楊宜樹 D-01-2828-0 Prof. Zhang

MATH3018 Assignment 2, due on Feb 12

(1) Let $A = \begin{pmatrix} 4 & 3 \\ 8 & 6 \\ 8 & -9 \end{pmatrix}$.

(a) Calculate $AA^\top$ and obtain its eigenvalues and eigenvectors.
(b) Calculate $A^\top A$ and obtain its eigenvalues and eigenvectors.
(c) Obtain the singular value decomposition of A.

(2) Show that every eigenvalue of a $k \times k$ positive definite matrix A is positive.
(Hint: Consider the definition of an eigenvalue, where $Au = \lambda u$. Multiply on the left by $u^\top$.)

(3) For any positive semi-definite matrix A, let $\lambda_1(A)$ denote its largest eigenvalue. Show that $x^\top Ax \leq \lambda_1(A)$ for any unit vector x. For which x does equality holds?

(4) Show that

$$\mathrm{Cov}\left(c_{11}X_1 + c_{12}X_2 + \cdots + c_{1p}X_p, c_{21}X_1 + c_{22}X_2 + \cdots + c_{2p}X_p\right) = \mathbf{c}_1^\top \Sigma \mathbf{c}_2$$

where $\mathbf{c}_1^\top = [c_{11}, c_{12}, \ldots, c_{1p}]$, $\mathbf{c}_2^\top = [c_{21}, c_{22}, \ldots, c_{2p}]$ and Σ is the population covariance matrix of $\mathbf{X} = (X_1, X_2, \cdots, X_p)^\top$.

(5) Let $X = (X_1, X_2)^T$ be a random vector. We are given $n = 3$ observations:

$$\mathbf{X} = \begin{pmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{pmatrix}$$

Define $b = (2, 3)^T$ and $c = (-1, 2)^T$. Find the following:
(i) sample means of $b^T X$ and $c^T X$,
(ii) sample variances of $b^T X$ and $c^T X$, respectively
(iii) sample covariance of $b^T X$ and $c^T X$

(6) Suppose the random vector $X = (X_1, X_2, X_3)^\top$ has covariance matrix

$$\Sigma = \begin{pmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{pmatrix}.$$

(a) Find the population correlation matrix of X.
(b) Find the covariance matrix of the random vector $(X_2 - 3, 2X_1 - X_2 + X_3 + 1)^\top$.

(1) Let $A = \begin{pmatrix} 4 & 3 \\ 8 & 6 \\ 8 & -9 \end{pmatrix}$.

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(a) Calculate $AA^\top$ and obtain its eigenvalues and eigenvectors.
(b) Calculate $A^\top A$ and obtain its eigenvalues and eigenvectors.
(c) Obtain the singular value decomposition of A.

Solution: (a):

$A^T = \begin{pmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{pmatrix}$ $\therefore AA^T = \begin{pmatrix} 4 & 3 \\ 8 & 6 \\ 8 & -9 \end{pmatrix}\begin{pmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{pmatrix} = \begin{pmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{pmatrix}$ $\therefore AA^T\vec{x} = \lambda\vec{x}$ to find λ and $\vec{x}$, where $\vec{x} \neq 0$ $\Rightarrow AA^T\vec{x} = \lambda I\vec{x} \Rightarrow (AA^T-\lambda I)\vec{x} = 0$ and $\vec{x} \neq 0$

For solution, if $AA^T - \lambda I$ is invertible, then $\vec{x} = (AA^T-\lambda I)^{-1}0 = 0$, contradiction, so $\cancel{AA}^T - \lambda I$ not invertible

$\therefore |AA^T - \lambda I| = 0 \Rightarrow \begin{vmatrix} 25-\lambda & 50 & 5 \\ 50 & 100-\lambda & 10 \\ 5 & 10 & 145-\lambda \end{vmatrix} = (25-\lambda)\cdot\begin{vmatrix} 100-\lambda & 10 \\ 10 & 145-\lambda \end{vmatrix} - 50\cdot\begin{vmatrix} 50 & 10 \\ 5 & 145-\lambda \end{vmatrix} + 5\cdot\begin{vmatrix} 50 & 100-\lambda \\ 5 & 10 \end{vmatrix}$

$= (25-\lambda)\cdot\big((100-\lambda)\cdot(145-\lambda)-100\big) - 50(7250-50\lambda-50) + 5(500-500+5\lambda)$

$= -\lambda^3 + 270\lambda^2 - 20525\lambda + 360000 - 360000 + 2500\lambda + 25\lambda$

$= -\lambda^3 + 270\lambda^2 - 18000\lambda = \lambda(-\lambda^2 + 270\lambda - 18000) = -\lambda(\lambda^2 - 270\lambda + 18000) = 0 \Leftrightarrow -\lambda(\lambda-120)(\lambda-150) = 0$

$\therefore \lambda_1 = 150, \lambda_2 = 120, \lambda_3 = 0$

when $\lambda_1 = 150$

$(AA^T - \lambda I)\vec{x} = \begin{bmatrix} -125 & 50 & 5 \\ 50 & -50 & 10 \\ 5 & 10 & -5 \end{bmatrix}\vec{x} = 0 \Rightarrow \begin{cases} -25x_1 + 10x_2 + x_3 = 0 \\ 5x_1 - 5x_2 + x_3 = 0 \\ -x_1 - 2x_2 + x_3 = 0 \end{cases} \Rightarrow \vec{x}_1 = x_2\begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{5}{2} \end{bmatrix}$ where $x_2 \neq 0 \Rightarrow \vec{x}_1 = \begin{bmatrix} \frac{\sqrt{30}}{30} \\ \frac{\sqrt{30}}{15} \\ \frac{\sqrt{30}}{6} \end{bmatrix}$

when $\lambda_2 = 120$

$(AA^T - \lambda I)\vec{x} = \begin{bmatrix} -95 & 50 & 5 \\ 50 & -20 & 10 \\ 5 & 10 & 25 \end{bmatrix}\vec{x} = 0 \Rightarrow \begin{cases} -19x_1 + 10x_2 + x_3 = 0 \\ 10x_1 - 4x_2 + 2x_3 = 0 \\ x_1 + 2x_2 + 5x_3 = 0 \end{cases} \Rightarrow \vec{x}_2 = \begin{bmatrix} -x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$, where $x_3 \neq 0 \Rightarrow \vec{x}_2 = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{bmatrix}$

when $\lambda_3 = 0$

$(AA^T - \lambda I)\vec{x} = AA^T\vec{x} = \begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix}\vec{x} = 0 \Rightarrow \begin{cases} 5x_1 + 10x_2 + x_3 = 0 \\ 5x_1 + 10x_2 + x_3 = 0 \\ x_1 + 2x_2 + 29x_3 = 0 \end{cases} \Rightarrow \begin{cases} 5x_1 + 10x_2 + x_3 = 0 \\ 5x_1 + 10x_2 + 145x_3 = 0 \end{cases} \Rightarrow \begin{cases} -2x_2 \\ x_2 \\ x_3 = 0 \end{cases} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, where $x_2 \neq 0$

$\therefore \vec{x}_3 = \begin{bmatrix} -\frac{2}{5}\sqrt{5} \\ \frac{\sqrt{5}}{5} \\ 0 \end{bmatrix}$

$\therefore$ for AA^T, $\lambda_1 = 150$ and $\vec{x}_1 = \begin{bmatrix} \frac{\sqrt{30}}{30} \\ \frac{\sqrt{30}}{15} \\ \frac{\sqrt{30}}{6} \end{bmatrix}$, $\lambda_2 = 120$ and $\vec{x}_2 = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{bmatrix}$, $\lambda_3 = 0$ and $\vec{x}_3 = \begin{bmatrix} -\frac{2}{5}\sqrt{5} \\ \frac{\sqrt{5}}{5} \\ 0 \end{bmatrix}$.

(b) $A^TA = \begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 8 & 6 \\ 8 & -9 \end{bmatrix} = \begin{bmatrix} 144 & -12 \\ -12 & 126 \end{bmatrix}$ $\therefore |A^TA - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} 144-\lambda & -12 \\ -12 & 126-\lambda \end{vmatrix} = 0 \Leftrightarrow (144-\lambda)(126-\lambda) - 144 = 0$

$\Rightarrow \lambda^2 - 270\lambda + 18000 = 0$ $\therefore \lambda_1 = 150, \lambda_2 = 120$ when $\lambda_1 = 150$ $(A^TA - \lambda I)\vec{x} = \begin{bmatrix} -6 & -12 \\ -12 & -24 \end{bmatrix}\vec{x} = 0 \Rightarrow \begin{cases} x_1 + 2x_2 = 0 \\ x_1 + 2x_2 = 0 \end{cases}$

$\Rightarrow \vec{x}_1 = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ where $x_2 \neq 0$ $\therefore \vec{x}_1 = \begin{bmatrix} -\frac{2}{5}\sqrt{5} \\ \frac{1}{5}\sqrt{5} \end{bmatrix}$. when $\lambda_2 = 120$: $(A^TA - \lambda I)\vec{x} = \begin{bmatrix} 24 & -12 \\ -12 & 6 \end{bmatrix}\vec{x} = 0$

$\Rightarrow \begin{cases} 2x_1 - x_2 = 0 \\ -12x_1 + 6x_2 = 0 \end{cases} \Rightarrow \begin{cases} 2x_1 - x_2 = 0 \\ 2x_1 - x_2 = 0 \end{cases} \Rightarrow \vec{x}_2 = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ where $x_1 \neq 0$ $\therefore \vec{x}_2 = \begin{bmatrix} \frac{\sqrt{5}}{5} \\ \frac{2}{5}\sqrt{5} \end{bmatrix}$

$\therefore$ for A^TA, $\lambda_1 = 150$ and $\vec{x}_1 = \begin{bmatrix} -\frac{2}{5}\sqrt{5} \\ \frac{1}{5}\sqrt{5} \end{bmatrix}$, $\lambda_2 = 120$ and $\vec{x}_2 = \begin{bmatrix} \frac{\sqrt{5}}{5} \\ \frac{2}{5}\sqrt{5} \end{bmatrix}$.

(c):

$A = USV^T$ where $A \in \mathbb{R}^{3\times2}$, $V \in \mathbb{R}^{3\times3}$, $S \in \mathbb{R}^{3\times2}$, $V \in \mathbb{R}^{2\times2}$, $V^T \in \mathbb{R}^{2\times2}$

here $V = \begin{bmatrix} -\frac{2}{5}\sqrt{5} & \frac{\sqrt{5}}{5} \\ \frac{1}{5}\sqrt{5} & \frac{2}{5}\sqrt{5} \end{bmatrix}$ $\therefore V^T = \begin{bmatrix} -\frac{2}{5}\sqrt{5} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & \frac{2}{5}\sqrt{5} \end{bmatrix}$, $U = \begin{bmatrix} \frac{\sqrt{30}}{30} & -\frac{\sqrt{6}}{6} & -\frac{2}{5}\sqrt{5} \\ \frac{\sqrt{30}}{15} & -\frac{\sqrt{6}}{3} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{30}}{6} & \frac{\sqrt{6}}{6} & 0 \end{bmatrix}$ $\therefore S = \begin{bmatrix} 150 & 0 \\ 0 & 120 \\ 0 & 0 \end{bmatrix}$

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$\therefore$ singular decomposition: $A = USV^T = \begin{bmatrix} \frac{\sqrt{30}}{30} & -\frac{\sqrt{6}}{6} & -\frac{2}{5\sqrt{5}} \\ \frac{\sqrt{30}}{15} & -\frac{\sqrt{6}}{3} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{30}}{6} & \frac{\sqrt{6}}{6} & 0 \end{bmatrix} \begin{bmatrix} 150 & 0 \\ 0 & 120 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{2}{5\sqrt{5}} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & \frac{2}{5\sqrt{5}} \end{bmatrix}$ o.

D.

(2) Show that every eigenvalue of a $k \times k$ positive definite matrix A is positive.
(Hint: Consider the definition of an eigenvalue, where $Au = \lambda u$. Multiply on the left by $u^\top$.)

Solution: since A is $k \times k$ positive definite, $x^T A x \geq 0$ for $\forall x^T \in R^{k\times 1}$ and equality holds only for $x^T = [0, 0, \cdots, 0]$
By such property we have $u^T A u > 0$ because u is a random eigenvalue, which means $u^T \neq [0, 0, \cdots, 0]$
when u is a random eigenvalue of A, we have: $Au = \lambda u$, multiply on the both sides of u^T on left.

$u^T \lambda u = u^T A u > 0$

$\therefore \lambda \|u\|_2^2 > 0$

since we have $\|u\|_2^2 = u^T u > 0$ because $u^T \neq [0, 0, \cdots, 0]$
Thus finally $\lambda > 0$ for each λ as eigenvalue.
hence shown. D.

(3) For any positive semi-definite matrix A, let $\lambda_1(A)$ denote its largest eigenvalue.
Show that $x^\top A x \leq \lambda_1(A)$ for any unit vector x. For which x does equality holds?

Solution:

$\because A$ is positive semi-definite matrix $\therefore A = U \Lambda U^T$ where $U^T U = I$ and $\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are eigenvalues
$U = [u_1 \; u_2 \cdots u_n]$ and $u_1, u_2, \cdots, u_n$ are orthogonal eigenvectors.

$\because x$ is a unit vector $\|x\| = 1$, we expand x in the eigenbasis:

$x = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$ where $c_i = u_i^T x$ and $c_1^2 + c_2^2 + \cdots + c_n^2 = 1$ (since $\|x\|^2 = 1$)

$\therefore x^T A x = x^T U \Lambda U^T x = (U^T x)^T \Lambda (U^T x)$, and let $y = U^T x$, $y = \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix} x = \begin{bmatrix} u_1^T x \\ u_2^T x \\ \vdots \\ u_n^T x \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = [c_1 \; c_2 \cdots c_n]^T$

$\therefore x^T A x = y^T \Lambda y = \sum_{i=1}^n \lambda_i c_i^2$

since $\lambda_1 \geq \lambda_i$ for $i \geq 2$

$\therefore \sum_{i=1}^n \lambda_i c_i^2 \leq \sum_{i=1}^n \lambda_1 c_i^2 = \lambda_1 \sum_{i=1}^n c_i^2 = \lambda_1$

$\therefore x^T A x \leq \lambda_1$

just shown $x^T A x \leq \lambda_1(A)$ for $\forall x$ and $\|x\| = 1$

equality holds when $\sum_{i=1}^n \lambda_i c_i^2 = \sum_{i=1}^n \lambda_1 c_i^2$, just $\sum_{i=1}^n (\lambda_i - \lambda_1) c_i^2 = 0$ or $\sum_{i=1}^n (\lambda_1 - \lambda_i) c_i^2 = 0$
since $\lambda_1 - \lambda_i \geq 0$ and $c_i^2 \geq 0$ $\therefore (\lambda_1 - \lambda_i) c_i^2 = 0$ for $\forall i$
for $\forall \lambda_i < \lambda_1$ $c_i^2 = 0$ $\therefore x = \sum_{i \in S} c_i u_i$ where $S = \{ i \mid \lambda_i = \lambda_1 \}$

D.

(4) Show that

$$\operatorname{Cov}(c_{11}X_1 + c_{12}X_2 + \cdots + c_{1p}X_p, c_{21}X_1 + c_{22}X_2 + \cdots + c_{2p}X_p) = \mathbf{c}_1^\top \Sigma \mathbf{c}_2$$

where $\mathbf{c}_1^\top = [c_{11}, c_{12}, \ldots, c_{1p}]$, $\mathbf{c}_2^\top = [c_{21}, c_{22}, \ldots, c_{2p}]$ and Σ is the population covariance matrix of $\mathbf{X} = (X_1, X_2, \cdots, X_p)^\top$.

Solution:

L.H.S. $= \operatorname{CoV}(c_{11}X_1 + c_{12}X_2 + \cdots + c_{1p}X_p, c_{21}X_1 + c_{22}X_2 + \cdots + c_{2p}X_p) = \sum_{i=1}^p \sum_{j=1}^p c_{1i} c_{2j} \operatorname{CoV}(X_i, X_j)$

R.H.S. $= c_1^T \Sigma c_2 = [c_{11}, c_{12}, \cdots, c_{1p}] \begin{bmatrix} \operatorname{CoV}_{11} & \operatorname{CoV}_{12} & \cdots & \operatorname{CoV}_{1p} \\ \operatorname{CoV}_{21} & \operatorname{CoV}_{22} & \cdots & \operatorname{CoV}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{CoV}_{p1} & \operatorname{CoV}_{p2} & \cdots & \operatorname{CoV}_{pp} \end{bmatrix} \begin{bmatrix} c_{21} \\ c_{22} \\ \vdots \\ c_{2p} \end{bmatrix} = \left[\sum_{i=1}^p c_{1i} \operatorname{CoV}(X_i, X_1) \; \sum_{i=1}^p c_{1i} \operatorname{CoV}(X_i, X_2) \cdots \sum_{i=1}^p c_{1i} \operatorname{CoV}(X_i, X_p) \right] \begin{bmatrix} c_{21} \\ c_{22} \\ \vdots \\ c_{2p} \end{bmatrix} = \sum_{i=1}^p \sum_{j=1}^p c_{1i} c_{2j} \operatorname{CoV}(X_i, X_j)$

by such computation, it is easy to show that L.H.S. = R.H.S.
shown. D.

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(5) Let $X = (X_1, X_2)^T$ be a random vector. We are given $n = 3$ observations:

$$\mathbf{X} = \begin{pmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{pmatrix}$$

Define $b = (2, 3)^T$ and $c = (-1, 2)^T$. Find the following:
(i) sample means of $b^T X$ and $c^T X$,
(ii) sample variances of $b^T X$ and $c^T X$, respectively
(iii) sample covariance of $b^T X$ and $c^T X$

Solution: (i): I: For $b^T=(2,3)$: First observation: $2\times9+3\times1=21$, Second observation: $2\times5+3\times3=19$, Third Observation: $2\times1+3\times2=8$

∴ Value: [21, 19, 8] ∴ Sample Mean of $b^TX = \frac{21+19+8}{3} = \frac{48}{3} = 16$

2: For $c^T=(-1,2)$: First Observation: $-1\times9+2\times1=-7$, Second Observation: $-1\times5+2\times3=1$, Third Observation: $-1\times1+2\times2=3$ ∴ Value: [-7, 1, 3] ∴ Sample Mean of c^TX: $\frac{-7+1+3}{3}=\frac{-3}{3}=-1$ 。

(ii): Sample Variance $=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})(x_i-\bar{x})^T=\frac{1}{2}\sum_{i=1}^{3}(x_i-\bar{x})(x_i-\bar{x})^T$

I: sample variances of $b^TX=\frac{1}{2}\left((21-16)^2+(19-16)^2+(8-16)^2\right)=\frac{1}{2}(25+9+64)=\frac{1}{2}\times98=49$

2: sample variances of $c^TX=\frac{1}{2}\left((-7-(-1))^2+(1-(-1))^2+(3-(-1))^2\right)=\frac{1}{2}\times(36+4+16)=\frac{1}{2}\times56=28$ 。

(iii): Sample Covariance $=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})^T=\frac{1}{2}\sum_{i=1}^{3}(x_i-\bar{x})(y_i-\bar{y})^T$

∴ sample covariance $(b^TX, c^TX)=\frac{1}{2}\left((21-16)(-7-(-1))+(19-16)(1-(-1))+(8-16)(3-(-1))\right)=\frac{1}{2}(-30+6-32)=-28$ 。

□.

(6) Suppose the random vector $X = (X_1, X_2, X_3)^\top$ has covariance matrix

$$\Sigma = \begin{pmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{pmatrix}.$$

(a) Find the population correlation matrix of X.
(b) Find the covariance matrix of the random vector $(X_2 - 3, 2X_1 - X_2 + X_3 + 1)^\top$.

Solution:

(a): $\rho = D_\sigma^{-1/2}\Sigma D_\sigma^{-1/2}$ where D_σ = diag of $\Sigma = \begin{pmatrix} 25 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}$ ∴ $\rho = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}\begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{5} & \frac{4}{15} \\ -\frac{1}{5} & 1 & \frac{1}{6} \\ \frac{4}{15} & \frac{1}{6} & 1 \end{bmatrix}$

(b): $Y = \begin{bmatrix} X_2 - 3 \\ 2X_1 - X_2 + X_3 + 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}X + \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

∴ $\Sigma_Y = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}\Sigma\begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ -7 & 135 \end{bmatrix}$

□.