HW4 solution

Problem 1:

$$v[n] = x[n]\,w[n],\quad x[n] = \cos(0.3\pi n),\quad w[n] = \begin{cases} 0.54 + 0.46\cos\frac{2\pi n}{N-1} \\ 0 & \text{else} \end{cases}$$

$$V_d(\omega) = \frac{1}{2\pi} X_d(\omega) \circledast W_d(\omega)$$

$$X_d(\omega) = \pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0) \text{ for } \omega \leq \pi, \quad \omega_0 = 0.3\pi$$

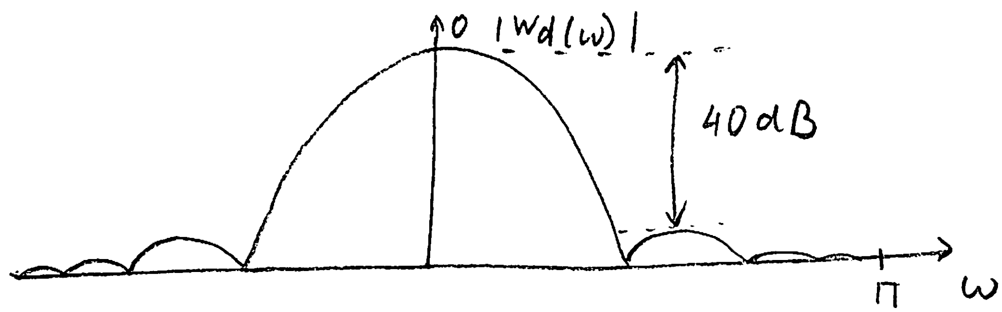
$$V_d(\omega) = \frac{1}{2} W_d(\omega - \omega_0) + \frac{1}{2} W_d(\omega + \omega_0)$$

$$|V_d(\omega)| \cong \frac{1}{2}|W_d(\omega - \omega_0)| + \frac{1}{2}|W_d(\omega + \omega_0)|$$

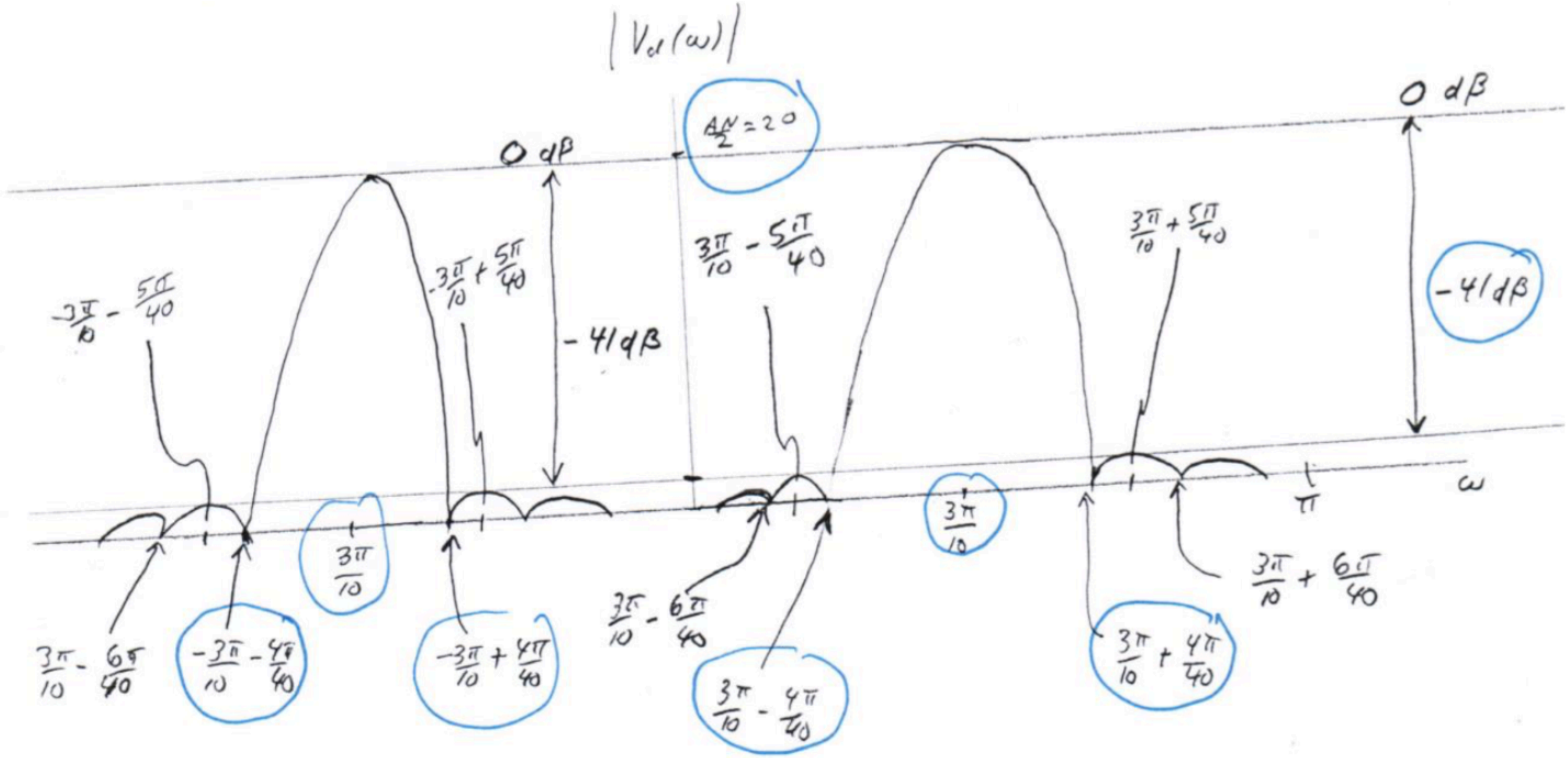
For the Hamming window, $W_d(\omega)$ has the following properties:

a) Main lobe width twice that of the truncation window, i.e $\frac{8\pi}{N}$ vs $\frac{4\pi}{N}$

b) Sidelobe $\approx -40\,dB$ below the main lobe



The resulting plot of $|V_d(\omega)|$ is shown below (circled are the quantities required in the problem)

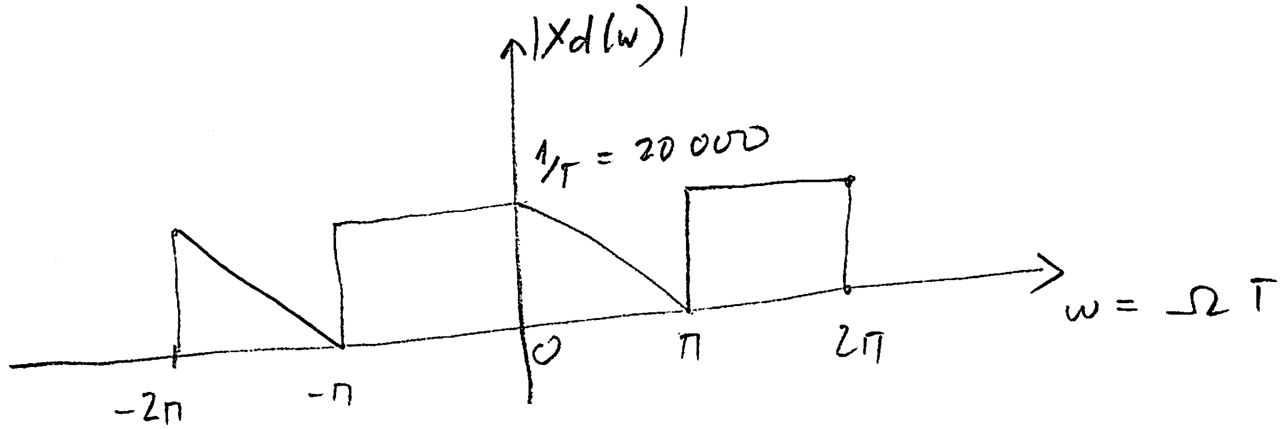

$|V_d(\omega)|$
$\frac{AN}{2} = 20$
0 dB
0 dB
-41 dB
-41 dB
ω
π
$-\frac{3\pi}{10} - \frac{5\pi}{40}$
$-\frac{3\pi}{10} + \frac{5\pi}{40}$
$\frac{3\pi}{10} - \frac{5\pi}{40}$
$\frac{3\pi}{10} + \frac{5\pi}{40}$
$\frac{3\pi}{10} - \frac{6\pi}{40}$
$-\frac{3\pi}{10} - \frac{4\pi}{40}$
$\frac{3\pi}{10}$
$-\frac{3\pi}{10} + \frac{4\pi}{40}$
$\frac{3\pi}{10} - \frac{6\pi}{40}$
$\frac{3\pi}{10} - \frac{4\pi}{40}$
$\frac{3\pi}{10}$
$\frac{3\pi}{10} + \frac{4\pi}{40}$
$\frac{3\pi}{10} + \frac{6\pi}{40}$

Problem 2:

a) (i) $T = \frac{1}{F_s} = \frac{1}{20000}$

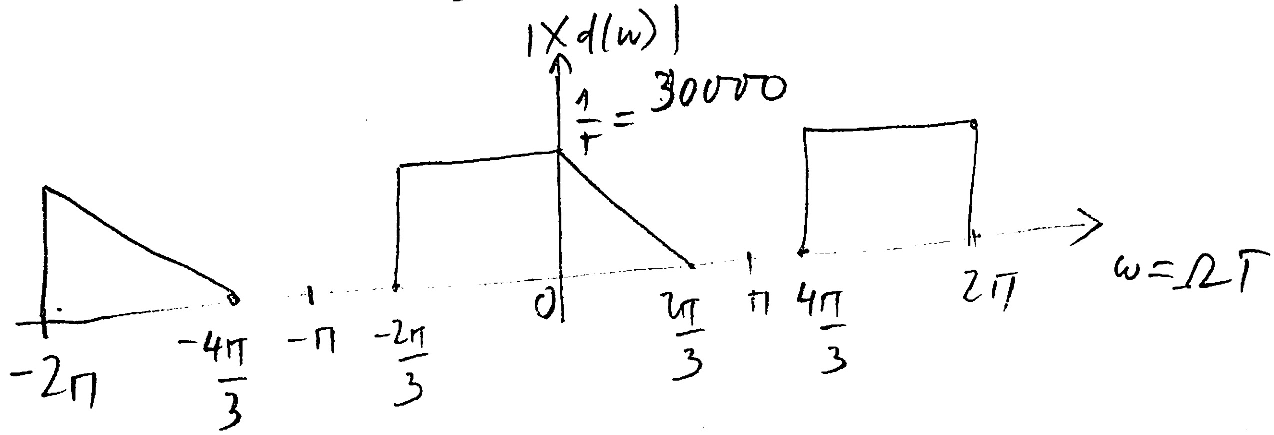
$$w = \pm \Omega T = \pm 20000\pi \, \frac{1}{20000} = \pm \pi$$

$$X_d(w) = \frac{1}{T} X_a\left(\frac{w}{T}\right)$$



(ii) $T = \frac{1}{F_s} = \frac{1}{30000}$

$$w = \pm \Omega T = \pm 20000\pi \, \frac{1}{30000} = \pm \frac{2\pi}{3}.$$



b) $B = 20000\pi$

$$\pi \geq BT$$

$$\frac{1}{T} \geq \frac{2\pi f_{max}}{\pi}$$

$$F_N \geq \frac{2\pi \, 10000}{\pi} = 20000 \text{ Hz}$$

Problem 3:

$$x_a(t) = \cos(\Omega_0 t) \longrightarrow \boxed{A/D} \longrightarrow x[n] = \cos\left[\frac{7\pi n}{12}\right]$$

$$\uparrow$$

$$T = \frac{1}{80} \ \frac{\text{sec}}{\text{sample}}$$

$$x[n] = x_a[nT] = \cos(\Omega_0 T n) = \cos((\Omega_0 T \pm 2\pi k)n), \ k \in \mathbb{Z}$$

$$0 \leq \Omega_0 T \pm 2\pi k = w_0 \leq \pi$$

$$\Omega_0 = \frac{w_0}{T} + \frac{2\pi k}{T} = \frac{7\pi}{12} 80 + 2\pi k \, 80$$

3 Lowest possible values:

$$\Omega_0 = \frac{7\pi}{12} 80 \quad (k = 0)$$

$$\Omega_0 = \frac{7\pi}{12} 80 + 160\pi \quad (k = 1)$$

$$\Omega_0 = \frac{7\pi}{12} 80 + 320\pi \quad (k = 2)$$

There are more possible values with larger values of k.

Problem 4:

$$x_a(t) = \cos(30\pi t)$$

$$x[n] = x_a\left(\frac{n}{40}\right) = \cos\left(\frac{3\pi}{4} n\right) = \frac{e^{j\frac{3\pi}{4}n} + e^{-j\frac{3\pi}{4}n}}{2} \quad (1)$$

DFT synthesis equation:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi k n}{N}}$$

$$= \frac{1}{16} \sum_{k=0}^{15} X[k] e^{j\frac{2\pi k n}{16}} \quad (2)$$

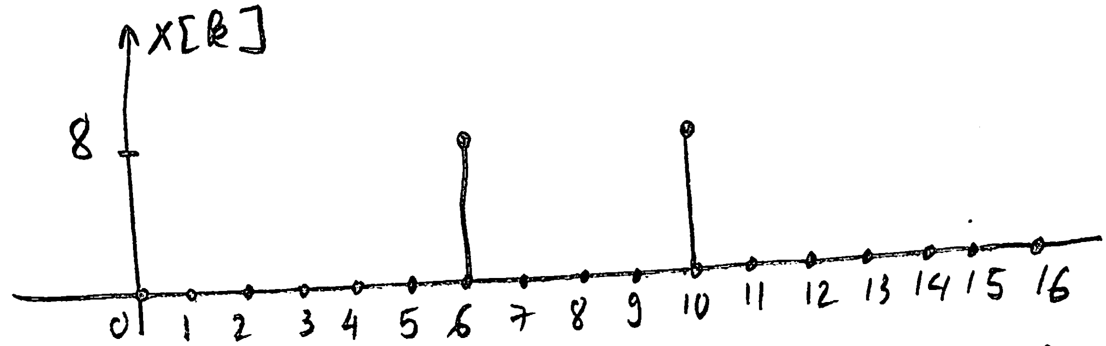
We can rewrite $x[n]$ in (1) as: $x[n] = \frac{8}{16} e^{j\frac{2\pi(6n)}{16}} + \frac{8}{16} e^{j\frac{2\pi(10n)}{16}}$

$$\left(e^{-j\frac{3\pi}{4}n} = e^{-j\left(\frac{3\pi}{4} - 2\pi\right)n} = e^{j\frac{5\pi}{4}n} = e^{j\frac{2\pi\, 10n}{16}}\right)$$

Matching this to the synthesis equation: (2)

$$X[6] = X[10] = 8$$

$X[k] = 0$, otherwise



The typical DFT spectrum of a sampled sinusoid looks like a pair of sampled shifted "digital sinc functions" with samples at multiples of $\frac{2\pi}{N}$. Here the sampling points fall on the nulls of the digital sinc functions, except for the samples at $k = 6, 10$ which fall on the peaks

Problem 5

$$x_a(t) = \cos(118\pi t) + 0.5\cos(120\pi t)$$

a) $\pi \geq BT$

$$F_s \geq \frac{2\pi f_{max}}{\pi}$$

$$F_N = 2 f_{max} = 120\ Hz$$

b) $N = 600,\ T = \frac{1}{200}\ s$

$$\pm w_0 = \pm \Omega_0 T = \pm 120\pi\left(\frac{1}{200}s\right) = \pm \frac{3\pi}{5}\ \frac{rad}{sample}$$

$$\pm w_1 = \pm \Omega_1 T = \pm 118\pi\left(\frac{1}{200}s\right) = \pm \frac{59\pi}{100}\ \frac{rad}{sample}$$

$$k = \frac{w_0 N}{2\pi} = \frac{3\pi(1024)}{(2\pi)5} = \frac{3}{10}(1024) \approx 307.2 \approx 307$$

$$k = N - 307 = 1024 - 307 = 717$$

$$k = \frac{w_1 N}{2\pi} = \frac{59}{200}(1024) \approx 302$$

$$k = N - 302 = 1024 - 302 = 722$$

c) There are two criterias we can use to find N

There are two criterias we can use to find N no overlap between the mainlobes corresponding to different sinusoids

Case i) no overlap between the mainlobes corresponding to different sinusoids

$$\Omega_1 T - \frac{2\pi}{N} > \Omega_0 T + \frac{2\pi}{N}$$

$$\Omega_1 T - \Omega_0 T > \frac{4\pi}{N}$$

$$N > \frac{4\pi}{(\Omega_1 - \Omega_0)T} = \frac{4\pi\, 200}{120\pi - 118\pi} = 400 \text{ samples.}$$

Case ii: No more than half overlap between the main lobes corresponding to different sinusoids

$$\Omega_1 T - \frac{2\pi}{N} > \Omega_0 T$$

$$(\Omega_1 - \Omega_0) T > \frac{2\pi}{N}$$

$$N > \frac{2\pi}{(\Omega_1 - \Omega_0) T} = \frac{2\pi \cdot 200}{(120\pi - 118\pi)} = 200 \text{ samples}.$$

5d) $F_N = 120$ Hz

$$T = \frac{1}{F_N} = \frac{1}{120} \text{ sec}$$

$$\Omega_1 - \Omega_0 = \frac{2\pi}{NT}$$, hence, with $N = 200$ from part c

$$\Omega_1 - \Omega_0 = \frac{2\pi}{200 \frac{1}{120}}$$

$$f = \frac{2\pi \cdot 120}{2\pi \cdot 200} = 0.6 \text{ Hz}.$$

e)

As Fs increased, all analog frequencies are squeezed in, and therefore come closer to each other in the Digital frequency domain. Therefore, assuming that the number of samples N is kept fixed so that the width of the main lobes of the peaks does not change, this degrades the resolution.

f)

(i) With N=600 and T=1/200, we know, from Part c, that all three sinusoids will be sufficiently separated to be resolved (the main lobes obtained in the spectral analysis will not overlap). So, the reason for missing the third peak is different. The magnitude of the third component is only 3% of that of the first component. Recall from Problem 1 that the sidelobes for the truncation window are as high as 20% of the mainlobe, and their magnitude falls off slowly -- only as 1/\omega. So, the main lobe corresponding to the third component may be obscured by overlapping sidelobes from the first component, and the third component will be missed.

(ii) Yes, the sidelobes with the Hamming window are more than 40 dB down from the main lobe (i.e, have magnitude smaller than 1% of the mainlobe). So, the sidelobes from the first two components will be much (about 3 times or more) smaller than the main lobe of the third component, so that the latter will be resolved.

Another problem that may arise, is due to the increased (doubled) width of the main lobe of the Hamming window compared to the truncation (rectangular) window. This will cause the main lobes of the first and second component to overlap, so that they may not be separated.

(iii) With a Hamming window, the main lobe width in the frequency domain is doubled relative to the rectangular window, so instead of N=400, now one needs N=800, according to the same resolution criterion.

Problem 6 :

a) $y[n] = x[3-n]$	linear	non-causal	shift varying
b) $y[n] = e^{j\pi n^2/4} x[n]$	linear	causal	shift varying
c) $y[n] = x[n^2]$	linear	non-causal	shift-varying
d) $y[n] = x[2n]$	linear	non-causal	shift-varying
e) $y[n] = x[n]+2$	non-linear	causal	shift-invariant
f) $y[n] = \sum_{m=-\infty}^{n+3} x[m-1]$	linear	non-causal	shift-invariant
g) $y[n] = \frac{x[n]}{x[5]}$	non-linear	non-causal	shift-varying.
h) $y[n-1] = x[n-1] + \tan(4)x[n] - 2y[n]$	linear	non-causal	shift-invariant
i) $y[n] = y[n-1] + \lvert x[n] \rvert$	non-linear	causal	shift-invariant

Problem 7:

a) $y[n] = 2y[n-2] + x[n-1]$

Let $x_1[n] \rightarrow y_1[n] = 2y_1[n-2] + x_1[n-1]$ (1)

$x_2[n] \rightarrow y_2[n] = 2y_2[n-2] + x_2[n-1]$. (2)

and let $\tilde{x}[n] = a x_1[n] + b x_2[n] \rightarrow \tilde{y}[n] = 2\tilde{y}[n-2] + \tilde{x}[n-1]$

$\tilde{y}[n] = 2\tilde{y}[n-2] + a x_1[n-1] + b x_2[n-1]$ (3)

now multiplying Equ.(1) by a, and Equ.(2) by b and add up, we get:

$a y_1[n] + b y_2[n] = 2a y_1[n-2] + a x_1[n-1] + 2b y_2[n-2]$
$+ b x_2[n-1]$ (4)

Comparing (3) & (4), we have that:

$\tilde{y}[n] = a y_1[n] + b y_2[n]$

$\therefore$ Linear

ii. $y[n]$ depends on past values of inputs $\therefore$ causal

iii. $x[n] \rightarrow y[n]$

$x[n-n_0] \rightarrow y_1[n] = 2y_1[n-2] + x[n-n_0-1]$

Let $n - n_0 = m$

$y_1[m+n_0] = 2y_1[m+n_0-2] + x[m-1]$

Since $y_1[m+n_0]$ and $y[m]$ satisfy the same equations:

$\Rightarrow y_1[n+n_0] = y[n] \Rightarrow y_1[n] = y[n-n_0] \Rightarrow$ shift invariance

7b) $y[n] = y[n+1] + x[n]$

Let $x_1[n] \rightarrow y_1[n] = y_1[n+1] + x_1[n]$

$x_2[n] \rightarrow y_2[n] = y_2[n+1] + x_2[n]$

let $\tilde{x}[n] = a x_1[n] + b x_2[n]$

$\tilde{x}[n] \rightarrow \tilde{y}[n] = \tilde{y}[n+1] + \tilde{x}[n]$ ①

$a y_1[n] + b y_2[n] = a y_1[n+1] + b y_2[n+1] + a x_1[n] + b x_2[n]$. ②

Comparing ① & ②: $\tilde{y}[n] = a y_1[n] + b y_2[n]$.

$\therefore$ the system is linear

$y[n]$ depends on future values $y[n+1]$

$\therefore$ the system is non-causal

$x[n] \rightarrow y[n]$

$x[n-n_0] \rightarrow y_1[n] = y_1[n+1] + x[n-n_0]$

Let $n - n_0 = m$ $(n = n_0 + m)$

$y_1[m+n_0] = y_1[m+n_0+1] + x[m]$.

$\Rightarrow y_1[n+n_0] = y[n] \Rightarrow y_1[n] = y[n-n_0]$.

$\therefore$ the system is invariant.

7c) $y[n] = \sum_{m=-2}^{n} x[m-n]\, 2^m$

let $x_1[n] \rightarrow y_1[n] = \sum_{m=-2}^{n} x_1[m-n]\, 2^m$

$x_2[n] \Rightarrow y_2[n] = \sum_{m=-3}^{n} x_2[m-n]\, 2^m$

now let $\tilde{x}[n] = a x_1[n] + b x_2[n] \rightarrow \tilde{y}[n] = \sum_{m=-2}^{n} (a x_1[m-n] + b x_2[m-n]) 2^m$

$= a \sum_{m=-2}^{n} x_1[m-n]\, 2^m + b \sum_{m=-2}^{n} x_2[m-n]\, 2^m = a y_1[n] + b y_2[n]$.

$\therefore$ the system is linear

The system is non-causal. Proof by counter-example.

$y[-1] = \sum_{m=-2}^{-1} x[m+1]\, 2^m$

$y[-1]$ depends on the future time values i.e. $x[0]$

iii $x[n] \rightarrow y[n] = \sum_{m=-2}^{n} x[m-n]\, 2^m$

$\tilde{x}[n] = x[n-n_0] \rightarrow \tilde{y}[n] = \sum_{m=-2}^{n} x[m-n+n_0]\, 2^m$

$\neq y[n-n_0] = \sum_{m=-2}^{n-n_0} x[m-n+n_0]\, 2^m$

$\therefore$ not shift invariant.