**University of Illinois** **Spring 2014**

# ECE 310: Problem Set 3: Problems and Solutions
## Discrete Fourier transform (DFT), DFT spectral analysis

**Due:** Wednesday February 12 at 6 p.m.
**Reading:** *310 Course Notes* Ch 2.5, 2.6
**Version:** 1.1

1. **[DFT]**
The discrete Fourier transform (DFT) of a finite-length sequence $x[n]$, defined only over the range $0 \leq n \leq N-1$, is given by

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} \ , \ 0 \leq k \leq N-1.$$

For each of the following finite length sequences, determine the corresponding DFT, $X[k]$.

(a) $x[n] = \delta[n-3] \ , \ 0 \leq n \leq 3$
**Solution:**

$$X[k] = \sum_{n=0}^{3} \delta[n-3] e^{-j\frac{2\pi kn}{4}} \ , \ 0 \leq k \leq 3.$$

$$X[k] = e^{-j\frac{3\pi k}{2}} \ , \ 0 \leq k \leq 3$$

$$X[k] = \{1, j, -1, -j\}$$

(b) $x[n] = \begin{cases} 1, & 0 \leq n \leq 2 \\ 0, & 3 \leq n \leq 5 \end{cases}$
**Solution:**

$$X[k] = \sum_{n=0}^{2} e^{-j\frac{\pi kn}{3}} = \frac{1-e^{-j\pi k}}{1-e^{\frac{-j\pi k}{3}}} = e^{-j\frac{\pi k}{3}} \frac{\sin\frac{\pi k}{2}}{\sin\frac{\pi k}{6}}, \quad k \neq 0$$

$$X[k] = \begin{cases} 3, & k = 0 \\ e^{-j\frac{\pi k}{3}} \frac{\sin\frac{\pi k}{2}}{\sin\frac{\pi k}{6}}, & 1 \leq k \leq 5 \end{cases}$$

(c) $x[n] = \cos\left(\frac{\pi n}{4}\right) \ , \ 0 \leq n \leq 7$
**Solution:**

$$X[k] = \sum_{n=0}^{7} \left[\frac{1}{2}\left(e^{j\frac{\pi n}{4}} + e^{-j\frac{\pi n}{4}}\right) e^{-j\frac{\pi kn}{4}}\right], \quad 0 \leq k \leq 7.$$

$$X[k] = \frac{1}{2}\left[\frac{1-e^{-j2\pi(k+1)}}{1-e^{-j\frac{\pi}{4}(k+1)}} + \frac{1-e^{-j2\pi(k-1)}}{1-e^{-j\frac{\pi}{4}(k-1)}}\right], \quad 0 \leq k \leq 7.$$

If $k \neq 1$ and $k \neq 7$, $X[k] = 0$.
If $k = 1$, $X[1] = \frac{1}{2}(8+0) = 4$.
If $k = 7$, $X[7] = \frac{1}{2}(0+8) = 4$.
Therefore, $X[k] = 4\delta[k-1] + 4\delta[k-7]$.

(d) $x[n] = \begin{cases} 1, & n \text{ even }, \ 0 \le n \le 6 \\ 0, & n \text{ odd }, \ 0 \le n \le 6 \end{cases}$

**Solution:** Since $x[n] = 1$ when $n$ is even, let $n = 2m$, $m = 0, 1, 2, 3$.

$$X[k] = \sum_{m=0}^{3} e^{-j\frac{2\pi k}{7}2m} = \frac{1 - e^{-j\frac{4\pi k}{7}\times 4}}{1 - e^{-j\frac{4\pi k}{7}}}, \ k \neq 0$$

$$X[k] = \begin{cases} 4, & k = 0 \\ \frac{1-e^{-j\frac{16\pi k}{7}}}{1-e^{-j\frac{4\pi k}{7}}}, & k \neq 0 \end{cases}$$

(e) Sketch the magnitude and phase for parts (a) and (b).

**Solution:**

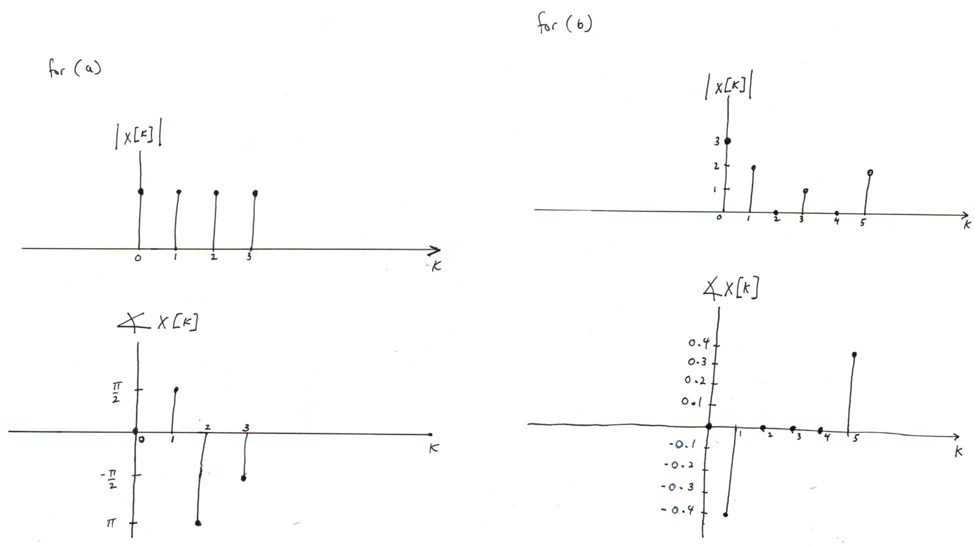


Figure 1: Magnitude and phase of 1(a-b)

2. **[Inverse DFT]**

(a) Find the inverse DFT of the sequence $X[k] = \{1, e^{-j\pi/2}, 0, e^{j\pi/2}\}$, where the first entry of $X[k]$ corresponds to $k = 0$.

**Solution:** N=4.

$$\begin{aligned} x[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi kn}{N}}, \ 0 \le n \le N-1. \\ &= \frac{1}{4}\left[X[0] + X[1]e^{j\frac{\pi n}{2}} + X[2]e^{j\pi n} + X[3]e^{j\frac{3\pi}{2}n}\right], \ 0 \le n \le N-1. \\ &= \frac{1}{4}\left[1 + e^{-j\pi/2}e^{j\frac{\pi n}{2}} + e^{j\pi/2}e^{j\frac{3\pi}{2}n}\right], \ 0 \le n \le N-1. \\ x[0] &= \frac{1}{4}, x[1] = \frac{3}{4}, x[2] = \frac{1}{4}, x[3] = -\frac{1}{4} \end{aligned}$$

(b) Without explicitly computing the inverse DFT sum, find the inverse DFT of the sequence $Y[k] = \{1, -1, 0, -1\}$,where the first entry of $Y[k]$ corresponds to $k = 0$, using your answer to part (a).

**Solution:**

Note: This problem uses the circular time shift property of the DFT:

$$x\left[\ll n - d \gg_N\right] \xleftrightarrow{\text{DFT}} X[k]e^{-j\frac{2\pi kd}{N}}.$$

Note that $Y[k] = X[k]e^{-j\frac{\pi}{2}k} = X[k]e^{-j\frac{2\pi k}{4}}$ is the modulation of $X[k]$, hence

$$y[n] = x\left[\ll n - 1 \gg_4\right] = \{-\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4}\}.$$

*Note that $\ll n \gg_N$ is used in the notes to denote $n$ mod $N$. An alternative notation is $[[n]]_N$. Both notations are considered eligible in this homework assignment.*

3. **[DFT of related sequences]**
Given the eight-point DFT of the sequence

$$x[n] = \begin{cases} 1, & 0 \le n \le 3 \\ 0, & 4 \le n \le 7 \end{cases}$$

compute the DFT of the following sequences (i.e., give your answer in terms of $X[k]$).

(a) $x_1[n] = \begin{cases} 1, & n = 0 \\ 0, & 1 \le n \le 4 \\ 1, & 5 \le n \le 7 \end{cases}$

**Solution:** Use circular time shift property: $x\left[\ll n - d \gg_N\right] \xleftrightarrow{\text{DFT}} X[k]e^{-j\frac{2\pi kd}{N}}$.
$x_1[n] = x[\ll n - 5 \gg_8] \xleftrightarrow{\text{DFT}} X[k]e^{-j\frac{5\pi k}{4}}$.

(b) $x_2[n] = \begin{cases} 0, & 0 \le n \le 2 \\ 1, & 3 \le n \le 6 \\ 0, & n = 7 \end{cases}$

**Solution:** Use circular time shift property: $x\left[\ll n - d \gg_N\right] \xleftrightarrow{\text{DFT}} X[k]e^{-j\frac{2\pi kd}{N}}$.
$x_2[n] = x[\ll n - 3 \gg_8] \xleftrightarrow{\text{DFT}} X[k]e^{-j\frac{3\pi k}{4}}$.

4. **[DFT of related sequences]**
Two finite-length sequences are defined as $\{x[n]\}_{n=0}^{7} = \{0, a, b, c, d, e, 0, 0\}$ and $\{y[n]\}_{n=0}^{7} = \{d, e, 0, 0, 0, a, b, c\}$. Determine the relation between $\{X[k]\}_{m=0}^{7}$ and $\{Y[k]\}_{m=0}^{7}$, the 8-point DFTs of the two sequences.

**Solution:** $y[n]$ is the circular shift of $x[n]$. By circular time shift property,

$$y[n] = x[\ll n - 4 \gg_N] \xleftrightarrow{\text{DFT}} Y[k] = X[k]e^{-j\frac{2\pi 4k}{8}} = (-1)^k X[k].$$

5. **[DFT of sampled signals]**
A real continuous-time signal $x_a(t)$ is bandlimited to frequencies below 5kHz, i.e., $X(\Omega) = 0$ for $|\Omega| \geq 2\pi(5000)$. The signal $x_a(t)$ is sampled with a sampling rate of 10,000 samples per second to produce a sequence $x[n] = x_a(nT)$ with $T = 10^{-4}$. Let $X[k]$ be the 1000-point DFT of $x[n]$. To what continuous-time frequency do the indices $k = 150$ and $k = 800$ correspond?

**Solution:** $T = 10^{-4}$, $N = 1000$. Continuous signal is bandlimited, $X_c(\Omega) = 0$ for $|\Omega| \geq 2\pi(5000)$. Hence sampled signal $X_d(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(\frac{\omega+2k\pi}{T}) = \frac{1}{T}X_c(\frac{\omega}{T})$ for $|\omega| < \pi$.

DFT $X[k] = X_d(\frac{2\pi k}{N})$, hence for $N$ even,

$$X[k] = \begin{cases} \frac{1}{T}X_c(\frac{2\pi k}{NT}), & 0 \leq k \leq \frac{N}{2} - 1 \\ \frac{1}{T}X_c(\frac{2\pi(k-N)}{NT}), & \frac{N}{2} \leq k \leq N-1 \end{cases}$$

The first half of DFT samples provide samples of $X_c(\Omega)$ for $\Omega > 0$, while the second half of the DFT gives samples of $X_c(\Omega)$ for $\Omega < 0$ (page 60 of the notes).

Therefore, index $k = 150$ corresponds to continous-time frequency $\Omega = \frac{2\pi k}{NT} = 3000\pi$, or $f = 1500Hz$. Index $k = 800$ corresponds to continous-time frequency $\Omega = \frac{2\pi(k-N)}{NT} = -4000\pi$, or $f = -2000Hz$.

6. **[Spectral analysis]**
Let $x[n] = \cos(\frac{2\pi n}{3})$ and $v[n]$ be the sequence obtained by applying a 32-point rectangular window to $x[n]$ before computing $V_d(\omega)$. Sketch $V_d(\omega)$ for $-\pi \leq \omega \leq \pi$, labeling the frequencies of all peaks and the first nulls on either side of the peak. In addition, label the amplitudes of the peaks and the strongest side lobe of each peak. If your sketch is approximate, indicate the approximation involved.

**Solution:** There Let $w[n] = \left\{ \begin{array}{ll} 1 & 0 \leq n < 32 \\ 0 & \text{otherwise} \end{array} \right.$. Then $v[n] = x[n]w[n]$.

$$\begin{aligned} V_d(\omega) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\nu)W_d(\omega-\nu)d\nu \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \pi\left(\delta(\nu - \frac{2\pi}{3}) + \delta(\nu + \frac{2\pi}{3})\right)\left(\frac{\sin(16(\omega-\nu))}{\sin((\omega-\nu)/2)}e^{-\frac{j31(\omega-\nu)}{2}}\right) \\ &= \frac{1}{2}\left(\frac{\sin(16(\omega-\frac{2\pi}{3}))}{\sin((\omega-\frac{2\pi}{3})/2)}e^{-\frac{j31(\omega-\frac{2\pi}{3})}{2}} + \frac{\sin(16(\omega+\frac{2\pi}{3}))}{\sin((\omega+\frac{2\pi}{3})/2)}e^{-\frac{j31(\omega+\frac{2\pi}{3})}{2}}\right) \end{aligned}$$

An alternative way of computing $V_d(\omega)$ is described in the example in page 51 of the notes. Applying the 32-point rectangular window is equivalent to truncating the sequence to 32 points. In our case, $v[n] = \cos\left(\frac{2\pi n}{3}\right) = \frac{1}{2}\left(e^{j\frac{2\pi n}{3}} + e^{-j\frac{2\pi n}{3}}\right)$, $0 \leq n < 32$. DTFT yields two finite geometric series, which sum to the following:

$$\begin{aligned} V_d(\omega) &= \sum_{n=0}^{31}\frac{1}{2}\left(e^{j\frac{2\pi n}{3}} + e^{-j\frac{2\pi n}{3}}\right)e^{-j\omega n} \\ &= \frac{1}{2}\left(\frac{1-e^{-j32(\omega-\frac{2\pi}{3})}}{1-e^{-j(\omega-\frac{2\pi}{3})}} + \frac{1-e^{-j32(\omega+\frac{2\pi}{3})}}{1-e^{-j(\omega+\frac{2\pi}{3})}}\right) \\ &= \frac{1}{2}\left(\frac{\sin(16(\omega-\frac{2\pi}{3}))}{\sin((\omega-\frac{2\pi}{3})/2)}e^{-\frac{j31(\omega-\frac{2\pi}{3})}{2}} + \frac{\sin(16(\omega+\frac{2\pi}{3}))}{\sin((\omega+\frac{2\pi}{3})/2)}e^{-\frac{j31(\omega+\frac{2\pi}{3})}{2}}\right) \end{aligned}$$

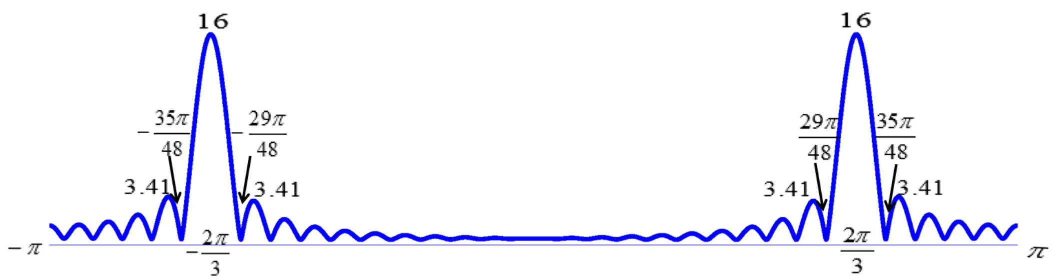


Figure 2: Magnitude of $V_d(\omega)$

$V_d(\omega)$ is the sum of two sinc functions centered at $\pm\frac{2\pi}{3}$. We use the following approximations:

(a) The expressioni $V_d(\omega)$ is complex, we will plot the magnitude $|V_d(\omega)|$.

(b) We will approximate $|V_d(\omega)|$ as the sum of the magnitudes of the two terms.

$$\begin{aligned}
|V_d(\omega)| &= \left|\frac{1}{2}\frac{\sin(16(\omega-\frac{2\pi}{3}))}{\sin((\omega-\frac{2\pi}{3})/2)}e^{-\frac{j31(\omega-\frac{2\pi}{3})}{2}} + \frac{1}{2}\frac{\sin(16(\omega+\frac{2\pi}{3}))}{\sin((\omega+\frac{2\pi}{3})/2)}e^{-\frac{j31(\omega+\frac{2\pi}{3})}{2}}\right| \\
&\approx \left|\frac{1}{2}\frac{\sin(16(\omega-\frac{2\pi}{3}))}{\sin((\omega-\frac{2\pi}{3})/2)}e^{-\frac{j31(\omega-\frac{2\pi}{3})}{2}}\right| + \left|\frac{1}{2}\frac{\sin(16(\omega+\frac{2\pi}{3}))}{\sin((\omega+\frac{2\pi}{3})/2)}e^{-\frac{j31(\omega+\frac{2\pi}{3})}{2}}\right| \\
&= \frac{1}{2}\left|\frac{\sin(16(\omega-\frac{2\pi}{3}))}{\sin((\omega-\frac{2\pi}{3})/2)}\right| + \frac{1}{2}\left|\frac{\sin(16(\omega+\frac{2\pi}{3}))}{\sin((\omega+\frac{2\pi}{3})/2)}\right|.
\end{aligned}$$

(c) The main contributions of these two sinc functions do not substantially overlap. We only consider the contribution of the first term for positive frequencies, contribution of the second term for negative frequencies.

Therefore, the peaks are located at $\pm\frac{2\pi}{3}$. The amplitude of each main lobe is $\lim_{\omega\to 0}\frac{1}{2}\left|\frac{\sin(16\omega)}{\sin(\omega/2)}\right| = 16$. The width of each main lobe is $\frac{4\pi}{N} = \frac{\pi}{8}$, and the first nulls on either side of peaks are $\pm\frac{2\pi}{3}\pm\frac{\pi}{16} = \{-\frac{35\pi}{48}, -\frac{29\pi}{48}, \frac{29\pi}{48}, \frac{35\pi}{48}\}$. The strongest side lobes are located at $\pm\frac{2\pi}{3}\pm\frac{3\pi}{32}$, and the amplitude of each of these side lobes is $\frac{1}{2}\left|\frac{\sin(16\times\frac{3\pi}{32})}{\sin(\frac{1}{2}\times\frac{3\pi}{32})}\right| \approx 3.41$. The magnitude $|V_d(\omega)|$ is shown in figure 2.