Preliminary:

Let r.v. X has a density function $p(x)$, then we have

① $p(x) \geq 0$, and $\int_{-\infty}^{+\infty} p(x)\,dx = 1$

② the c.d.f. of X is

$$F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} p(t)\,dt$$

$$p(x) = F'(x).$$

If $X \sim N(\mu, \sigma^2)$, its probability density function (p.d.f) is

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

Hence, we know

$$\int_{-\infty}^{+\infty} p(x)\,dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx = 1$$

$$\Rightarrow \int_{-\infty}^{+\infty} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx = \sqrt{2\pi\sigma^2}, \quad \forall \mu \in \mathbb{R},\ \sigma \neq 0$$

┌ By some calculations, we also obtain that

$$\int_{-\infty}^{+\infty} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx \overset{\frac{x-\mu}{\sigma}=y}{=\!=} \int_{-\infty}^{+\infty} \exp\left\{-\frac{y^2}{2}\right\} \sigma\,dy$$

$$\underset{y=\sqrt{2z}}{\overset{\frac{y^2}{2}=z}{=\!=}} 2\sigma \int_{0}^{+\infty} \exp\{-z\} \frac{1}{\sqrt{2}} \cdot z^{-\frac{1}{2}}\,dz$$

$$= \sqrt{2}\sigma \int_{0}^{+\infty} z^{-\frac{1}{2}} \exp\{-z\}\,dz$$

$$= \sqrt{2}\sigma\, \Gamma\left(\frac{1}{2}\right) \overset{\Gamma(\frac{1}{2})=\sqrt{\pi}}{=} \sqrt{2\pi\sigma^2}$$ ┘

Question: Let

$$y \mid x \sim N(\mu_1, \sigma_1^2)$$
$$z \mid y \sim N(\mu_2 y, \sigma_2^2)$$

where $\mu_1, \mu_2 \in \mathbb{R}$, $\sigma_1^2, \sigma_2^2 > 0$. Show that

$$z \mid x \sim N(\mu_1\mu_2, \sigma_1^2\mu_2^2 + \sigma_2^2)$$

Solve: we know that

$$p(z|x) = \int_{-\infty}^{+\infty} p(z|y)\, p(y|x)\, dy$$

$$= \frac{1}{\sqrt{2\pi\sigma_1^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}} \int_{-\infty}^{+\infty} \exp\left\{ -\frac{(z-\mu_2 y)^2}{2\sigma_2^2} - \frac{(y-\mu_1)^2}{2\sigma_1^2} \right\} dy \qquad (1)$$

Step. 1.

$$-\frac{(z-\mu_2 y)^2}{2\sigma_2^2} - \frac{(y-\mu_1)^2}{2\sigma_1^2} = -\frac{\sigma_1^2(z-\mu_2 y)^2 + \sigma_2^2(y-\mu_1)^2}{2\sigma_1^2\sigma_2^2}$$

$$= -\frac{(\sigma_1^2\mu_2^2+\sigma_2^2)y^2 - 2(\sigma_1^2\mu_2 z + \sigma_2^2\mu_1)y + \sigma_1^2 z^2 + \sigma_2^2\mu_1^2}{2\sigma_1^2\sigma_2^2}$$

$$= -\frac{(\sigma_1^2\mu_2^2+\sigma_2^2)\left[y - \frac{\sigma_1^2\mu_2 z + \sigma_2^2\mu_1}{\sigma_1^2\mu_2^2+\sigma_2^2}\right]^2 + \sigma_1^2 z^2 + \sigma_2^2\mu_1^2 - \frac{(\sigma_1^2\mu_2 z + \sigma_2^2\mu_1)^2}{\sigma_1^2\mu_2^2+\sigma_2^2}}{2\sigma_1^2\sigma_2^2} \qquad (2)$$

Observe that

$$\sigma_1^2 z^2 + \sigma_2^2\mu_1^2 - \frac{(\sigma_1^2\mu_2 z + \sigma_2^2\mu_1)^2}{\sigma_1^2\mu_2^2+\sigma_2^2} = \frac{\sigma_1^2\sigma_2^2\,(z-\mu_1\mu_2)^2}{\sigma_1^2\mu_2^2+\sigma_2^2} \qquad (3)$$

Step. 2. combing (1), (2) and (3), we have

$$p(z|x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left\{ -\frac{(z-\mu_1\mu_2)^2}{2(\sigma_1^2\mu_2^2+\sigma_2^2)} \right\} \underbrace{\int_{-\infty}^{+\infty} \exp\left\{ -\frac{\left[y - \frac{\sigma_1^2\mu_2 z + \sigma_2^2\mu_1}{\sigma_1^2\mu_2^2+\sigma_2^2}\right]^2}{2\sigma_1^2\sigma_2^2/(\sigma_1^2\mu_2^2+\sigma_2^2)} \right\} dy}_{=\sqrt{2\pi}\cdot\sqrt{\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2\mu_2^2+\sigma_2^2}}}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{ -\frac{(z-\mu_1\mu_2)^2}{2(\sigma_1^2\mu_2^2+\sigma_2^2)} \right\} \sqrt{2\pi} \cdot \sqrt{\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2\mu_2^2+\sigma_2^2}}$$

$$= \frac{1}{\sqrt{2\pi(\sigma_1^2\mu_2^2+\sigma_2^2)}} \exp\left\{ -\frac{(z-\mu_1\mu_2)^2}{2(\sigma_1^2\mu_2^2+\sigma_2^2)} \right\}$$

That is,

$$z|x \sim N(\mu_1\mu_2,\ \sigma_1^2\mu_2^2+\sigma_2^2)$$

Question: Let $\pi(\theta) = \frac{1}{C} \exp\left\{ -\frac{1}{2}(\theta + \cos\theta)^2 \right\}$, where $C = \int e^{-\frac{1}{2}(\theta+\cos\theta)^2} d\theta$.

Derive its normal approximation.

Solve:

$$\log \pi(\theta) = -\frac{1}{2}(\theta + \cos\theta)^2 - \log C$$

$$[\log \pi(\theta)]' = -(\theta + \cos\theta)(1 - \sin\theta)$$

$$[\log \pi(\theta)]'' = -(1-\sin\theta)^2 + (\theta + \cos\theta)\cos\theta$$

Let $[\log \pi(\theta)]' = 0$, we have

$$\theta_1 \approx -0.739, \quad \theta_2 = \frac{\pi}{2}.$$

Due to

$$[\log \pi(\theta)]''\Big|_{\theta=\theta_2} = 0,$$

we use θ_1 to derive its normal approximation.

$$[\log \pi(\theta)]''\Big|_{\theta=\theta_1} \approx -[1+\sin(0.739)]^2 \approx -2.8$$

Hence,

$$\pi(\theta) \propto \exp\left\{ -1.4(\theta + 0.739)^2 \right\}$$