Signals & Systems
EEEN 30110

Electronic, Electrical and Communications Engineering

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Online resources such as Khan Academy, Wolfram and Wikipedia are likely to be far more useful to you than textbooks. In particular in the case of Khan Academy most of the information will be incorrect, but the inaccuracies are in the right direction. The decision is to make it simple, not to make it right. For a first pass it is an excellent resource and many of you are likely to find it far more useful than my notes. To a large extent I take up where Salman Khan leaves off.

Overview of Topics

Mathematical background and problem statement.

Laplace transform and solution of linear constant-coefficient ordinary differential equations.

Special case of steady-state response to sinusoidal input. Fourier transform.

Solution of linear constant-coefficient partial differential equations. Fourier series.

Discrete-time systems.

Section 0 – Mathematical background and problem statement

- Modelling
- From models to linear constant coefficient differential equations

Models

Fundamentally this module is about solving *ordinary differential equations*. In general this has proved very difficult. There is really only one generous class of equations of this kind which can be solved, these are the *linear, constant coefficient ordinary differential equations*. To be more precise this module is about the best method yet discovered to solve this special class of differential equations. Before presenting it I wish to give some indication as to why it is important to have such a method and also some means to judge that it is the best.

Models

In order to obtain scientific predictions of system behaviour one must have a set of mathematical equations which approximately describe this behaviour. You will probably never have a set of equations which exactly describe the behaviour although you may believe that you do.

Such a set of equations is called a *model* of the system. Fundamental sciences (such as physics and chemistry) are concerned with finding such models.

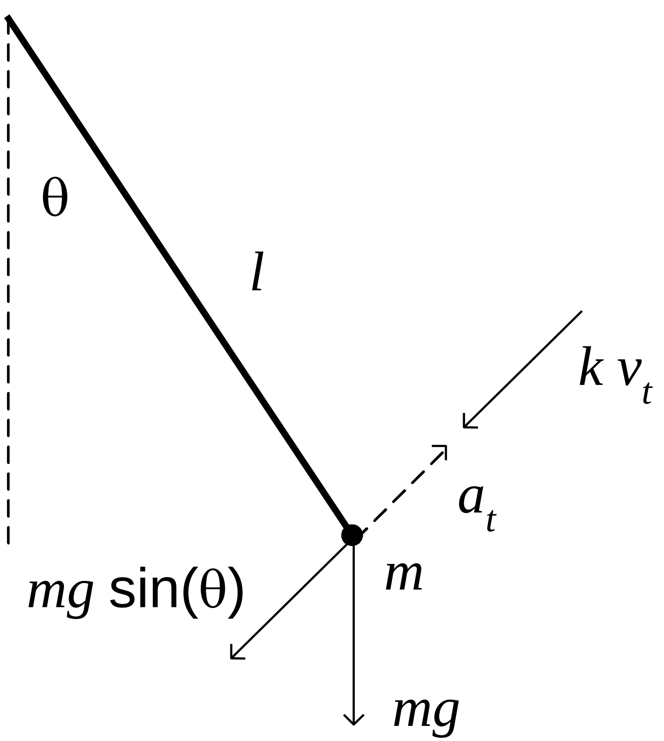
Models

There are two aspects to the problem of finding a model for the behaviour of a system. The first aspect is theoretical. Basic science must suggest the mathematical form that such a model should take. Usually this is subject to a large number of assumptions so it comes with a strong caveat. The first theoretical model will contain within it a number of parameters (such as mass, resistance, *etc*.). The second aspect is experimental. By performing suitable experiments values for these parameters must be chosen, ideally to give the closest possible fit between the predicted and actual behaviour of the system. In practice these two sub-problems may be (and commonly are) combined into one.

Models: Example 1

Consider for example the simple gravity pendulum.

Assume the rod to be sufficiently stiff that it does not bow. Assume the mass of the bobbin to be much larger than the rest of the mass of the system. Assume the surroundings and mounting to be still and not subject to buffeting, tremor, *etc*. Assume a very simple friction model.



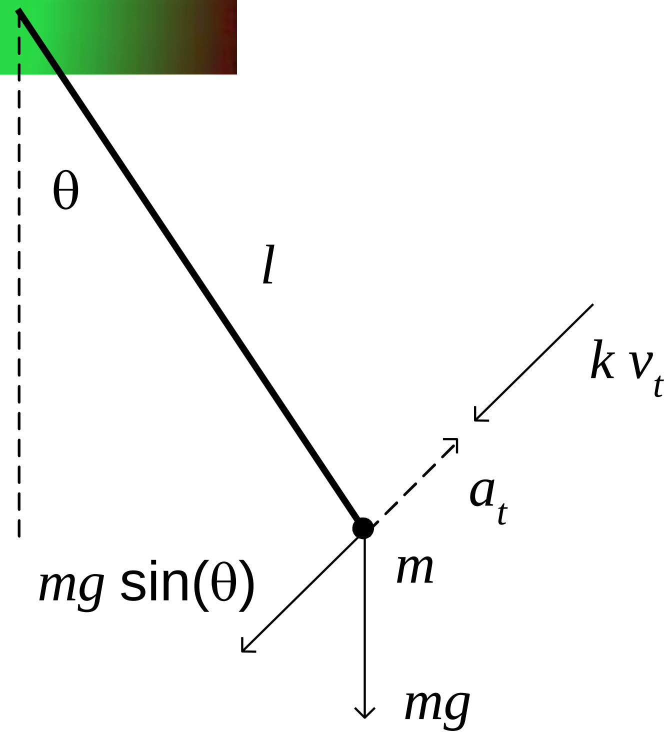
Models: Example 1

This illustrates an important general fact. We make a large number of assumptions straight away. For example we assume also that the mass of the bobbin is constant. Of course this is false. Surfaces tarnish and joints (such as the mounting joint) wear. In the long run the system performance will degrade. This may be a gradual process or it may be a rather sudden one where some component breaks. But if our interest is in how the system behaves before it breaks and if we only consider a short timeframe then the assumption is usually justified.

Models: Example 1

The simple gravity pendulum may be analysed by employing basic mechanics. There are at least four approaches which may be taken: force and linear momentum; torque and angular momentum; energy and dissipation; principle of stationary action. Of course we are entirely ignoring a host of other physical mechanisms such as acoustics and thermodynamics. Again this is typical. We usually determine what are the dominant physical mechanisms involved and just model those. This leads to an approximate model, but indeed there is no other type of model.

Models: Example 1



$$a_t = l\ddot{\theta} \quad , \quad v_t = l\dot{\theta}$$

Performance is approximated by:

$$ml\ddot{\theta} = -mg\sin(\theta) - kl\dot{\theta}$$

We may consider the problem where the bobbin is pulled to a certain angle, allowed to come to rest at this angle and then released. We obtain the following initial conditions:

$$\theta(0) = \theta_0 \quad , \quad \dot{\theta}(0) = 0$$

Models: Example 1

This illustrates another important and fairly general fact. The mathematical model provided by basic science is commonly an ordinary differential equation (or possibly a system of such equations). Moreover, a differential equation in and of itself does not provide a complete solution of a system problem. One must also specify initial conditions. I should note that the assumption that the angle has come to rest prior to release is plausible but will never be exactly true. There will always be some residual movement.

Models: Example 1

The first aspect has led to the following model:

$$ml\ddot{\theta} + kl\dot{\theta} + mg\sin(\theta) = 0$$

$$\theta(0) = \theta_0 \quad , \quad \dot{\theta}(0) = 0.$$

Parameters *m*, *k*, *g* and *l* all appear. The second aspect of modelling involves selecting suitable values for these model parameters.

Models: Example 1

The problem above can be solved in the special case where the friction is absent, i.e. $k = 0$. The solution dates from 1898 (Helmholtz *et al.*) and it involves a special function denoted *sn* which is one of a set of special functions known collectively as *Jacobi's elliptic functions*. In the more general case where friction is present, even the relatively simply friction considered here, the situation is less satisfactory from the theoretical perspective and we cannot leave it there.

Models: Example 1

A very significant simplification may be achieved if we observe that there exists one special set of initial conditions under which we may easily solve the equation. Namely if

$$\theta(0) = 0 \quad , \quad \dot{\theta}(0) = 0$$

then the special solution $\theta(t) = 0$ results. This solution (where nothing is changing) is called an *equilibrium* solution. One of the most important things to know about an equilibrium solution is whether or not it is *stable*, i.e. whether a small change in initial conditions can result in a not so small long term change in the solution.

Models: Example 1

The question of stability of this equilibrium solution is concerned with how the pendulum behaves when

$$\theta(t) \quad , \quad \dot{\theta}(t)$$

are small. You might often see in this scenario where θ is assumed small a statement like:

$$\sin(\theta) \cong \theta \, .$$

This is true, although the "proofs" offered to support the statement rarely are.

Models: Example 1

Thus replacing the difficult sinusoidal term yields:

$$ml\ddot{\theta} + kl\dot{\theta} + mg\theta = 0$$

$$\theta(0) = \theta_0 \quad , \quad \dot{\theta}(0) = 0$$

where of course the initial angle θ_0 must be small for the approximate/simplified equation to be in any way meaningful as a description of the system. This ordinary differential equation is much easier than the equation above. It is *linear*. It is commonly the case that close to equilibria systems may be approximately described by linear ODEs.

Models: Example 1

The pendulum example is very typical in many ways but it is atypical in at least one important way. The external intervention to the system (human or otherwise) amounts to the setting of the initial condition only. Otherwise the system progresses on its own, governed by its own physical laws and disconnected (courtesy of many of our less plausible assumptions) from the rest of the universe.

Models: Example 1

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Models: Example 2

Consider for example the launch of a rocket. Upward acceleration is provided by the thrust from the main engine(s). This is opposed by gravity and drag. A simple characterisation is:

$$m\ddot{h} = T - mg - \left(\frac{\rho C_d A}{2}\right)\left(\dot{h}\right)^2$$

where m denotes the mass, T denotes thrust, C_d denotes the drag coefficient, A denotes the transverse cross-sectional area, h denotes the altitude, ρ denotes air density and g denotes gravitational acceleration.

Models: Example 2

$$m\ddot{h} = T - mg - \left(\frac{\rho C_d A}{2}\right)\left(\dot{h}\right)^2$$

This apparently somewhat simple equation hides numerous issues. Gravitational acceleration depends upon the inverse square law, hence *g* decreases as *h* increases. Air density decreases with altitude but the rate of decrease depends upon pressure, temperature, humidity and other factors. Thrust depends on throttle setting and on air density among other things. Even with a fixed throttle setting it will generally decrease with altitude.

Models: Example 2

$$m\ddot{h} = T - mg - \left(\frac{\rho C_d A}{2}\right)\left(\dot{h}\right)^2$$

A significant portion of the mass of a rocket is comprised of the propellant (often called the fuel). Propellant is exhausted (i.e. expelled through the main engine) during launch. It follows that the mass decreases significantly with time. The formula for the drag is approximate at the best of times. To employ it one must recognise that the drag coefficient will not be constant. At the very least a significant change takes place at trans-sonic speeds.

Models: Example 2

$$m\ddot{h} = T - mg - \left(\frac{\rho C_d A}{2}\right)\left(\dot{h}\right)^2$$

The definition of trans-sonic speeds (i.e. speeds neighbouring the speed of sound) depends upon the speed of sound itself, and this also depends upon air density, among other factors. The cross-sectional area A typically does not remain constant because of *staging*. At some point a stage of the launch comes to a close and a part of the rocket is jettisoned. As well as involving a rather sudden drop in mass this can also involve a sudden change in cross-sectional area.

Models: Example 2

$$m\ddot{h} = T - mg - \left(\frac{\rho C_d A}{2}\right)\left(\dot{h}\right)^2$$

So every “parameter” in this equation is actually changing and many change with altitude *h*. Some of these changes are rather unpredictable. The air density depends for example on the moisture content of the air currently surrounding the rocket and of course we cannot know that in advance for every point along the trajectory. Moreover we have not modelled unpredictable disturbances, due to wind for example, at all.

Models: Example 2

$$m\ddot{h} = T - mg - \left(\frac{\rho C_d A}{2}\right)\left(\dot{h}\right)^2$$

There is one way in which this example is more typical of engineering systems than the pendulum. External intervention in the form of the thrust affects the rocket throughout the flight. We as humans do not simply set the initial conditions and then let the system go. We actually fly the craft and the setting of the thrust (usually by adjustment of the throttle) is one of our principle mechanisms for controlling that flight.

Models: Example 2

$$m\ddot{h} = T - mg - \left(\frac{\rho C_d A}{2}\right)\left(\dot{h}\right)^2$$

Of course an actual rocket is more complicated still since it is not a point mass. It is an object with actual physical dimensions. In terms of mechanics it can rotate or spin. Rotation about certain axes can be catastrophic. The message is somewhat dispiriting: real things are messy and difficult. We cannot exactly model them and we cannot analytically solve the equations which even approximate their behaviour.

Models: Example 3

Consider for example an elementary one-tank liquid level system. Assuming a high pressure, turbulent output flow, the system may be characterised as follows :

$$A\dot{h} = Q_{in} - Q_{out} \quad , \quad Q_{out} = C_d A_d \sqrt{2gh}$$

where Q_{in} denotes the controlled (i.e. pumped) input flow, Q_{out} denotes the output flow, A is the tank cross-sectional area (assumed constant w.r.t. height), A_d denotes the outlet pipe cross-sectional area, C_d denotes a coefficient characterising the outlet.

Models: Example 3

$$A\dot{h} = Q_{in} - C_d A_d \sqrt{2gh}$$

represents yet another equation which is difficult to solve in general. A critical idea is that there is likely to be some desired value h_r of the liquid level. This might be called the *rated value* or the *operating point*. What value must the input flow attain in order for the liquid level to remain at the rated value? Well if h is unchanging at the value h_r then:

$$0 = Q_{in,r} - C_d A_d \sqrt{2gh_r} \qquad \text{i.e.} \qquad Q_{in,r} = C_d A_d \sqrt{2gh_r}$$

Models: Example 3

$$A\dot{h} = Q_{in} - C_d A_d \sqrt{2gh}$$

Now suppose that the liquid level is close to the rated value. It may have been disturbed from it for some unforeseen reason. Introduce *offset variables* (or *signal values*)

$$Q_{in} = Q_{in,r} + \tilde{q}_{in} \quad , \quad h = h_r + \tilde{h}$$

$$A\dot{h} = A\dot{\tilde{h}} = C_d A_d \sqrt{2gh_r} + \tilde{q}_{in} - C_d A_d \sqrt{2g\left(h_r + \tilde{h}\right)}$$

$$A\dot{\tilde{h}} = C_d A_d \sqrt{2gh_r} + \tilde{q}_{in} - C_d A_d \sqrt{2gh_r}\sqrt{1 + \frac{\tilde{h}}{h_r}}$$

Models: Example 3

Since the liquid level is assumed close to the rated value it follows that the ratio $\tilde{h}/h_r$ is small. It is therefore possible (by employing the Maclaurin expansion) to deduce that:

$$\sqrt{1+\frac{\tilde{h}}{h_r}} \cong 1+\tfrac{1}{2}\left(\frac{\tilde{h}}{h_r}\right)$$

$$A\dot{\tilde{h}} \cong C_d A_d \sqrt{2gh_r} + \tilde{q}_{in} - C_d A_d \sqrt{2gh_r} - \frac{C_d A_d \sqrt{2gh_r}}{2}\left(\frac{\tilde{h}}{h_r}\right)$$

Models: Example 3

This simplifies to:

$$\dot{\tilde{h}} = \frac{1}{A}\tilde{q}_{in} - \left(\frac{C_d A_d \sqrt{g}}{\sqrt{2h_r A}}\right)\tilde{h}$$

$$\dot{\tilde{h}} + \alpha\tilde{h} = \gamma\tilde{q}_{in} \quad , \quad \alpha = \frac{C_d A_d \sqrt{g}}{\sqrt{2h_r A}} > 0 \quad , \quad \gamma = \frac{1}{A} > 0$$

This is a far simpler equation in part because again it is *linear*. The parameter α also does not change with time, so in fact the equation is linear and has constant coefficients.

Models: Example 3

$$\dot{\tilde{h}} + \alpha\tilde{h} = \gamma\tilde{q}_{in} \quad , \quad \alpha > 0$$

Now of course this equation only approximately describes the performance of the liquid-level system provided it operates close to the rated level. There are some rather deep theoretical results relating the behaviour of the system close to the rated value and the behaviour of the approximate linear system. But these results only prove what is intuitively obvious, that the behaviour of the actual level $h(t)$ is quantitatively the same as $h_r + \tilde{h}(t)$.

Models: Example 3

This example begins to make a very significant point. In this module we will talk almost exclusively about systems which are described by linear differential equations whose coefficients do not change with time. We will also argue that such systems *do not exist*. Where these “linear, time-invariant” systems come from is as approximations to the behaviour of actual systems in terms of their deviations from the operating point, assuming that those deviations remain small, i.e. that those systems remain close to rated values or operating points.

Models: Example 3

In practice if there is a desired operating point of a system, we may add to the system other systems called *controllers* whose sole (or at least main) purpose is to ensure that the system does indeed remain close to the operating point. We obtain a virtuous circle. Close to the operating point the system behaves like a linear, constant coefficient (*aka* time-invariant) system. Based on this description we design controllers which ensure that the system stays close to the operating point where this simpler description is valid.

Models: Example 3

It is possible to rather satisfactorily analyse the "offset" system, i.e. the system described by:

$$\dot{\tilde{h}} + \alpha \tilde{h} = \gamma \tilde{q}_{in}(t) \qquad t \geq 0.$$

Towards this end we note a consequence of linearity. Suppose that a particular solution of the equation is

$$\tilde{h}_p(t) \qquad \text{i.e.} \qquad \dot{\tilde{h}}_p + \alpha \tilde{h}_p = \gamma \tilde{q}_{in}(t) \qquad t \geq 0.$$

Consider the associated *homogeneous* equation:

$$\dot{\tilde{h}} + \alpha \tilde{h} = 0 \qquad t \geq 0$$

Models: Example 3

Suppose that a particular solution of the homogeneous equation is $\tilde{h}_0(t)$

$$\text{i.e.} \qquad \dot{\tilde{h}}_0 + \alpha \tilde{h}_0 = 0 \qquad t \geq 0.$$

It follows that another solution of the equation is:

$$\tilde{h}_0(t) + \tilde{h}_p(t) \qquad t \geq 0.$$

Moreover this is the general solution of the equation for if $\tilde{h}_1(t)$ and $\tilde{h}_2(t)$ are two solutions of the equation then $\tilde{h}_1(t) - \tilde{h}_2(t)$ is a solution of the homogeneous equation. This is true of linear equations in general.

Models: Example 3

Important observation #1:

The general solution of a linear ordinary differential equation equals the sum of a particular solution and a solution of the associated homogeneous equation.

Models: Example 3

General solutions of the associated homogeneous equations are readily come by. For the present example the homogeneous equation is:

$$\dot{\tilde{h}} + \alpha \tilde{h} = 0 \qquad t \geq 0.$$

The key idea is to recognise the pivotal role of the exponential function. Specifically look for a solution of the form $\tilde{h}(t) = ce^{\beta t} \qquad t \geq 0.$

$$\dot{\tilde{h}} + \alpha \tilde{h} = c\beta e^{\beta t} + \alpha c e^{\beta t} = (\beta + \alpha) c e^{\beta t} = 0 \qquad t \geq 0.$$

$$\text{i.e.} \quad \beta = -\alpha.$$

Models: Example 3

So a solution of the homogeneous equation is

$$\tilde{h}_0(t) = ce^{-\alpha t} \qquad t \geq 0$$

for arbitrary choice of constant c. This solution is of course an exponential function. The *exponent* is $-\alpha$. The homogeneous equation

$$\dot{\tilde{h}} + \alpha\tilde{h} = 0 \qquad t \geq 0$$

may be written as: $D\tilde{h} + \alpha\tilde{h} = (D + \alpha)\tilde{h} = 0 \qquad t \geq 0$

where the symbol D denotes the operation of differentiation.

Models: Example 3

A key observation is that there is, associated with the homogeneous equation, a polynomial, namely the polynomial:

$$s+\alpha$$

which is obtained by formally replacing the differentiation operation by the symbol D and subsequently replacing the symbol D by the symbol s. The important point is that the root of this polynomial, namely $-\alpha$ is the exponent of the exponential appearing in the general solution to the homogeneous equation.

Models: Example 3

What is true in the case of this special linear, constant-coefficient equation is true in general.

Important observation #2:

The general solution of a linear, constant coefficient, homogeneous, ordinary differential equation almost always equals a sum of exponentials. Moreover the exponents of the exponentials appearing in this sum are the roots of a polynomial which may be acquired from the equation itself in an elementary and systematic manner.

Models: Example 3

Now there is a very clever and rather general idea for determining a particular solution of a linear equation when a solution to the associated homogeneous equation is known. It is called the *method of variation of parameters*. For the equation

$$\dot{\tilde{h}} + \alpha\tilde{h} = \gamma\tilde{q}_{in}(t) \qquad t \geq 0$$

we have the solution to the associated homogenous equation: $ce^{-\alpha t} \qquad t \geq 0.$

The method of variation of parameters suggest looking for a particular solution of the form: $y(t)e^{-\alpha t}$

Models: Example 3

Putting this purported solution into the equation yields:

$$\dot{y}e^{-\alpha t} - \alpha y e^{-\alpha t} + \alpha y e^{-\alpha t} = \gamma \tilde{q}_{in}(t) \qquad t \geq 0$$

$$\dot{y} = \gamma \tilde{q}_{in}(t) e^{\alpha t} \qquad t \geq 0.$$

$$y(t) - y(0) = \gamma \int_0^t \tilde{q}_{in}(\tau) e^{\alpha \tau} d\tau \qquad t \geq 0.$$

$$\tilde{h}_p(t) = y(t)e^{-\alpha t} = y(0)e^{-\alpha t} + \gamma e^{-\alpha t} \int_0^t \tilde{q}_{in}(\tau) e^{\alpha \tau} d\tau \qquad t \geq 0.$$

Models: Example 3

Without loss of generality we select $y(0) = 0$. The general solution to the equation becomes:

$$\tilde{h}(t) = c\,e^{-\alpha t} + \int_0^t \gamma \tilde{q}_{in}(\tau) e^{-\alpha(t-\tau)} d\tau \qquad t \geq 0.$$

The associated approximate solution to the liquid-level system is:

$$h(t) \cong h_r + \tilde{h}(t) = h_r + c\,e^{-\alpha t} + \int_0^t \gamma \tilde{q}_{in}(\tau) e^{-\alpha(t-\tau)} d\tau \qquad t \geq 0$$

provided the offset from the rated value remains small.

Models: Example 3

In engineering systems we generally control inputs such as $Q_{in}(t)$, setting them so that, disturbances and perturbations notwithstanding the system does indeed remain close to the desired or rated value. The assumption that the system operates close to some desired operating point is in general a much safer assumption to make of engineering systems because there exists an entire control infrastructure whose *raison d'être* is to ensure that this assumption holds. In the electric utility for example a slew of governors cajole the grid frequency to be 50Hz.

Models: Example 3

The upshot is that linear systems, which are really only approximate descriptions of system performance close to some desired location, actually provide a rather effective and useful set of models for practical engineering systems. Indeed it is broadly accurate to assert that much of engineering is based upon approximate linear models. There is, however, a rather important caveat. The approximate models which emerge in this manner may be linear, but they do not necessarily have constant coefficients. Moreover they also may not be linear.

Models: Example 3

The correct phrase which should be employed for these approximate models, supposedly valid close to a desired operating point, is *first approximation*. Although this phrase is employed in western literature it is not employed sufficiently often. In western literature it is far more common to see the phrase *linearisation* (actually *linearization*) and that is in the rather small percentage of cases where any mention at all is made of the approximation process. More commonly everything is assumed linear from the moment of its introduction.

Models: Example 3

You will probably have seen this kind of thing with Ohm's law for example. There are two laws named after Georg Ohm, but one of them, that the voltage across a resistor is proportional to the current flowing through it, is far more common so that relatively few people are aware of the ambiguity. As stated I have in fact misquoted the more famous Ohm's law. In fact the statement of the law which I have given is preposterous.

Models: Example 3

To demonstrate suppose I have a 1 kΩ resistor. If I place 1 V across it then 1 mA of current will flow through it. The electrical power delivered to the device will be 1 mW. This is not a large amount of power. The device will mainly react to this infusion of electrical energy by heating up, but it can easily shed this heat by radiation and conduction to its surroundings. If however I place 1 GV across the device then 1 MA of current will flow through it. The electrical power delivered will be 1×10^{15} W. This is an immense amount of power, far in excess of what the device can shed in the form of heat lost to its surroundings. Long before I have raised the voltage to this level the device will simply have melted.

Models: Example 3

Actually the more famous Ohm's law states that the voltage across a resistor is in direct proportion to the current flowing through it *when all other things remain unchanged*. It is this rider, commonly and somewhat sloppily omitted, which permits the law meaning, when examples such as the previous slide suggest that it must be nonsense. When I place 1 GV across a 1 kΩ resistor, or rather when I increase the applied voltage towards this value, I find that I simply cannot remove the resulting thermal energy fast enough. All other things do not remain unchanged. Rather the temperature of the device steadily increases and, strictly speaking, about this situation Ohm's law has nothing whatsoever to say.

Models: Example 3

Such serious issues notwithstanding a resistor will commonly be modelled by the linear equation $V = RI$ for some constant value of resistance R. When the *modus operandi* of modelling is to introduce all elements as linear components or rather to linearise from the outset, then the resulting model will inevitably be linear. Unfortunately the linear model may also be completely invalid. Some systems do not possess first approximations which are linear and these systems are not esoteric, purely academic systems, they are actual real systems which occur in very large numbers of engineering products (in mobile phones for example).

Models: Example 3

Even if the first approximation is linear it may not be constant-coefficient. In the case of the rocket for example we may have a desired or target trajectory for the flight. By looking for the equations which describe the deviation of the actual trajectory from this target, subject to the assumption that these deviations are small (an assumption known in the field of electronics as the *small signal approximation*) we may be able to find an approximate set of linear equations describing this deviation. However, many components of the "true" equations depend upon altitude, i.e. upon the current location on the trajectory, and therefore upon time which parameterises this trajectory. It follows that the first approximation may be linear, but it varies with time.

Models: Example 3

We have solved the first approximation equations for the liquid-level system.

$$\dot{\tilde{h}} + \alpha \tilde{h} = \gamma \tilde{q}_{in} \quad , \quad \alpha > 0$$

$$\tilde{h}(t) = c\, e^{-\alpha t} + \int_0^t \gamma \tilde{q}_{in}(\tau) e^{-\alpha(t-\tau)} d\tau \qquad t \geq 0.$$

For some special cases of the forcing term $\tilde{q}_{in}(t)$ we may demonstrate another very powerful method of solution.

Consider the special case:

Models: Example 3

$$\lim_{t\to\infty} \tilde{h}(t)e^{-st} = 0.$$

I note that if the function $\tilde{h}(t)$ is uniformly bounded in modulus for $t \geq 0$ then this limit equation holds for every complex number s for which

$$\mathrm{Re}(s) > 0.$$

Models: Example 3

$$\int_0^\infty \tilde{h}(t)e^{-st}dt = \frac{\tilde{h}(0)}{(s+\alpha)} + \frac{1}{(s+\alpha)(s+1)}$$

We note that the polynomial $s+\alpha$ has emerged. It is possible, using the *partial fraction expansion* to rewrite as

$$\int_0^\infty \tilde{h}(t)e^{-st}dt = \frac{\tilde{h}(0)}{(s+\alpha)} - \frac{1/_{\alpha-1}}{(s+\alpha)} + \frac{1/_{\alpha-1}}{(s+1)}$$

$$\int_0^\infty \tilde{h}(t)e^{-st}dt = \frac{\tilde{h}(0)-\left(1/_{\alpha-1}\right)}{(s+\alpha)} + \frac{1/_{\alpha-1}}{(s+1)}$$

Models: Example 3

$$\int_0^\infty \tilde{h}(t)e^{-st}dt = \left(\tilde{h}(0) - \left(\tfrac{1}{\alpha-1}\right)\right)\int_0^\infty e^{-\alpha t}e^{-st}dt + \left(\tfrac{1}{\alpha-1}\right)\int_0^\infty e^{-t}e^{-st}dt$$

$$\int_0^\infty \tilde{h}(t)e^{-st}dt = \int_0^\infty \left(\left(\tilde{h}(0) - \left(\tfrac{1}{\alpha-1}\right)\right)e^{-\alpha t} + \left(\tfrac{1}{\alpha-1}\right)e^{-t}\right)e^{-st}dt.$$

Lerch's cancellation law establishes that:

$$\tilde{h}(t) = \left(\tilde{h}(0) - \left(\tfrac{1}{\alpha-1}\right)\right)e^{-\alpha t} + \left(\tfrac{1}{\alpha-1}\right)e^{-t}.$$

We have obtained the general solution by a quite ingenious method. This method is called the *Laplace transform method*.

Models: Example 3

As it has been applied in the above case the Laplace transform method is based upon five ideas. Firstly we note that if $\dot{\tilde{h}} + \alpha\tilde{h} - \gamma\tilde{q}_{in}(t) = 0 \quad t \geq 0$

then $$\int_0^\infty \left(\dot{\tilde{h}} + \alpha\tilde{h} - \gamma\tilde{q}_{in}(t) \right) w(t)\,dt = 0$$

for *any* weighting function $w(t)$ chosen such that the integral converges. In general the integral will converge provided the integrand goes to zero sufficiently rapidly. A decaying exponential does indeed converge to zero rapidly so the choice of weighting function $exp(-\sigma t)$ is somewhat reasonable.

Models: Example 3

Since
$$\int_0^\infty \left(\dot{\tilde{h}} + \alpha \tilde{h} - \gamma \tilde{q}_{in}(t) \right) w(t)\,dt = 0$$
for any weighting function $w(t)$ chosen such that the integral converges we may observe that it holds, not for a single weighting function, but for a whole class of weighting functions. If all of the individual integrals converge then

$$\int_0^\infty \dot{\tilde{h}}\, w(t)\,dt + \alpha \int_0^\infty \tilde{h}\, w(t)\,dt = \int_0^\infty \gamma \tilde{q}_{in}(t) w(t)\,dt$$

Models: Example 3

The next idea is to process the first of these integrals by using the method of integration known as *integration by parts*. This is the inverse of the product rule of differentiation being derived by combining the product rule with the fundamental theorem of calculus.

$$\int_0^\infty \dot{\tilde{h}}\, w(t)\,dt = \tilde{h}(t)w(t)\Big|_0^\infty - \int_0^\infty \tilde{h}\,\dot{w}(t)\,dt$$

For a general weighting function $w(t)$ this formula will be messy. Is there a choice of the weighting function which will simplify it?

Models: Example 3

Firstly I should note that the weighting function has been chosen so that, among other things, the integral

$$\int_0^\infty \tilde{h}(t)w(t)dt$$

converges. A necessary condition for this convergence is that the integrand converge to zero, i.e.

$$\lim_{t\to\infty}\tilde{h}(t)w(t)=0$$

$$\int_0^\infty \dot{\tilde{h}}\,w(t)dt=-\tilde{h}(0)w(0)-\int_0^\infty \tilde{h}\,\dot{w}(t)dt$$

Models: Example 3

Now it was noted from their first introduction that the exponential functions possess a particularly notable property from the perspective of the calculus

$$\frac{de^{\lambda t}}{dt} = \lambda e^{\lambda t}$$

for any exponent λ, real or complex. It has taken us as a specie a long time to fully appreciate the consequences of this simple observation, and indeed it may well be that we have yet to fully appreciate them. For the present we may simply note the fact, but there are now some very deep explanations and interpretations of it.

Models: Example 3

If the weighting function is chosen as an exponential function, *exp*(λt), then

$$\dot{w}(t) = \lambda w(t) \quad , \quad w(0) = 1$$

$$\int_0^\infty \dot{\tilde{h}}\, w(t)\,dt = -\tilde{h}(0) - \lambda \int_0^\infty \tilde{h}\, w(t)\,dt$$

for any exponent λ, real or complex. Since we want certain integrals to converge we feel that the weighting functions chosen should converge to zero reasonably rapidly. If

$$\lambda = \sigma + j\omega \quad \text{then} \quad e^{\lambda t} = e^{\sigma t} e^{j\omega t}$$

Note: I denote the square root of -1 by *j* not *i*.

Models: Example 3

For every real number t, the complex number, $exp(j\omega t)$ has modulus equal to unity. Accordingly it does not converge to zero. It follows that

$$e^{\lambda t} = e^{\sigma t} e^{j\omega t}$$

converges to zero only if the real part, σ, of the exponent λ is negative. We deduce that if we select for our class of weighting functions the exponential functions $exp(\lambda t)$ then we should restrict the exponents λ so that $Re(\lambda) = \sigma < 0$. It has been found convenient to write $\lambda = -s$ and restrict complex s to have positive real part instead.

Models: Example 3

We deduce that a potentially good choice of the class of weighting functions is to extract them from the set of exponential functions, *exp*(-*st*), where the complex numbers *s* have positive real part. We may indeed have to restrict the class of weighting functions further but this much at least we elect to do. With this choice

$$\int_0^\infty \dot{\tilde{h}}\, w(t)\, dt = -\tilde{h}(0) + s\int_0^\infty \tilde{h}\, w(t)\, dt$$

$$s\int_0^\infty \tilde{h}\, e^{-st}\, dt - \tilde{h}(0) + \alpha\int_0^\infty \tilde{h}\, e^{-st}\, dt = \int_0^\infty \gamma\, \tilde{q}_{in}(t) e^{-st}\, dt.$$

Models: Example 3

The next step is simple:

$$\int_0^\infty \tilde{h}\, e^{-st}\, dt = \frac{\tilde{h}(0)}{s+\alpha} + \frac{\int_0^\infty \gamma \tilde{q}_{in}(t) e^{-st}\, dt}{s+\alpha}$$

where it seems we should further restrict s so that it does not equal $-\alpha$. The final steps depend upon the nature of the input $Q_{in}(t)$ and we shall not take them here. A few points may be made however. Evidently the integral

$$\int_0^\infty \tilde{q}_{in}(t) e^{-st}\, dt$$

plays a significant role. Of course the value of the integral depends on s, i.e. it is a function of s.

Models: Example 3

Recognising this we write:

$$\tilde{Q}_{in}(s) = \int_0^\infty \tilde{q}_{in}(t) e^{-st}\, dt.$$

Although it is not quite correct to do so it is common enough to call this function $\tilde{Q}_{in}(s)$ the *Laplace transform* of the input $\tilde{q}_{in}(t)$. So we obtain

$$\int_0^\infty \tilde{h} e^{-st}\, dt = \frac{\tilde{h}(0)}{s+\alpha} + \frac{\gamma \tilde{Q}_{in}(s)}{s+\alpha}.$$

Models: Example 3

Another key idea in the Laplace transform method is based upon the very simple observation that, should s be chosen such that the integral converges then

$$\int_0^{\infty} e^{-\alpha t} e^{-st}\, dt = \frac{1}{s+\alpha}$$

$$\int_0^{\infty} \tilde{h} e^{-st}\, dt = \tilde{h}(0) \int_0^{\infty} e^{-\alpha t} e^{-st} dt + \frac{\gamma \tilde{Q}_{in}(s)}{s+\alpha}$$

Models: Example 3

If it is possible to find $f(t)$ such that

$$\frac{\gamma \tilde{Q}_{in}(s)}{s+\alpha} = \int_0^\infty f(t) e^{-st}\, dt$$

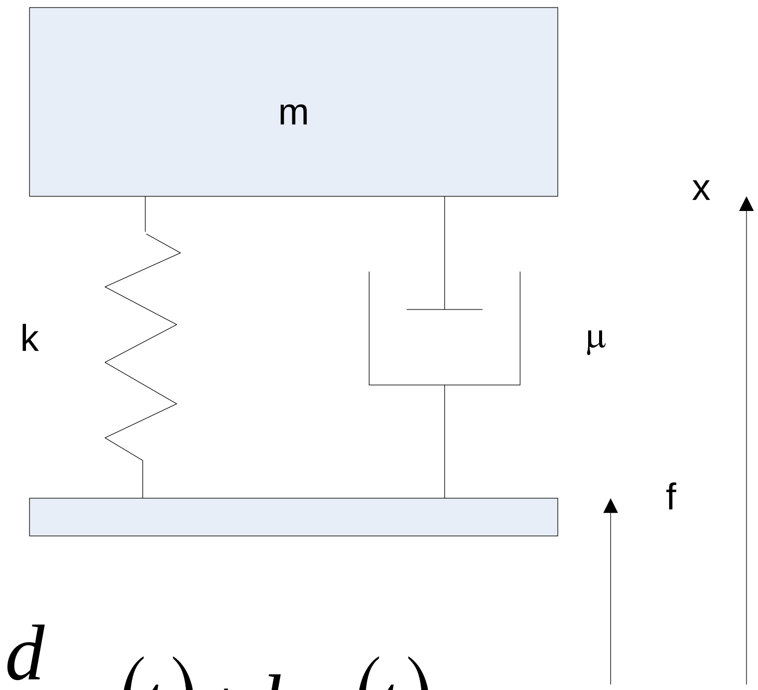
then $$\int_0^\infty \tilde{h} e^{-st}\, dt = \tilde{h}(0) \int_0^\infty e^{-\alpha t} e^{-st} dt + \int_0^\infty f(t) e^{-st} dt.$$

Finally Lerch’s cancellation law possibly gives (there are a few conditions to check):

$$\tilde{h}(t) = \tilde{h}(0) e^{-\alpha t} + f(t).$$

Models: Example 4

Consider a preliminary model of a vehicle suspension system.



$$m\ddot{x} = -\mu\left(\dot{x} - \dot{f}\right) - k\left(x - f\right)$$

$$\mu \frac{d}{dt} f(t) + k\, f(t) = m \frac{d^2}{dt^2} x(t) + \mu \frac{d}{dt} x(t) + k\, x(t)$$

$$x(0) = x_0 \qquad , \qquad \frac{d}{dt} x(0) = v_0$$

f = road disturbance

Models: Example 4

Evidently this is a highly simplified model of what is usually referred to as a *quarter car*. The spring models both the suspension spring and the tyre. The damper models the shock absorber. A shock absorber commonly acts like a spring-damper combination but the spring-like behaviour has again been absorbed into the one spring in the model. The mass equals a quarter of the vehicle mass at least. The deflection f represents the undulation of the road surface above a fixed reference. Linear models are employed for the spring and damper, which as noted is unrealistic.

Models: Example 4

The equation obtained is very similar to that obtained for the approximate model of the pendulum except for the presence of a forcing term, namely the road disturbance. This illustrates a very important point: mathematics is actually universal. The same mathematical model can describe an abundance of physically quite distinct systems. The model is a linear, constant-coefficient ordinary differential equation.

Models: Example 4

In reality the vehicle travels along the road so that the road disturbance *f* is a function of position along the road. Given the velocity along the road of the vehicle however we may find the position of the vehicle at any given time so that the road disturbance can be translated into a function of time. In the model presented we allow our co-ordinate system to travel along the road with the vehicle. Accordingly the vehicle is motionless relative to the horizontal of this moving co-ordinate system and the actual movement is manifest in the model as a time varying road disturbance.

Models: Example 4

A valuable feature of linearity is as follows. Suppose the road disturbance f can be expressed as a sum of several disturbances. There might for example be a sudden jump corresponding to a seam or pothole, there might be a smooth undulation, there might be a rather jittery, possibly random component representing the bumpy nature of the surface of a real road. Accordingly:

$$f(t) = \sum_i f_i(t)$$

Models: Example 4

It follows that the general solution of the equation is given by:

$$x(t) = x_0(t) + \sum_i x_i(t)$$

where $x_0(t)$ is a solution to the associated homogenous equation and $x_i(t)$ is a particular solution to the non-homogeneous equation:

$$\mu \frac{d}{dt} f_i(t) + k\, f_i(t) = m \frac{d^2}{dt^2} x_i(t) + \mu \frac{d}{dt} x_i(t) + k\, x_i(t)$$

Models: Example 4

Important point #3

We can simplify the problem of finding the general response of a forced linear system. It is equal to the response to the homogeneous equation (i.e. the unforced system) added to the particular response to each individual component of the forcing term.

Models: Example 4

This observation leads to a significant problem. Given a general input or forcing term (i.e. road disturbance in this case) is there some obvious way of identifying the “components” of this input? Of course the “components” should represent something that is physically important (a seam for example in the present case) but, if possible it would seem reasonable to try also to select the components so that the problem of finding the associated component of the particular system response ($x_i(t)$ in this case) is simplified.

Models: Example 4

Although again there are very deep explanations and interpretations available we may simply observe one mathematical "component" for which the associated particular solution may be rather easily found. Consider the equation for the special input $\cos(\omega_0 t)$. We seek a particular solution to the equation:

$$m\frac{d^2}{dt^2}x(t)+\mu\frac{d}{dt}x(t)+k\,x(t)=\mu\frac{d}{dt}\cos(\omega_0 t)+k\cos(\omega_0 t)$$

Models: Example 4

Suppose we simply guess that there may be a solution of the form $A\cos(\omega_0 t+\phi)$. Then

$$mA\left(-\omega_0^2\right)\cos\left(\omega_0 t+\phi\right)-\mu A\left(\omega_0\right)\sin\left(\omega_0 t+\phi\right)+k\cos\left(\omega_0 t+\phi\right)$$
$$=-\mu A\left(\omega_0\right)\sin\left(\omega_0 t\right)+k\cos\left(\omega_0 t\right)$$

This equation is a bit tedious to work with in the present form, however there is a very elegant idea which greatly simplifies the task. We must recall two wonderful formulae:

$$\cos(\theta)=\frac{e^{j\theta}+e^{-j\theta}}{2} \quad , \quad \sin(\theta)=\frac{e^{j\theta}-e^{-j\theta}}{2j}$$

Models: Example 4

Now the idea is to replace all of the trigonometric functions by their expansions in terms of complex exponential functions:

$$\frac{mA\left(-\omega_0^2\right)}{2}e^{j\omega_0 t}e^{j\phi}+\frac{mA\left(-\omega_0^2\right)}{2}e^{-j\omega_0 t}e^{-j\phi}-\frac{\mu A\left(\omega_0\right)}{2j}e^{j\omega_0 t}e^{j\phi}$$

$$+\frac{\mu A\left(\omega_0\right)}{2j}e^{-j\omega_0 t}e^{-j\phi}+\frac{kA}{2}e^{j\omega_0 t}e^{j\phi}+\frac{kA}{2}e^{-j\omega_0 t}e^{-j\phi}=$$

$$-\frac{\mu\left(\omega_0\right)}{2j}e^{j\omega_0 t}+\frac{\mu\left(\omega_0\right)}{2j}e^{-j\omega_0 t}+\frac{k}{2}e^{j\omega_0 t}+\frac{k}{2}e^{-j\omega_0 t}$$

This looks worse than before but I can rearrange:

Models: Example 4

$$\left(\frac{mA\left(-\omega_0^2\right)}{2}e^{j\phi}-\frac{\mu A\left(\omega_0\right)}{2j}e^{j\phi}+\frac{kA}{2}e^{j\phi}+\frac{\mu\left(\omega_0\right)}{2j}-\frac{k}{2}\right)e^{j\omega_0 t}=$$

$$\left(-\frac{mA\left(-\omega_0^2\right)}{2}e^{-j\phi}-\frac{\mu A\left(\omega_0\right)}{2j}e^{-j\phi}-\frac{kA}{2}e^{-j\phi}+\frac{\mu\left(\omega_0\right)}{2j}+\frac{k}{2}\right)e^{-j\omega_0 t}$$

Let:

$$z_1=\left(\frac{mA\left(-\omega_0^2\right)}{2}e^{j\phi}-\frac{\mu A\left(\omega_0\right)}{2j}e^{j\phi}+\frac{kA}{2}e^{j\phi}+\frac{\mu\left(\omega_0\right)}{2j}-\frac{k}{2}\right)$$

$$z_2=\left(-\frac{mA\left(-\omega_0^2\right)}{2}e^{-j\phi}-\frac{\mu A\left(\omega_0\right)}{2j}e^{-j\phi}-\frac{kA}{2}e^{-j\phi}+\frac{\mu\left(\omega_0\right)}{2j}+\frac{k}{2}\right)$$

Models: Example 4

The equation becomes:

$$z_1 e^{j\omega_0 t} = z_2 e^{-j\omega_0 t} \quad \text{i.e.} \quad z_1 e^{j2\omega_0 t} = z_2$$

where z_1 and z_2 are certain fixed complex numbers and where this equation must hold for all times t. Evidently the left hand side changes with time t if z_1 is non-zero. The right hand side of course is fixed. From this incompatibility I deduce that z_1 and therefore also z_2 must both be zero. It is reasonably clear that

$$z_2 = -\bar{z}_1$$

Models: Example 4

It follows that the equation becomes:

$$z_1 = 0$$

$$mA\left(-\omega_0^2\right)e^{j\phi} + \mu A\left(j\omega_0\right)e^{j\phi} + kAe^{j\phi} = \mu\left(j\omega_0\right) + k$$

$$Ae^{j\phi} = \frac{\mu\left(j\omega_0\right) + k}{m\left(-\omega_0^2\right) + \mu\left(j\omega_0\right) + k}$$

$$= \frac{\mu\left(j\omega_0\right) + k}{m\left(j\omega_0\right)^2 + \mu\left(j\omega_0\right) + k}$$

Models: Example 4

So the particular solution when the forcing term is a sinusoid $\cos(\omega_0 t)$ (i.e. a unit amplitude co-sinusoid of radian frequency ω_0 rad/sec) is also a co-sinusoid of radian frequency ω_0 rad/sec, namely $A\cos(\omega_0 t+\phi)$. The *amplitude* A and *phase* ϕ are described by the equation:

$$Ae^{j\phi} = \frac{\mu\left(j\omega_0\right)+k}{m\left(j\omega_0\right)^2+\mu\left(j\omega_0\right)\ +k}$$

An elegant and fairly elementary solution to the problem is obtained provided we employ complex numbers.

Models: Example 4

$$Ae^{j\phi} = \frac{\mu\left(j\omega_0\right) + k}{m\left(j\omega_0\right)^2 + \mu\left(j\omega_0\right) + k}$$

For a given frequency ω_0 and given system parameters m, μ and k the right hand side is just a complex number. The amplitude A of the sinusoidal particular solution is then obtained as the modulus of this complex number. The phase ϕ of the sinusoidal particular solution is likewise obtained as an argument of this complex number. All of this occurs more generally.

Models: Example 4

Important observation #4:

A particular solution to a linear, constant-coefficient ordinary differential equation with a forcing term which is a co-sinusoid $\cos(\omega_0 t)$ (i.e. a unit amplitude co-sinusoid of radian frequency ω_0 rad/sec) is also a co-sinusoid, $A\cos(\omega_0 t + \phi)$, of radian frequency ω_0 rad/sec. The amplitude A and phase ϕ of this particular solution may be obtained by a systematic process directly from the equation.

Models: Example 4

$$Ae^{j\phi} = \frac{\mu(j\omega_0)+k}{m(j\omega_0)^2+\mu(j\omega_0)\ +k} = \frac{j\frac{\mu}{m}\omega_0+\frac{k}{m}}{\frac{k}{m}-\omega_0^2+j\frac{\mu}{m}\omega_0}$$

$$A = \frac{\sqrt{\left(\frac{k}{m}\right)^2+\left(\frac{\mu}{m}\right)^2\omega_0^2}}{\sqrt{\left(\frac{k}{m}-\omega_0^2\right)^2+\left(\frac{\mu}{m}\right)^2\omega_0^2}}$$

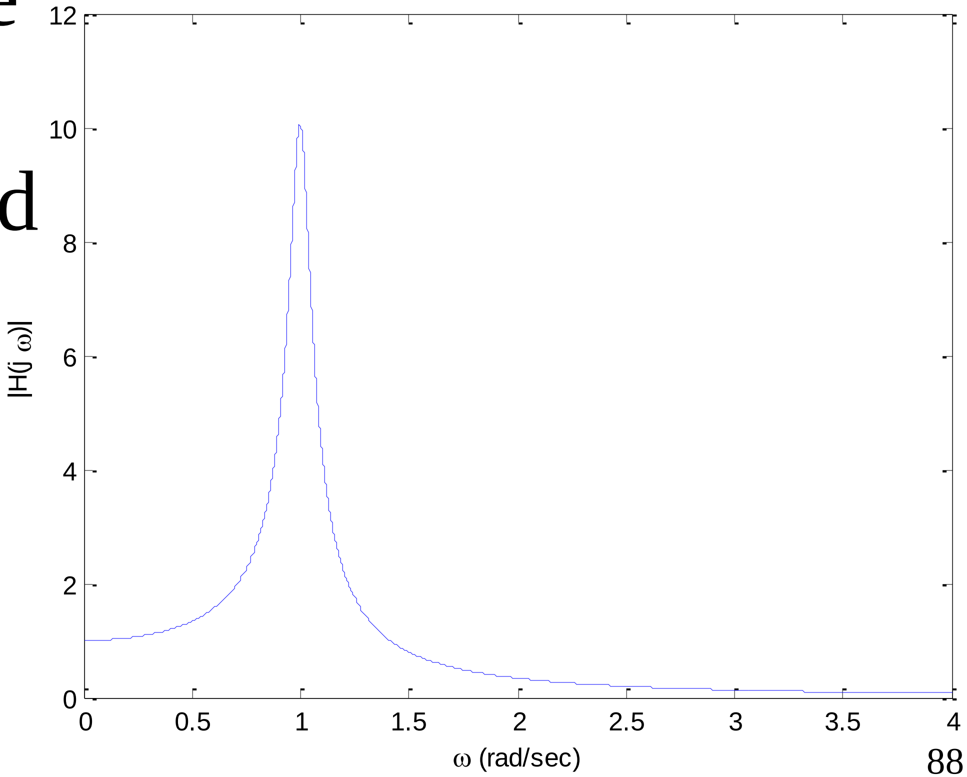
We find that the amplitude of the particular co-sinusoidal solution varies with the frequency ω_0 of the forcing term. At zero frequency (DC) the amplitude is 1. At very large frequencies the amplitude falls towards 0.

Models: Example 4

$$\frac{\sqrt{\left(\frac{k}{m}\right)^2 + \left(\frac{\mu}{m}\right)^2 \omega^2}}{\sqrt{\left(\frac{k}{m} - \omega^2\right)^2 + \left(\frac{\mu}{m}\right)^2 \omega^2}}$$

We may plot the amplitude vs frequency. In the plot I have taken $k = 1$, $m = 1$ and $\mu = 0.1$.

The plot reveals the important phenomenon of *resonance*.



Models: Example 4

The plot on the previous slide is a special case of an *amplitude response plot* in turn a part of the *frequency response*. We see that at a frequency of 0.998 rad/sec the amplitude of the response of the suspension system is about 10.06, whereas that of the input component was 1. The system amplifies an approximately 1 rad/sec frequency approximately tenfold. This is the phenomenon of resonance. All structures have resonant frequencies, i.e. frequencies which are amplified by the structure. Resonance is the curse of bridges, tall buildings and vehicle chasses and suspension systems.

Models: Example 4

We see also that at a reasonably high frequency the amplitude of the response of the suspension system is almost zero, whereas that of the input component was 1. The system attenuates high frequencies. This is ubiquitous. No systems can react infinitely rapidly. Accordingly no systems can react at all to infinitely high frequencies.

Models: Example 4

Of course the input to a system, and in particular the input to the suspension system of the present example will not be a co-sinusoid. The sinusoids are a very useful set of mathematical functions, but it is false to suggest that they actually exist as real signals, for example as an actual road disturbance. For one sinusoids are periodic and therefore go on forever. Roads of course do not. Should we care that the system has a resonance at about 1 rad/sec if the input to the system can never be a sinusoid at all, let alone a sinusoid of this particular frequency?

Models: Example 4

The answer is unequivocally that yes we should care. Although an input (the road disturbance in this case) may not be a sinusoid it transpires that it may contain a component which is a sinusoid. By way of illustration consider the following signal:

$$f(t) = \begin{cases} e^{-t} & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases}$$

It is apparent that this signal is not a sinusoid.

Models: Example 4

It is a standard application of *contour integration* to establish that:

$$f(t) = \tfrac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{1+j\omega} e^{j\omega t} d\omega = \tfrac{1}{\pi} \int_{0}^{+\infty} \frac{1}{\sqrt{1+\omega^2}} \cos\left(\omega t - \tan^{-1}(\omega)\right) d\omega$$

The integration is achieved by considering the contour integral:

$$\tfrac{1}{2\pi j} \oint \frac{1}{1+z} e^{zt} dz$$

Two different contours are employed, one for $t > 0$ and one for $t < 0$. Subsequently *Cauchy's residue theorem* is applied. The calculation for $t = 0$ is more straightforward and may be achieved directly.

Models: Example 4

The formula:

$$f(t) = \frac{1}{\pi} \int_0^{+\infty} \frac{1}{\sqrt{1+\omega^2}} \cos\left(\omega t - \tan^{-1}(\omega)\right) d\omega$$

establishes that the signal $f(t)$, although it is not even periodic, may in fact be expressed as a "sum" of co-sinusoids of all non-negative frequencies. Granted the "sum" is a bit unusual, not only is it not a finite sum, it is not even a countably infinite sum. In fact it is an uncountable infinite sum, although we do not call such things sums, we call them integrals.

Models: Example 4

In fact a very large number of real functions can be written as:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\omega)e^{j\omega t}\,d\omega = \frac{1}{\pi}\int_{0}^{+\infty} \left|F(j\omega)\right| \cos\left(\omega t + \arg\left(F(j\omega)\right)\right)d\omega$$

for a suitable associated function $F(j\omega)$. This function is called the *Fourier transform* of the function $f(t)$. The two expansions of this formula establish that the real function $f(t)$ is really some kind of superposition of the complex exponential functions $exp(j\omega t)$ or alternatively the co-sinusoids.

Models: Example 4

Important observation #5:

Although a real signal will not be periodic it may commonly be expressed as a “sum” of sinusoids. Accordingly the signal may nonetheless have a component which is a sinusoid at a certain frequency.

Models: Example 4

The equation:

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(j\omega)e^{j\omega t}d\omega=\frac{1}{\pi}\int_{0}^{+\infty}\left|F(j\omega)\right|\cos\left(\omega t+\arg\left(F(j\omega)\right)\right)d\omega$$

is interpreted as describing the *frequency content* or *spectrum* of the signal $f(t)$. In short, if the Fourier transform $F(j\omega)$ is not zero at the frequency ω then we interpret this as saying that some of the energy of the signal takes the form of a component of the signal which is equal to a sinusoid of frequency ω. Indeed the modulus of $F(j\omega)$ expresses the weighting of or degree to which this frequency is present.

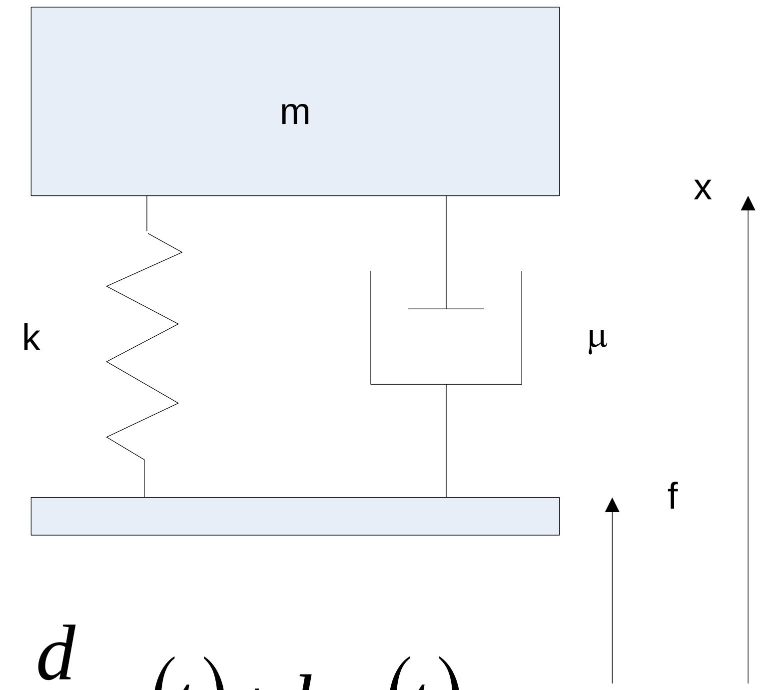
Models: Example 4

In the case of the suspension system with a resonance at about 1 rad/sec it will not matter that the road disturbance cannot be a sinusoid. If the spectrum of the road disturbance indicates that there is a significant component of that disturbance at frequencies at or close to 1 rad/sec then this component of the disturbance will be amplified by the system. In short we actually experience resonance occurring in real systems notwithstanding that sinusoids do not really exist.

Models: Example 4

Now recall the model of the suspension system.

$$m\ddot{x} = -\mu\left(\dot{x} - \dot{f}\right) - k\left(x - f\right)$$



$$\mu \frac{d}{dt} f(t) + k\, f(t) = m \frac{d^2}{dt^2} x(t) + \mu \frac{d}{dt} x(t) + k\, x(t)$$

f = road disturbance

$$x(0) = x_0 \qquad , \qquad \frac{d}{dt} x(0) = v_0$$

Models: Example 4

Suppose we attempt to solve this equation by the two methods of solution considered above in the case of the liquid-level system. To be specific take parameter values $m = 1$, $k = 1$ and $\mu = 0.1$ as above.

$$0.1\frac{d}{dt}f(t)+f(t)=\frac{d^2}{dt^2}x(t)+0.1\frac{d}{dt}x(t)+x(t)$$

$$x(0)=x_0 \qquad , \qquad \frac{d}{dt}x(0)=v_0$$

Models: Example 4

The first method requires firstly that we solve the associated homogeneous equation. I will also impose the known initial conditions:

$$\frac{d^2}{dt^2}x(t)+0.1\frac{d}{dt}x(t)+x(t)=0$$

$$x(0)=x_0 \qquad , \qquad \frac{d}{dt}x(0)=v_0$$

As the system is linear, constant-coefficient we look for a solution of the form $exp(\beta t)$, i.e.

$$\left(\beta^2+0.1\beta+1\right)\exp(\beta t)=0$$

Models: Example 4

A solution of this form exists provided the exponent β is a root of the polynomial:

$$s^2 + 0.1s + 1$$

This polynomial has two complex roots which are of course conjugates:

$$\beta_1 \, , \, \beta_2 = -0.05 \pm j\sqrt{\frac{3.99}{4}}$$

By linearity we obtain a general solution to the homogeneous equation:

$$x_0(t) = c_1 \exp(\beta_1 t) + c_2 \exp(\beta_2 t).$$

Models: Example 4

There are two arbitrary coefficients in this general solution formula. This is sufficient to permit us to tailor it to meet any given initial conditions:

$$x_0(0) = c_1 + c_2 \quad , \quad \dot{x}_0(0) = c_1\beta_1 + c_2\beta_2$$

$$\begin{bmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_0(0) \\ \dot{x}_0(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$$

which is solvable if the determinant of the matrix is non-zero, i.e. if β_1 and β_2 are not equal.

Models: Example 4

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\beta_2 - \beta_1} \begin{bmatrix} \beta_2 & -1 \\ -\beta_1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} = \frac{1}{\beta_2 - \beta_1} \begin{bmatrix} \beta_2 x_0 - v_0 \\ -\beta_1 x_0 + v_0 \end{bmatrix}$$

$$x_0(t) = \left(\frac{\beta_2 x_0 - v_0}{\beta_2 - \beta_1} \right) \exp(\beta_1 t) + \left(\frac{-\beta_1 x_0 + v_0}{\beta_2 - \beta_1} \right) \exp(\beta_2 t).$$

Now we require a particular solution satisfying zero initial conditions and we apply the method of variation of parameters:

$$x_p(t) = c_1(t) \exp(\beta_1 t) + c_2(t) \exp(\beta_2 t).$$

Models: Example 4

$$\ddot{x}_p(t)+0.1\dot{x}_p(t)+x_p(t)=$$
$$\ddot{c}_1\exp(\beta_1 t)+2\beta_1\dot{c}_1\exp(\beta_1 t)+\beta_1^2 c_1\exp(\beta_1 t)+$$
$$\ddot{c}_2\exp(\beta_2 t)+2\beta_2\dot{c}_2\exp(\beta_2 t)+\beta_2^2 c_2\exp(\beta_2 t)+$$
$$0.1\dot{c}_1\exp(\beta_1 t)+0.1\beta_1 c_1\exp(\beta_1 t)+$$
$$0.1\dot{c}_2\exp(\beta_2 t)+0.1\beta_2 c_2\exp(\beta_2 t)+$$
$$c_1\exp(\beta_1 t)+c_2\exp(\beta_2 t)=0.1\dot{f}(t)+f(t)$$

$$\ddot{c}_1\exp(\beta_1 t)+2\beta_1\dot{c}_1\exp(\beta_1 t)+\ddot{c}_2\exp(\beta_2 t)+$$
$$2\beta_2\dot{c}_2\exp(\beta_2 t)+0.1\dot{c}_1\exp(\beta_1 t)+0.1\dot{c}_2\exp(\beta_2 t)=0.1\dot{f}(t)+f(t)$$

Models: Example 4

This equation does not uniquely define the functions c_1 and c_2. It seems that the method has failed. The additional clever idea is to constrain these functions. In this case we require:

$$\dot{c}_1(t)\exp(\beta_1 t)+\dot{c}_2(t)\exp(\beta_2 t)=0$$

$$\ddot{c}_1(t)\exp(\beta_1 t)+\dot{c}_1(t)\beta_1\exp(\beta_1 t)+\ddot{c}_2(t)\exp(\beta_2 t)+\dot{c}_2(t)\beta_2\exp(\beta_2 t)=0$$

$$\begin{aligned}&\ddot{c}_1\exp(\beta_1 t)+2\beta_1\dot{c}_1\exp(\beta_1 t)+\ddot{c}_2\exp(\beta_2 t)+\\&2\beta_2\dot{c}_2\exp(\beta_2 t)+0.1\dot{c}_1\exp(\beta_1 t)+0.1\dot{c}_2\exp(\beta_2 t)\\&=\beta_1\dot{c}_1\exp(\beta_1 t)+\beta_2\dot{c}_2\exp(\beta_2 t)=0.1\dot{f}(t)+f(t)\end{aligned}$$

Models: Example 4

$$\begin{bmatrix} \exp(\beta_1 t) & \exp(\beta_2 t) \\ \beta_1 \exp(\beta_1 t) & \beta_2 \exp(\beta_2 t) \end{bmatrix} \begin{bmatrix} \dot{c}_1 \\ \dot{c}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.1\dot{f}(t) + f(t) \end{bmatrix}$$

$$\begin{bmatrix} \dot{c}_1 \\ \dot{c}_2 \end{bmatrix} = \frac{1}{(\beta_2 - \beta_1)\exp((\beta_1 + \beta_2)t)} \begin{bmatrix} \beta_2 \exp(\beta_2 t) & -\exp(\beta_2 t) \\ -\beta_1 \exp(\beta_1 t) & \exp(\beta_1 t) \end{bmatrix} \begin{bmatrix} 0 \\ 0.1\dot{f}(t) + f(t) \end{bmatrix}$$

The determinant appearing here, $(\beta_2 - \beta_1)\exp((\beta_1 + \beta_2)t)$ is given a special name. It is called the *Wronskian*.

$$x_p(t) = -e^{\beta_1 t} \int_0^t \frac{e^{-\beta_1 \tau}\left(0.1\dot{f}(\tau) + f(\tau)\right)}{(\beta_2 - \beta_1)} d\tau + e^{\beta_2 t} \int_0^t \frac{e^{-\beta_2 \tau}\left(0.1\dot{f}(\tau) + f(\tau)\right)}{(\beta_2 - \beta_1)} d\tau.$$

Models: Example 4

$$x_p(t)=\int_0^t \frac{e^{\beta_2(t-\tau)}-e^{\beta_1(t-\tau)}}{(\beta_2-\beta_1)} f(\tau)d\tau+(0.1)\int_0^t \frac{e^{\beta_2(t-\tau)}-e^{\beta_1(t-\tau)}}{(\beta_2-\beta_1)} \dot{f}(\tau)d\tau.$$

$$x_p(t)=\int_0^t \frac{e^{\beta_2(t-\tau)}-e^{\beta_1(t-\tau)}}{(\beta_2-\beta_1)} f(\tau)d\tau+(0.1)\left(\left.\frac{e^{\beta_2(t-\tau)}-e^{\beta_1(t-\tau)}}{(\beta_2-\beta_1)} f(\tau)\right|_0^t\right)$$

$$-(0.1)\int_0^t \frac{-\beta_2 e^{\beta_2(t-\tau)}+\beta_1 e^{\beta_1(t-\tau)}}{(\beta_2-\beta_1)} f(\tau)d\tau.$$

$$x_p(t)=\int_0^t \frac{e^{\beta_2(t-\tau)}-e^{\beta_1(t-\tau)}}{(\beta_2-\beta_1)} f(\tau)d\tau-(0.1)\int_0^t \frac{-\beta_2 e^{\beta_2(t-\tau)}+\beta_1 e^{\beta_1(t-\tau)}}{(\beta_2-\beta_1)} f(\tau)d\tau$$

where $f(0) = 0$ is assumed.

Models: Example 4

After a fairly tedious application of integration by parts we conclude that

$$x_p(t) = \int_0^t h(t-\tau) f(\tau) d\tau.$$

for suitable function $h(t)$. This is the second time that the method of variation of parameters has given a solution having this form. The reason is because the method always gives a solution having this form, although it can be very tedious actually using the method to find this solution.

Models: Example 4

Important fact #5

In general the particular solution of linear constant coefficient ordinary differential equation with forcing term $f(t)$ having zero initial conditions takes the form:

$$x_p(t) = \int_0^t h(t-\tau) f(\tau) d\tau.$$

One view of the Laplace transform method is that it comprises an often rather efficient procedure (and certainly more efficient than the method of variation of parameters) for calculating the function h appearing in this general formula for the particular solution.

Models: Example 4

Now consider the Laplace transform method:

$$\frac{d^2}{dt^2}x(t)+0.1\frac{d}{dt}x(t)+x(t)=0.1\frac{d}{dt}f(t)+f(t)$$

$$x(0)=x_0 \qquad , \qquad \frac{d}{dt}x(0)=v_0$$

$$\int_0^\infty \frac{d^2}{dt^2}x(t)e^{-st}dt+0.1\int_0^\infty \frac{d}{dt}x(t)e^{-st}dt+\int_0^\infty x(t)e^{-st}dt$$

$$=0.1\int_0^\infty \frac{d}{dt}f(t)e^{-st}+dt\int_0^\infty f(t)e^{-st}dt.$$

Apply integration by parts:

Models: Example 4

$$-v_0 + s\int_0^\infty \frac{d}{dt}x(t)e^{-st}dt + 0.1\left(-x_0 + s\int_0^\infty x(t)e^{-st}dt\right) + \int_0^\infty x(t)e^{-st}dt$$

$$= 0.1s\int_0^\infty f(t)e^{-st}dt + \int_0^\infty f(t)e^{-st}dt.$$

$$-v_0 + s\left(-x_0 + s\int_0^\infty x(t)e^{-st}dt\right) + 0.1\left(-x_0 + s\int_0^\infty x(t)e^{-st}dt\right) + \int_0^\infty x(t)e^{-st}dt = (0.1s+1)\int_0^\infty f(t)e^{-st}dt.$$

where it is assumed that s is chosen such that all of the integrals converge and therefore all of the integrands converge to zero. Let

$$X(s) = \int_0^\infty x(t)e^{-st}dt \quad , \quad F(s) = \int_0^\infty f(t)e^{-st}dt$$

Models: Example 4

$$-v_0 + s(-x_0 + sX(s)) + 0.1(-x_0 + sX(s)) + X(s) = (0.1s+1)F(s).$$

$$(s^2 + 0.1s + 1)X(s) - v_0 - (s+0.1)x_0 = (0.1s+1)F(s).$$

$$X(s) = \frac{v_0 + (s+0.1)x_0}{(s^2+0.1s+1)} + \frac{(0.1s+1)F(s)}{(s^2+0.1s+1)}.$$

$$X(s) = \frac{v_0 + (s+0.1)x_0}{(s-\beta_1)(s-\beta_2)} + \frac{(0.1s+1)F(s)}{(s^2+0.1s+1)}.$$

Models: Example 4

The quantity multiplying the function $F(s)$, namely

$$\frac{0.1s+1}{\left(s^2+0.1s+1\right)}$$

has substantially appeared above, albeit with the complex variable s replaced by $j\omega_0$. It appeared in the formula:

$$Ae^{j\phi}=\frac{\mu\left(j\omega_0\right)+k}{m\left(j\omega_0\right)^2+\mu\left(j\omega_0\right)\ +k}$$

for the amplitude and phase of the particular co-sinusoidal solution in the event that the forcing term was a unit amplitude co-sinusoid of radian frequency ω_0 (rad/sec).

Models: Example 4

$$X(s)=\left(\frac{v_0+(s+0.1)x_0}{\beta_2-\beta_1}\right)\left(-\frac{1}{s-\beta_1}+\frac{1}{s-\beta_2}\right)+\frac{(0.1s+1)F(s)}{(s^2+0.1s+1)}.$$

$$X(s)=\int_0^\infty\left(\frac{v_0+(s+0.1)x_0}{\beta_2-\beta_1}\right)\left(-e^{\beta_1 t}+e^{\beta_2 t}\right)dt+\int_0^\infty g(t)e^{-st}dt$$

$$\text{if}\quad \int_0^\infty g(t)e^{-st}dt=\frac{(0.1s+1)F(s)}{s^2+0.1s+1}$$

$$x(t)=\left(\frac{v_0+(s+0.1)x_0}{\beta_2-\beta_1}\right)\left(-e^{\beta_1 t}+e^{\beta_2 t}\right)+g(t).$$

Models:

There are significant generalisations of almost everything which has been said above, particularly in the case of example 4. This module is largely about solving the equations which comprise the first approximation close to an operating point, assuming that this first approximation is linear and constant-coefficient. We will consider also some deeper explanations and interpretations of the ideas considered above.