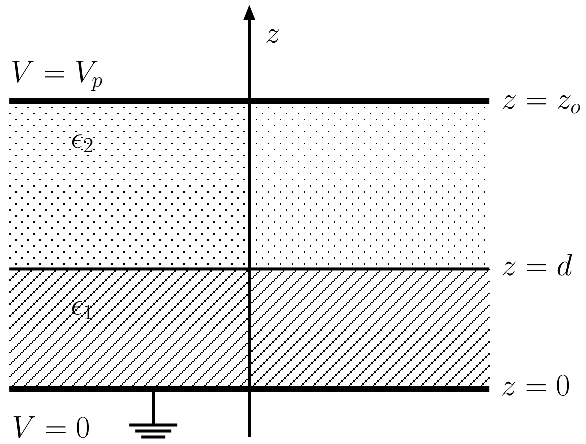
ECE 329-Spring2013 — Homework 4 — Solution — Due: Feb. 12, 2013, 5PM

1. Two infinite perfectly conducting plates at $z = 0$ and $z = z_o$ are kept at potentials $V = 0$ and $V = V_p$, respectively. The region between the plates is filled with two slabs of perfect dielectric materials having permittivities ϵ_1 for $0 < z < d$ (region 1) and ϵ_2 for $d < z < z_o$(region 2), as shown in the following figure.



a) Using Laplace's equation, $\nabla^2 V = 0$, and due to the symmetry of the problem, the potential in the dielectrics takes the form

$$V(z) = \begin{cases} a_1 z + b_1 & \text{for} \quad 0 < z < d \\ a_2 z + b_2 & \text{for} \quad d < z < z_o. \end{cases}$$

Therefore, the associated electric field is

$$\vec{E}(z) = -\nabla V = \begin{cases} -a_1\,\hat{z} & \text{for} \quad 0 < z < d \\ -a_2\,\hat{z} & \text{for} \quad d < z < z_o. \end{cases}$$

Since $V = 0$ at $z = 0$, we get $b_1 = 0$. In addition, given that $V = V_p$ at $z = z_o$, we have

$$a_2 z_o + b_2 = V_p.$$

We also know that V must be continuous at the interface $z = d$, thus, we have

$$a_2 d + b_2 - a_1 d = 0.$$

Maxwell's boundary conditions tell us that the normal component of the electric flux density $\vec{D}(z)$ must be continuous across the interface $z = d$. Because of this we can write that

$$\hat{n} \cdot (\vec{D}_1 - \vec{D}_2) = (-\hat{z}) \cdot (\epsilon_1(-a_1\hat{z}) - \epsilon_2(-a_2\hat{z})) = \epsilon_1 a_1 - \epsilon_2 a_2 = 0.$$

Combining these three equations, we find

$$a_1 = \frac{\epsilon_2 V_p}{z_o\epsilon_1 + d\,(\epsilon_2 - \epsilon_1)},$$
$$a_2 = \frac{\epsilon_1 V_p}{z_o\epsilon_1 + d\,(\epsilon_2 - \epsilon_1)},$$
$$b_2 = \frac{d\,(\epsilon_2 - \epsilon_1)\,V_p}{z_o\epsilon_1 + d\,(\epsilon_2 - \epsilon_1)},$$

and, in consequence, the electric potential in the two dielectrics is

$$V(z) = \begin{cases} \frac{\epsilon_2 V_p}{z_o\epsilon_1 + d(\epsilon_2 - \epsilon_1)} z & 0 < z < d, \\ \frac{\epsilon_1 V_p}{z_o\epsilon_1 + d(\epsilon_2 - \epsilon_1)} z + \frac{d(\epsilon_2 - \epsilon_1) V_p}{z_o\epsilon_1 + d(\epsilon_2 - \epsilon_1)} & d < z < z_o. \end{cases}$$

b) The electric field inside the dielectrics is given by

$$\vec{E}(z) = \begin{cases} -\frac{\epsilon_2 V_p}{z_o\epsilon_1 + d(\epsilon_2 - \epsilon_1)}\,\hat{z} & 0 < z < d, \\ -\frac{\epsilon_1 V_p}{z_o\epsilon_1 + d(\epsilon_2 - \epsilon_1)}\,\hat{z} & d < z < z_o. \end{cases}$$

Given that $z_o = 4d = 2\,\mathrm{m}$,$\epsilon_1 = 3\epsilon_o$, $\epsilon_2 = \epsilon_o$, and $\mathbf{E}(0 < z < d) = -5\hat{z}$ V/m, we can find that

$$-\frac{\epsilon_2 V_p}{z_o\epsilon_1 + d\,(\epsilon_2 - \epsilon_1)}\hat{z} = -5\hat{z}$$

So V_p=25 V.

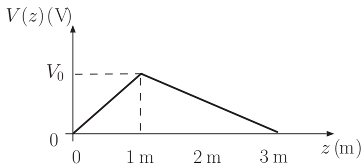
c) Given the parameters in part (b) above, the electric filed can be calculated as :

$$\vec{E}(z) = \begin{cases} -5\,\hat{z}\,\frac{\mathrm{V}}{\mathrm{m}} & 0 < z < d, \\ -15\,\hat{z}\,\frac{\mathrm{V}}{\mathrm{m}} & d < z < z_o. \end{cases}$$

The surface charge density at $z = z_o$ is given by

$$\rho_S\big|_{z=z_o} = \vec{D}\cdot\hat{n}\Big|_{z=z_o} = D_z(z_o)\hat{z}\cdot(-\hat{z}) = -D_z(z_0) = -\epsilon_2 E_z(z_o) = 15\epsilon_o\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

2. Two conducting plates that are grounded and have zero potential are positioned on $z = 0$ and $z = 3\,\mathrm{m}$ surfaces. On $z = 1\,\mathrm{m}$ surface, there is a uniform and static surface charge of $6\,\mathrm{C/m^2}$. The permittivity of the region $z < 1\,\mathrm{m}$ is ϵ_o, whereas it is $2\epsilon_o$ in the region $z > 1\,\mathrm{m}$. To determine surface charge densities at $z = 0$ and $z = 3\,\mathrm{m}$, we proceed as follows: Let V_o denote the electrostatic potential at $z = 1$. According to Laplace's equation, $V(z)$ needs to be a linear function of z as shown in the following figure



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and can be expressed as

$$V(z) = \begin{cases} V_0 z & ,0 < z < 1\,\mathrm{m} \\ -\frac{V_0}{2}z + \frac{3}{2}V_0 & ,1\,\mathrm{m} < z > 3\,\mathrm{m}. \end{cases}$$

The corresponding electric field is

$$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right) = \begin{cases} -V_0\hat{z}\,\frac{\mathrm{V}}{\mathrm{m}} & ,0 < z < 1\,\mathrm{m} \\ \frac{V_0}{2}\hat{z}\,\frac{\mathrm{V}}{\mathrm{m}} & ,1\,\mathrm{m} < z < 3\,\mathrm{m}. \end{cases}$$

which implies,

$$\mathbf{D} = \epsilon\mathbf{E} = \begin{cases} -\epsilon_0 V_0\hat{z}\,\frac{\mathrm{V}}{\mathrm{m}} & ,0 < z < 1\,\mathrm{m} \\ \epsilon_0 V_0\hat{z}\,\frac{\mathrm{V}}{\mathrm{m}} & ,1\,\mathrm{m} < z < 3\,\mathrm{m}. \end{cases}$$

To calculate V_o, we apply boundary condition (BC) on $z = 1$ surface,

$$\hat{z}\cdot(\mathbf{D}_{z>1} - \mathbf{D}_{z<1}) = D_{z>1} - D_{z<1} = 6\,\frac{\mathrm{C}}{\mathrm{m}^2},$$

$$\epsilon_o V_o - (-\epsilon_o V_o) = 6\,\frac{\mathrm{C}}{\mathrm{m}^2},$$

$$V_o = \frac{3}{\epsilon_o}\,\mathrm{V}.$$

To calculate surface charge density on $z = 0$, apply BC on $z = 0$ surface,

$$D_{z<1} = \rho_{z=0},$$

$$-\epsilon_o V_o = \rho_{z=0},$$

$$\rho_{z=0} = -3\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

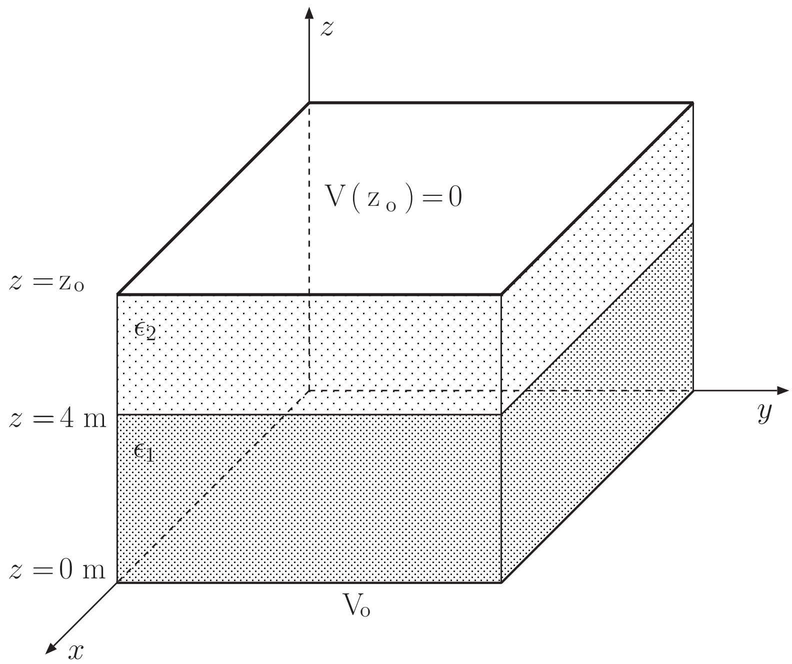
To calculate surface charge density on $z = 3$ m, apply BC on $z = 3$ surface,

$$-D_{z>1} = \rho_{z=3},$$

$$-\epsilon_o V_o = \rho_{z=3},$$

$$\rho_{z=3} = -3\,\frac{\mathrm{C}}{\mathrm{m}^2}.$$

3. The region between two infinite, parallel, conducting plates is filled by two dielectrics having permittivities ϵ_1 and ϵ_2 as shown in the next figure.



In addition, the electric field between the plates is known to be

$$\vec{E}(z) = \begin{cases} \frac{4\epsilon_2}{\epsilon_1 + 8\epsilon_2}\,\hat{z}\,\frac{\mathrm{V}}{\mathrm{m}}, & 0 < z < d, \\ \frac{4\epsilon_1}{\epsilon_1 + 8\epsilon_2}\,\hat{z}\,\frac{\mathrm{V}}{\mathrm{m}}, & d < z < z_o. \end{cases}$$

a) First, let us verify that the component of the displacement field normal to the interface at $z = d = 4m$ (which in this case is the z-component) satisfies Maxwell's boundary condition

$$D_z\big|_{z=d^-} = D_z\big|_{z=d^+}.$$

Since

$$D_z|_{z=d^-} = \epsilon_1 \cdot E_z|_{z=d^-} = \frac{4\epsilon_1\epsilon_2}{\epsilon_1 + 8\epsilon_2}$$

and

$$D_z|_{z=d^+} = \epsilon_2 \cdot E_z|_{z=d^+} = \frac{4\epsilon_1\epsilon_2}{\epsilon_1 + 8\epsilon_2},$$

then the field given above satisfies Maxwell's condition.

b) The electrostatic potential $V(z)$ for $0 < z < z_o$ is given by $V(z) = -\int_0^z E_z dz$. Integrating the electric field $E(z)$, we can find that

$$V(z) = \begin{cases} V_o - \frac{4\epsilon_2}{\epsilon_1+8\epsilon_2} z \,\mathrm{V}, & 0 < z < d, \\ V_o - \frac{4\epsilon_1}{\epsilon_1+8\epsilon_2} z - \frac{16(\epsilon_2-\epsilon_1)}{\epsilon_1+8\epsilon_2} \,\mathrm{V}, & d < z < z_o. \end{cases}$$

c) Given that $\rho_s = 4\epsilon_o \,\mathrm{C/m^2}$ at $z = 0\,\mathrm{m}$, and applying Maxwell's boundary condition $\rho_s = D_z|_{z=0^+}$, we have that

$$4\epsilon_o = \frac{4\epsilon_1\epsilon_2}{\epsilon_1 + 8\epsilon_2}.$$

Replacing $\epsilon_2 = 2\epsilon_o$ and solving for ϵ_1, we get

$$\epsilon_1 = 16\epsilon_o.$$

d) Given that $V_o = 3\,\mathrm{V}$, we can write that

$$3 - \frac{4\epsilon_1}{\epsilon_1 + 8\epsilon_2} z_o - \frac{16(\epsilon_2 - \epsilon_1)}{\epsilon_1 + 8\epsilon_2} = 0$$

$$z_o = \frac{19\epsilon_1 + 8\epsilon_2}{4\epsilon_1}$$

Substituting $\epsilon_1 = 16\epsilon_o$ and $\epsilon_2 = 2\epsilon_o$ and solving for z_o, we obtain

$$z_o = 5\,\mathrm{m}.$$

So the thickness of region 2 is 1m.

e) Laplace's equation

$$\nabla^2 V = 0$$

comes from the assumption that the permittivity is constant in space. In our case, however, there are two dielectrics in the region $0 < z < z_o$, which implies that the permittivity function is discontinuous. Because of this, $V(z)$ does not satisfy Laplace's equation at all points in the region (in fact, in this particular case, the equation is not satisfied only at $z = d$).

f) The capacitance of the structure is given by $C = \frac{Q}{V}$, where Q is the tota l charge accumulated on the top plate (with area A) and V is the voltage difference between the plates. Denoting by ρ_s the surface charge density on the plate, we have that $Q = \rho_s A$. In addition, since the voltage between the plates can be computed as $V = |E_{1z}|\, d + |E_{2z}|\,(z_o - d)$, we have that

$$C = \frac{Q}{V} = \frac{\rho_s A}{|E_{1z}|\, d + |E_{2z}|\,(z_o - d)}.$$

Using Maxwell's boundary conditions, we find that $\rho_s = \epsilon_2\,|E_{2z}| = \epsilon_1\,|E_{1z}|$, then, the capacitance can be re-expressed as

$$C = \frac{A}{d/\epsilon_1 + (z_o - d)/\epsilon_2} = \frac{\epsilon_1\epsilon_2 A}{z_o\epsilon_1 + (\epsilon_2 - \epsilon_1)d}.$$

Note: The same result can be obtained by combining the capacitances of each dielectric in series. Given that

$$C_1 = \epsilon_1 \frac{A}{d} \qquad \text{and} \qquad C_2 = \epsilon_2 \frac{A}{z_o - d},$$

we have that

$$C = \left(C_1^{-1} + C_2^{-1}\right)^{-1} = \left(\frac{d}{\epsilon_1 A} + \frac{z_o - d}{\epsilon_2 A}\right)^{-1} = \left(\frac{z_o \epsilon_1 + (\epsilon_2 - \epsilon_1)\, d}{\epsilon_1 \epsilon_2 A}\right)^{-1} = \frac{\epsilon_1 \epsilon_2 A}{z_o \epsilon_1 + (\epsilon_2 - \epsilon_1)\, d}.$$

4. For a wire of copper with conductivity $\sigma = 5.8 \times 10^7\,\mathrm{S/m}$ and free-electron density $N_e = 8.45 \times 10^{28}\,\mathrm{m}^{-3}$, let us compute the following quantities.

   a) The resistance of a conducting wire with transverse area A and length d is given by $R = \frac{d}{\sigma A}$. For a cylindrical wire of radius $r = 1.4 \times 10^{-3}\,\mathrm{m}$ and length $d = 180\,\mathrm{m}$, we get

   $$R = \frac{d}{\sigma \pi r^2} = 0.504\,\Omega.$$

   b) The voltage drop across the wire is given by $V = E \cdot d = R \cdot I$, where I is the current flowing along the wire. Assuming $I = 1\,\mathrm{A}$, the electric field within the wire is

   $$E = \frac{R \cdot I}{d} = 2.8 \times 10^{-3}\,\frac{\mathrm{V}}{\mathrm{m}}.$$

   c) Since $\vec{J} = \sigma \vec{E} = N_e q_e \vec{v}_e$ where $q_e = -1.6 \times 10^{-19}\,\mathrm{C}$, we have that the mean speed of an electron is given by

   $$|\vec{v}_e| = \frac{\sigma \left|\vec{E}\right|}{N_e\, |q_e|} = 1.201 \times 10^{-5}\,\frac{\mathrm{m}}{\mathrm{s}}.$$

   d) Finally, the time it would take an electron to travel from one end of the wire to the other is simply

   $$t = \frac{d}{|\vec{v}_e|} = 1.5 \times 10^7\,\mathrm{s}.$$

5. Given is a pair of metallic spherical shells of radii a and $b > a$. The medium between the shells has permittivity $\epsilon = 2\epsilon_o$ and conductivity $\sigma = 2 \times 10^{-6}\,\mathrm{S/m}$.

   a) Integral form of Gauss's law states that

   $$\epsilon \oint \mathbf{E} \cdot d\mathbf{S} = Q.$$

   Consider a sphere of radius r (where $a < r < b$). Applying Gauss's law to this sphere, we get

   $$4\pi r^2 E_r = \frac{Q}{\epsilon},$$

   $$\mathbf{E} = \frac{Q}{4\pi\epsilon r^2}\hat{r}.$$

   Now, applyling $V = \int \vec{E} \cdot d\vec{l}$ and integrating from a to b, we get

   $$V_{ab} = V_a - V_b = -\int_b^a \mathbf{E} \cdot \hat{r} dr = \int_a^b \mathbf{E} \cdot \hat{r} dr = \int_a^b \frac{Q}{4\pi\epsilon r^2} dr,$$

   $$V = \frac{Q}{4\pi\epsilon}\left(\frac{b-a}{ab}\right).$$

Since $CV = Q$, we obtain

$$\frac{1}{C} = \frac{1}{4\pi\epsilon}(\frac{b-a}{ab}),$$

implying

$$C = 4\pi\epsilon(\frac{ab}{b-a}).$$

b) Capacitance of a metallic shell of radius $a = 1\,\text{m}$ and $b \to \infty$ can be obtained as follows:

$$C = 4\pi\epsilon(\frac{ab}{b-a}) = 4\pi\epsilon\frac{a}{(1-\frac{a}{b})}.$$

Now taking limit $b \to \infty$, we get

$$C = 4\pi\epsilon a = 4 \times \pi \times 2 \times 8.85 \times 10^{-12} \times 1,$$

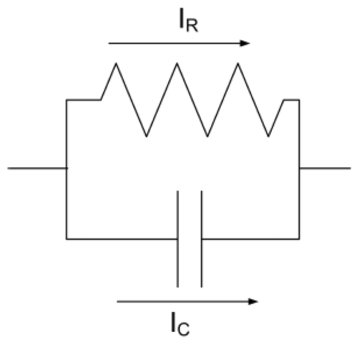
$$C = 222.4\,\text{pF}.$$

c) Conductance of the metallic shell is

$$G = \frac{\sigma}{\epsilon}C = \frac{\sigma}{\epsilon} \times 4\pi\epsilon a = 2 \times 10^{-6} \times 4\pi \times 1 = 8\pi \times 10^{-6}$$

$$G = 8\pi \times 10^{-6} = 25.13\,\mu\text{S}.$$

d) The RC equivalent circuit that represents the physical system described above is a parallel R and C circuit as shown below,



Here, resistance $R = \frac{1}{G}$ is in parallel with capacitance C. The two terminals represent the surface of the spherical conducting shell and a point at infinity. Because the voltage across both elements is the same (equal to the voltage between the shell and infinity), they must be in parallel. In the given situation, no source is connected across the terminals, only an initial voltage is present, caused by the initial charge on the shell. The charge on the capacitor is $Q = CV$ and the equation describing the current flow can be rewritten in terms of the charge on the shell as follows. We start applying KCL on the node on the left,

$$I_R + I_C = 0,$$

$$GV + C\frac{\partial V}{\partial t} = 0,$$

$$\frac{GQ}{C} + \frac{\partial Q}{\partial t} = 0.$$

e) Initial condition for the differential equation in part (d) is that $Q(t) = 1\,\mathrm{C}$ on the inner shell at $t = 0$. To obtain an expression for $Q(t)$ in $t > 0$, we will solve the differential equation as follows,

$$\frac{\partial Q}{\partial t} + \frac{GQ}{C} = 0,$$

which has general solution

$$Q(t) = a_1 e^{-\frac{G}{C}t} + a_2.$$

The initial condition can be written as $Q(0) = 1\,\mathrm{C}$ and at $t \to \infty$ we expect no charge $Q(\infty) = 0$. This conditions lead to $a_2 = 0$ and $a_1 = 1\,\mathrm{C}$.

$$\therefore Q(t) = e^{-(\frac{G}{C}t)}\ \mathrm{C}.$$