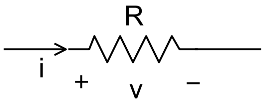
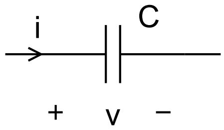
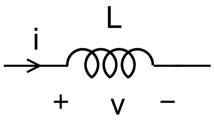
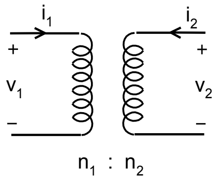
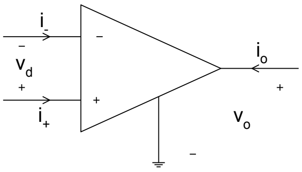
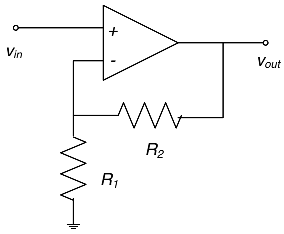
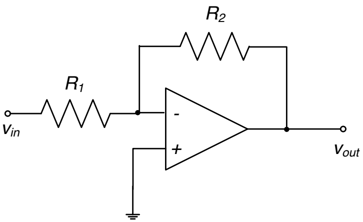
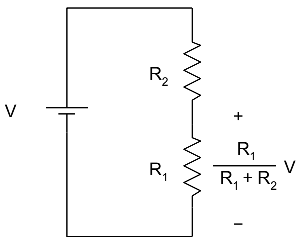
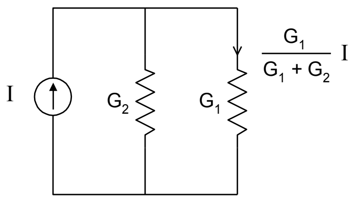
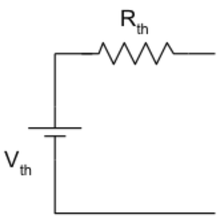
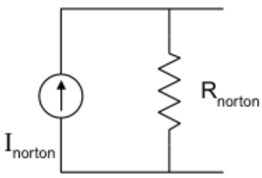
CIRCUIT THEORY EEN30020

REVIEW NOTES

Elementary Circuit Theory

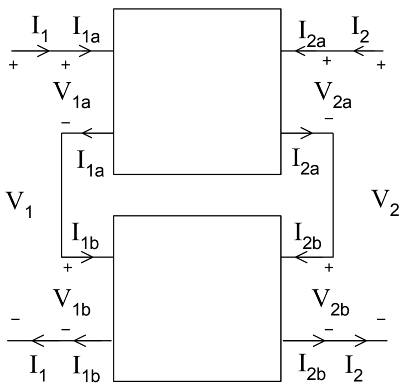
Resistor 	$v = Ri$ $i = Gv$ $Z = R$	Capacitor 	$q = Cv$ $i = C\dfrac{dv}{dt}$ $Z = \dfrac{1}{j\omega C}$
Inductor 	$\phi = Li$ $v = L\dfrac{di}{dt}$ $Z = j\omega L$	Transformer 	$\dfrac{v_2}{v_1} = \dfrac{n_2}{n_1}$ $\dfrac{i_2}{i_1} = -\dfrac{n_1}{n_2}$
Op amp 	$i_- = 0$ $i_+ = 0$ $v_d = 0 \quad \Leftrightarrow \quad v_- = v_+$	Op amp (non-inverting) 	$v_{out} = \dfrac{R_1 + R_2}{R_1} v_{in}$
Op amp (inverting) 	$A_v = \dfrac{v_{out}}{v_{in}} = -\dfrac{R_2}{R_1}$		

Voltage division 	$\dfrac{R_1}{R_1 + R_2} V$	Current division 	$\dfrac{R_2}{R_1 + R_2} I$ $\dfrac{G_1}{G_1 + G_2} I$
Thevenin and Norton equivalents  		$I_{norton} = \dfrac{V_{th}}{R_{th}}$ $R_{norton} = R_{th}$	

Two-Ports

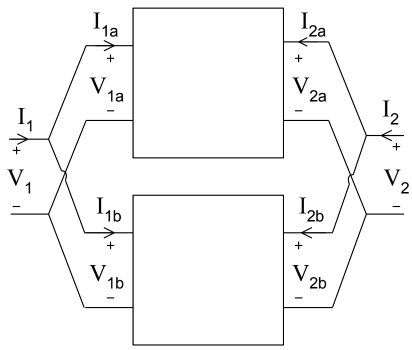
Imedance Matrix	Admittance Matrix
$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$ $\begin{matrix} V_1 = z_{11}I_1 + z_{12}I_2 \\ V_2 = z_{21}I_1 + z_{22}I_2 \end{matrix}$	$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ $\begin{matrix} I_1 = y_{11}V_1 + y_{12}V_2 \\ I_2 = y_{21}V_1 + y_{22}V_2 \end{matrix}$
$z_{11} = \left.\frac{\mathrm{V}_1}{\mathrm{I}_1}\right\|_{\mathrm{I}_2=0}$ $z_{21} = \left.\frac{\mathrm{V}_2}{\mathrm{I}_1}\right\|_{\mathrm{I}_2=0}$	$y_{11} = \left.\frac{I_1}{V_1}\right\|_{V_2=0}$ $y_{21} = \left.\frac{I_2}{V_1}\right\|_{V_2=0}$
$z_{12} = \left.\frac{V_1}{I_2}\right\|_{I_1=0}$ $z_{22} = \left.\frac{V_2}{I_2}\right\|_{I_1=0}$	$y_{12} = \left.\frac{I_1}{V_2}\right\|_{V_1=0}$ $y_{22} = \left.\frac{I_2}{V_2}\right\|_{V_1=0}$
Hybrid Matrix	Inverse Hybrid Matrix
$\begin{pmatrix} V_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} I_1 \\ V_2 \end{pmatrix}$ $\begin{matrix} V_1 = h_{11}I_1 + h_{12}V_2 \\ I_2 = h_{21}I_1 + h_{22}V_2 \end{matrix}$	$\begin{pmatrix} I_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} V_1 \\ I_2 \end{pmatrix}$ $\begin{matrix} I_1 = g_{11}V_1 + g_{12}I_2 \\ V_2 = g_{21}V_1 + g_{22}I_2 \end{matrix}$
$h_{11} = \left.\frac{V_1}{I_1}\right\|_{V_2=0}$ $h_{12} = \left.\frac{V_1}{V_2}\right\|_{I_1=0}$	$g_{11} = \left.\frac{I_1}{V_1}\right\|_{I_2=0}$ $g_{12} = \left.\frac{I_1}{I_2}\right\|_{V_1=0}$
$h_{21} = \left.\frac{I_2}{I_1}\right\|_{V_2=0}$ $h_{22} = \left.\frac{I_2}{V_2}\right\|_{I_1=0}$	$g_{21} = \left.\frac{V_2}{V_1}\right\|_{I_2=0}$ $g_{22} = \left.\frac{V_2}{I_2}\right\|_{V_1=0}$
Transmission Matrix	Inverse Transmission Matrix:
$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} V_2 \\ \text{-}I_2 \end{pmatrix}$ $\begin{matrix} V_1 = AV_2 - BI_2 \\ I_1 = CV_2 - DI_2 \end{matrix}$	$\begin{pmatrix} V_2 \\ -I_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}\begin{pmatrix} V_1 \\ I_1 \end{pmatrix}$
$A = \left.\frac{V_1}{V_2}\right\|_{I_2=0}$ $B = -\left.\frac{V_1}{I_2}\right\|_{V_2=0}$	
$C = \left.\frac{I_1}{V_2}\right\|_{I_2=0}$ $D = -\left.\frac{I_1}{I_2}\right\|_{V_2=0}$	

Two-Port Interconnections



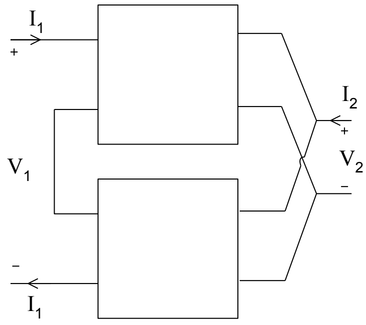
series connection ⇒

add Z matrices



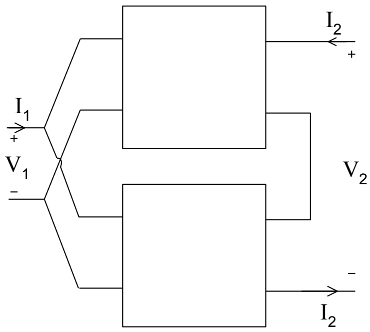
parallel connection ⇒

add Y matrices



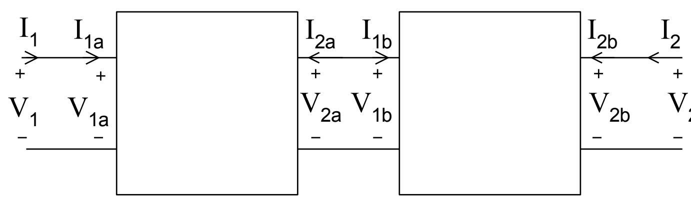
series-parallel connection ⇒

add H matrices



parallel-series connection ⇒

add G matrices



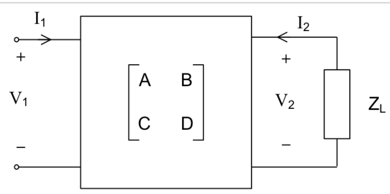
cascade connection ⇒

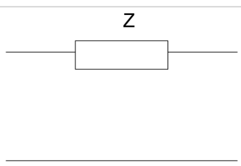
multiply T matrices

Input impednace in terms of the ABCD parameters:

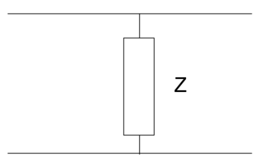
$$Z_{in} = \frac{V_1}{I_1} = \frac{AZ_L + B}{CZ_L + D}$$

$$H(s) = \frac{1}{A + B + C + D}$$



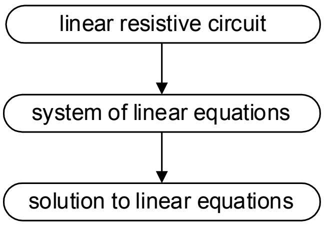


$$T_a = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}$$



$$T_b = \begin{pmatrix} 1 & 0 \\ 1/Z & 1 \end{pmatrix}$$

Circuit Simulation



Nodal Analysis (NA) and ***Modified Nodal Analysis (MNA)***

(1) Choose a voltage reference node and label the branches.

(2) Write KCL for each node except the reference. Use the convention: if a current leaves a node, assign the plus sign to this current. If the current enters a node, assign the minus sign to this current. Rearrange the equation resulting from KCL in the form:

$$\Sigma \text{ branch currents} = \Sigma \text{ independent current sources}$$

(3) Use the branch equations to eliminate as many branch currents as possible. Any branch current not eliminated at this step remains in the equations as an additional variable.

(4) Write the branch equations corresponding to the remaining branch currents.

In ***MNA***:

- A branch current is always introduced as an additional variable if it flows through a voltage source (independent or controlled).
- In addition, if we are applying MNA in the sinusoidal steady state, we keep the branch current if it flows through an inductor.
- A branch current is introduced if any circuit element is controlled by that current.
- A branch current is introduced if it is requested as an output variable in a simulation environment).

The ***Newton-Raphson algorithm*** estimates the solution to the nonlinear equation $F(v) = 0$ by iterating the equation

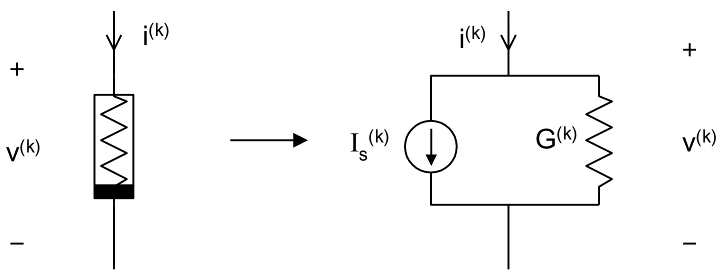
$$v^{(k+1)} = v^{(k)} - \frac{F(v^{(k)})}{F'(v^{(k)})}$$

from an initial estimate $v^{(0)}$.

Applied to a nonlinear circuit element described by the nonlinear current-voltage characteristic

$$i = g(v)$$

the Newton-Raphson algorithm results in the ***companion model*** or the ***linearised model*** for the nonlinear resistor (diode):



where

$$G^{(k)} = g'(v^{(k)}) \qquad \text{and} \qquad I_s^{(k)} = g(v^{(k)}) - g'(v^{(k)})v^{(k)}$$

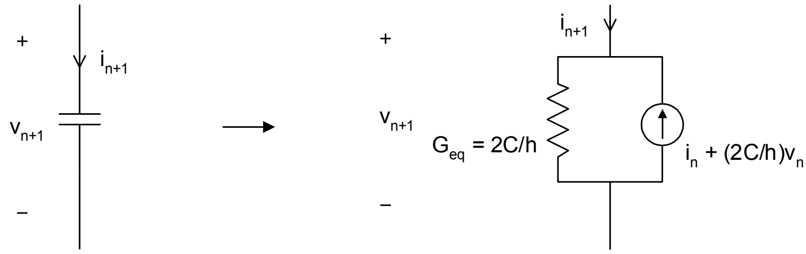
The ***trapezoidal rule*** is a numerical integration technique:

$$x_{n+1} = x_n + \frac{1}{2}h\left(\dot{x}_n + \dot{x}_{n+1}\right)$$

Applied to a capacitor

$$i = C\frac{dv}{dt}$$

it results in the ***companion model*** of the capacitor

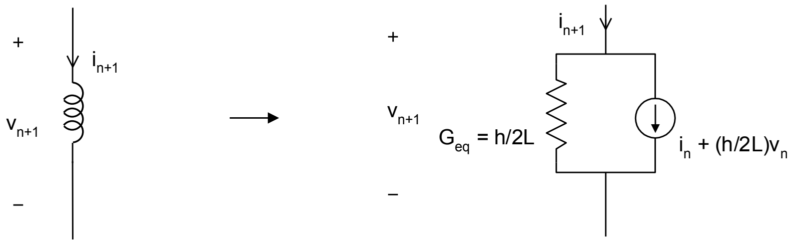


$$i_{n+1} = \frac{2C}{h}v_{n+1} - \left(i_n + \frac{2C}{h}v_n\right)$$

Applied to an inductor

$$v = L\frac{di}{dt}$$

it results in the ***companion model*** of the capacitor



$$i_{n+1} = \frac{h}{2L}v_{n+1} + \left(i_n + \frac{h}{2L}v_n\right)$$

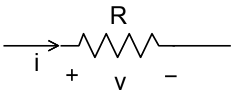
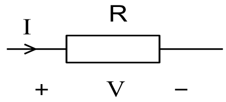
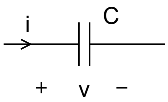
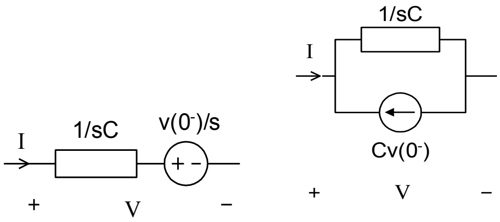
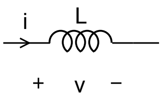
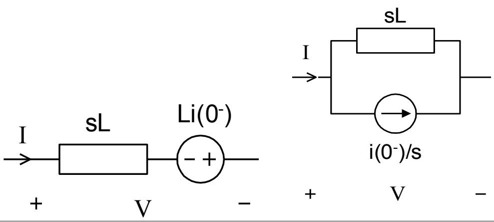
Laplace Transform

f(t)	*F(s)*	*f(t)*	*F(s)*
K (const) or *u(t)* (unit step function)	*K/s* *1/s*	e^{at}	$\frac{1}{s-a}$
cos ωt	$\frac{s}{s^2+\omega^2}$	*sin ωt*	$\frac{\omega}{s^2+\omega^2}$
$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$	$\int_{0^-}^{t} f(\tau)d\tau$	$\frac{1}{s}F(s)$
$e^{-\alpha t}f(t)$	$F(s+\alpha)$	$f(t-\tau)u(t-\tau)$	$e^{-s\tau}F(s)$

Partial Fraction Expansion

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s-p_1)(s-p_2)\ldots(s-p_m)} = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_m}{s-p_m}$$

where $k_i = (s-p_i)\frac{N(s)}{D(s)}\Big|_{s=p_i}$

Time domain	Frequency domain
$v(t) = R\,i(t)$ 	$V(s) = R\,I(s)$ 
$i(t) = C\dfrac{dv(t)}{dt}$ 	$I(s) = sC\,V(s) - C\,V(0^-)$ 
$v(t) = L\dfrac{di(t)}{dt}$ 	$V(s) = sL\,I(s) - L\,i(0^-)$ 

Zero-State and Zero-Input Responses:

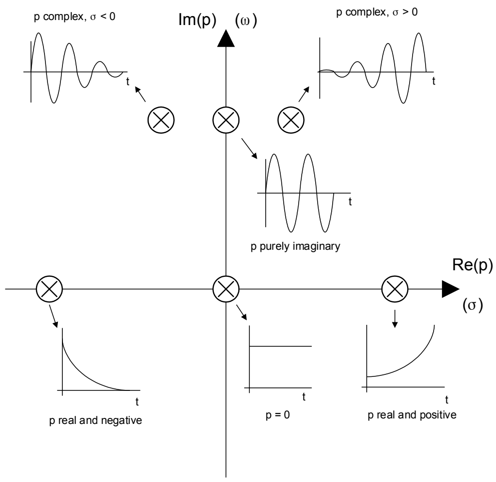
The ***zero-input response*** is due to the initial conditions acting alone, with all independent sources set to zero.

The ***zero-state response*** is due to the independent sources acting alone with all initial conditions set to zero.

Natural frequencies can be found:

- From the equation Det $\underline{M} = 0$, where $\underline{M}$ is the MNA matrix of the circuit
- As the poles of the transfer function *H(s)*.

Stable circuits have natural frequencies p_i such that $Re\,(p_i) < 0$.



The ***zero-state response*** can be found from the network (transfer) function:

$$\mathcal{L}\{\text{response}\} = \text{Network Function} \cdot \mathcal{L}\{\text{input}\}$$

The Laplace Transform vs. Phasor Analysis

Phasor analysis is a particular case of the Laplace transform, and the Laplace transform is a more general technique. Use phasor analysis only if you are asked to find the ***steady state response*** of a stable circuit <u>and</u> the independent current and/or voltage sources driving the circuit are ***sinusoidal***;

Use the Laplace transform in all other cases.

Frequency Response

The output voltage in the time domain:

$$v_{out}(t) = |H(j\omega)| \cdot |V_{in}(j\omega)| \sin(\omega t + \angle H(j\omega) + \angle V_{in}(j\omega))$$

The ***Bode plot*** (or diagram) is a simple technique for obtaining an approximate plot of the magnitude and phase angle of a transfer function *H(s)* evaluated at $s = j\omega$. The diagram consists of two plots – a plot of the logarithm of the magnitude of $H(j\omega)$ and a plot of the phase angle of $H(j\omega)$, both plotted against (positive) frequency, with a logarithmic scale for the frequency axis. Ensure that you understand how to derive the Bode plots for simple transfer functions.

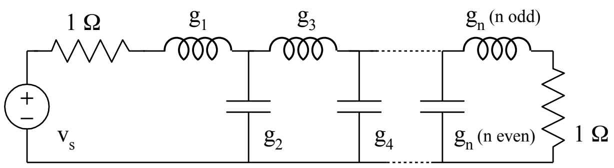
Filter Design

The n[th] order Butterworth low-pass magnitude response with cutoff frequency ω_c is given by the equation

$$|H(j\omega)|^2 = \frac{1}{1+\left(\frac{\omega}{\omega_c}\right)^{2n}}$$

An ***RLC circuit*** realisation of the n[th] order normalised Butterworth low-pass filter (ω_c=1rad/s) is shown below, with element values given by

$$g_r = 2\sin\left((2r-1)\frac{\pi}{2n}\right)$$



Frequency transformations:

Low-pass to high-pass: $j\omega \rightarrow \frac{\omega_0}{j\omega}$

Low-pass to band-pass: $j\omega \rightarrow \beta\left(\frac{j\omega}{\omega_0} + \frac{\omega_0}{j\omega}\right)$

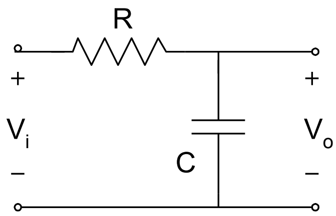
The Butterworth polynomials for order 1 to 6 for ***active filter design*** are given in the following table:

	Butterworth polynomial *D(s)*	Transfer function $\|H(jw)\|^2=1/\|D(s)\|^2$
1	$s+1$	$\frac{1}{1+\omega^2}$
2	$s^2+1.414s+1$	$\frac{1}{1+\omega^4}$
3	$(s+1)(s^2+s+1)$	$\frac{1}{1+\omega^6}$
4	$(s^2+0.765s+1)(s^2+1.848s+1)$	$\frac{1}{1+\omega^8}$
5	$(s+1)(s^2+0.618s+1)(s^2+1.618s+1)$	$\frac{1}{1+\omega^{10}}$
6	$(s^2+0.518s+1)(s^2+1.414s+1)(s^2+1.932s+1)$	$\frac{1}{1+\omega^{12}}$

Fist and second order transfer functions:

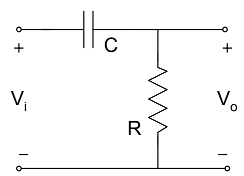
low-pass	$\dfrac{A}{s^2+a_1 s+a_0}$	which is often written in the form	$\dfrac{k\omega_n^2}{s^2+\left(\dfrac{\omega_n}{Q}\right)s+\omega_n^2}$
high-pass	$\dfrac{As^2}{s^2+a_1 s+a_0}$	which is often written in the form	$\dfrac{ks^2}{s^2+\left(\dfrac{\omega_n}{Q}\right)s+\omega_n^2}$
band-pass	$\dfrac{As}{s^2+a_1 s+a_0}$	which is often written in the form	$\dfrac{k\left(\dfrac{\omega_n}{Q}\right)s}{s^2+\left(\dfrac{\omega_n}{Q}\right)s+\omega_n^2}$

First order low-pass section



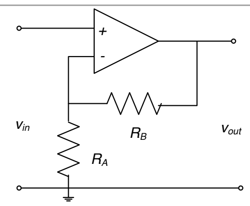
$$\frac{V_o(s)}{V_i(s)}=\frac{\dfrac{1}{RC}}{s+\dfrac{1}{RC}}$$

First order high-pass section



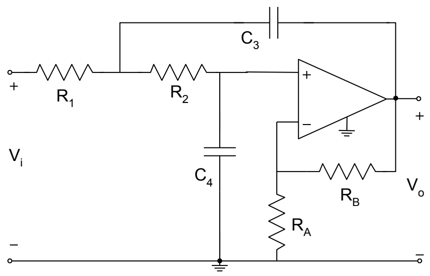
$$\frac{V_o(s)}{V_i(s)}=\frac{s}{s+\dfrac{1}{RC}}$$

Buffer



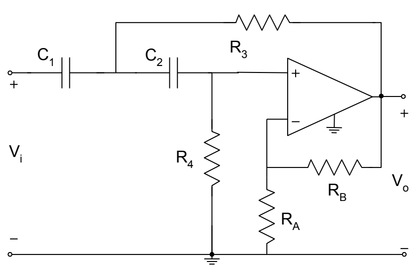
$$V_{out}=\left(1+\frac{R_B}{R_A}\right)V_{in}$$

Low-pass Sallen-Key filter



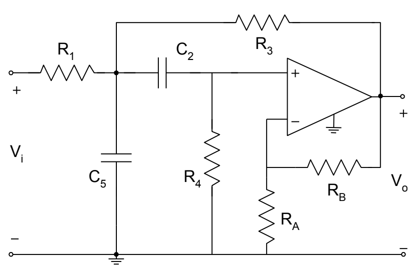
$$\frac{V_o(s)}{V_i(s)}=\frac{K\dfrac{1}{R_1R_2C_3C_4}}{s^2+s\left(\dfrac{1}{R_1C_3}+\dfrac{1}{R_2C_3}+\dfrac{1}{R_2C_4}-\dfrac{K}{R_2C_4}\right)+\dfrac{1}{R_1R_2C_3C_4}}$$

High-pass Sallen-Key filter



$$\frac{V_o(s)}{V_i(s)}=\frac{Ks^2}{s^2+s\left(\dfrac{1}{R_4C_2}+\dfrac{1}{R_4C_1}+\dfrac{(1-K)}{R_3C_1}\right)+\dfrac{1}{C_1C_2R_3R_4}}$$

Band-pass Sallen-Key filter



$$\frac{V_o(s)}{V_i(s)}=\frac{(K/R_1C_5)s}{s^2+s\left(\dfrac{1}{R_1C_5}+\dfrac{(1-K)}{R_3C_5}+\dfrac{1}{R_4C_5}+\dfrac{1}{R_4C_2}\right)+\dfrac{1/R_1+1/R_3}{R_4C_2C_5}}$$