ECE 310 HW5 Soln

1\. (a) $y[n] = x[n] \circledast h[n] = (\delta[n+1] + 2\delta[n] + 3\,\delta[n-1]) \circledast h[n]$

$$= h[n+1] + 2h[n] + 3h[n+1]$$

$$= \{1, \underset{\uparrow}{0}, -1, 1, 2, 0, 0\} + \{0, \underset{\uparrow}{2}, 0, -1, 2, 0\} + \{0, \underset{\uparrow}{0}, 3, 0, -3, 3, 6\}$$

$$= \{1, \underset{\uparrow}{2}, 2, -1, 1, 7, 6\}$$

(b) $x[n] = n^3(u[n+2] - u[n-10])$, $h[n] = \{\underset{\uparrow}{-1}, 2, 3\}$

$x[n] = n^3$ for $-2 \le n \le 9$

$h[0] = -1, \quad h[1] = 2, \quad h[2] = 3$

$\therefore y[n]$ is non-zero when $-2 \le n \le 11$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k] = -x[n] + 2x[n-1] + 3x[n-2]$$

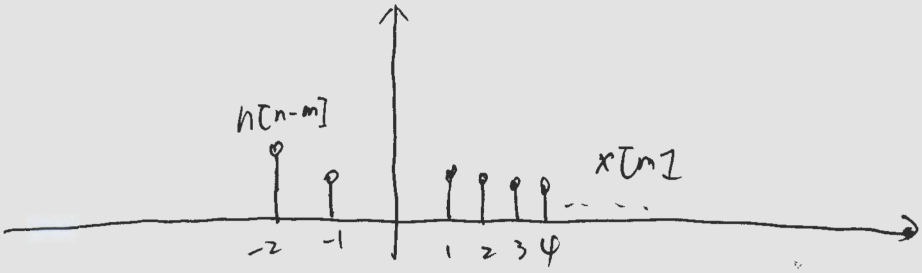
$$= -n^3(u[n+2] - u[n-10]) + 2(n-1)^3(u[n+1] - u[n-11]) + 3(n-2)^3(u[n] - u[n-12])$$

$$= (-n^3 + 2(n-1)^3 + 3(n-2)^3)(u[n] - u[n-10]) + -8\cdot\delta[n+2] + (-17)\,\delta[n+1] + (2(9\hat{}3) + 3(8\hat{}3))\,\delta[n-10]$$

$$+ 3(9\hat{}3)\,\delta[n-11]$$

c) $x[n] = n\,0.5^{n}u[n]$ $\quad$ $x[n] \neq 0$ for $n \geq 1$

$h[n] = n(u[n] - u[n-3])$ $\quad$ $h[n] \neq 0$ for $0 \leq n \leq 2$.



$y[2] = x[1]\cdot h[1] = 0.5\cdot 1 = 0.5$

$y[n] = x[n-1]\cdot h[1] + x[n-2]\cdot h[2] = x[n-1] + 2x[n-2]$ for $n \geq 3$

d) $x[n] = (-1)^{-n}u[n]$ $\qquad$ $h[n] = e^{-n}u[n-2]$

$$x[n] \circledast h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} (-1)^{-k} e^{-n+k}\,u[n-k-2]$$

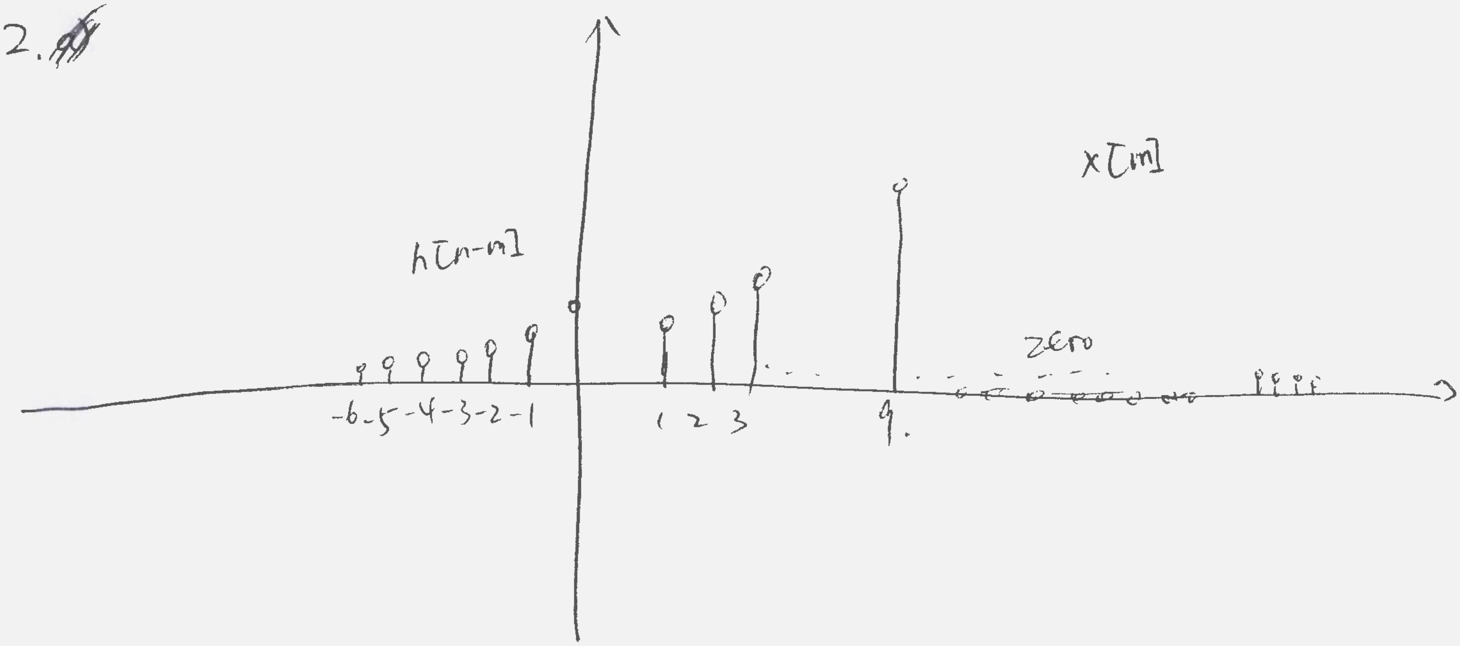
$$= \sum_{k=0}^{n-2} (-1)^{k} e^{-(n-k)}$$

$$= e^{-n} \sum_{k=0}^{n-2} (-e)^{k}$$

$$= e^{-n}\left(\frac{1-(-e)^{n-1}}{1+e}\right)$$

$$= \left(\frac{1}{1+e}\right)e^{-n} + \frac{(-1)^{n}\cdot e^{-1}}{1+e} \qquad \text{for } n \geq 2$$

2.



non-zero overlap between x[m] & h[n-m] for $1 \le n \le 15$ and $30 \le n \le \infty$.

Because both x[n] and h[n] are non-negative, will never get cancellations in the convolution sum $\Rightarrow$ y[n] is non-zero whenever there is overlap between x[m] and h[n-m]

i.e. $n \in [1, 15] \cup [30, \infty)$

Problem 3

First, if the system is LSI, then it is causal if and only if h[n] = 0 for n<0.

This takes care of 1-(f) (causal).

Second regardless of linearity or shift invariance, if h[n] is not equal to zero for n<0, this means that the unit pulse response appears before the input appears (because the input is a unit pulse at n=0) So, the system must be non-causal. This takes care of 1(b) and 1(c) (non causal).

In the other cases 1(a), 1(d) and 1(e), it is true that h[n] = 0 for n<0, so we can not rule out the possibility that they are causal, but because these systems are not LSI, we can not conclude that the systems are actually causal. Hence, in all these cases causality can not be determined from the given information.

4. $x[n]=u[n-3] \to \boxed{LSI} \to y[n]=\left(\frac{1}{2}\right)^n u[n-5]=\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{n-5}u[n-5]$

$x[n]=u[n-3]$

$x[n]=\delta[n-3]+\delta[n-4]+\delta[n-5]+\dots$

$x[n-1]=u[n-4]=\delta[n-4]+\delta[n-5]+\dots$

$x[n]-x[n-1]=\delta[n-3] \to \boxed{LSI} \to h[n-3]=\cancel{y[n-3]-y[n-4]}\ y[n]-y[n-1]$

$h[n-3]=\left(\frac{1}{2}\right)^n u[n-5]-\left(\frac{1}{2}\right)^{n-1}u[n-6]=\left(\frac{1}{2}\right)^n\left(u[n-5]-2u[n-6]\right)$

$h[n]=\left(\frac{1}{2}\right)^{n+3}\left(u[n-2]-2u[n-3]\right)$

5. a) $x[n] = \begin{cases} [1, 0, -2, 3] & 0 \le n \le 3 \\ 0 & \text{otherwise.} \end{cases}$

$$x[n] = \delta[n] - 2\delta[n-2] + 3\delta[n-3]$$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} (\delta[n] - 2\delta[n-2] + 3\delta[n-3]) z^{-n}$$

$$= 1 - 2z^{-2} + 3z^{-3}, \quad \text{ROC: } |z| > 0 \quad \text{DTFT exists}$$

b) $x[n] = 3^n u[n] + 0.5^n u[n-3]$

$$= 3^n u[n] + (\tfrac{1}{2})^3 (\tfrac{1}{2})^{n-3} u[n-3]$$

$$X(z) = \frac{z}{z-3} + \frac{1}{8} z^{-3} \cdot \frac{z}{z-\frac{1}{2}}, \quad \text{ROC: } |z| > 3, \text{ DTFT does not exist}$$

c) $x[n] = (\tfrac{1}{2})^{n-1} \sin(\tfrac{n\pi}{4} + \tfrac{\pi}{3}) u[n-2]$

$$x[n] = (\tfrac{1}{2})(\tfrac{1}{2})^{n-2} \sin(\tfrac{(n-2)\pi}{4} + \tfrac{5}{6}\pi) u[n-2]$$

Let us consider a general case,

$$y[n] = a^n \sin(w_0 n + \phi) u[n] = \frac{1}{2i} a^n (e^{iw_0 n} e^{i\phi} - e^{-iw_0 n} e^{-i\phi}) u[n]$$

$$Y(z) = (\tfrac{1}{2i} e^{i\phi}) \cdot \frac{1}{1 - \frac{a}{z} \cdot e^{iw_0}} - (\tfrac{1}{2i} e^{-i\phi}) \cdot \frac{1}{1 - \frac{a}{z} e^{-iw_0}}$$

$$= \frac{\frac{1}{2i} z^2 (e^{i\phi} - e^{-i\phi}) + \frac{1}{2i} az (e^{i(w_0 - \phi)} - e^{-i(w_0 - \phi)})}{z^2 - az(e^{iw_0} + e^{-iw_0}) + a^2}$$

$$= \frac{z^2 \sin(\phi) + az \sin(w_0 - \phi)}{z^2 - 2az\cos(w_0) + a^2}$$

Sub $a = \frac{1}{2}$, $w_0 = \frac{\pi}{4}$, $\phi = \frac{5\pi}{6}$.

$$Y(z) = \frac{\frac{1}{2} z^2 - \frac{1}{2} z \sin(-\frac{7\pi}{12})}{z^2 - \frac{\sqrt{2}}{4} z + \frac{1}{4}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

using delay property

$$X(z) = \frac{1}{2}\cdot z^{-2}\left(\frac{\frac{1}{2}z^{2} + \frac{1}{2}z\sin(\frac{5\pi}{12})}{z^{2} - \frac{\sqrt{2}}{4}z + \frac{1}{4}}\right) = \frac{1+z^{-1}\sin(\frac{5\pi}{12})}{4z^{2}-\sqrt{2}z+1}, \quad |z| > \frac{1}{2}$$

DTFT exists.

d) $x[n] = n\left(\frac{1}{2}\right)^{n} u[n-3]$

$$n x_1[n] \longleftrightarrow -z\left(\frac{d}{dz}X_1(z)\right)$$

let $x_1[n] = \left(\frac{1}{2}\right)^{n} u[n-3] = \frac{1}{8}\left(\frac{1}{2}\right)^{n-3} u[n-3]$

$$X_1(z) = \frac{1}{8}\cdot\frac{z^{-3}}{1-\frac{1}{2z}} = \frac{1}{8}\cdot\frac{z^{-2}}{z-\frac{1}{2}}, \quad \text{ROC}: |z| > \frac{1}{2}$$

$$\frac{d}{dz}X_1(z) = \frac{1}{8}\cdot\frac{-2z^{-3}}{z-\frac{1}{2}} - \frac{1}{8}\frac{z^{-2}}{(z-\frac{1}{2})^{2}}$$

$$= \frac{1}{8}\cdot\frac{-3z^{-2}+z^{-3}}{(z-\frac{1}{2})^{2}}$$

$$-z\frac{d}{dz}X_1(z) = \frac{1}{8}\cdot\frac{3z^{-1}-z^{-2}}{(z-\frac{1}{2})^{2}}, \quad \text{ROC } |z| > \frac{1}{2}, \text{ DTFT exists}$$

$$= \frac{z^{-2}}{8}\cdot\frac{(3z-1)}{(z-\frac{1}{2})^{2}}$$

e) $x[n] = 2^n (u[n] - u[n-30])$

$$= 2^n u[n] - 2^n u[n-30] = 2^n u[n] - 2^{30} 2^{n-30} u[n-30]$$

$$X(z) = \frac{1}{1-\frac{2}{z}} - 2^{30} \frac{1}{1-\frac{2}{z}} z^{-30}$$

$$= \frac{z}{z-2} - 2^{30} \frac{z^{-29}}{z-2} \qquad ROC: |z| > 2$$

$$= \frac{z - 2^{30} z^{-29}}{z-2}, \qquad ROC: |z| > 2, \text{ DTFT does not exist.}$$

6. a) $$X(z) = \frac{z^{-3}}{1+az^{-1}}$$

$$= z^{-3} \frac{z}{z+a}, \quad |z| > a$$

$$\therefore x[n] = a^{n-3} u[n-3]$$

b) $X(z) = \dfrac{z^2+z}{z^2-2z-3}$

$$= \frac{z(z+1)}{(z-3)(z+1)}$$

$$= \frac{z}{z-3}$$

$\therefore x[n] = 3^n$

c) $X(z) = \dfrac{1}{(1-0.5z^{-1})(1-0.2z^{-1})}$

$$X(z) = \frac{z^2}{(z-0.5)(z-0.2)}$$

$$\frac{X(z)}{z} = \frac{z}{(z-0.5)(z-0.2)} = \frac{A}{z-0.5} + \frac{B}{z-0.2}$$

$$A = \left.\frac{z}{z-0.2}\right|_{z=0.5} \qquad B = \left.\frac{z}{z-0.5}\right|_{z=0.2}$$

$$A = \frac{5}{3} \qquad B = -\frac{2}{3}$$

$$\frac{X(z)}{z} = \frac{\frac{5}{3}}{z-0.5} - \frac{\frac{2}{3}}{z-0.2}$$

$$X(z) = \frac{\frac{5}{3}z}{z-0.5} - \frac{\frac{2}{3}z}{z-0.2}$$

$$x[n] = \left(\frac{5}{3}(0.5)^n - \frac{2}{3}(0.2)^n\right)u[n]$$

d) $X(z) = \dfrac{1}{(z-0.5)^2(z-0.25)} = \dfrac{A}{z-0.5} + \dfrac{B}{(z-0.5)^2} + \dfrac{C}{z-0.25}$

$C = \dfrac{1}{(z-0.5)^2}\Big|_{z=0.25}$ $\qquad B = \dfrac{1}{z-0.5}\Big|_{z=0.5}$

$C = 16$ $\qquad B = 4$

To find A, fill in the known constants to solve for A.

$1 = A(z-0.5)(z-0.25) + B(z-0.25) + C(z-0.5)^2$

$= A(z^2 - 0.75z + 0.125) + B(z-0.25) + C(z^2 - z + 0.25)$

$\boxed{\begin{array}{l}\text{Alternative way to compute } A \\ A = \dfrac{d}{dz}\dfrac{1}{z-0.25}\Big|_{z=0.5} = -\dfrac{1}{(z-0.25)^2}\Big|_{z=0.5} = -16.\end{array}}$

term z

$0 = -0.75A + B - C$

$0 = -0.75A + 4 - 16$

$A = -\frac{4}{3}(12) = -16$

$\boxed{\begin{array}{c}\text{Recall:} \\ na^n u[n] \longleftrightarrow \dfrac{az}{(z-a)^2},\ |z| > |a|\end{array}}$

$X(z) = \dfrac{-16z \cdot z^{-1}}{z-0.5} + \dfrac{4z \cdot z^{-1}}{(z-0.5)^2} + \dfrac{16z \cdot z^{-1}}{z-0.25}$

$\therefore\ x[n] = -16(0.5)^{n-1}u[n-1] + 8(n-1)(\tfrac{1}{2})^{n-1}u[n-1] + 64(0.25)^{n-1}u[n-1]$

e) $X(z) = \dfrac{1}{z^3 + \frac{1}{8}}$

Find poles

$$z^3 + \frac{1}{8} = 0$$

$$z^3 = -\frac{1}{8} = \cancel{e^{i\pi+2\pi k}}\ e^{i(\pi + 2\pi k)} \cdot \frac{1}{8}$$

$$z = \frac{1}{2} e^{i\left(\frac{\pi}{3} + \frac{2}{3}\pi k\right)}, \quad k = 0, 1, 2, \ldots$$

$$= \frac{1}{2} e^{i\frac{\pi}{3}},\ \frac{1}{2} e^{i\pi},\ \frac{1}{2} e^{-i\pi}.$$

$$X(z) = \frac{1}{(z+\frac{1}{2})(z - \frac{1}{2}e^{i\frac{\pi}{3}})(z - \frac{1}{2}e^{-i\frac{\pi}{3}})} = \frac{A}{z+\frac{1}{2}} + \frac{B}{z - \frac{1}{2}e^{i\pi/3}} + \frac{C}{z - \frac{1}{2}e^{-i\pi/3}}$$

$$A = \left.\frac{1}{z^2 - \frac{z}{2} + \frac{1}{4}}\right|_{z=-\frac{1}{2}}$$

$$A = \frac{3}{4}$$

$$B = \left.\frac{1}{(z+\frac{1}{2})(z - \frac{1}{2}e^{-i\pi/3})}\right|_{z = e^{i\pi/3}}$$

$$= \frac{1}{(e^{i\pi/3} + \frac{1}{2})(e^{i\pi/3} - \frac{1}{2}e^{-i\pi/3})}$$

$$= \frac{1}{(1 + i\frac{\sqrt{3}}{2})\, i\frac{\sqrt{3}}{2}}$$

$$= \frac{-\frac{3}{2} + i\sqrt{3}}{2\left(\frac{21}{4}\right)}$$

$$B = \frac{-3 + i2\sqrt{3}}{21}$$

$$C = B^* = \frac{-3 - i2\sqrt{3}}{21}$$

$$B = \frac{-3 + i2\sqrt{3}}{21} = \frac{1}{\sqrt{21}} e^{i(\phi+\pi)} \qquad \phi = -\tan^{-1}\left(\frac{2\sqrt{3}}{3}\right)$$

$$C = \frac{1}{\sqrt{21}} e^{i(\pi-\phi)}$$

$$X(z) = \frac{\frac{3}{4}}{z+\frac{1}{2}} + \frac{B}{z-\frac{1}{2}e^{i\pi/3}} + \frac{B^*}{z-\frac{1}{2}e^{-i\pi/3}}$$

$$X(z) = \frac{\frac{3}{4} z \cdot z^{-1}}{z+\frac{1}{2}} + \frac{Bz \cdot z^{-1}}{z-\frac{1}{2}e^{i\pi/3}} + \frac{B^* z \cdot z^{-1}}{z-\frac{1}{2}e^{-i\pi/3}}$$

$$x[n] = \frac{3}{4}\left(-\frac{1}{2}\right)^{n-1} u[n-1] + B\left(\frac{1}{2}e^{i\pi/3}\right)^{n-1} u[n-1] + B^*\left(\frac{1}{2}e^{-i\pi/3}\right)^{n-1} u[n-1]$$

$$= \frac{3}{4}\left(-\frac{1}{2}\right)^{n-1} u[n-1] + \frac{1}{2\sqrt{21}}\left[e^{i(\phi+\pi)} e^{i\frac{\pi}{3}(n-1)} + e^{-i(\phi+\pi)} e^{-i\frac{\pi}{3}(n-1)}\right] u[n-1]$$

$$= \frac{3}{4}\left(-\frac{1}{2}\right)^{n-1} u[n-1] + \frac{1}{\sqrt{21}} \cos\left[\frac{\pi}{3}(n-1) + \pi + \phi\right] u[n-1]$$

$$\phi = \tan^{-1}\left(\frac{2\sqrt{3}}{3}\right)$$

c) $X(z) = \dfrac{1}{(1+0.5z^{-1})(1-0.2z^{-1})}$

$$= \frac{z^2}{(z+0.5)(z-0.2)}$$

$$\frac{X(z)}{z} = \frac{z}{(z+0.5)(z-0.2)} = \frac{A}{z+0.5} + \frac{B}{z-0.2}$$

$$A = \frac{z}{z-0.2}\bigg|_{z=-0.5} \qquad B = \frac{z}{z+0.5}\bigg|_{z=0.2}$$

$$A = \frac{5}{7} \qquad B = \frac{2}{7}$$

$$\frac{X(z)}{z} = \frac{\frac{5}{7}}{z+0.5} + \frac{\frac{2}{7}}{z-0.2}$$

$$X(z) = \frac{5}{7}\frac{z}{z+0.5} + \frac{2}{7}\frac{z}{z-0.2}$$

$$x[n] = \left(\frac{5}{7}(-0.5)^n + \frac{2}{7}(0.2)^n\right)u[n]$$

d) $X(z) = \dfrac{1}{(z+0.5)^2(z-0.25)} = \dfrac{A}{z+0.5} + \dfrac{B}{(z+0.5)^2} + \dfrac{C}{z-0.25}$

$$C = \frac{1}{(z+0.5)^2}\bigg|_{z=0.25} = \frac{16}{9} \qquad B = \frac{1}{z-0.25}\bigg|_{z=-0.5} = -\frac{4}{3}$$

$$A = \frac{d}{dz}\frac{1}{z-0.25}\bigg|_{z=-0.5} = -\frac{1}{(z-0.25)^2}\bigg|_{z=-0.5} = -\frac{16}{9}$$

$$\therefore X(z) = -\frac{16}{9}\frac{z\cdot z^{-1}}{z+0.5} - \frac{4}{3}\frac{z\cdot z^{-1}}{(z+0.5)^2} + \frac{16}{9}\frac{z\cdot z^{-1}}{z-0.25}$$

$$x[n] = -\frac{16}{9}(-0.5)^{n-1}u[n-1] - \frac{8}{3}(n-1)\left(-\frac{1}{2}\right)^{n-1}u[n-1] + \frac{64}{9}(0.25)^{n-1}u[n-1]$$