Question 8: Solution:

$X \sim N(\mu, \sigma^2) \Rightarrow P(X=x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}$, hence by $\int_{-\infty}^{+\infty} p(x)\,dx = 1$ we know: $\int_{-\infty}^{+\infty} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}dx = \sqrt{2\pi\sigma^2}$, $\forall \mu \in R$, $\sigma \neq 0$

Thus by the Question we have $p(z|x) = \int_{-\infty}^{+\infty} p(z|y)p(y|x)\,dy = \frac{1}{\sqrt{2\pi\sigma_1^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}} \int_{-\infty}^{+\infty} \exp\{-\frac{(z-\mu_2 y)^2}{2\sigma_2^2} - \frac{(y-\mu_1)^2}{2\sigma_1^2}\}dy$. (1)

Step I: $-\frac{(z-\mu_2 y)^2}{2\sigma_2^2} - \frac{(y-\mu_1)^2}{2\sigma_1^2} = -\frac{\sigma_1^2(z-\mu_2 y)^2 + \sigma_2^2(y-\mu_1)^2}{2\sigma_1^2\sigma_2^2} = -\frac{(\mu_2^2\sigma_1^2+\sigma_2^2)y^2 - 2(\mu_2 z\sigma_1^2 + \mu_1\sigma_2^2)y + \sigma_1^2 z^2 + \sigma_2^2\mu_1^2}{2\sigma_1^2\sigma_2^2} = -\frac{(\sigma_1^2\mu_2^2+\sigma_2^2)\left[y - \frac{\sigma_1^2\mu_2 z + \mu_1\sigma_2^2}{\sigma_1^2\mu_2^2+\sigma_2^2}\right]^2 + \sigma_1^2 z^2 + \sigma_2^2\mu_1^2 - \frac{(\sigma_1^2\mu_2 z + \sigma_2^2\mu_1)^2}{\sigma_1^2\mu_2^2+\sigma_2^2}}{2\sigma_1^2\sigma_2^2}$ (2)

Observe that:

$\sigma_1^2 z^2 + \sigma_2^2\mu_1^2 - \frac{(\sigma_1^2\mu_2 z + \sigma_2^2\mu_1)^2}{\sigma_1^2\mu_2^2+\sigma_2^2} = \frac{\cancel{\sigma_1^4 z^2\mu_2^2} + \sigma_1^2\sigma_2^2 z^2 + \sigma_1^2\sigma_2^2\mu_1^2\mu_2^2 + \cancel{\sigma_2^4\mu_1^2} - \cancel{\sigma_1^4\mu_2^2 z^2} - \cancel{\sigma_2^4\mu_1^2} - 2\sigma_1^2\sigma_2^2\mu_1\mu_2 z}{\sigma_1^2\mu_2^2+\sigma_2^2} = \frac{\sigma_1^2\sigma_2^2(z-\mu_1\mu_2)^2}{\sigma_1^2\mu_2^2+\sigma_2^2}$ (3)

Combining (1),(2),(3), we have: $p(z|x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\{-\frac{(z-\mu_1\mu_2)^2}{2(\sigma_1^2\mu_2^2+\sigma_2^2)}\} \underbrace{\int_{-\infty}^{+\infty} \exp\{-\frac{\left(y - \frac{\sigma_1^2\mu_2 z + \sigma_2^2\mu_1}{\sigma_1^2\mu_2^2+\sigma_2^2}\right)^2}{2\sigma_1^2\sigma_2^2/(\sigma_1^2\mu_2^2+\sigma_2^2)}\}dy}$

$= \frac{1}{2\pi\sigma_1\sigma_2} \exp\{-\frac{(z-\mu_1\mu_2)^2}{2(\sigma_1^2\mu_2^2+\sigma_2^2)}\} \cdot \underbrace{\sqrt{2\pi} \cdot \sqrt{\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2\mu_2^2+\sigma_2^2}}}$

$= \frac{1}{\sqrt{2\pi(\sigma_1^2\mu_2^2+\sigma_2^2)}} \exp\{-\frac{(z-\mu_1\mu_2)^2}{2(\sigma_1^2\mu_2^2+\sigma_2^2)}\}$ That is. $z|x \sim N(\mu_1\mu_2, \sigma_1^2\mu_2^2+\sigma_2^2)$