MATH3018 Assignment 2, due on Feb 12

(1) Let $A = \begin{pmatrix} 4 & 3 \\ 8 & 6 \\ 8 & -9 \end{pmatrix}$.

(a) Calculate $AA^\top$ and obtain its eigenvalues and eigenvectors.
(b) Calculate $A^\top A$ and obtain its eigenvalues and eigenvectors.
(c) Obtain the singular value decomposition of A.

(2) Show that every eigenvalue of a $k \times k$ positive definite matrix A is positive.
(Hint: Consider the definition of an eigenvalue, where $Au = \lambda u$. Multiply on the left by $u^\top$.)

(3) For any positive semi-definite matrix A, let $\lambda_1(A)$ denote its largest eigenvalue. Show that $x^\top Ax \leq \lambda_1(A)$ for any unit vector x. For which x does equality holds?

(4) Show that

$$\mathrm{Cov}\left(c_{11}X_1 + c_{12}X_2 + \cdots + c_{1p}X_p, c_{21}X_1 + c_{22}X_2 + \cdots + c_{2p}X_p\right) = \mathbf{c}_1^\top \Sigma \mathbf{c}_2$$

where $\mathbf{c}_1^\top = [c_{11}, c_{12}, \ldots, c_{1p}]$, $\mathbf{c}_2^\top = [c_{21}, c_{22}, \ldots, c_{2p}]$ and Σ is the population covariance matrix of $\mathbf{X} = (X_1, X_2, \cdots, X_p)^\top$.

(5) Let $X = (X_1, X_2)^T$ be a random vector. We are given $n = 3$ observations:

$$\mathbf{X} = \begin{pmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{pmatrix}$$

Define $b = (2, 3)^T$ and $c = (-1, 2)^T$. Find the following:
(i) sample means of $b^T X$ and $c^T X$,
(ii) sample variances of $b^T X$ and $c^T X$, respectively
(iii) sample covariance of $b^T X$ and $c^T X$

(6) Suppose the random vector $X = (X_1, X_2, X_3)^\top$ has covariance matrix

$$\Sigma = \begin{pmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{pmatrix}.$$

(a) Find the population correlation matrix of X.
(b) Find the covariance matrix of the random vector $(X_2 - 3, 2X_1 - X_2 + X_3 + 1)^\top$.