CIRCUIT THEORY EEEN30020

Tutorial 3: The Laplace Transform,
Natural Frequencies and Network Functions

Problem Set and Workbook

The aim is to reinforce your knowledge on the Laplace transform, zero-input and zero-state responses and transfer function. The Laplace transform allows us to analyse the stability of circuits (through their natural frequencies) and to understand their frequency response (through the transfer function).

At the end of the tutorial, you will receive a short **test.** You must answer the test and return it to the Module Coordinator or Teaching Assistants. You can use any materials, for example, the lecture notes. You can answer the test with your Homework/Lab team partner. Allow ten minutes for the test.

Solve Problems 1 and 3 first. If any time left – solve Problem 2.

1. If $v_i(t) = e^{-t}$ V and $v_C(0^-) = 2$ V find i(t) for $t \geq 0$ in the circuit of figure 1. What are the natural frequencies of the circuit? Is this circuit stable?

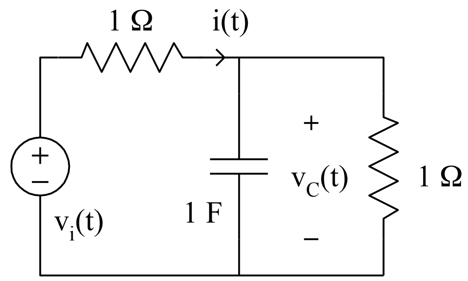


Figure 1.

2. In the circuit of Question 1, find i(t) in steady state if $v_i(t) = \sin 10t$ V and $v_C(0^-) = 1$ V.

3. In the following circuit all resistances are 1 kΩ and both capacitances are 1 mF. Show that the transfer function $H(s) = V_2(s) / V_1(s)$ is $-2s/(s^2 + s + 1)$. Plot the poles and zeros of this transfer function in the complex plane, and predict the form of the magnitude response $|H(j\omega)|$ of the circuit. Verify your predictions at $\omega = 0$ and as $\omega \rightarrow \infty$ by examination of the circuit. Plot the magnitude response in dB for frequencies on a log scale from 0.01 to 100 rad/s.

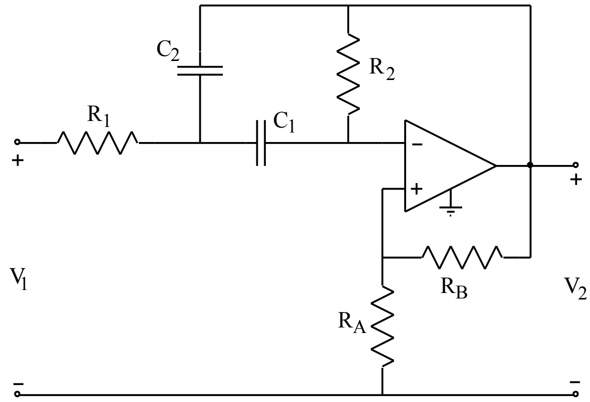


Figure 2.

Useful Formulae and Notes

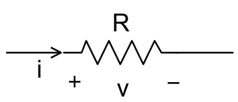
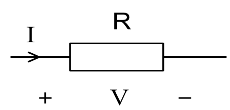
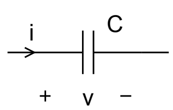
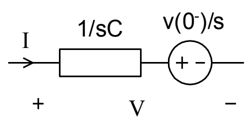
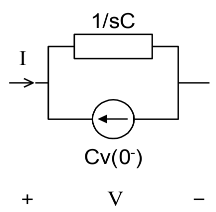
Laplace Transform

f(t)	*F(s)*	*f(t)*	*F(s)*
K (const) or $u(t)$ (unit step function)	K/s $1/s$	e^{at}	$\dfrac{1}{s-a}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\dfrac{df(t)}{dt}$	$sF(s)-f(0^-)$	$\int_{0^-}^{t} f(\tau)d\tau$	$\dfrac{1}{s}F(s)$
$e^{-\alpha t}f(t)$	$F(s+\alpha)$	$f(t-\tau)u(t-\tau)$	$e^{-s\tau}F(s)$

Partial Fraction Expansion

$$F(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{(s-p_1)(s-p_2)\dots(s-p_m)}=\frac{k_1}{s-p_1}+\frac{k_2}{s-p_2}+\cdots+\frac{k_m}{s-p_m}$$

$$\text{where } k_i=(s-p_i)\left.\frac{N(s)}{D(s)}\right|_{s=p_i}$$

Time domain	Frequency domain
$v(t)=Ri(t)$ 	$V(s)=RI(s)$ 
$i(t)=C\dfrac{dv(t)}{dt}$ 	$I(s)=sCV(s)-CV(0^-)$  

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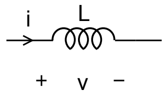
$$v(t) = L\frac{di(t)}{dt}$$



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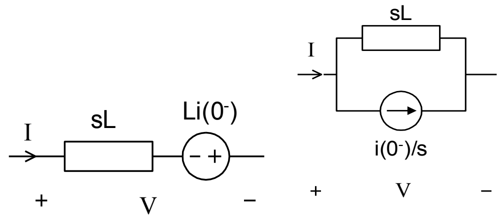
$$V(s) = sLI(s) - Li(0^-)$$



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Zero-State and Zero-Input Responses:

The ***zero-input response*** is due to the initial conditions acting alone, with all independent sources set to zero.

The ***zero-state response*** is due to the independent sources acting alone with all initial conditions set to zero.

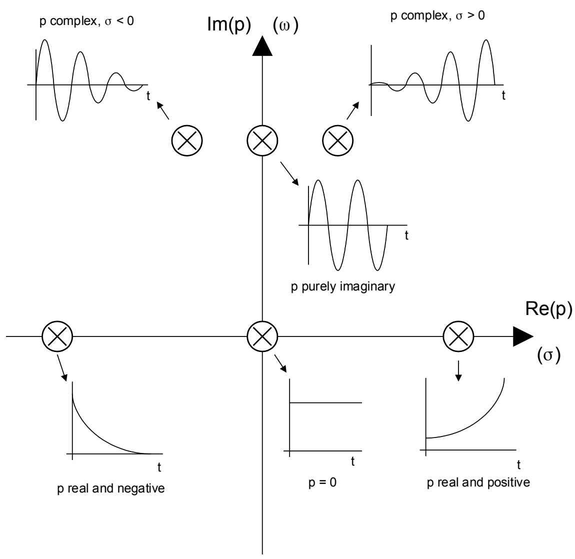
Natural frequencies can be found:

- From the equation Det $\underline{M} = 0$, where $\underline{M}$ is the MNA matrix of the circuit

  or

- As the poles of the transfer function *H(s)*.

Stable circuits have natural frequencies p_i such that $Re\,(p_i) < 0$.



The ***zero-state response*** can be found from the network (transfer) function:

$$\mathcal{L}\{\text{response}\} = \text{Network Function} \cdot \mathcal{L}\{\text{input}\}$$

The Laplace Transform vs. Phasor Analysis

Phasor analysis is a particular case of the Laplace transform, and the Laplace transform is a more general technique.

Use phasor analysis only if

- you are asked to find the ***steady state response*** of a stable circuit;
  <u>and</u>
- the independent current and/or voltage sources driving the circuit are ***sinusoidal***;

Use the Laplace transform in all other cases.

Workbook

1. If $v_i(t) = e^{-t}$ V and $v_C(0^-) = 2$ V find i(t) for $t \geq 0$ in the circuit of figure 1. What are the natural frequencies of the circuit? Is this circuit stable?

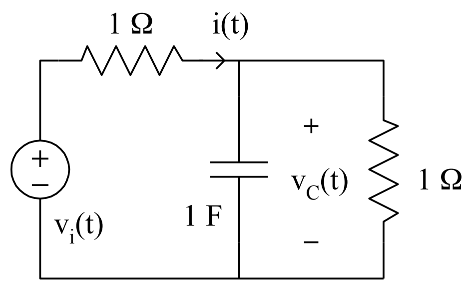


Figure 1.

Solution: This is a circuit that contains a dynamical element (capacitor). We have two options – phasor analysis and the Laplace transform. Thus, we start by understanding which of the two can be used in this case. We note that

The voltage source is not sinusoidal

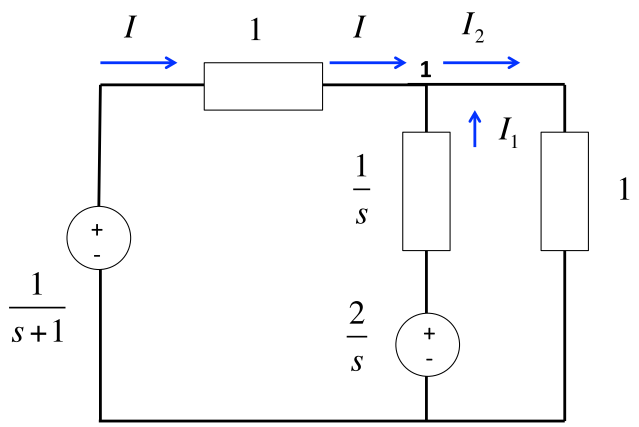
and

The initial voltage across the capacitor is given.

Thus,

We will use the Laplace transform.

The next step is to transform the circuit to the Laplace domain (complex frequency domain):



Note how

- The input voltage source e^{-t} is transformed.
- The initial voltage across the capacitor results in a voltage source in the circuit.

We must find *I(s)* and transform it back to the time domain.

Write KCL for node 1:

(1)

Write KVL-1 for the (smaller) loop that includes the input voltage source, the impedance 1, the impedance 1/s and the voltage source 2/s:

(2)

Write KVL-2 for the (larger) loop that includes the input voltage source, the impedance 1 and the other impedance 1:

(3)

You have three equations and you must resolve then to find the current I. One way to find this current is to undertake the following steps:

1. Express I_2 from the third equation (KVL-2);
2. Express I_1 from the first equation (KCL). In this expression, use I_2 that you just found from KVL-2.
3. Substitute I_1 and I_2 in the second equation (KVL-1) and express I from it.
   Note that $1/(s(s+1))=1/s - 1/(s+1)$.

There are other ways to resolve the circuit. As an exercise, try to apply another approach to find the current I.

You should obtain the following answer:

$$I(s) = -\frac{1}{s+2}$$

Write its inverse Laplace transform in order to obtain the current in the time domain:

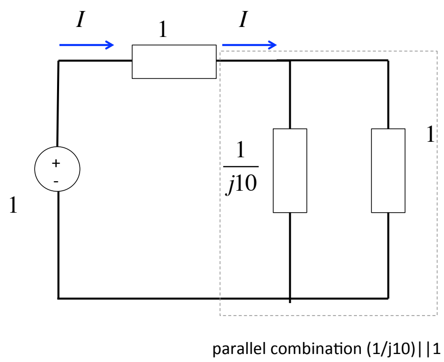
Sometimes, the natural frequencies of a circuit are evident from the solution, and there is no need to assemble the MNA matrix or to find the transfer function. This circuit contains one capacitor, and so we anticipate no more than one natural frequency. By inspecting the answer, we see that the pole of *I(s)* is:

and this is the natural frequency of the circuit. Is the circuit stable?

2. In the circuit of Question 1, find i(t) in steady state if $v_i(t) = \sin 10t$ V and $v_C(0^-) = 1$ V.

Solution: We just found that the circuit is ***stable.*** We also see that now the input voltage source is ***sinusoidal***: $v_i(t) = \sin 10t$. In this particular case, we can apply ***phasor analysis***.

Transforming to the phasor domain (frequency domain):



Note the difference between the circuit in the phasor domain and in the Laplace domain. Also note that this circuit is easier to analyse by combining the two impedances 1/j10 and 1.

Find the current phasor *I*:

You should obtain

$$I = \frac{1 + j10}{2 + j10}$$

Write it in the standard form of a complex number (Re(Z) + j Im (Z)):

Find

$|I| =$

$\angle I =$

And so

$I = |I| \sin(10t + \angle I) =$

3. In the following circuit all resistances are 1 kΩ and both capacitances are 1 mF. Show that the transfer function $H(s) = V_2(s) / V_1(s)$ is $-2s/(s^2 + s + 1)$. Plot the poles and zeros of this transfer function in the complex plane, and predict the form of the magnitude response $|H(j\omega)|$ of the circuit. Verify your predictions at $\omega = 0$ and as $\omega \to \infty$ by examination of the circuit. Plot the magnitude response in dB for frequencies on a log scale from 0.01 to 100 rad/s.

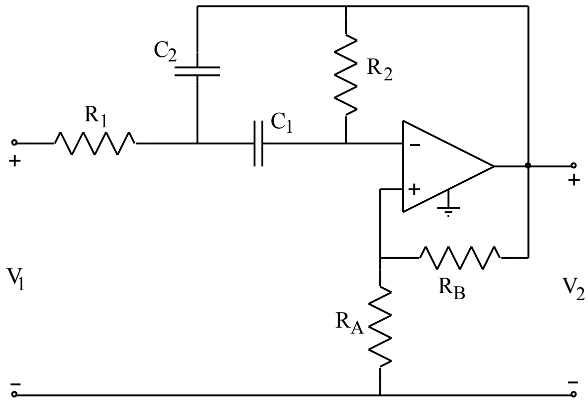
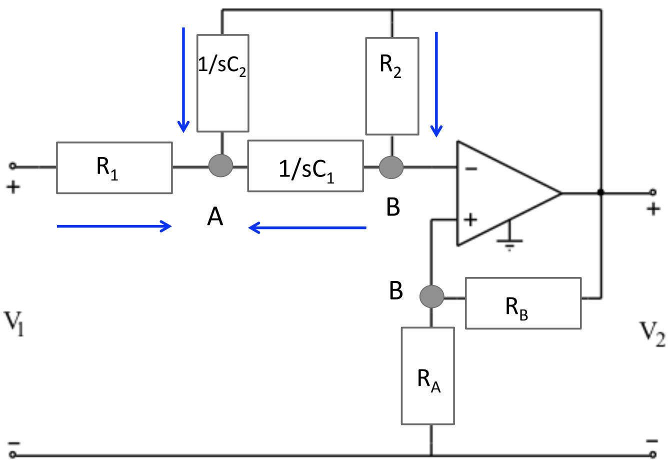


Figure 2.

Solution: We are asked to find the transfer function, and the input voltage source is not specified. Thus, we transform the circuit to the Laplace domain. Since we are to find the transfer function of the circuit, we do not include initial condition generators.



By voltage division, we find that

$$V_B = \frac{R_A}{R_B + R_A} V_2 = \frac{V_2}{2}$$

Write KCL at node A (take into account that $C_1 R_1 = C_2 R_1 = 1$):

Write KCL at node B (take into account that $C_1R_2 = C_2R_2 = 1$):

You should obtain the following expression for V_A from KCL at node B:

$$V_A = V_2\left(\frac{1}{2} - \frac{1}{2s}\right)$$

Substitute V_A to KCL at node A and find the ratio $H(s) = V_2/V_1$ from this expression:

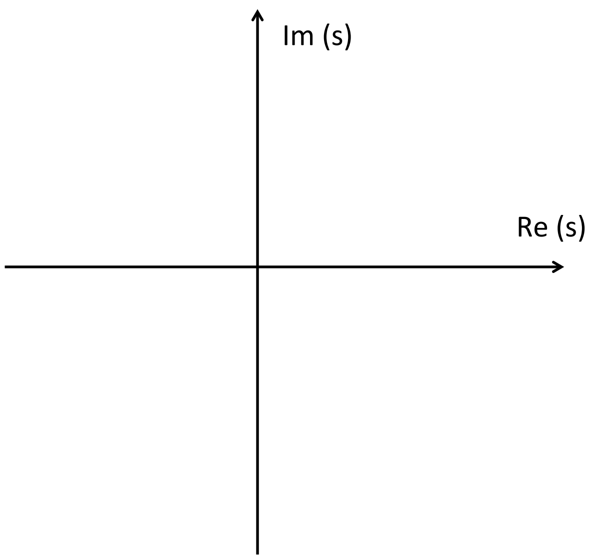
You should obtain that

$$H(s) = -\frac{2s}{s^2 + s + 1}$$

The zero of the transfer function:

The poles of the transfer function:

Plot them in the complex frequency plane:



Now we investigate the frequency response of the circuit in sinusoidal steady-state. We replace s → jω:

$$H(s) = -\frac{2s}{s^2 + s + 1} \quad \Rightarrow \quad H(j\omega) = -\frac{2j\omega}{-\omega^2 + j\omega + 1}$$

What is the magnitude of the response at ω = 0 (at dc)?

At what frequency do we observe a peak of the magnitude response?

We will use a simple and useful technique to plot the magnitude of the transfer function. The magnitude of the transfer function (the gain) is:

$|H(j\omega)|^2 =$

And we evaluate the gain as follows:

ω	20 Log $_{10}$ $\lvert H(j\omega)\rvert^2$
0	
0.01	
0.1	
1	
10	
100	
∞	

Plot these points in the following figure: