MATH3018 Assignment 3, due on Feb 21

1. Suppose that the random vector $X = (X_1, X_2)^T$ has a bivariate normal distribution with $\mathrm{Var}(X_1) = \mathrm{Var}(X_2)$. Show that $X_1 + X_2$ and $X_1 - X_2$ are independent random variables.

2. Suppose $X = (X_1, X_2, X_3)^T$ follows $N_3(\mu, \Sigma)$, where

$$\mu = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}, \Sigma = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

   Justify whether each of the following pairs is independent or not:
   (a) X_1 and X_2.
   (b) X_2 and X_3.
   (c) $(X_1, X_2)^T$ and X_3.
   (d) $X_1 + X_2$ and X_3.
   (e) X_2 and $-2X_1 + X_2 - X_3$.

3. Let $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$ be distributed as $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}$, $\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}$, and $|\boldsymbol{\Sigma}_{22}| > 0$. Then the conditional distribution of $\mathbf{X}_1$, given that $\mathbf{X}_2 = \mathbf{x}_2$, is normal and has mean $\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2)$ and covariance $\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}$

   (Hint: consider taking the linear transformation $\mathbf{AX}$ where $\mathbf{A} = \begin{pmatrix} \mathbf{I}_q & -\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1} \\ 0_{(p-q)\times q} & \mathbf{I}_{(p-q)} \end{pmatrix}$, and where q denotes the dimension of $\mathbf{X}_1$.)

4. Suppose $Y = (Y_1, Y_2, Y_3, Y_4)^T$ follows $N_4(\mu, \Sigma)$, where

$$\mu = \begin{pmatrix} 1 \\ 2 \\ 3 \\ -2 \end{pmatrix}, \Sigma = \begin{pmatrix} 4 & 2 & -1 & 2 \\ 2 & 6 & 3 & -2 \\ -1 & 3 & 5 & -4 \\ 2 & -2 & -4 & 4 \end{pmatrix}.$$

   Find the following :
   (a) The marginal distribution of $(Y_1, Y_3)^T$.
   (b) The marginal distribution of Y_2.
   (c) The distribution of $Z = Y_1 + 2Y_2 - Y_3 + 3Y_4$.
   (d) The conditional distribution of $X_1 = (Y_1, Y_2)^T$ given that $X_2 = (Y_3, Y_4)^T = (y_3, y_4)^T = x_2$.

5. If two normal variables (or vectors) have a covariance of zero, are they necessarily to be independent?

   (Hint: Consider $X_1 \sim \mathcal{N}(0, 1)$, and $X_2 = ZX_1$ for some Z independent of X_1 and has distribution $P(Z = \pm 1) = 1/2$. Verify that X_1, X_2 are both normal, but they are not independent.)