University of Illinois **Spring 2014**

ECE 310: Problem Set 5: Problems and Solutions
Linear and shift invariant systems; Convolution; Impulse response

Due: Wednesday February 26 at 6 p.m.
Reading: *310 Course Notes* Ch 3.3–3.9
Version: 1.2

1. **[System properties]**
Determine if the systems characterized by the following relations are, with respect to the input,

(i) linear or non-linear (ii) causal or non-causal (iii) shift-invariant or shift-varying

Assume that the input is zero before $n = 0$ and that the initial conditions of the systems are all set to zero. **Justify** your answers with proofs or counter-examples.

(a) $y[n] = y[n-3] + x[n] + 5x[n-2]$
Solution: The system is linear. Assume that

$$y_1[n] = y_1[n-3] + x_1[n] + 5x_1[n-2]$$

and

$$y_2[n] = y_2[n-3] + x_2[n] + 5x_2[n-2].$$

If the input is $x_3[n] = a_1x_1[n] + a_2x_2[n]$ (a_1 and a_2 are real numbers), then the output is given by

$$y_3[n] = y_3[n-3] + (a_1x_1[n] + a_2x_2[n]) + 5(a_1x_1[n-2] + a_2x_2[n-2]).$$

We also know that

$$(a_1y_1[n]+a_2y_2[n]) = (a_1y_1[n-3]+a_2y_2[n-3])+(a_1x_1[n]+a_2x_2[n])+5(a_1x_1[n-2]+a_2x_2[n-2]).$$

Hence $y_3[n] = a_1y_1[n] + a_2y_2[n]$.
The system is causal. $y[n]$ depends only on $x[m], m \leq n$.
The system is shift-invariant. If the input is $x_0[n] = x[n-n_0]$, then the output is given by $y_0[n] = y_0[n-3] + x[n-n_0] + 5x[n-n_0-2]$. We also know that $y[n-n_0] = y[n-n_0-3] + x[n-n_0] + 5x[n-n_0-2]$. Hence $y_0[n] = y[n-n_0]$.

(b) $y[n-1] = x[|n|] - 2y[n]$
Solution: The system is linear. Assume that

$$y_1[n-1] = x_1[|n|] - 2y_1[n]$$

and

$$y_2[n-1] = x_2[|n|] - 2y_2[n],$$

then

$$(a_1y_1[n-1] + a_2y_2[n-1]) = (a_1x_1[|n|] + a_2x_2[|n|]) - 2(a_1y_1[n] + a_2y_2[n])$$

for all real numbers a_1 and a_2. If the input is $x_3[n] = a_1x_1[n] + a_2x_2[n]$, the output $y_3[n]$ is given by

$$y_3[n-1] = (a_1x_1[|n|] + a_2x_2[|n|]) - 2y_3[n].$$

Hence $y_3[n] = a_1y_1[n] + a_2y_2[n]$.
The system is non-causal. $y[n] = -\frac{1}{2}y[n-1] + \frac{1}{2}x[|n|]$, hence $y[-1] = -\frac{1}{2}y[-2] + \frac{1}{2}x[1]$ depends on $x[1]$.
The system is shift-varying. Counter example: Let $x_1[n] = \delta[n]$,

$$y_1[n] = \frac{1}{2}(-\frac{1}{2})^n u[n].$$

Let $x_2[n] = x_1[n-1] = \delta[n-1]$, then

$$y_2[n] = \frac{1}{2}(-\frac{1}{2})^{n+1}u[n+1] + \frac{1}{2}(-\frac{1}{2})^{n-1}u[n-1].$$

Hence $y_2[0] = -\frac{1}{4} \neq 0 = y_1[0-1]$. Also, let $x_3[n] = x_1[n+1] = \delta[n+1]$, then $y_3[n] = 0$. Hence $y_3[0] = 0 \neq -\frac{1}{4} = y_1[0+1]$.

(c) $y[n-1] = \cos\left(\frac{\pi}{3n}\right)x[n-1] + \sqrt{2}x[n] - 3y[n]$

Solution: The system can also be described by

$$y[n] = -\frac{1}{3}y[n-1] + \frac{1}{3}\cos\left(\frac{\pi}{3n}\right)x[n-1] + \frac{\sqrt{2}}{3}x[n].$$

The system is linear. Assume that

$$y_1[n] = -\frac{1}{3}y_1[n-1] + \frac{1}{3}\cos\left(\frac{\pi}{3n}\right)x_1[n-1] + \frac{\sqrt{2}}{3}x_1[n]$$

and

$$y_2[n] = -\frac{1}{3}y_2[n-1] + \frac{1}{3}\cos\left(\frac{\pi}{3n}\right)x_2[n-1] + \frac{\sqrt{2}}{3}x_2[n],$$

then $(a_1y_1[n] + a_2y_2[n]) = -\frac{1}{3}(a_1y_1[n-1] + a_2y_2[n-1]) + \frac{1}{3}\cos\left(\frac{\pi}{3n}\right)(a_1x_1[n-1] + a_2x_2[n-1]) + \frac{\sqrt{2}}{3}(a_1x_1[n] + a_2x_2[n])$ for all real numbers a_1 and a_2. If the input is $x_3[n] = a_1x_1[n] + a_2x_2[n]$, the output $y_3[n]$ is given by

$$y_3[n] = -\frac{1}{3}y_3[n-1] + \frac{1}{3}\cos\left(\frac{\pi}{3n}\right)(a_1x_1[n-1] + a_2x_2[n-1]) + \frac{\sqrt{2}}{3}(a_1x_1[n] + a_2x_2[n]).$$

Hence $y_3[n] = a_1y_1[n] + a_2y_2[n]$.
The system is causal. If the system is implemented using $y[n] = -\frac{1}{3}y[n-1] + \frac{1}{3}\cos\left(\frac{\pi}{3n}\right)x[n-1] + \frac{\sqrt{2}}{3}x[n]$, then $y[n]$ depends only on $x[m], m \leq n$.
The system is shift-varying. If the input is $x_0[n] = x[n-n_0]$, then the output is given by

$$\begin{aligned} y_0[n] &= -\frac{1}{3}y_0[n-1] + \frac{1}{3}\cos\left(\frac{\pi}{3n}\right)x_0[n-1] + \frac{\sqrt{2}}{3}x_0[n] \\ &= -\frac{1}{3}y_0[n-1] + \frac{1}{3}\cos\left(\frac{\pi}{3n}\right)x[n-n_0-1] + \frac{\sqrt{2}}{3}x[n-n_0]. \end{aligned}$$

However,

$$y[n-n_0] = -\frac{1}{3}y[n-n_0-1] + \frac{1}{3}\cos\left(\frac{\pi}{3(n-n_0)}\right)x[n-n_0-1] + \frac{\sqrt{2}}{3}x[n-n_0].$$

Hence $y_0[n] \neq y[n-n_0]$.

2. **[System properties]**
Determine if the following systems are

(i) linear or non-linear (ii) causal or non-causal (iii) shift-invariant or shift-varying

Assume that the input is zero before $n = 0$ and that the initial conditions of the systems are all set to zero. Justification is **not** needed.

Note: L - Linear, NL - Non-linear, C - Causal, NC - Non-causal, SI - Shift-invariant, SV - Shift-varying.

(a) $y[n-2] = x^2[n] + 3y[n]$
Solution: NL,C,SI

(b) $y[n] = x[-n+2]$
Solution: L,NC,SV

(c) $y[n] = \left(\frac{1}{4}\right)^{|n|} x[n]$
Solution: L,C,SV

(d) $y[n] = \sum_{m=-\infty}^{n+1} x[m]$
Solution: L,NC,SI

(e) $y[n] = \frac{x[n]}{x[2]}$
Solution: NL,NC,SV

(f) $y[n-1] = x[n-1] + \tan(4)x[n] - \cos(0.4\pi n)y[n]$
Solution: L,C,SV

(g) $y[n] = x[12n]$
Solution: L,NC,SV

3. **[Graphical convolution]**
Let $x[n] = 3\delta[n] + 2\delta[n-1] + \delta[n-2]$ and $h[n] = 4\delta[n] + \delta[n-2] - 2\delta[n-3] + 3\delta[n-4]$. Compute the convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

by the following steps.

(a) First determine the values of n for which $y[n]$ is non-zero.
Solution: The length of x is 3. The length of h is 5. Hence the length of their convolution is $3 + 5 - 1 = 7$. The values of n for which $y[n]$ is non-zero are $0, 1, 2, ..., 6$.

(b) Now plot $h[n-k]$ vs. k for each value of n that you determined in part (a) above.
Solution: We plot $h[-k]$ vs. k, $h[1-k]$ vs. k, etc. (figure 1)

(c) Now, determine $y[n]$ by multiplying $x[k]$ by $h[n-k]$ and summing the result for each value of n.
Solution: Given $x[k]$ and $h[n-k]$ depicted in figure 1, for $n = 0, 1, 2..., 6$, we have that
$y[0] = 3 \times 4 = 12.$
$y[1] = 3 \times 0 + 2 \times 4 = 8.$
$y[2] = 3 \times 1 + 2 \times 0 + 1 \times 4 = 7.$
$y[3] = 3 \times (-2) + 2 \times 1 + 1 \times 0 = -4.$

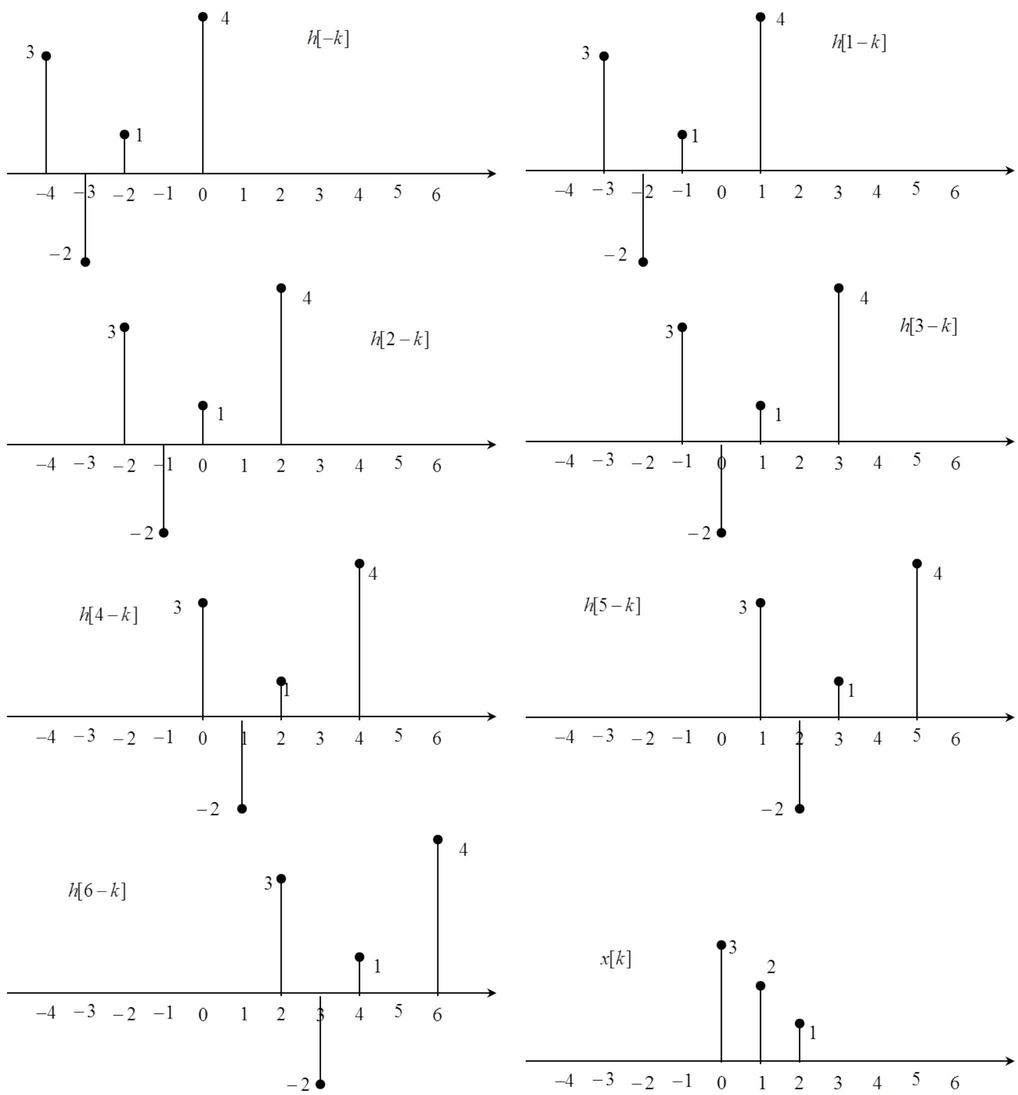

Figure 1: Problem 3

$y[4] = 3 \times 3 + 2 \times (-2) + 1 \times 1 = 6.$
$y[0] = 2 \times 3 + 1 \times (-2) = 4.$
$y[0] = 1 \times 3 = 3.$

4. **[Discrete convolution]**
   Let $x[n] = (0.9)^n u[n]$, $h[n] = (0.7)^n u[n]$, $w[n] = u[n] - u[n-10]$, find the discrete-time convolution of the following:

   (a) $y_1[n] = x[n] * h[n]$
   Solution:

$$\begin{aligned} y[n] = &\sum_{k=-\infty}^{\infty} h[n-k]x[k] \\ =&(0.7)^n \sum_{k=0}^{n} (0.7)^{-k}(0.9)^k u[n] \\ =&(0.7)^n \frac{1-\left(\frac{.9}{.7}\right)^{n+1}}{1-\left(\frac{.9}{.7}\right)} u[n] \\ =&(4.5(0.9)^n - 3.5(0.7)^n)u[n] \end{aligned}$$

   (b) $y_2[n] = x[n] * w[n]$
   Solution:

$$\begin{aligned} y_2[n] = &\sum_{k=-\infty}^{\infty} x[n-k]w[k] \\ =&(0.9)^n \sum_{k=0}^{9} (0.9)^{-k} u[n-k] \\ =&\begin{cases} 0, & n < 0 \\ (0.9)^n \sum_{k=0}^{n} (0.9)^{-k}, & 0 \le n \le 9 \\ (0.9)^n \sum_{k=0}^{9} (0.9)^{-k}, & n \ge 10 \end{cases} = \begin{cases} 0, & n < 0 \\ (0.9)^n \frac{1-0.9^{-(n+1)}}{1-0.9^{-1}}, & 0 \le n \le 9 \\ (0.9)^n \frac{1-0.9^{-10}}{1-0.9^{-1}}, & n \ge 10 \end{cases} \\ =&\begin{cases} 0, & n < 0 \\ 10(1-(0.9)^{n+1}), & 0 \le n \le 9 \\ 10(0.9)^{n-9}(1-(0.9)^{10}), & n \ge 10 \end{cases} \end{aligned}$$

   (c) $y[n] = x[n] * v[n]$, where $v[n] = \frac{1}{5}h[n-1] + \frac{1}{10}w[n+9]$
   Solution: First, notice that:

$$y[n] = x[n] * v[n] = \frac{1}{5}(x[n] * h[n-1]) + \frac{1}{10}(x[n] * w[n+9]) = \frac{1}{5}y_1[n-1] + \frac{1}{10}y_2[n+9]$$

Second, substitution and appropriately modifying (shift and scale) the results from part (a) and (b) to

$$\frac{1}{5}y_2[n-1] = \begin{cases} 0, & n < 1 \\ 0.9^n - 0.7^n, & n \geq 1 \end{cases}$$

$$\frac{1}{10}y_2[n+9] = \begin{cases} 0, & n < -9 \\ (1-0.9^{n+10}), & -9 \leq n \leq 0 \\ (0.9)^n(1-(0.9)^{10}), & n \geq 1 \end{cases}$$

Combinding the two results together gives:

$$y[n] = \begin{cases} 0, & n < -9 \\ 1-(0.9)^{n+10}, & -9 \leq n \leq 0 \\ 0.9^n - 0.7^n + (0.9)^n(1-0.9^{10}). & n \geq 1 \end{cases},$$

$$y[n] = \begin{cases} 0, & n < -9 \\ 1-(0.9)^{n+10}, & -9 \leq n \leq 0 \\ 0.9^n(2-(0.9)^{10}) - 0.7^n & n \geq 1 \end{cases},$$

(d) $y_0[n] = s[n] * t[n]$, where $s[n] = x[n-3]$ and $t[n] = h[n+1]$

Solution:

$y_0[n] = s[n] * t[n] = x[n-3] * h[n+1] = y_1[n-3+1] = y_1[n-2]$

$$5((0.9)^{n-1} - (0.7)^{n-1})u[n-2] = \begin{cases} 0, & n < 2 \\ 5((0.9)^{n-1} - 0.7^{n-1}), & n \geq 2 \end{cases}$$

Hint: For parts (c) and (d), there is an easier way to compute the discrete-time convolution than using the convolution sum directly.

5. **[Unit-pule response]**
Assume that the zero-state response of a linear shift-invariant (LSI) system to the input $x[n] = 4^{-n}u[n]$ is $y[n] = \left(\frac{1}{5}\right)^n u[n]$. Use the system properties of linearity and shift-invariance to find the system's response $h[n]$ to a unit pulse input ($x[n] = \delta[n]$), which is also called the unit-pulse response or impulse response of a discrete-time system.

Solution:

Time-domain method:

- Due to shift-invariance, for the input

$$x_1[n] = x[n-1] = 4^{-(n-1)}u[n-1] = 4 \times 4^{-n}u[n-1],$$

the response is

$$y_1[n] = y[n-1] = \left(\frac{1}{5}\right)^{(n-1)} u[n-1].$$

- Due to linearity (homogeneity), for the input

$$x_2[n] = \frac{1}{4}x_1[n] = 4^{-n}u[n-1],$$

the response is

$$y_2[n] = \frac{1}{4}y_1[n] = \frac{1}{4}\left(\frac{1}{5}\right)^{(n-1)} u[n-1].$$

- Due to linearity, for the input

$$x_3[n] = x[n] - x_2[n] = \delta[n],$$

the response is

$$y_3[n] = y[n] - y_2[n] = \left(\frac{1}{5}\right)^n u[n] - \frac{1}{4}\left(\frac{1}{5}\right)^{(n-1)} u[n-1]$$

Hence, impulse response

$$\begin{aligned} h[n] = y_3[n] &= \left(\frac{1}{5}\right)^n u[n] - \frac{1}{4}\left(\frac{1}{5}\right)^{(n-1)} u[n-1] \\ &= \left(\frac{1}{5}\right)^n u[n] - \frac{5}{4}\left(\frac{1}{5}\right)^n u[n-1] \\ &= \delta[n] - \frac{1}{4}\left(\frac{1}{5}\right)^n u[n-1] \\ &= \frac{5}{4}\delta[n] - \frac{1}{4}\left(\frac{1}{5}\right)^n u[n]. \end{aligned}$$

Frequency-domain method:

For LSI system, $y[n] = x[n] * h[n]$. In frequency domain, the DTFTs have the following relation:

$$Y_d(\omega) = X_d(\omega)H_d(\omega)$$

The DTFT of the input is $X_d(\omega) = \frac{1}{1-e^{-j\omega}/4}$. The DTFT of the output is $Y_d(\omega) = \frac{1}{1-e^{-j\omega}/5}$. Hence the DTFT of the impulse response is

$$H_d(\omega) = \frac{Y_d(\omega)}{X_d(\omega)} = \frac{1-e^{-j\omega}/4}{1-e^{-j\omega}/5} = \frac{1}{1-e^{-j\omega}/5} - \frac{1}{4}\frac{1}{1-e^{-j\omega}/5}e^{-j\omega}.$$

The impulse response $h[n]$ is the inverse DTFT of the $H_d(\omega)$, which is

$$h[n] = \left(\frac{1}{5}\right)^n u[n] - \frac{1}{4}\left(\frac{1}{5}\right)^{(n-1)} u[n-1].$$

6. **[Difference equations and unit-pulse response]**
Given a causal, linear shift-invariant system characterized by

$$y[n] - \frac{1}{4}y[n-1] = 4x[n] + 3x[n-1],$$

find the impulse response $h[n]$ (i.e., determine $y[n]$ when $x[n] = \delta[n]$).

Solution: Setting $x[n] = \delta[n]$,

$$h[n] - \frac{1}{4}h[n-1] = 4\delta[n] + 3\delta[n-1]$$

Also, since the system is causal, $h[n] = 0$ for $n < 0$.

1. $n = 0: \quad h[0] - \frac{1}{4}h[-1] = 4 \Rightarrow h[0] = 4$
2. $n = 1: \quad h[1] - \frac{1}{4}h[0] = 3 \Rightarrow h[1] = 4$
3. $n \geq 2: \quad h[n] - \frac{1}{4}h[n-1] = 0 \Rightarrow z - \frac{1}{4} = 0 \Rightarrow z = \frac{1}{4}$

   $\Rightarrow h[n] = C\left(\frac{1}{4}\right)^n, n \geq 2$

   Therefore,

$$h[n] = \begin{cases} 4, & n = 0 \\ 16\left(\frac{1}{4}\right)^n, & n \geq 1 \end{cases}$$

$$h[n] = -12\delta[n] + 16\left(\tfrac{1}{4}\right)^n u[n]$$

Note: Frequency-domain method. Take the DTFT of the equation $y[n] - \frac{1}{4}y[n-1] = 4x[n] + 3x[n-1]$,

$$Y_d(\omega)(1 - \frac{1}{4}e^{-j\omega}) = X_d(\omega)(4 + 3e^{-j\omega}).$$

This is a LSI system,

$$H_d(\omega) = \frac{Y_d(\omega)}{X_d(\omega)} = \frac{4 + 3e^{-j\omega}}{1 - \frac{1}{4}e^{-j\omega}}.$$

Take the Inverse DTFT, we have that

$$h[n] = 4\left(\frac{1}{4}\right)^n u[n] + 3\left(\frac{1}{4}\right)^{n-1} u[n-1] = -12\delta[n] + 16\left(\frac{1}{4}\right)^n u[n].$$