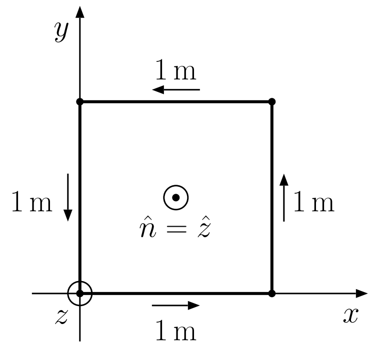
1. Given the time-varying magnetic field $\vec{B} = B_0\,(t\sin(\omega t)\,\hat{y} + \cos(\omega t)\,\hat{z})$ Wb/m^2, we can apply Faraday's law to compute the emf $\mathcal{E}$ around the following closed paths. Since the closed paths are not varying in time and the magnetic field $\vec{B}$ is independent of position, we can rewrite Faraday's law as follows

$$\mathcal{E} = \oint_C \vec{E}\cdot d\vec{l} = -\frac{d\vec{B}}{dt}\cdot\int_S d\vec{S} = \left(-\frac{d\vec{B}}{dt}\cdot\hat{n}\right)\text{Area},$$

where $\hat{n}$ is the unit vector normal to the surface, and

$$\frac{d\vec{B}}{dt} = B_0\left((\sin(\omega t) + \omega t\,\cos(\omega t))\,\hat{y} - \omega\sin(\omega t)\,\hat{z}\right).$$

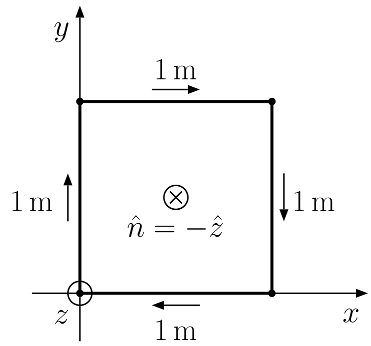
a) For the rectangular path shown in the figure below,



the area of the enclosed surface S is $1\,\text{m}^2$ and the unit vector normal to S is $\hat{n} = \hat{z}$. Therefore, the electromotive force is

$$\mathcal{E} = -\frac{d\vec{B}}{dt}\cdot\hat{z} = B_0\omega\sin(\omega t)\,\text{V}.$$

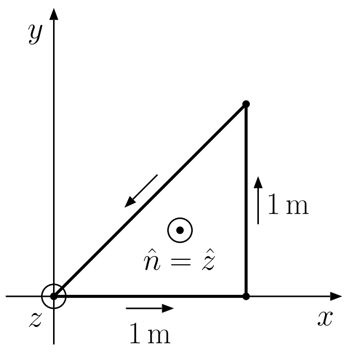
b) If we consider the same rectangle of part (a), but the direction of the path is reversed, we have that $\hat{n} = -\hat{z}$.



As a result, the electromotive force is

$$\mathcal{E} = -\frac{d\vec{B}}{dt}\cdot(-\hat{z}) = -B_0\omega\sin(\omega t)\,\text{V}.$$

c) For the triangular path shown in the figure below,



the area of the enclosed surface S is $\frac{1}{2}\,\mathrm{m}^2$ and the unit vector normal to S is $\hat{n} = \hat{z}$. Then, the electromotive force is

$$\mathcal{E} = -\frac{d\vec{B}}{dt} \cdot \hat{z} \cdot \frac{1}{2} = \frac{1}{2} B_0 \omega \sin(\omega t)\,\mathrm{V}.$$

2. Consider a square loop of wire free to rotate around the $\hat{x}$ axis that has some finite resistance R with $2\,\mathrm{cm}^2$ surface area and located in a region of constant magnetic field $\vec{B} = 4\hat{z}\,\mathrm{Wb/m}^2$. For the **dS** vector given in the figure, the associated contour (which follows the right-hand-rule) is in the clockwise direction when viewed looking down from the $+z$ direction.

   a) If $\theta = 0°$, the differential surface vector is $d\vec{S} = -\hat{z}dS$, and therefore, the magnetic flux is

   $$\Psi = \int_S \vec{B} \cdot d\vec{S} = \vec{B} \cdot \int_S d\vec{S} = 4\hat{z}\,\frac{\mathrm{Wb}}{\mathrm{m}^2} \cdot (-2 \times 10^{-4}\,\mathrm{m}^2\,\hat{z}) = -8 \times 10^{-4}\mathrm{Wb}.$$

   b) For any angle θ, the differential area is $d\vec{S} = -\left(\cos\theta\hat{z} + \sin\theta\hat{y}\right) dS$, and therefore, the magnetic flux is

   $$\Psi = \int_S \vec{B} \cdot d\vec{S} = 4\hat{z} \cdot \left(-2 \times 10^{-4} \left(\cos\theta\hat{z} + \sin\theta\hat{y}\right)\right) = -8 \times 10^{-4} \cos\theta\,\mathrm{Wb}.$$

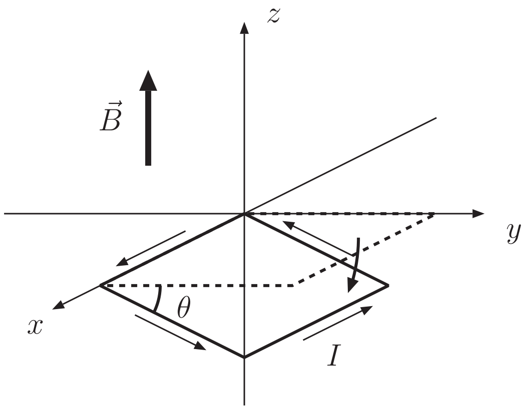
   c) The induced emf ($\mathcal{E} = \int_C \vec{E} \cdot d\vec{l}$) can be found to be

   $$\mathcal{E} = -\frac{d\Psi}{dt} = -8 \times 10^{-4} \sin\theta \frac{d\theta}{dt}.$$

   Thus, considering $\theta = \frac{\pi}{4}$ rad and $\frac{d\theta}{dt} = \pi\,\mathrm{rad/s}$, we get

   $$\mathcal{E}(\theta = \frac{\pi}{4}) = -8 \times 10^{-4} \left(\sin\frac{\pi}{4}\right)(\pi) = -4\sqrt{2}\pi \times 10^{-4}\,\mathrm{V}.$$

   d) At the time when $\theta = \frac{\pi}{4}$, the induced emf $\mathcal{E}$ is negative, therefore, the current on the wire at the same instant flows in the direction of the contour as it is shown in the figure below. We also could have found the direction of the induced current applying Lenz's rule, which states that the induced current generates an induced magnetic field that opposes the change in magnetic flux caused by the rotation of the loop. In this case, $\frac{d\theta}{dt} > 0$, such that θ is getting greater with increasing time and the loop is moving downwards away from the $z = 0$ plane from its location at $\theta = \frac{\pi}{4}$. Since the magnetic flux in the $+z$ direction through the loop is decreasing, a counterclockwise current (when viewed from above) is required to generate an induced magnetic field having a positive B_z component.

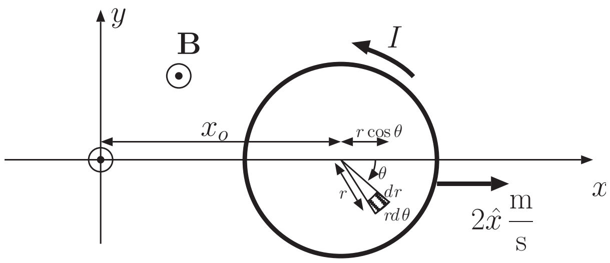


e) Since opposite $d\vec{S}$ is used, the contour direction is also opposite (i.e., it is now counterclockwise when viewed from above). The calculated emf has the opposite algebraic sign: $\mathcal{E}= 4\sqrt{2}\pi \times 10^{-4}$ V, and since $\mathcal{E} > 0$, the current must flow in the same direction as the contour, that is, counterclockwise when viewed from above. Note that the direction of the induced current is the same regardless of the **dS** orientation used to find it.

3. Consider a conducting wire loop of radius $r = 1\,\mathrm{m}$ moving with velocity $v = 2\hat{x}\,\mathrm{m/s}$ in a region where

$$B(x, y, z) = \hat{z}25 \times 10^{-6}(1 + \frac{x}{L})\,\mathrm{T}.$$

The geometry of the problem is shown in the figure below at some time t_0, when the loop has moved to position $x_0 = 2t_0$.



a) Given the described geometry, we have magnetic flux as

$$\Psi = \int_S \vec{B} \cdot d\vec{S} \;=\; \int_0^r \int_0^{2\pi} 25 \times 10^{-6}(1 + \frac{x_o + r\cos\theta}{L})rd\theta dr,$$

If the magnetic field strength varied significantly over the area of the loop surface, we would have to calculate this integral explicitly with r and θ expressed in terms of x. However, the scale size of magnetic field variation is much smaller than the size of the loop, such that B_z can be treated as a constant:

$$\Phi = 25 \times 10^{-6}\left(1 + \frac{x_o}{L}\right)\int_S dS = 25 \times 10^{-6}\left(1 + \frac{x_o}{L}\right)\pi r^2$$

Thus, the emf $\mathcal{E}$ is

$$\mathcal{E} = -\frac{d\Psi}{dt} = -25\pi r^2 \times 10^{-6} \times \frac{1}{L} \times \frac{dx_o}{dt} = -25\pi \times 10^{-6} \times \frac{1}{1000} \times 2$$
$$= -157.08\,\mathrm{nV}.$$

b) The magnitude of the loop current with resistance $2\,\Omega$ is

$$I = \frac{|\mathcal{E}|}{R} = 78.54\,\mathrm{nA}.$$

4. Two concentric circular wire loops of radii $a = 10\,\text{cm}$ and $b = 0.25\,\text{cm}$ are placed on the $x-y$ plane, with their centers at the origin.

   a) The current in the outer loop can be calculated as follows

$$I_a = \frac{V}{R} = \frac{V}{\frac{2\pi a}{\sigma \pi r^2}} = \frac{5}{\frac{2\times 0.1}{4\times 10^7 \times 0.001^2}} = 1000\,(\text{A}).$$

   b) Refering to the Lecture 13, the magnetic flux density generated by the outer loop only has z component along the z axis

$$B_z = \frac{\mu_o I_a a^2}{2(a^2+z^2)^{\frac{3}{2}}}.$$

   At the origin, $B_z(z=0) = \frac{\mu_o I_a}{2a}\,(\text{Wb/m}^2)$. Since $b << a$, we assume the B_z across the inner loop is constant. Thus,

$$\Psi_{a\to b} = \int_S B\cdot dS = \int_S B_z(z=0)\hat{z}\cdot\hat{z}dS = \frac{\mu_o I_a}{2a}\pi b^2 = \frac{\mu_o \pi b^2}{2a} I_a\,(\text{Wb}).$$

   c) The numerical value of $L_{a\to b}$ is

$$L_{a\to b} = \frac{\Psi_{a\to b}}{I_a} = \frac{\mu_o \pi b^2}{2a} = 1.23\times 10^{-10}\,(\text{H}).$$

   d) Applying induced emf formula $\mathcal{E} = -\frac{d\Psi}{dt}$, we get

$$\mathcal{E}_{a\to b} = -\frac{d\Psi_{a\to b}}{dt} = -\frac{d}{dt}\int_S B\cdot dS = -\frac{d}{dt}(B_z A_b) = -A_b \frac{d}{dt} B_z.$$

   Here

$$B_z = \frac{\mu_o I_a a^2}{2(a^2+z^2)^{\frac{3}{2}}},$$

   Therefore,

$$\frac{d}{dt}B_z = -\frac{3}{2}\frac{\mu_o I_a a^2 z}{(a^2+z^2)^{\frac{5}{2}}}\frac{d}{dt}z = -\frac{3}{2}\frac{\mu_o I_a a^2 z}{(a^2+z^2)^{\frac{5}{2}}} v_z.$$

   Finally,

$$\mathcal{E}_{a\to b} = \frac{3}{2}\frac{\mu_o I_a a^2 z \pi b^2}{(a^2+z^2)^{\frac{5}{2}}}\,(\text{V}).$$

   If we consider, $z = v_z t$, we can write $\mathcal{E}_{a\to b}$ as a function of time t as (here $v_z = 1$)

$$\mathcal{E}_{a\to b}(t) = \frac{3}{2}\frac{\mu_o I_a a^2 t \pi b^2}{(a^2+t^2)^{\frac{5}{2}}}\,(\text{V}).$$

   e) We know that the induced emf $\mathcal{E}$ is given by

$$\mathcal{E} = -\frac{d\Psi_t}{dt},$$

   where Ψ_t = total flux and also the current I is given by

$$I = \frac{\mathcal{E}}{R}.$$

   From the above two equations, the induced current in the inner loop will be

$$I_b = -\frac{1}{R}\frac{d\Psi_t}{dt},$$

where $\Psi_t = \Psi_b + \Psi_s$. Here $\Psi_b = -\int \mathcal{E}_{a\to b}(t)\, dt$ and $\Psi_s = LI_b$ ($L = 0.25\,\mu$H). Thus,

$$I_b = -\frac{1}{R}\frac{d\Psi_b}{dt} - \frac{L}{R}\frac{dI_b}{dt},$$

rearranging the above equation will give us the differential equation as

$$\frac{dI_b}{dt} + \frac{R}{L}I_b = -\frac{1}{L}\frac{d\Psi_b}{dt},$$

where right hand side is the emf $\mathcal{E}_{a\to b}(t)$ of part (d) divided by L.

5. (a) At time t, area of the loop$=L(Z_0+V_0t)$; for clockwise contour (following ABCD), $d\vec{S} = dS(-\hat{x})$.

$$\begin{aligned}
\Psi &= \int_S \vec{B}\cdot d\vec{S} \\
&= B_0L(Z_0 + V_0t) \\
\mathcal{E} &= -\frac{d\Psi}{dt} = -B_0LV_0\,(V)
\end{aligned}$$

(b) $\mathcal{E}<0$ so the current flows opposite to the original contour, i.e. along DCBA (counterclockwise),

$$I_0 = \frac{\mathcal{E}}{R} = \frac{-B_0LV_0}{R}\,(A)$$

(c) Lorentz force per unit length

$$\begin{aligned}
d\overrightarrow{F} &= I_0 d\overrightarrow{l} \times \overrightarrow{B} \\
&= I_0(dy\hat{y}) \times B_0(-\hat{x}) \\
&= I_0B_0dy(\hat{z})
\end{aligned}$$

Therefore the total $\overrightarrow{F}$

$$\overrightarrow{F} = \int dF = \int_0^L I_0B_0dy(\hat{z}) = I_0B_0L(\hat{z})\,(N)$$

For I_0 as above

$$\overrightarrow{F} = \frac{B_0^2L^2V_0}{R}(-\hat{z})\,(N)$$

(d) Since the current flows along the contour ABCD, $I'$is in the $+\hat{y}$ direction in the armature. Similar to part (c),

$$\begin{aligned}
\overrightarrow{F} &= I'\hat{y} \times B_0(-\hat{x})L \\
&= I'B_0L(\hat{z})\,(N) \\
\overrightarrow{a} &= \frac{I'B_0L}{M}(\hat{z})\,(m/s^2)
\end{aligned}$$

Alternatively, we could find the total force by superposing the Lorentz forces on each charge in the armature:

$$\vec{F} = (q\vec{v} \times \vec{B_0})NLA$$

where N is the number density of the charges in the armature, A is its cross-sectional area, and LA is its volume. Since $\vec{J} = qN\vec{v} = \frac{I'}{A}\hat{z}$, we can arrive at the same result above.