Q_3: solution:

$P(\theta X_1 + (1-\theta)X_2 \le x) = P(X_1 \le x \mid \theta = 1)P(\theta=1) + P(X_2 \le x \mid \theta = 0)P(\theta=0) = P \cdot P(X_1 \le x \mid \theta=1) + (1-P)P(X_2 \le x \mid \theta=0)$

$\xlongequal{X_1, X_2 \text{ are independent of } \theta} P \cdot P(X_1 \le x) + (1-P)P(X_2 \le x)$

$\therefore P(\theta X_1 + (1-\theta)X_2 = x) = \dfrac{d[P(\theta X_1 + (1-\theta)X_2 \le x)]}{dx} = \dfrac{d[P P(X_1 \le x) + (1-P)P(X_2 \le x)]}{dx} = P\dfrac{dP(X_1 \le x)}{dx} + (1-P)\dfrac{dP(X_2 \le x)}{dx}$

$= P P(X_1 = x) + (1-P)P(X_2 = x) = \dfrac{P}{\sqrt{2\pi}\sigma_1}\exp\left(-\dfrac{(x-\mu_1)^2}{2\sigma_1^2}\right) + \dfrac{1-P}{\sqrt{2\pi}\sigma_2}\exp\left(-\dfrac{(x-\mu_2)^2}{2\sigma_2^2}\right)$

because $X_1 \sim N(\mu, \sigma_1^2)$; $X_2 \sim N(\mu, \sigma_2^2)$