Q1: Chapman-Kolmogorov relation: Suppose that $P^m(x,y) = P(\theta^{(m)}=y \mid \theta^{(0)}=x)$. Show that:
(1) $P^{m+1} = P^m \cdot P$ and $P^m = \underbrace{P \cdots P}_{m}$ / (2) $P^{m+n} = P^m \cdot P^n$ for all m,n

Solve (1): For any $x,y \in S$, we have that by Markov's Property:

$$P^{m+1}(x,y) = P(\theta^{(m+1)}=y \mid \theta^{(0)}=x) = \sum_{z\in S} P(\theta^{(m+1)}=y \mid \theta^{(1)}=z, \theta^{(0)}=x) P(\theta^{(1)}=z \mid \theta^{(0)}=x)$$

$$\overset{\text{Markov}}{=\!=\!=} \sum_{z\in S} P(\theta^{(m+1)}=y \mid \theta^{(1)}=z) P(\theta^{(1)}=z \mid \theta^{(0)}=x)$$

$$\overset{\text{time homogeneous}}{=\!=\!=\!=\!=} \sum_{z\in S} P(\theta^{(m)}=y \mid \theta^{(0)}=z) P(\theta^{(1)}=z \mid \theta^{(0)}=x)$$

$$=\!=\!=\!= \sum_{z\in S} P^m(z,y) P(x,z)$$

$$\Longrightarrow P^{m+1} = P^m \cdot P$$

Moreover, by induction,

$$P^{m+1} = P^m \cdot P = P^{m-1} \cdot P \cdot P = \cdots = \overbrace{P \cdots P}^{m}$$

Solve (2):

Observe that for any $x,y \in S$, and $m,n \in \mathbb{N}$.

$$P^{m+n}(x,y) = P(\theta^{(m+n)}=y \mid \theta^{(0)}=x) = \sum_{z\in S} P(\theta^{(m+n)}=y, \theta^{(n)}=z \mid \theta^{(0)}=x) = \sum_{z\in S} P(\theta^{(m+n)}=y, \theta^{(n)}=z, \theta^{(0)}=x) / P(\theta^{(0)}=x)$$

$$= \sum_{z\in S} \frac{P(\theta^{(m+n)}=y, \theta^{(n)}=z, \theta^{(0)}=x)}{P(\theta^{(n)}=z, \theta^{(0)}=x)} \cdot \frac{P(\theta^{(n)}=z, \theta^{(0)}=x)}{P(\theta^{(0)}=x)} \overset{\text{Markov}}{=\!=\!=} \sum_{z\in S} P(\theta^{(m+n)}=y \mid \theta^{(n)}=z) P(\theta^{(n)}=z \mid \theta^{(0)}=x)$$

$$\overset{\text{time homogeneous}}{=\!=\!=\!=\!=} \sum_{z\in S} P(\theta^{(m)}=y \mid \theta^{(0)}=z) P(\theta^{(n)}=z \mid \theta^{(0)}=x) = \sum_{z\in S} P^m(z,y) P^n(x,z)$$ hence we have

$$P^{m+n} = P^m + P^n$$

D.

Q2: $P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1/3 & 2/3 \end{bmatrix}$ $\quad P^2(2,0) = P(\theta^{(2)}=0 \mid \theta^{(0)}=2) = \sum_{j=0}^{2} P(\theta^{(2)}=0 \mid \theta^{(1)}=j) P(\theta^{(1)}=j \mid \theta^{(0)}=2)$

$$= P(0,0)P(2,0) + P(1,0)P(2,1) + P(2,0)P(2,2) = \tfrac{1}{2}\cdot 0 + \tfrac{1}{2}\times\tfrac{1}{3} + 0\cdot\tfrac{2}{3} = \tfrac{1}{6} > \tfrac{1}{12}$$

D.