University of Illinois **Spring 2014**

ECE 310: Problem Set 1: Problems and Solutions
DSP overview, Continuous-time (CT) and discrete-time (DT) signals, Complex numbers, Impulses

Due: Wednesday January 29 at 6 p.m.
Reading: *310 Course Notes* Ch 1, Appendix A, Appendix D

1. **[Complex Variables]**
   Evaluate and represent your final answer in both Cartesian and polar forms. Try to simplify as far as you can by hand, without using a calculator or computer.

   (a) $(3\angle 150°) + (5\angle -60°) + (4\angle 120°)$
   Solution:

$$\begin{aligned}
&(3\angle 150°) + (5\angle -60°) + (4\angle 120°)\\
=&3(-\frac{\sqrt{3}}{2} + j\frac{1}{2}) + 5(\frac{1}{2} - j\frac{\sqrt{3}}{2}) + 4(-\frac{1}{2} + j\frac{\sqrt{3}}{2})\\
=&(-\frac{3\sqrt{3}}{2} + \frac{1}{2}) + j(\frac{3}{2} - \frac{\sqrt{3}}{2})\\
\approx&2.1918e^{j2.8481}
\end{aligned}$$

   (b) $\frac{(-1+j)^5}{1+j}$
   Solution:

$$\begin{aligned}
&\frac{(-1+j)^5}{1+j}\\
=&\frac{(\sqrt{2}\angle 135°)^5}{\sqrt{2}\angle 45°}\\
=&4\angle 90°\\
=&-4j
\end{aligned}$$

   (c) $\frac{(5\angle 60°)}{2j} + \frac{\sqrt{2}e^{j\pi}}{2-j}$
   Solution:

$$\begin{aligned}
&\frac{(5\angle 60°)}{2j} + \frac{\sqrt{2}e^{j\pi}}{2-j}\\
=&\frac{5/2 + j5\sqrt{3}/2}{2j} + \frac{-\sqrt{2}(2+j)}{5}\\
=&(\frac{5\sqrt{3}}{4} - \frac{2\sqrt{2}}{5}) + j(-\frac{5}{4} - \frac{\sqrt{2}}{5})\\
\approx&2.2153e^{-j0.7642}
\end{aligned}$$

(d) $(\frac{-1+j3}{1-j} + \frac{3+j}{1+j2})^n$
Solution:

$$
\begin{aligned}
&(\frac{-1+j3}{1-j} + \frac{3+j}{1+j2})^n \\
=&(-2+j+1-j)^n \\
=&(-1)^n \\
=&e^{jn\pi}
\end{aligned}
$$

2. **[Magnitude and Phase]**
Derive close form expressions for the magnitude and phase of the function $G(\omega)$ of the real variable ω, where $G(\omega) = 1 - e^{-j\omega}$, and sketch (by hand) the magnitude and phase over the interval $\omega \in [-\pi, \pi]$. Label your plots.

Solution:

$$
\begin{aligned}
G(\omega) &= 1 - e^{-j\omega} \\
&= e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2}) \\
&= e^{-j\omega/2} 2j \sin(\omega/2)
\end{aligned}
$$

Magnitude:

$$|G(\omega)| = 2|\sin(\omega/2)|$$

Phase:

$$\arg(G(\omega)) = \begin{cases} \frac{\pi}{2} - \frac{\omega}{2}, & \text{if } \omega > 0 \\ -\frac{\pi}{2} - \frac{\omega}{2}, & \text{if } \omega < 0 \end{cases}$$

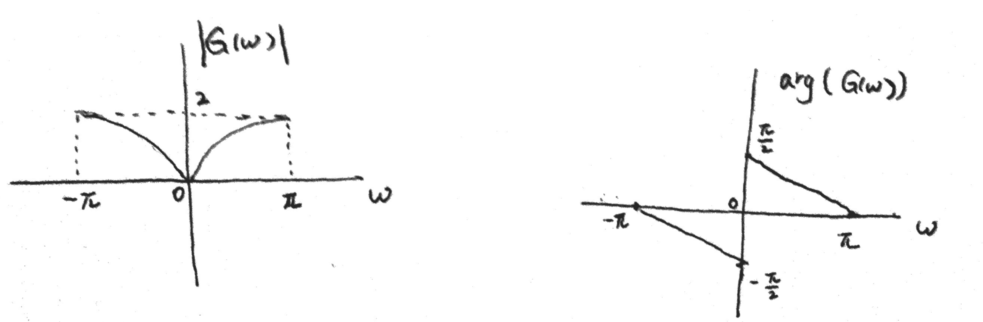


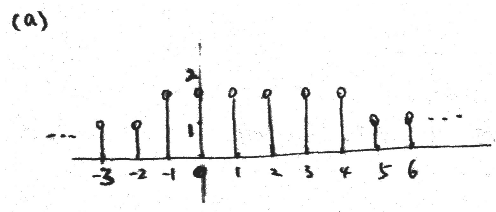
Figure 1: Magnitude and phase of $G(\omega)$

3. **[Discrete-Time Signals]**
Sketch the following signals ($u[n]$ is the unit step function in the discrete-time variable n):

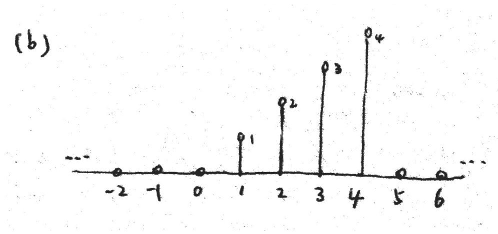
(a) $u[n+1] + u[-n+4]$
Solution: See figure 2(a).

(b) $n\ (u[n] - u[n-5])$
Solution: See figure 2(b).

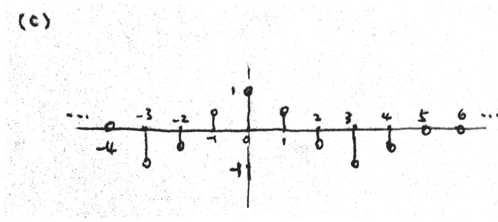
(c) $\cos(\frac{n\pi}{3})\ u[-n+4]\ u[n+3]$
Solution: See figure 2(c).



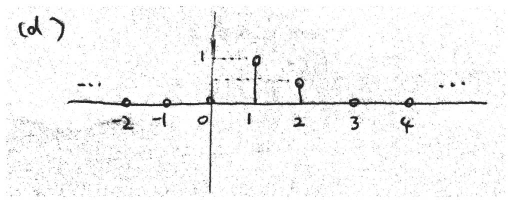
(a)



(b)



(c)



(d)

Figure 2: Problem 3

(d) $(\frac{1}{2})^n\ u[n-1]\ u[-n+2]$
Solution: See figure 2(d).

4. **[Periodic Signals]**
Determine if each of the following signals is periodic. If the signal is periodic, determine its fundamental period.

(a) $x[n] = \sin(\frac{\pi n}{5})$
Solution: Let $N\pi/5 = 2k\pi$. Fundamental period $N = 10$ $(k = 1)$.

(b) $x[n] = \cos(\frac{\pi n}{5})$
Solution: Let $N\pi/5 = 2k\pi$. Fundamental period $N = 10$ $(k = 1)$.

(c) $x[n] = \cos(\frac{3\pi n}{5})$
Solution: Let $3N\pi/5 = 2k\pi$. Fundamental period $N = 10$ $(k = 3)$.

(d) $x[n] = e^{j\pi n/3}$
Solution: Let $N\pi/3 = 2k\pi$. Fundamental period $N = 6$ $(k = 1)$.

5. **[Impulses]**
Evaluate the following integrals, where $\delta(t)$ is the Dirac delta function and $u(t)$ is the unit step function:

(a) $\int_{-\infty}^{\infty}(t^2+t-9)\ \delta(t+2)\ dt$
Solution: $\int_{-\infty}^{\infty}(t^2+t-9)\ \delta(t+2)\ dt = (t^2+t-9)|_{t=-2} = -7$.

(b) $\int_{3}^{\infty}(t^2+t-9)\ \delta(t-2)\ dt$
Solution: $\int_{3}^{\infty}(t^2+t-9)\ \delta(t-2)\ dt = 0$.

(c) $\int_{-\infty}^{3}(t^2+t-9)\ \delta(t-2)\ dt$
Solution: $\int_{-\infty}^{3}(t^2+t-9)\ \delta(t-2)\ dt = (t^2+t-9)|_{t=2} = -3$.

(d) $\int_{-\infty}^{1}(t^2+t-9)\ \delta(3t-2)\ dt$

Solution:

$$\int_{-\infty}^{1}(t^2+t-9)\ \delta(3t-2)\ dt = \frac{1}{3}\int_{-\infty}^{1}(t^2+t-9)\ \delta(t-\frac{2}{3})\ dt = \frac{1}{3}(t^2+t-9)|_{t=\frac{2}{3}} = -\frac{71}{27}.$$

(e) $[e^{-t}u(t)] * \delta(3t-2)$, where $*$ is the convolution

Solution:

$$[e^{-t}u(t)] * \delta(3t-2) = \int_{-\infty}^{\infty} e^{-(t-\tau)}u(t-\tau)\delta(3\tau-2)\ d\tau = \frac{1}{3}e^{-(t-\frac{2}{3})}u(t-\frac{2}{3}).$$

6. **[Fourier Transform]**
Determine the Fourier transform of the following signals. Note that $u(t)$ denotes the unite step in continuous time.

(a) $\delta(3t+2)$

Solution:

$$\mathcal{F}[\delta(3t+2)] = \frac{1}{3}\int_{-\infty}^{\infty}\delta(t+\frac{2}{3})e^{-j\Omega t}\ dt = \frac{1}{3}e^{j\frac{2}{3}\Omega}.$$

(b) $\cos(2\Omega_0 t+\phi)$, where Ω and ϕ are known real numbers.

Solution:

$$\begin{aligned}\mathcal{F}[\cos(2\Omega_0 t+\phi)] &= \mathcal{F}\left[\frac{1}{2}e^{j(2\Omega_0 t+\phi)} + \frac{1}{2}e^{-j(2\Omega_0 t+\phi)}\right]\\ &= \frac{\pi}{2}\left[e^{j\phi}\delta(\Omega-2\Omega_0) + e^{-j\phi}\delta(\Omega+2\Omega_0)\right].\end{aligned}$$

(c) $e^{-a|t|}$

Solution:

$$\begin{aligned}\mathcal{F}\left[e^{-a|t|}\right] &= \int_{-\infty}^{0} e^{at}e^{-j\Omega t}\ dt + \int_{0}^{\infty} e^{-at}e^{-j\Omega t}\ dt\\ &= \frac{1}{a+j\Omega} + \frac{1}{a-j\Omega}\\ &= \frac{2a}{a^2+\Omega^2}.\end{aligned}$$

(d) $u(t)-u(t-T)$, where T is a known real number.

Solution:

$$\begin{aligned}
\mathcal{F}\left[u(t)-u(t-T)\right] &= \int_0^T e^{-j\Omega t}\ dt \\
&= -\left.\frac{e^{-j\Omega t}}{j\Omega}\right|_0^T \\
&= -\frac{1}{j\Omega}\left(e^{-j\Omega T}-1\right) \\
&= \frac{1}{j\Omega}e^{-j\Omega T/2}\left(e^{j\Omega T/2}-e^{-j\Omega T/2}\right) \\
&= Te^{-j\Omega T/2}\frac{\sin\left(\frac{\Omega T}{2}\right)}{\frac{\Omega T}{2}} \\
&= Te^{-j\Omega T/2}\operatorname{sinc}\left(\frac{\Omega T}{2}\right)
\end{aligned}$$

(e) $(u(t-1)-u(t-6))\ e^{j2\pi t}$

Solution: Use the result in last part,

$$\mathcal{F}\left[(u(t)-u(t-5))\right] = 5e^{-j\frac{5}{2}\Omega}\operatorname{sinc}\left(\frac{5\Omega}{2}\right).$$

Shift by 1,

$$\mathcal{F}\left[(u(t-1)-u(t-6))\right] = (5e^{-j\frac{5}{2}\Omega}\operatorname{sinc}\left(\frac{5\Omega}{2}\right))e^{-j\Omega} = 5e^{-j\frac{7}{2}\Omega}\operatorname{sinc}\left(\frac{5\Omega}{2}\right).$$

Modulate,

$$\begin{aligned}
\mathcal{F}\left[(u(t-1)-u(t-6))e^{j2\pi t}\right] &= 5e^{-j\frac{7}{2}(\Omega-2\pi)}\operatorname{sinc}\left(\frac{5}{2}(\Omega-2\pi)\right) \\
&= -5e^{-j\frac{7}{2}\Omega}\operatorname{sinc}\left(\frac{5}{2}(\Omega-2\pi)\right).
\end{aligned}$$