ECE 310: Lecture 30–31 : Fast Fourier Transforms (FFT)

Discrete Fourier Transform :

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn} \qquad 0 \le k \le N-1$$

$$\text{where} \quad W_N = e^{-j2\pi/N}$$

Consider $N = 4$ $\quad x[n] = \{x_0, x_1, x_2, x_3\}$

$$x[k] = x_0 e^{-j\frac{2\pi k}{4}\cdot 0} + x_1 e^{-j\frac{2\pi k}{4}\cdot 1} + x_2 e^{-j\frac{2\pi k}{4}\cdot 2} + x_3 e^{-j\frac{2\pi}{4}\cdot k\cdot 3}$$

$$0 \le k \le N-1$$

For computing each sample we need :

1. N Complex multiplications
2. $N-1$ Complex additions

1 complex multiplication = 4 Real multiplications
1 complex addition = 2 Real additions

Note W_N has some properties that can be exploited :

1. Symmetry Property :

$$\boxed{W_N^{k+N/2} = -W_N^{k}}$$

2. Periodicity :

$$\boxed{W_N^{k+N} = W_N^{k}}$$

FFT exploits these properties to compute DFT efficiently.

1. Radix-2 Decimation-in time FFT algorithm :

Let, $\quad N = 2^m$

Idea: 1. Break the input into two groups (Size $\frac{N}{2}$) where one group corresponding to even numbered and odd numbered samples.

2. Compute $\frac{N}{2}$ length DFT of these two sequences.

3. Combine Size $\frac{N}{2}$ DFT's to Calculate N-point DFT.

Advantage : Multiplication of each half of the sequence

$$= \frac{N}{2} \cdot \frac{N}{2} = \frac{N^2}{4}$$

$$\Rightarrow \text{Total multiplication} = \frac{N^2}{4} + \frac{N^2}{4}$$

$$\Rightarrow \boxed{\text{Total Multiplication} = \frac{N^2}{2}}$$

Let $y(n) = x[2n]$

$\qquad\qquad\qquad 0 \le n \le \frac{N}{2} - 1$

$z(n) = x[2n+1]$

To show : $\{X(k)\}_{k=0}^{N-1}$ can be obtained from the

$\frac{N}{2}$ DFT's $\{Y(k)\}_{k=0}^{\frac{N}{2}-1}$ and $\{Z(k)\}_{k=0}^{\frac{N}{2}-1}$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x[2n]\, W_N^{2kn} + x(2n+1)\, W_N^{(2n+1)k} , \quad 0 \le k \le \frac{N}{2}-1$$

$$= \sum_{n=0}^{\frac{N}{2}-1} y(n)\, W_{\frac{N}{2}}^{kn} + W_N^k \sum_{n=0}^{\frac{N}{2}-1} z(n)\, W_{\frac{N}{2}}^{nk} , \quad 0 \le k \le \frac{N}{2}-1$$

$$\Rightarrow \boxed{X(k) = Y(k) + W_N^k Z(k)} \qquad 0 \le k \le \frac{N}{2}-1$$

Obtaining $X(k)$ for $k > \frac{N}{2} - 1$:

$$X\left[k + \frac{N}{2}\right] = \sum_{n=0}^{\frac{N}{2}-1} y[n]\, W_{\frac{N}{2}}^{n[k+\frac{N}{2}]} + W_N^{k+\frac{N}{2}} \sum_{n=0}^{\frac{N}{2}-1} z(n)\, W_{\frac{N}{2}}^{n[k+\frac{N}{2}]}$$

$$W_{\frac{N}{2}}^{n[k+\frac{N}{2}]} = W_{\frac{N}{2}}^{nk} \cdot W_{\frac{N}{2}}^{kN/2} = W_{\frac{N}{2}}^{nk}$$

$$W_N^{k+\frac{N}{2}} = W_N^k \cdot W_N^{N/2} = -W_N^k$$

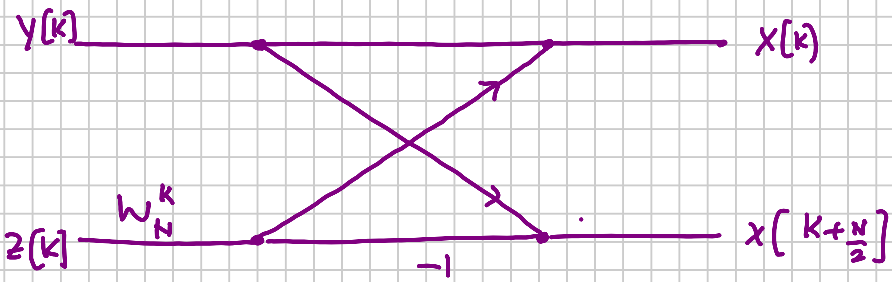
$$X\left[k + \frac{N}{2}\right] = \sum_{n=0}^{\frac{N}{2}-1} y(n)\, W_{\frac{N}{2}}^{nk} - W_N^k \sum_{n=0}^{\frac{N}{2}-1} z(n)\, W_{\frac{N}{2}}^{nk}$$

$$\boxed{X\left[k + \frac{N}{2}\right] = Y(k) - W_N^k Z(k)}$$

Hence the N length DFT can be computed as follows:

$$\boxed{\begin{aligned} X[k] &= y[k] + W_N^k z[k] \\ X[k+\tfrac{N}{2}] &= y[k] - W_N^k z[k] \end{aligned} \quad , \quad 0 \le k \le \tfrac{N}{2}-1} \qquad (1)$$

The above set of equations can be represented as the following flow graph:



Butterfly Computation of decimation in time FFT algorithm.

The decimation in time can be performed once more on the sequences y[n] and z[n]. The decimation in time can be repeated again and again till the data sequence is reduced to length one sequence.

Example: Length 2 sequence

$$x = [x[0] \;\; x[1]] \Rightarrow \boxed{N = 2}$$

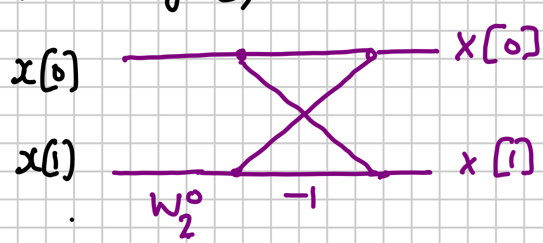
(1) Decimate to length-one sequence

$$y[n] = x[0] \quad , \quad z[n] = x[1]$$

(2) Compute DFT of y[n] and z[n]

$$\Rightarrow \quad Y[k] = x[0] \quad \& \quad Z[k] = x[1]$$

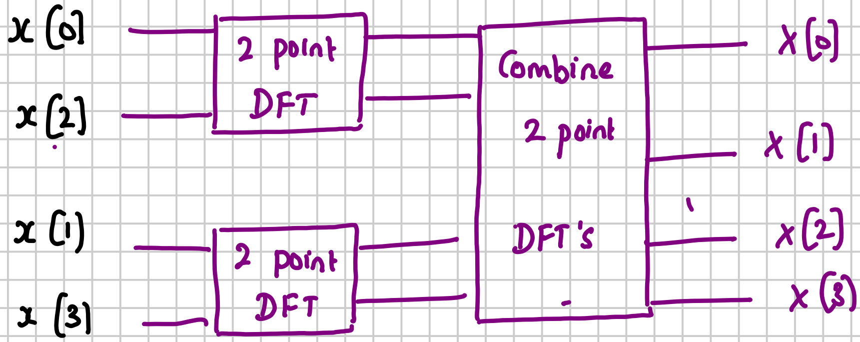
(3) Combine: (using (1))

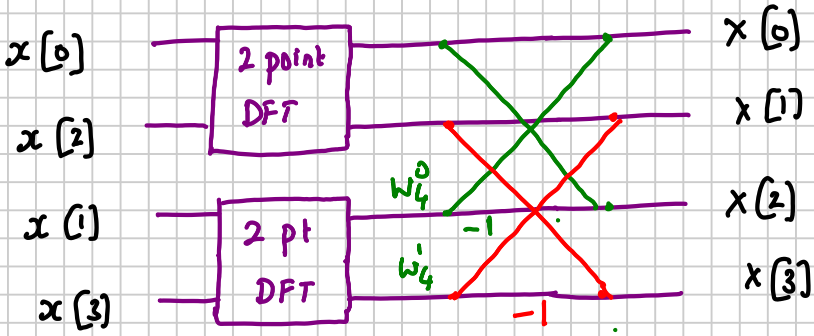


Example : Length 4 sequence : $[x[0] \quad x[1] \quad x[2] \quad x[3]]$

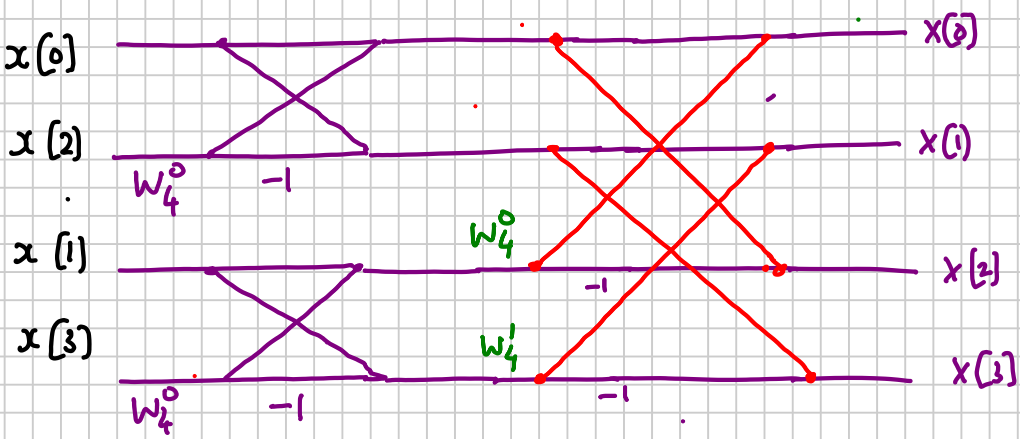
$y[n] = [x[0] \quad x[2]]$ $\qquad z[n] : [x[1] \quad x[3]]$

Four point DFT can be obtained as follows:





Butterfly Structure :



Example : Length 8 FFT

$x[n] = [x[0] \quad x[1] \quad x[2] \quad x[3] \quad x[4] \quad x[5] \quad x[6] \quad x[7]]$

1. $y[n] = [x[0] \quad x[2] \quad x[4] \quad x[6]]$

$z[n] = [x[1] \quad x[3] \quad x[5] \quad x[7]]$

② Decimate $y[n]$ and $z[n]$

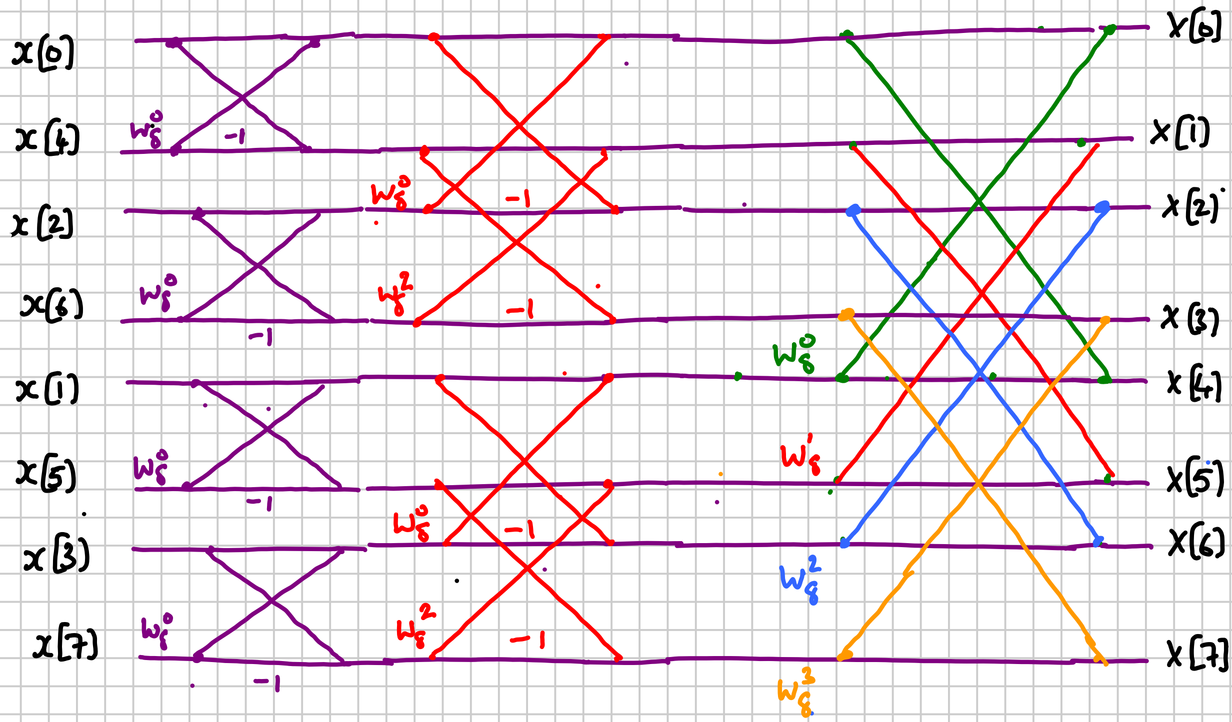
$$y_1[n] = [\, x[0] \quad x[4] \,]$$

$$z_1[n] = [\, x[2] \quad x[6] \,]$$

$$y_2[n] = [\, x[1] \quad x[5] \,]$$

$$z_2[n] = [\, x[3] \quad x[7] \,]$$

FFT Structure :



① Note the order of inputs ⇒ They are bit reversed

Input	Binary form	Reverse	Sample No.
0	000	000	$x[0]$
1	001	100	$x[4]$
2	010	010	$x[2]$
3	011	110	$x[6]$
4	100	001	$x[1]$
5	101	101	$x[5]$

② No. of stages : $N = 2^m$

⇒ no. of stages = $\log_2 N = m$

⇒ $\boxed{\text{multiplications} = \frac{N}{2} \log_2 N}$

$$\boxed{\text{Additions} = N \log_2 N}$$

Computation Comparison : Direct DFT and FFT

$$N = 2^{14}$$

$$\Rightarrow \quad N^2 = (2^{14})^2 = 268,435,456$$

$$\boxed{\text{DFT} \rightarrow 268,435,456}$$

$$\frac{N}{2} \log_2 N = \frac{2^{14}}{2} \log_2 2^{14}$$

$$\Rightarrow \quad \frac{N}{2} \log_2 N = 114688$$

$$\boxed{\text{FFT} \rightarrow 114688}$$

$$\text{Savings factor} = \frac{268435456}{114688} = 2340$$

Decimation In Frequency FFT :

Break the input sequence into two halves and then compute the even and odd parts.

$$X[k] = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{kn}$$

$$X[k] = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + W_N^{kN/2} \sum_{n=0}^{\frac{N}{2}-1} x\left[n + \frac{N}{2}\right] W_N^{kn}$$

$$X[k] = \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{kn} + W_N^{kN/2} \sum_{n=0}^{\frac{N}{2}-1} x\left[n + \frac{N}{2}\right] W_N^{kn}$$

$$W_N^{kN/2} = (-1)^k$$

$$X[k] = \sum_{n=0}^{\frac{N}{2}-1} \left[x[n] + (-1)^k x\left[n + \frac{N}{2}\right]\right] W_N^{kn}$$

Split (decimate) $X[k]$ into even and odd-numbered Samples

$$\boxed{X[2k] = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x\left[n+\tfrac{N}{2}\right]\right] W_{\frac{N}{2}}^{kn}}$$

$$\boxed{0 \le k \le \tfrac{N}{2}-1}$$

$$\boxed{X[2k+1] = \sum_{n=0}^{\frac{N}{2}-1} \left\{\left[x(n) - x\left(n+\tfrac{N}{2}\right)\right] W_N^{n}\right\} W_{\frac{N}{2}}^{kn}}$$

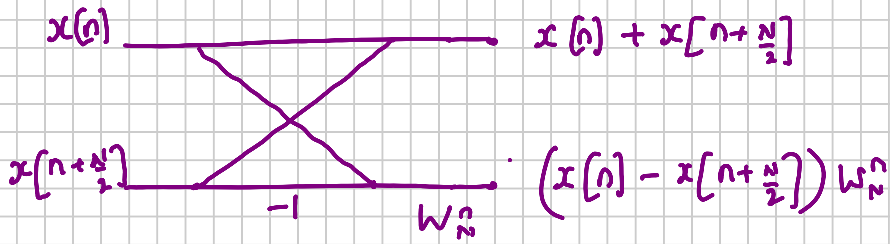
Let $y_1(n) = x(n) + x\left[n+\frac{N}{2}\right]$

$y_2[n] = \left(x(n) - x\left(n+\frac{N}{2}\right)\right) W_N^n$

$X[k]$ can be obtained from $\frac{N}{2}$-pt DFT of $y_1(n)$ and $y_2(n)$.

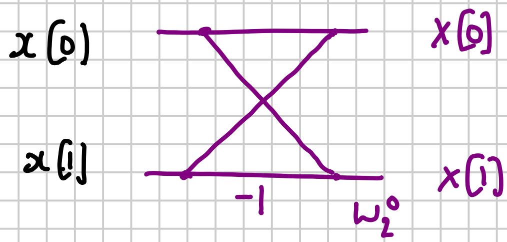
The procedure can be repeated for $X[k]$ and $X[2k+1]$.

Butterfly Computation for Decimation-in-frequency:



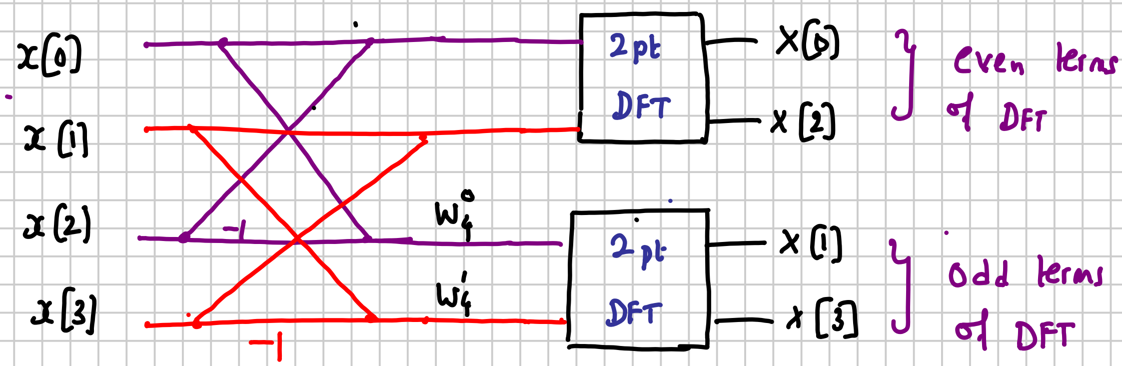
Example: $N=2$

$x : [x(0) \ x(1)]$

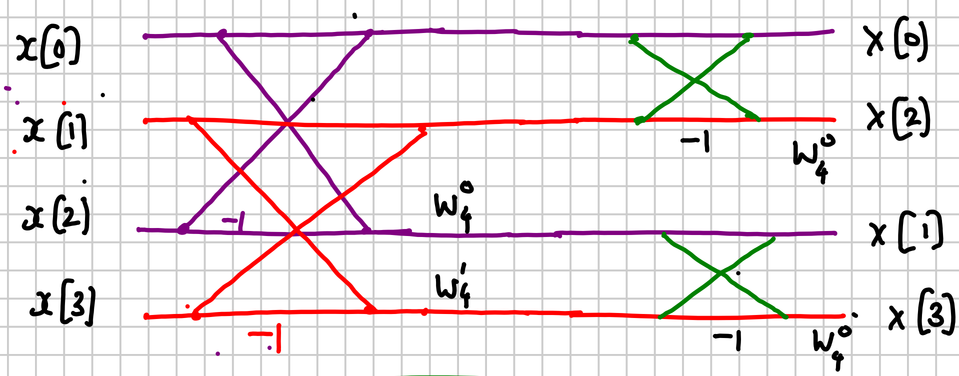


Example: N = 4

$$x[n] = [\,x[0]\ \ x[1]\ \ x[2]\ \ x[3]\,]$$



⇒



Note: The outputs are bit-reversed.

Example: $\boxed{N = 8}$