Problem 1

$$x[n] = 2^{-n} u[n-2] = \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^{n-2} u[n-2]$$

$$y[n] = (1 + 3^{-n}) u[n] = u[n] + \left(\tfrac{1}{3}\right)^n u[n]$$

The z-transforms of $x[n]$ and $y[n]$ are:

$$X(z) = \frac{1}{4} z^{-2} \frac{z}{z-\frac{1}{2}} \qquad \text{ROC: } |z| > \frac{1}{2}$$

$$Y(z) = \frac{z}{z-1} + \frac{z}{z-\frac{1}{3}} \qquad \text{ROC: } |z| > 1$$

The z-transform of $x[n] * y[n]$ is:

$$X(z)\,Y(z) = \frac{1}{4} z^{-2} \left(\frac{z}{z-\frac{1}{2}}\right)\left(\frac{z}{z-1} + \frac{z}{z-\frac{1}{3}}\right) = \frac{1}{4}\left(\frac{1}{z-\frac{1}{2}}\right)\left(\frac{1}{z-1} + \frac{1}{z-\frac{1}{3}}\right)$$

Using partial fraction expansion: ROC: $|z| > 1$

$$X(z)\,Y(z) = \frac{1}{4}\left(\frac{4}{z-\frac{1}{2}} - \frac{6}{z-\frac{1}{3}} + \frac{2}{z-1}\right)$$

$$= \frac{1}{4} z^{-1}\left(\frac{4z}{z-\frac{1}{2}} - \frac{6z}{z-\frac{1}{3}} + \frac{2z}{z-1}\right)$$

Find $x[n] * y[n]$ using the delay property:

$$x[n] * y[n] = \left(\tfrac{1}{2}\right)^{n-1} u[n-1] - \frac{3}{2}\left(\tfrac{1}{3}\right)^{n-1} u[n-1] + \frac{1}{2} u[n-1]$$

Problem 2

$$y[n] - \frac{5}{6} y[n-1] + \frac{1}{6} y[n-2] = x[n] - \frac{1}{2} x[n-1]$$

(a) applying the z-transform on both sides:

$$Y(z) - \frac{5}{6} z^{-1} Y(z) + \frac{1}{6} z^{-2} Y(z) = X(z) - \frac{1}{2} z^{-1} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{1}{2} z^{-1}}{1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}} = \frac{z(z - \frac{1}{2})}{z^2 - \frac{5}{6} z + \frac{1}{6}} = \frac{z(z-\frac{1}{2})}{(z-\frac{1}{2})(z-\frac{1}{3})}$$

$$= \frac{z}{z - \frac{1}{3}}$$ ROC: $|z| > \frac{1}{3}$ because system is causal

$$h[n] = \left(\frac{1}{3}\right)^n u[n]$$

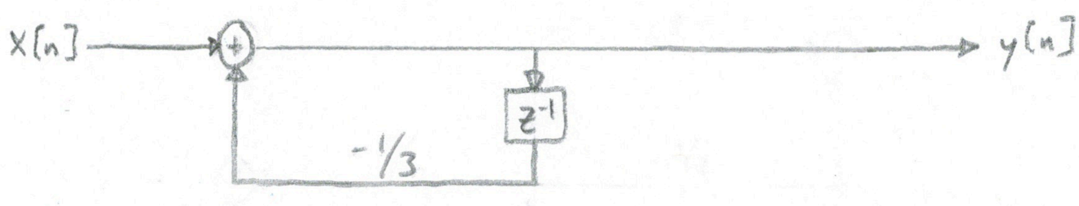
(b) $x[n] = \left(\frac{1}{3}\right)^n u[n] \iff X(z) = \frac{z}{z - \frac{1}{3}}$ ROC: $|z| > \frac{1}{3}$

$$Y(z) = H(z)\,X(z) = \left(\frac{z}{z-\frac{1}{3}}\right)^2 = 3z \; \frac{\frac{1}{3} z}{(z - \frac{1}{3})^2}$$

Using the advance property:

$$y[n] = 3\,(n+1) \left(\frac{1}{3}\right)^{n+1} u[n+1] = (n+1) \left(\frac{1}{3}\right)^n u[n+1]$$

(c) Direct Form II realization for the system:



$$H(z) = \frac{z}{z - \frac{1}{3}} = \frac{1}{1 - \frac{1}{3} z^{-1}}$$

$$y[n] - \frac{1}{3} y[n-1] = x[n]$$

Problem 3

(a) $y[n] = x^5[n] + n\left(\frac{1}{3}\right)^n$

The term $n\left(\frac{1}{3}\right)^n$ is unbounded for any input $x[n]$ as $n \to -\infty$. Therefore the system is not BIBO stable.

(b) $y[n] = \tan(x[n])$

Let $x[n] = \left(\frac{\pi}{2} - \frac{1}{n}\right)u[n]$, then $|x[n]| < \frac{\pi}{2}$ is bounded. However, $\lim_{x \to \frac{\pi}{2}} \tan(x) = \infty$, therefore the system is not BIBO stable.

(c) $y[n] = n\cos(x[n])$

Let $x[n] = 0 \;\forall n$, then $|x[n]| = 0$ is bounded. Next, $y[n] = n\cos(x[n]) = n\cos(0) = n$ is unbounded. Therefore, the system is not BIBO stable.

(d) $y[n] = x[n] * h[n]$ where $h[n] = \begin{cases} 0 & n < 0 \\ 10^{100} & 0 \le n \le 10^{10} \\ e^{-0.01n} & 10^{10} < n < \infty \end{cases}$

This is an LSI system.

An LSI system is BIBO stable iff $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{10^{10}} 10^{100} + \sum_{n=10^{10}+1}^{\infty} e^{-0.01n} < (10^{10}+1)\,10^{100} + \sum_{n=0}^{\infty} e^{-0.01n}$$

$$= (10^{10}+1)\,10^{100} + \frac{1}{1-e^{-0.01}} < \infty$$

Therefore the system is BIBO stable.

Problem 4

(a) $H(z) = \dfrac{z+10}{z^2+\frac{1}{4}} = \dfrac{z+10}{(z+j\frac{1}{2})(z-j\frac{1}{2})}$

ROC: $|z| > \frac{1}{2}$ includes the unit circle. Therefore, the system is BIBO stable.

(b) $H(z) = \dfrac{z+10}{z^2-\frac{3}{2}z+\frac{1}{2}} = \dfrac{z+10}{(z-1)(z-\frac{1}{2})}$

ROC: $|z| > 1$ does not include the unit circle. Therefore, the system is not BIBO stable.

Let $x[n] = u[n]$, then $X(z) = \dfrac{z}{z-1}$ and

$$Y(z) = \frac{z(z+10)}{(z-1)^2(z-\frac{1}{2})}, \quad \text{ROC: } |z| > 1$$

$$= z(z+10)\left(\frac{4}{z-\frac{1}{2}} - \frac{4}{z-1} + \frac{2}{(z-1)^2}\right)$$

Let $Y_1(z) = \dfrac{20z}{(z-1)^2}$, then $Y(z) = Y_1(z) +$ other terms

$y_1[n] = 20\, n\, u[n]$, so $y[n]$ is unbounded.

(c) $H(z) = \dfrac{z-10}{z-3}$

ROC: $|z| > 3$ does not include the unit circle. Therefore, the system is not BIBO stable.

Let $x[n] = \delta[n]$, then $X(z) = 1$

$$Y(z) = H(z)X(z) = \frac{z-10}{z-3}, \quad \text{ROC: } |z| > 3$$

$y[n]$ contains the term $-10\,(3)^n u[n]$, which is unbounded.

Problem 4

(d) $H(z) = \dfrac{z+1}{z^2+j} = \dfrac{z+1}{(z+e^{j\frac{\pi}{4}})(z-e^{j\frac{\pi}{4}})}$

ROC: $|z|>1$ does not include the unit circle. Therefore, the system is not BIBO stable.

Let $x[n] = \cos(\frac{\pi}{4}n)u[n]$, then $X(z) = \dfrac{z(z-\frac{\sqrt{2}}{2})}{z^2-\sqrt{2}z+1}$

$$X(z) = \frac{z(z-\frac{\sqrt{2}}{2})}{(z-e^{j\frac{\pi}{4}})(z-e^{-j\frac{\pi}{4}})}$$

$$Y(z) = H(z)X(z) = \frac{z(z+1)(z-\frac{\sqrt{2}}{2})}{(z+e^{j\frac{\pi}{4}})(z-e^{-j\frac{\pi}{4}})(z-e^{j\frac{\pi}{4}})^2}$$

after partial fraction expansion, $Y(z)$ will have a term $Y_1(z) = \dfrac{ce^{j\frac{\pi}{4}}z}{(z-e^{j\frac{\pi}{4}})^2}$ and

$y_1[n] = c\,n(e^{j\frac{\pi}{4}})^n u[n]$ which is unbounded.

Problem 5

$$x[n] = \frac{1}{2}\left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{3}\right)^n u[n-1] = \frac{1}{2}\left(\frac{1}{3}\right)^n u[n] - \frac{1}{3}\left(\frac{1}{3}\right)^{n-1} u[n-1]$$

$$y[n] = \left(\frac{1}{2}\right)^n u[n]$$

(a) The z-transforms of x[n] and y[n] are:

$$X(z) = \frac{1}{2}\,\frac{z}{z-\frac{1}{3}} - \frac{1}{3}z^{-1}\frac{z}{z-\frac{1}{3}} = 1/2\left(\frac{z-\frac{2}{3}}{z-\frac{1}{3}}\right) \qquad \text{ROC: } |z| > \frac{1}{3}$$

$$Y(z) = \frac{z}{z-\frac{1}{2}} \qquad \text{ROC: } |z| > \frac{1}{2}$$

$$H(z) = \frac{Y(z)}{X(z)} = 2\,\frac{z(z-\frac{1}{3})}{(z-\frac{1}{2})(z-\frac{2}{3})}$$

Using partial fraction expansion:

$$H(z) = 4\left(\frac{z}{z-\frac{2}{3}}\right) - 2\left(\frac{z}{z-\frac{1}{3}}\right)$$

The only possible region of convergence is $|z| > \frac{2}{3}$, so the impulse response is unique.

$$h[n] = 4\left(\frac{2}{3}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[n]$$

(b) The ROC for this system contains the unit circle, therefore the system is BIBO stable.

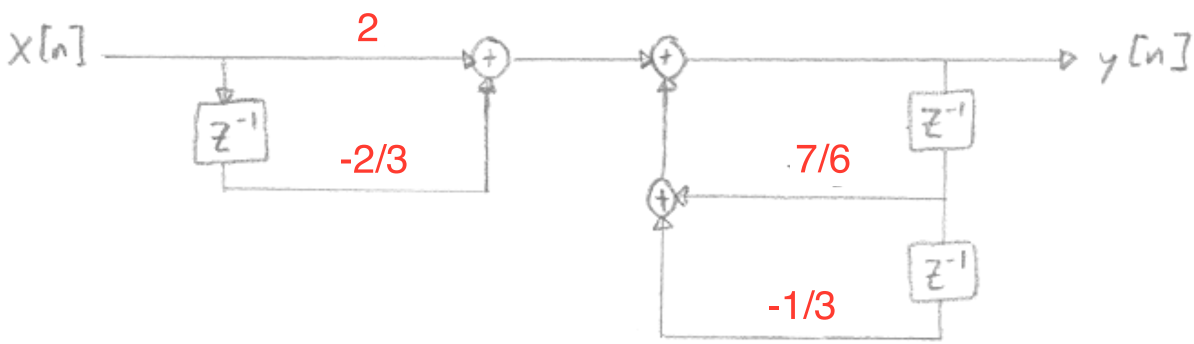
(c) $$H(z) = 2\,\frac{1-\frac{1}{3}z^{-1}}{1-\frac{7}{6}z^{-1}+\frac{1}{3}z^{-2}}$$

$$Y(z)\left(1-\frac{7}{6}z^{-1}+\frac{1}{3}z^{-2}\right) = 2\,X(z)\left(1-\frac{1}{3}z^{-1}\right)$$

$$y[n] - \frac{7}{6}y[n-1] + \frac{1}{3}y[n-2] = 2\,x[n] - 2/3\,x[n-2]$$

Problem 5

Direct Form I:



Direct Form II: