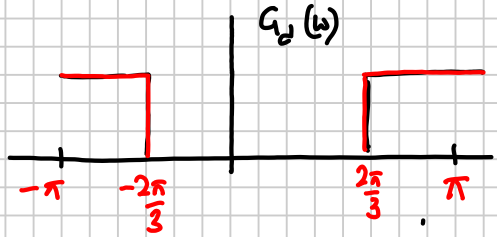
① Truncation : $h[n] = \frac{1}{4}\,\text{Sinc}\left(\frac{\pi}{4}\left(n-\frac{29}{2}\right)\right) \quad 0 \leq n \leq 29$

② Hamming Window:

$$h[n] = \frac{1}{4}\,\text{Sinc}\left(\frac{\pi}{4}\left(n-\frac{29}{2}\right)\right)\left[0.54 - 0.46\cos\frac{2\pi n}{30}\right]$$

$$0 \leq n \leq 29$$

② Design high pass FIR, length $N = 61$, $\omega_c = \frac{2\pi}{3}$



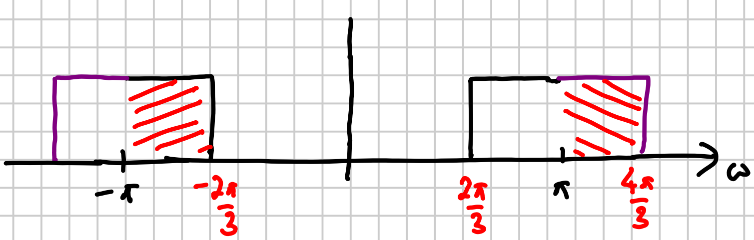
$\boxed{N = 61} \longrightarrow$ Odd length $\qquad \boxed{M = \frac{N-1}{2} = 30}$

Type I GLP can be used

$\Rightarrow$ Even symmetry

$$D(\omega) = \begin{cases} 1\, e^{j\omega M} & \frac{2\pi}{3} \leq |\omega| \leq \pi \\ 0 & \text{else} \end{cases}$$

$$d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} D(\omega)\, e^{j\omega n}\, d\omega.$$



$$\Rightarrow d[n] = \frac{1}{2\pi}\int_{\frac{2\pi}{3}}^{4\pi/3} D(\omega)\, e^{j\omega n}\, d\omega$$

$$\Rightarrow d[n] = \frac{1}{2\pi}\int_{\frac{2\pi}{3}}^{4\pi/3} e^{j\omega M}\, e^{j\omega n}\, d\omega$$

$$\Rightarrow \boxed{d[n] = \left(\frac{-1}{2}\right)^n \text{Sinc}\, \frac{\pi}{3}(n-30)}$$

① Truncate :

$$h[n] = \left(\frac{-1}{3}\right)^n \text{Sinc}\, \frac{\pi}{3}(n-30) \qquad 0 \le n \le 60$$

② Hamming :

$$h[n] = \left(-\frac{1}{3}\right)^n \text{Sinc}\, \frac{\pi}{3}(n-30) \cdot \left[0.54 - 0.46 \cos \frac{2\pi n}{60}\right]$$

③ Design HPF , length $N = 62$, $\omega_c = \frac{2\pi}{3}$

Here , $\boxed{N = 62}$ $\Rightarrow$ Even $\boxed{M = \frac{N-1}{2} = \frac{61}{2}}$

$\Rightarrow$ Type III FIR $\Rightarrow$ Odd Symmetry

In case of odd Symmetry :

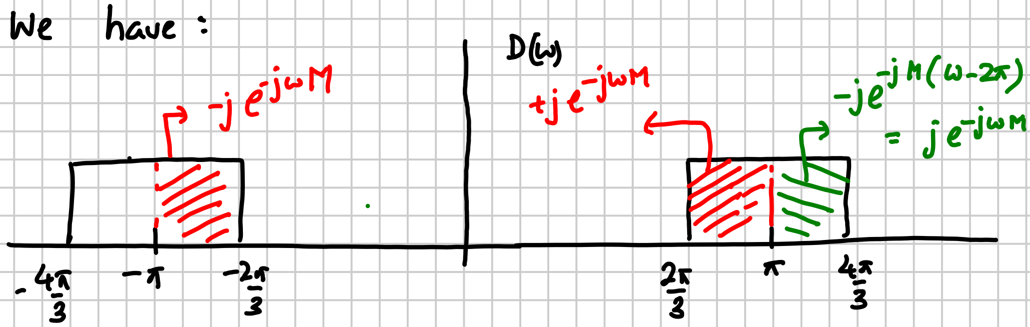
$$D(\omega) = R(\omega)\, e^{j\angle D(\omega)}$$

$R(\omega)$: Combination of Sines .

$\Rightarrow$ $R(\omega)$ changes sign at $\omega = 0$

$$\Rightarrow \quad D(\omega) = \begin{cases} e^{j(\pi/2 - \omega M)} & \frac{2\pi}{3} \le \omega \le \pi \\ 0 & |\omega| < 2\pi/3 \\ e^{j(-\pi/2 - \omega M)} & -\pi \le \omega \le -\frac{2\pi}{3} \end{cases}$$

We have :



$\Rightarrow$ $D(\omega)$ is Same in $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$

$$\Rightarrow \quad d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} j e^{-j\omega M} d\omega$$

$$\Rightarrow \boxed{d(n) = (-1)^n \frac{1}{3} \operatorname{Sinc} \frac{\pi}{3}(n-M)}$$

① Truncation:

$$h(n) = (-1)^n \frac{1}{3} \operatorname{Sinc} \frac{\pi}{3}(n-M) \quad 0 \le n \le 61$$

② Hamming:

$$h(n) = (-1)^n \frac{1}{3} \operatorname{Sinc}\left[\frac{\pi}{3}(n-M)\right]\left[0.54 - 0.46 \cos \frac{2\pi n}{61}\right] \quad 0 \le n \le 61$$