1. The magnetic field of a plane wave propagating in a non-magnetic material ($\mu = \mu_o = 4\pi \times 10^{-7}$ H/m) is given by

$$\mathbf{H} = 25\, e^{-z} \cos(8\pi \cdot 10^6 t - \sqrt{3}z - \frac{\pi}{3})\, \hat{x}\, \frac{\text{A}}{\text{m}}.$$

By inspection, we have the wave frequency $\omega = 8\pi \cdot 10^6$ rad/s, the attenuation constant $\alpha = 1\,\text{m}^{-1}$, and the wavenumber $\beta = \sqrt{3}\,\text{m}^{-1}$. Now, let us determine the following parameters.

a) Propagation constant $\gamma = \alpha + j\beta = 1 + j\sqrt{3} = 2e^{j\frac{\pi}{3}}\,\text{m}^{-1}$.
Intrinsic impedance:

$$\eta = \frac{\mu j\omega}{\gamma} = \frac{4\pi \times 10^{-7} \times j8\pi \times 10^6}{2e^{j\frac{\pi}{3}}} = \frac{8\pi^2}{5} e^{j\frac{\pi}{6}}\,\Omega.$$

b) $\frac{\gamma}{\eta} = \sigma + j\omega\epsilon = \frac{2e^{-j\frac{\pi}{3}}}{\frac{8}{5}\pi^2 e^{j\frac{\pi}{6}}} = \frac{5}{4\pi^2} e^{j\frac{\pi}{6}} = \frac{5}{4\pi^2}\left(\frac{\sqrt{3}}{2} + j\frac{1}{2}\right)$. Therefore, the permittivity is:

$$\epsilon = \frac{1}{\omega}\text{Im}\left\{\frac{\gamma}{\eta}\right\} = \frac{1}{8\pi \times 10^6}\frac{5}{8\pi^2} = \frac{5 \times 10^{-6}}{64\pi^3} \approx 2.52 \times 10^{-9}\,\frac{\text{F}}{\text{m}},$$

and the conductivity is

$$\sigma = \text{Re}\left\{\frac{\gamma}{\eta}\right\} = \frac{5\sqrt{3}}{8\pi^2} \approx 0.1097\,\frac{\text{S}}{\text{m}}.$$

c) Magnetic field phasor

$$\tilde{\mathbf{H}} = 25\, e^{-z} e^{-j(\sqrt{3}z + \frac{\pi}{3})}\, \hat{x}\, \frac{\text{A}}{\text{m}}.$$

d) Electric field phasor

$$\begin{aligned}\tilde{\mathbf{E}} =& \ -\hat{y}\eta \cdot 25e^{-z}e^{-j(\sqrt{3}z+\frac{\pi}{3})} = -\hat{y}(\frac{8\pi^2}{5}e^{j\frac{\pi}{6}}) \cdot 25e^{-z}e^{-j(\sqrt{3}z+\frac{\pi}{3})} \\ =& \ 40\,\pi^2 e^{-z}e^{-j(\sqrt{3}z+\frac{\pi}{3}+\frac{5\pi}{6})}\,\hat{y} = 40\,\pi^2 e^{-z}e^{j(-\sqrt{3}z+\frac{5\pi}{6})}\,\hat{y}\,\frac{\text{V}}{\text{m}}.\end{aligned}$$

e) Time-averaged Poynting vector

$$\begin{aligned}\langle\mathbf{S}\rangle &= \frac{1}{2}\text{Re}\left\{\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*\right\} = \frac{1}{2}\text{Re}\left\{10^3\pi^2 e^{-2z}e^{j\frac{\pi}{6}}\,\hat{z}\right\} \\ &= 250\sqrt{3}\,\pi^2 e^{-2z} = 4273.66\,e^{-2z}\,\frac{\text{W}}{\text{m}^2}.\end{aligned}$$

f) Poynting's theorem in integral form states that

$$\oint_S \tilde{\text{S}} \cdot d\vec{S} = -\int_V \sigma E^2 dV - \frac{d}{dt}\int_V \frac{1}{2}\epsilon E^2 dV - \frac{d}{dt}\int_V \frac{1}{2}\mu H^2 dV.$$

The time average of the previous equation simplifies to

$$\oint_S \langle\mathbf{S}\rangle \cdot d\vec{S} = -\int_V \sigma\left\langle E^2\right\rangle dV,$$

where $P_d = \int_V \sigma\left\langle E^2\right\rangle dV$ is the time averaged dissipated power. Note that, after taking the average, the derivative terms became zero simply because averaged quantities are constant in time. Either side of this equation can be used to calculate the power dissipated in the cubic volume. It is easier to use the left side that corresponds to a closed surface integral taken over the sides of the cube. In this problem, the Poynting vector is parallel to z-direction, thus, only the two faces perpendicular to $\hat{z}$ will contribute to the integral. As a result, we get

$$\oint_S \langle \mathbf{S} \rangle \cdot d\vec{S} = \int_{z=0} \langle \mathbf{S} \rangle \cdot d\vec{S} + \int_{z=1} \langle \mathbf{S} \rangle \cdot d\vec{S} = -\int_{z=0} \langle \mathbf{S} \rangle \, dS + \int_{z=1} \langle \mathbf{S} \rangle \, dS$$
$$= -4273.66 \, (1 - e^{-2}) \, \mathrm{W}.$$

The minus sign implies that the material absorbs (dissipates) energy.

2. The electric field phasor of a plane wave is given by

$$\tilde{\mathbf{E}} = 4e^{-(0.0001+j0.2)z} \hat{x} \, \frac{\mathrm{V}}{\mathrm{m}}.$$

Clearly, $\gamma = \alpha + j\beta = 0.0001 + j0.2 \, \mathrm{m}^{-1}$. Let us consider $\mu = 4\mu_o = 16\pi \times 10^{-7} \, \mathrm{H/m}$ and $v_p = 10^8 \, \mathrm{m/s}$ in order to determine the following parameters.

a) We know that the propagation velocity, also called phase velocity, is $v_p = \frac{\omega}{\beta}$. Therefore, the wave frequency $f = \frac{\beta}{2\pi} v_p = \frac{0.2}{2\pi} \times 10^8 = \frac{1}{\pi} \times 10^7 = 3.183 \times 10^6 \, \mathrm{Hz}$.

b) Wavelength $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.2} = 10\pi = 31.416 \, \mathrm{m}$.

c) We know that the propagation velocity is $v_p = \frac{1}{\sqrt{\mu\epsilon}}$. Therefore, the dielectric constant is

$$\epsilon_r = \frac{\epsilon}{\epsilon_o} = \frac{1}{\mu \epsilon_o v_p^2} = \frac{1}{(16\pi \times 10^{-7}) \times \left(\frac{1}{36\pi} \times 10^{-9}\right) \times 10^{16}} = \frac{9}{4} = 2.25.$$

d) We will consider the attenuation constant α for the case of an imperfect dielectric: $\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$. Therefore, the conductivity is

$$\sigma = \frac{2\alpha}{\sqrt{\frac{\mu}{\epsilon}}} = \frac{2\alpha}{\eta_o}\sqrt{\frac{\epsilon_r}{\mu_r}} = \frac{2 \times 0.0001}{120\pi} \times \sqrt{\frac{9/4}{4}} = \frac{10^{-5}}{8\pi} = 3.98 \times 10^{-7} \, \mathrm{S/m}.$$

e) The distance at which the peak amplitude of $\tilde{\mathbf{E}}$ is attenuated by $1/e$ is at

$$z = \delta = \frac{1}{\alpha} = 10^4 \, \mathrm{m}.$$

f) The magnetic field phasor is

$$\tilde{\mathbf{H}} = \frac{4}{\eta} e^{-(0.0001+j0.2)z} \hat{y} = \frac{4}{\eta_o}\sqrt{\frac{\epsilon_r}{\mu_r}} e^{-(0.0001+j0.2)z} \hat{y}$$
$$= \frac{1}{40\pi} e^{-(0.0001+j0.2)z} \hat{y} \, \frac{\mathrm{A}}{\mathrm{m}}.$$

3. In conducting media, the wave propagation constant $\gamma = \alpha + j\beta$ and intrinsic impedance η are given by

$$\gamma = \sqrt{(j\omega\mu)(\sigma + j\omega\epsilon)} \quad \text{and} \quad \eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}.$$

Let us consider a medium with $\mu = \mu_o = 4\pi \times 10^{-7} \, \mathrm{H/m}$, $\epsilon = 81\epsilon_o \approx \frac{9}{4\pi} \times 10^{-9} \, \mathrm{F/m}$, and $\sigma = 4 \, \mathrm{S/m}$.

a) Since $\frac{\sigma}{\omega\epsilon} = \frac{4}{2\pi \times 5 \times 10^3 \cdot \frac{9}{4\pi} \times 10^{-9}} = \frac{8}{45} \times 10^6 \gg 1$, we use the approximations for good conductors.

$$\alpha = \beta \approx \sqrt{\pi f \mu \sigma} = \sqrt{\pi \times 5 \times 10^3 \cdot 4\pi \times 10^{-7} \cdot 4} = 4\sqrt{5}\pi \times 10^{-2} \, \mathrm{m}^{-1}.$$

Therefore the propagation constant is

$$\gamma = \alpha + j\beta = 0.089\pi \, (1 + j) \, \mathrm{m}^{-1} \approx 0.281 + j0.281 \, \mathrm{m}^{-1},$$

the wavelength is

$$\lambda = \frac{2\pi}{\beta} = 22.36\,\mathrm{m},$$

the penetration depth

$$\delta = \frac{1}{0.281} = 3.56\,\mathrm{m},$$

and the intrinsic impedance is

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}} e^{j\pi/4} = \sqrt{\frac{2\pi \times 5 \times 10^3 \times 4\pi \times 10^{-7}}{4}} \left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) = \sqrt{5}\pi \times 10^{-2}\,(1+j)\ \Omega.$$

b) Since $\frac{\sigma}{\omega\epsilon} = \frac{4}{2\pi\times 5\times 10^9 \cdot \frac{9}{4\pi}\times 10^{-9}} = \frac{8}{45}$, approximations can't be used. The propagation constant is

$$\gamma = \sqrt{j\omega\mu\,(\sigma + j\omega\epsilon)} = \sqrt{(j2\pi \times 5 \times 10^9 \times 4\pi \times 10^{-7})(4 + j2\pi \times 5 \times 10^9 \times \frac{9}{4\pi} \times 10^{-9})}$$
$$= 20\pi\sqrt{j40 - 225} = 83.56 + j946.2\,\mathrm{m}^{-1}.$$

Since $\alpha = 83.56\,\mathrm{m}^{-1}$ and $\beta = 946.2\,\mathrm{m}^{-1}$, the penetration depth and the wavelength are

$$\delta = \frac{1}{\alpha} = 1.2 \times 10^{-2}\,\mathrm{m} \quad \text{and} \quad \lambda = \frac{2\pi}{\beta} = 6.6 \times 10^{-3}\,\mathrm{m}.$$

Finally, the intrinsic impedance is

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = \sqrt{\frac{j2\pi \times 5 \times 10^9 \times 4\pi \times 10^{-7}}{4 + j2\pi \times 5 \times 10^9 \times \frac{9}{4\pi} \times 10^{-9}}}$$
$$= 40\pi\sqrt{\frac{j}{1.6 + j9}} = 41.4 + j3.65\,\Omega.$$

c) In the ocean, $\sigma = 4\,\mathrm{S/m}$, $\epsilon_r = 81$, and $\mu_r = 1$. For $\omega = 40\pi \times 10^3\,\mathrm{rad/s}$, the propagation constant is

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = \sqrt{j0.158 \times (4 + j9.01 \times 10^{-5})} = 0.562 + j0.562\,\mathrm{m}^{-1}.$$

Since $\alpha = 0.562\,\mathrm{m}^{-1}$, the distance at which a submarine should be located in order to receive at least 0.1% of the amplitude of an EM signal transmitted from a ship located at the surface can be calculated as follows

$$e^{-\alpha z} = 0.001 \quad \rightarrow \quad z = -\frac{1}{\alpha}\ln(0.001) = 12.3\,\mathrm{m}.$$

4. The TEM wave is propagating through a good conductor:

   a) $\alpha = \beta = 10\pi\,\mathrm{m}^{-1}$. So the propagation constant $\gamma = \alpha + j\beta = 10\pi(1+j)\,\mathrm{m}^{-1}$.

   b) $v_p = \frac{\omega}{\beta} = \frac{4\pi\times 10^8}{10\pi} = 4 \times 10^7\,\mathrm{m/s}$. The wave is travelling in $\hat{y}$ direction.

   c) The magnetic polarization is in $\hat{x}$ direction.

   d) $|\,\eta\,| = \frac{|\mathbf{E}|}{|\mathbf{H}|} = \frac{100}{50} = 2$; $\angle\eta = \frac{\pi}{4}$ for good conductor. Therefore, $\eta = 2e^{j\pi/4}\Omega$.

   e) $\phi + \frac{\pi}{3} = \frac{\pi}{4}$. So $\phi = -\frac{\pi}{12}\,\mathrm{rad}$.

5. A plane wave $\mathbf{H} = 2\sin(\omega t + \beta x)\hat{y} - 2\cos(\omega t + \beta x)\hat{z}\,\mathrm{A/m}$ is propagating in $-\hat{x}$ direction:

   a) $\eta_1 = \sqrt{\frac{\mu_o}{\varepsilon_o}}$, $\eta_2 = \sqrt{\frac{\mu_o}{2.25\varepsilon_o}} = \frac{2}{3}\eta_1$. Therefore,

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = -\frac{1}{5}$$

$$\tau = 1 + \Gamma = \frac{4}{5}$$

b) $\beta_1 = \omega\sqrt{\mu_o\varepsilon_o}$, $\beta_2 = \omega\sqrt{\mu_o 2.25\varepsilon_o} = 1.5\beta_1$

$$\begin{aligned} \tilde{\mathbf{H}} &= 2e^{j\beta x}(-j\hat{y} - \hat{z}) \\ \tilde{\mathbf{E}} &= \eta 2e^{j\beta x}(-j\hat{z} + \hat{y}) \end{aligned}$$

Thus,

$$\begin{aligned} \tilde{\mathbf{E}}_\mathbf{i} &= \eta_1 2e^{j\beta_1 x}(-j\hat{z} + \hat{y})\,\mathrm{V/m} \\ \tilde{\mathbf{E}}_\mathbf{r} &= \Gamma\tilde{\mathbf{E}}_\mathbf{i}(\text{with } -x \to x) = -\frac{2}{5}\eta_1 e^{-j\beta_1 x}(-j\hat{z} + \hat{y})\,\mathrm{V/m} \\ \tilde{\mathbf{E}}_\mathbf{t} &= \tau\tilde{\mathbf{E}}_\mathbf{i}(\text{with } \eta_1 \to \eta_2 \text{and } \beta_1 \to \beta_2) = \frac{8}{5}\eta_2 e^{j\beta_2 x}(-j\hat{z} + \hat{y})\,\mathrm{V/m} \end{aligned}$$

c) The corresponding **H** fields are:

$$\begin{aligned} \tilde{\mathbf{H}}_\mathbf{i} &= 2e^{j\beta_1 x}(-j\hat{y} - \hat{z})\,\mathrm{A/m} \\ \tilde{\mathbf{H}}_\mathbf{r} &= \Gamma\tilde{\mathbf{H}}_\mathbf{i}(\text{with } -x \to x) = -\frac{2}{5}e^{-j\beta_1 x}(-j\hat{y} - \hat{z})\,\mathrm{A/m} \\ \tilde{\mathbf{H}}_\mathbf{t} &= \tau\tilde{\mathbf{H}}_\mathbf{i}(\text{with } \beta_1 \to \beta_2) = \frac{8}{5}e^{j\beta_2 x}(-j\hat{y} - \hat{z})\,\mathrm{A/m} \end{aligned}$$

d) The time-average Poynting vectors of the incident, reflected, and transmitted waves are

$$\begin{aligned} \langle \mathbf{S}_i \rangle &= \frac{1}{2}\mathrm{Re}\left\{ \tilde{\mathbf{E}}_i \times \tilde{\mathbf{H}}_i{}^* \right\} = 4\eta_1\,(-\hat{x})\,\frac{\mathrm{W}}{\mathrm{m}^2}, \\ \langle \mathbf{S}_r \rangle &= \frac{1}{2}\mathrm{Re}\left\{ \tilde{\mathbf{E}}_r \times \tilde{\mathbf{H}}_r{}^* \right\} = 4\eta_1\Gamma^2\,\hat{x}\,\frac{\mathrm{W}}{\mathrm{m}^2}, \\ \langle \mathbf{S}_t \rangle &= \frac{1}{2}\mathrm{Re}\left\{ \tilde{\mathbf{E}}_t \times \tilde{\mathbf{H}}_t{}^* \right\} = 4\eta_2\tau^2\,(-\hat{x})\,\frac{\mathrm{W}}{\mathrm{m}^2}. \end{aligned}$$

Note that the energy conservation is conserved. The power density transported by the incident wave is distributed between the reflected and transmitted waves, such that

$$|\langle \mathbf{S}_i \rangle| = |\langle \mathbf{S}_r \rangle| + |\langle \mathbf{S}_t \rangle|,$$

therefore, it can be shown that

$$\Gamma^2 + \frac{\eta_1}{\eta_2}\tau^2 = 1.$$

e) The polarization of the incident and transmitted waves is LHCP - left hand circular polarized. The polarization of the reflected wave is RHCP - right hand circular polarized.

f) Reflectance $|\Gamma|^2$ expressed in dB units. If we take the ratio of the reflected power and the incident power and expressed that in dB we have

$$\begin{aligned} 10\log_{10}\frac{|\langle \mathbf{S}_r \rangle|}{|\langle \mathbf{S}_i \rangle|} &= 10\log_{10}\left|\Gamma^2\right| = 10\log_{10}\frac{1}{25} = -20\log_{10}5 = -20\log_{10}\frac{10}{2} \\ &= -2\,(10\log_{10}10 - 10\log_{10}2) = -13.98\,\mathrm{dB}. \end{aligned}$$

Therefore, the reflectance of an interface indicates the ratio of the reflected and the incident powers. For this case, the reflected signal level is $\sim 14\,\mathrm{dB}$ below the incident signal level — equivalent to 4% in power and 20% in amplitude compared to the incident signal.