University of Illinois **Spring 2014**

ECE 310: Problem Set 4: Problems and Solutions
Sampling, Ideal A/D converter, Aliasing, Analog frequency response of a digital system

Due: Wednesday February 19 at 6 p.m.
Reading: *310 Course Notes* Ch 3.1, 3.2, 5.3
Version: 1.1

1. **[Sampling and Aliasing]**
A continuous-time signal $y(t) = \cos(\Omega_0 t)$ was sampled at a rate of 512 samples/sec in order to obtain the following discrete-time signal:

$$y[n] = \cos\left(\frac{\pi}{128}n\right).$$

Find three possible different values of Ω_0 that can produce the same sequence $y[n]$ above.

Solution: Recall that $y[n] = y(nT)$, thus $\cos\left(\frac{\pi}{128}n\right) = \cos\left(\Omega_0 nT\right)$, where $T = \frac{1}{512}sec.$ Therefore, $\Omega_0 T = \frac{\pi}{128} + 2k\pi$, where k is an integer. Take $k = 0, 1, 2$, we get three possible values for Ω_0:

$\Omega_0 = \frac{\pi}{128} \times 512 = 4\pi.$

$\Omega_0 = (\frac{\pi}{128} + 2\pi) \times 512 = 1028\pi.$

$\Omega_0 = (\frac{\pi}{128} + 4\pi) \times 512 = 2052\pi.$

A more general solution is: $\Omega_0 = \pm 4\pi + 1024k\pi$, where $k \in \mathbb{Z}$.

2. **[Sampling and Aliasing]**
A continuous-time signal $v(t) = \sin(15\pi t) + \cos(60\pi t)$ was sampled with a period T in order to obtain the following discrete-time signal:

$$v[n] = \sin\left(\frac{\pi}{3}n\right) + \cos\left(\frac{4\pi}{3}n\right).$$

(a) Find a period T consistent with the continuous-time signal and its discrete-time counterpart.

Solution: For information consistency, we want $v[n] = v(nT)$. Thus, from the given expression of $v[n]$ and $v(t)$

$$\sin\left(\frac{\pi}{3}n\right) + \cos\left(\frac{4\pi}{3}n\right) = \sin(15\pi nT) + \cos(60\pi nT)$$

which leads to $T = \frac{1}{45}$ sec.

(b) Is this T found in part (a) unique? If so, expound. If not, specify another period T consistent with the signals.

Solution: No, the choice in part (a) is not unique. The relationship in part (a) involves 2π periodic functions which means

$$\sin\left(\frac{\pi}{3}n\right) = \sin\left(n\left(\frac{\pi}{3} + 2\pi k_1\right)\right)$$

and

$$\cos\left(\frac{4\pi}{3}n\right) = \cos\left(n\left(\frac{4\pi}{3} + 2\pi k_2\right)\right)$$

where $k_1, k_2 \in \mathbb{Z}$. Let

$$n\left(\frac{\pi}{3} + 2\pi k_1\right) = 15\pi nT, \quad n\left(\frac{4\pi}{3} + 2\pi k_2\right) = 60\pi nT,$$

we have $k_2 = 4k_1$, $T = \frac{1+6k_1}{45}$, where $k_1 \in \mathbb{Z}$. Therefore, all periods of the form $T = \frac{1+6k}{45}$ ($k \in \mathbb{Z}$) are consistent.

3. **[Sampling and Aliasing]**
The continuous-time signal $g(t) = \cos(250\pi t)$ was sampled with a period T in order to obtain the following discrete-time signal:

$$g[n] = g(nT).$$

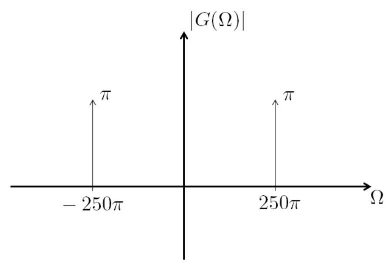
(a) Compute and sketch the magnitude of the continuous-time Fourier transform of $g(t)$ and the discrete-time Fourier transform of $g[n]$ for $T = 2ms$ and $T = 5ms$.

Solution: Let $G(\Omega)$ be the continuous-time Fourier transform of $g(t)$, then:

$$\begin{aligned} G(\Omega) &= \mathcal{F}\left\{\frac{1}{2}e^{j250\pi t} + \frac{1}{2}e^{-j250\pi t}\right\} \\ &= \frac{1}{2}[2\pi\delta(\Omega - 250\pi) + 2\pi\delta(\Omega + 250\pi)] \\ &= \pi[\delta(\Omega - 250\pi) + \delta(\Omega + 250\pi)]. \end{aligned}$$

Hence,

$$|G(\Omega)| = \pi[\delta(\Omega - 250\pi) + \delta(\Omega + 250\pi)].$$



Discrete-time signal is $g[n] = \cos(250\pi nT)$. Then:

$$\begin{aligned} G(\omega) &= DTFT\{\cos(250\pi nT)\} \\ &= \sum_{k=-\infty}^{\infty} \pi[\delta(\omega - 250\pi T + 2k\pi) + \delta(\omega + 250\pi T + 2k\pi)]. \end{aligned}$$

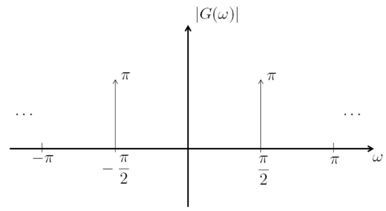
Hence,

$$|G(\omega)| = \sum_{k=-\infty}^{\infty} \pi[\delta(\omega - 250\pi T + 2k\pi) + \delta(\omega + 250\pi T + 2k\pi)].$$

For $T = 2ms$,

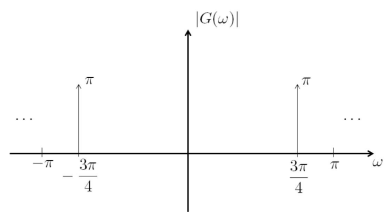
$$|G(\omega)| = \sum_{k=-\infty}^{\infty} \pi[\delta(\omega - \frac{\pi}{2} + 2k\pi) + \delta(\omega + \frac{\pi}{2} + 2k\pi)].$$

For $T = 5ms$,



$$|G(\omega)| = \sum_{k=-\infty}^{\infty} \pi[\delta(\omega - \frac{5\pi}{4} + 2k\pi) + \delta(\omega + \frac{5\pi}{4} + 2k\pi)]$$

$$= \sum_{k=-\infty}^{\infty} \pi[\delta(\omega + \frac{3\pi}{4} + 2k\pi) + \delta(\omega - \frac{3\pi}{4} + 2k\pi)].$$



(b) Find the maximum sampling period T_{max} such that aliasing does not occur.

Solution: The continuous-time signal is bandlimited, $|G(\Omega)| = 0$ for $|\Omega| > \Omega_0 = 250\pi$. Hence the Nyquist sampling rate is $\frac{\pi}{\Omega_0} = \frac{1}{250}s = 4ms$. To avoid aliasing, the maximum sampling period $T_{max} < 4ms$.

4. **[Sampling and Aliasing]**
A sequence $x[n] = x_a(nT)$, with $T = \frac{2\pi}{\Omega_0}$, is generated from an analog signal $x_a(t)$ with the following Fourier transform (figure 1).

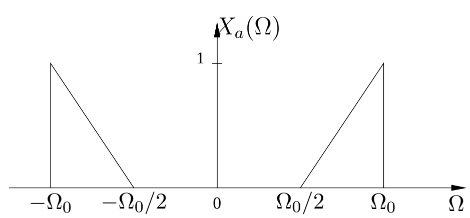


Figure 1: Problem 4

(a) Sketch the DTFT of the sequence $x[n] = x_a(nT)$.

Solution:

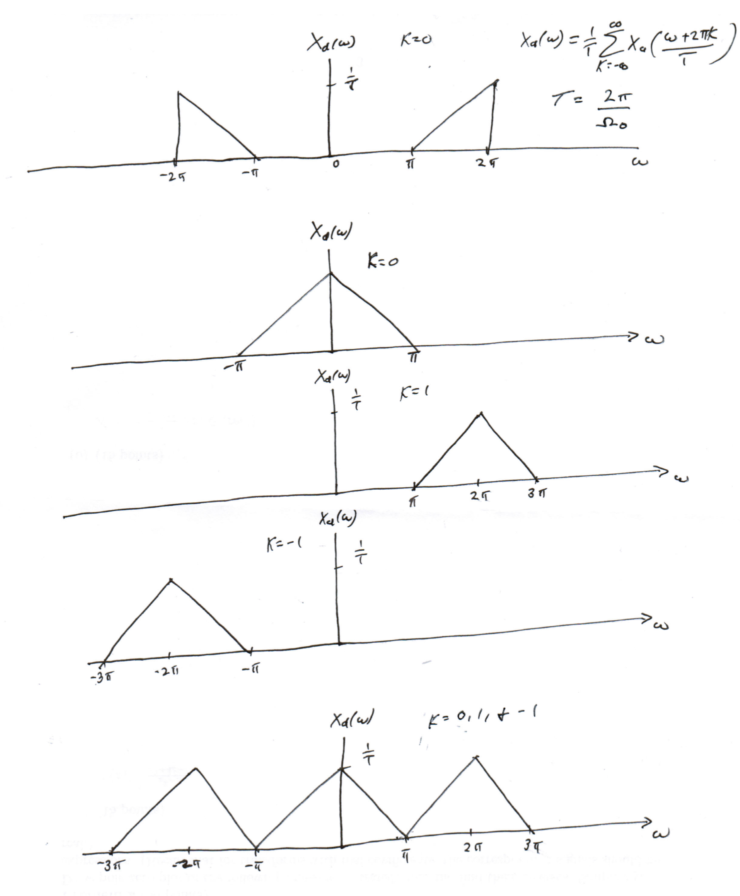


Figure 2: Solution to Problem 4

(b) In terms of Ω_0, for what range of values of T can $x_a(t)$ be reconstructed from the sampled sequence $x[n] = x_a(nT)$.

Solution: $T < \frac{\pi}{\Omega_0}$.

5. **[CT and DT Systems]**
An analog low-pass filter(LPF) $H_a(\Omega)$ is shown in figure 3 together with the Fourier transform $X_a(\Omega)$ of an input signal $x_a(t)$.

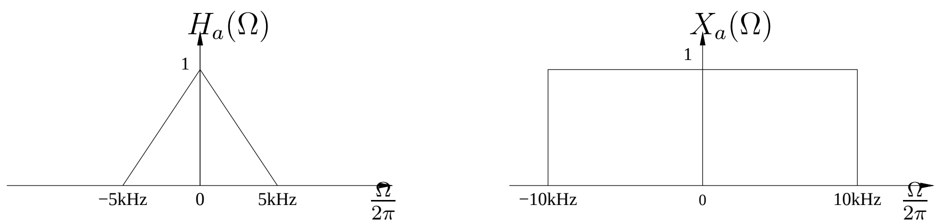


Figure 3: Problem 5

A digital system is to be designed using uniform sampling and an ideal D/A that will produce the same output as the analog LPF system for the given input $x_a(t)$.

(a) What is the largest sampling period T possible? Note that, in this system, it is possible to allow certain aliasing at the sampler and yet still have the overall system behave like the desired analog system.

Solution: Let $y_a(t)$ be the output of the analog LPF system. Its Fourier transform $Y_a(\Omega) = X_a(\Omega)H_a(\Omega) = H_a(\Omega)$. Note that it is bandlimited and $|Y_a(\Omega)| = 0$ for $|\Omega| > 2\pi(5000)$. Hence the Nyquist sampling rate is $\frac{\pi}{2\pi(5000)} = \frac{1}{10000} = 0.1ms$. The largest sampling period $T = 0.1ms$.

Although there is aliasing at the sampler, the overall system can behave like the desired analog system, as will be demonstrated in part (b).

(b) Determine the filter in digital system $H_d(\omega)$ for the T in part (a).

Solution: After sampling at the rate of $T = 0.1ms$, the discrete-time signal is $x[n]$. The DTFT is

$$X_d(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(\frac{\omega + 2k\pi}{T}\right).$$

Due to aliasing,

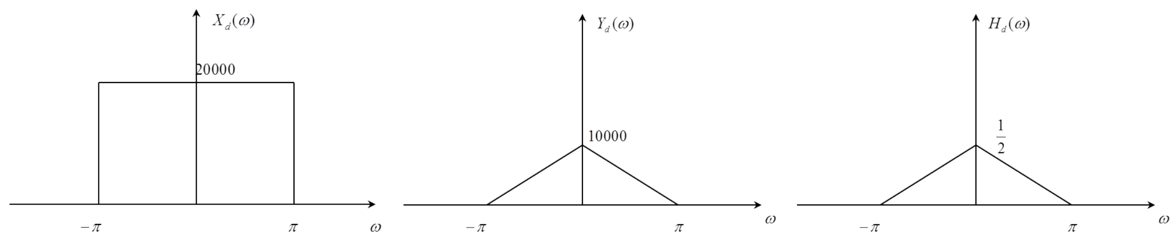
$$X_d(\omega) = \frac{2}{T} = 20000, \ |\omega| < \pi.$$

The output of the digital system $y[n]$ can reproduce $Y_a(\Omega)$ using an ideal D/A, hence $y[n]$ can be obtained by sampling $y_a(t)$ at the rate of $T = 0.1ms$. The DTFT of $y[n]$ is

$$Y_d(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} Y_a\left(\frac{\omega + 2k\pi}{T}\right).$$

$$Y_d(\omega) = \frac{1}{T}Y_a\left(\frac{\omega}{T}\right), \ |\omega| < \pi.$$

The digital filter $H_d(\omega) = \frac{Y_d(\omega)}{X_d(\omega)}$. $X_d(\omega), Y_d(\omega)$ and $H_d(\omega)$ are shown as follows.



6. **[CT and DT Systems]**
A speech signal $x_a(t)$ is assumed to be bandlimited to 10 kHz. It is desired to filter this signal with a bandpass filter that will pass all the frequencies between 300 Hz to 5 kHz by using a digital filter sandwiched between an A/D and an ideal D/A.

   (a) Determine the Nyquist sampling rate for the input signal.
   Solution: $F_N = 20kHz,\ T = 50\mu s.$

   (b) Sketch the necessary $H_d(\omega)$ using the Nyquist sampling rate.
   Solution:
   $\omega_1 = \Omega_1 T = \frac{2\pi(300)}{20000} = \frac{3\pi}{100}.$

   $\omega_2 = \Omega_2 T = \frac{2\pi(5000)}{20000} = \frac{\pi}{2}.$

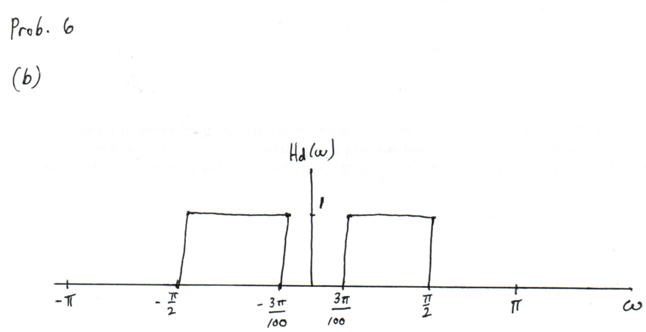


Figure 4: Solution to Problem 6b

   (c) Find the largest sampling period T for which the A/D, $H_d(\omega)$, and D/A can perform the desired filtering function.
   Solution: Some aliasing of the input signal is allowed with the condition that the minimum alaising frequency is greater then the cutoff frequency of the filter. Lowest frequency of aliased component: $2\pi - T\Omega_{max}$. Hence, to prevent the aliased components from appearing at the output need them to fall in the stopband of the filter:

$$2\pi - T\Omega_{max} > \omega_2$$

where the cutoff frequency ω_2 of the filter is given by: $\omega_2 = \Omega_2 T$. Then:

$$
\begin{aligned}
2\pi - 2\pi \times 10000T &\geq 2\pi \times 5000T \\
1 &\geq 15000T \\
\therefore T = \frac{1}{15000} &\approx 66.67\mu s
\end{aligned}
$$

(d) Based on T from part (c), sketch the necessary $H_d(\omega)$.

Solution:

$\omega_1 = \Omega_1 T = \frac{2\pi(300)}{15000} = \frac{\pi}{25}$

$\omega_2 = \Omega_2 T = \frac{2\pi(5000)}{15000} = \frac{2\pi}{3}$

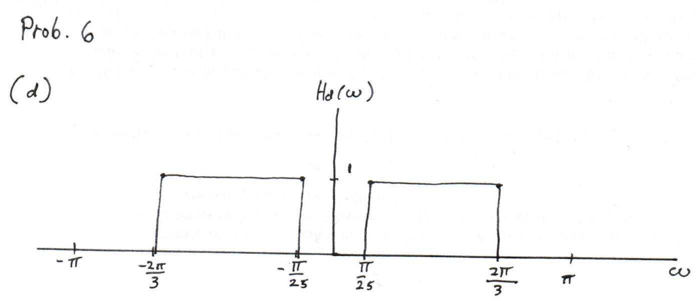


Figure 5: Solution to Problem 6d

7. **[CT and DT Systems]**
For the digital system pictured in figure 6, let $T = \frac{1}{4\times 10^6}$. The input and digital filter are shown in figure 7.

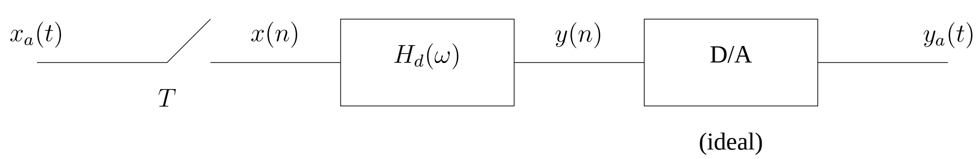


Figure 6: Problem 7, system

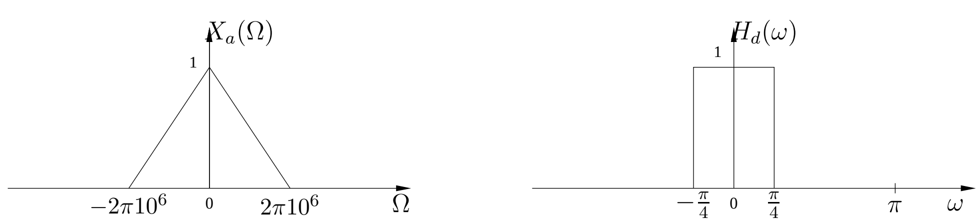


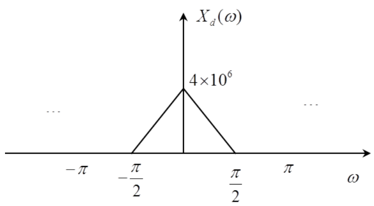
Figure 7: Problem 7, input and digital filter

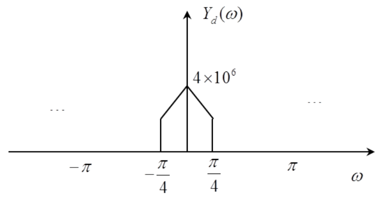
(a) Sketch $X_d(\omega)$, $Y_d(\omega)$, and $Y_a(\Omega)$. ($X_d(\omega)$ and $Y_d(\omega)$ are the DTFTs of $x[n]$ and $y[n]$, respectively. $Y_a(\Omega)$ is the CTFT of $y_a(t)$.)
Solution:

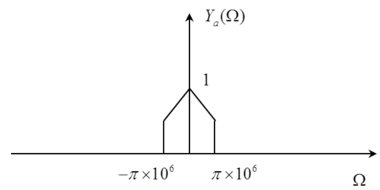
$$X_d(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(\frac{\omega+2k\pi}{T}\right).$$

$$Y_d(\omega) = X_d(\omega)H_d(\omega).$$

$$Y_a(\Omega) = \begin{cases} TY_d(\Omega T) & |\Omega| < \frac{\pi}{T} \\ 0 & else \end{cases}.$$







(b) Now, suppose that the ideal D/A is mismatched to the true sampling period. Suppose it uses a period $T/2$ instead of T and simply reads in every element of the $y[n]$ sequence twice in a row. That is, the D/A, in effect, uses the input

$$\{w[n]\} = \{..., y[0], y[0], y[1], y[1], y[2], y[2], y[3], y[3], ...\}.$$

Sketch both $W_d(\omega)$ (DTFT of input of D/A $w[n]$) and the resulting $Y_a(\Omega)$.
Solution: Note that $w[n]$ is the sum of two signals:

$$w[n] = w_0[n] + w_1[n].$$
$$\{w_0[n]\} = \{..., y[0], 0, y[1], 0, y[2], 0, y[3], 0, ...\}.$$
$$\{w_1[n]\} = \{..., 0, y[0], 0, y[1], 0, y[2], 0, y[3], ...\}.$$

$$\begin{aligned} W_{d0}(\omega) &= \sum_{n=-\infty}^{\infty} w_0[n]e^{-j\omega n} \\ &= \sum_{m=-\infty}^{\infty} w_0[2m]e^{-j\omega 2m} \\ &= \sum_{m=-\infty}^{\infty} y[m]e^{-j(2\omega)m} \\ &= Y_d(2\omega). \end{aligned}$$

$$w_1[n] = w_0[n-1] \longleftrightarrow W_{d1}(\omega) = W_{d0}(\omega)e^{-j\omega} = Y_d(2\omega)e^{-j\omega}.$$

Hence,

$$\begin{aligned} W_d(\omega) &= W_{d0}(\omega) + W_{d1}(\omega) \\ &= Y_d(2\omega)(1+e^{-j\omega}) \\ &= 2Y_d(2\omega)\cos\left(\frac{\omega}{2}\right)e^{-j\frac{\omega}{2}} \\ |W_d(\omega)| &= 2Y_d(2\omega)\cos\left(\frac{\omega}{2}\right) \end{aligned}$$

D/A uses the mismatched sampling rate $T/2$. Hence the resulting $Y_a(\Omega)$ is

$$Y_a(\Omega) = \begin{cases} \frac{T}{2} W_d(\Omega \frac{T}{2}) & |\Omega| < \frac{2\pi}{T} \\ 0 & else \end{cases}.$$