Algorithm Improvement Documentation

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1 Introduction

This document summarizes the reasoning employed in `new_algorithm.ipynb`, where we argue that $X_{\text{transformed}} = X_{\text{transformed\_1}}$. Our scheme is

$$
\begin{aligned}
\tilde{X} = H^* D^* X^* &= (H_k \otimes H_m) \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_k \end{bmatrix} \begin{bmatrix} X_{1,\text{loc}} \\ X_{2,\text{loc}} \\ \vdots \\ X_{k,\text{loc}} \end{bmatrix} \\
&= (H_k I_k \otimes I_m H_m) \begin{bmatrix} D_1 X_{1,\text{loc}} \\ D_2 X_{2,\text{loc}} \\ \vdots \\ D_k X_{k,\text{loc}} \end{bmatrix} = (H_k \otimes I_m)(I_k \otimes H_m) \begin{bmatrix} D_1 X_{1,\text{loc}} \\ D_2 X_{2,\text{loc}} \\ \vdots \\ D_k X_{k,\text{loc}} \end{bmatrix} \\
&= (H_k \otimes I_m) \begin{bmatrix} H_m & 0 & \cdots & 0 \\ 0 & H_m & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & H_m \end{bmatrix}_{k \text{ blocks}} \begin{bmatrix} D_1 X_{1,\text{loc}} \\ D_2 X_{2,\text{loc}} \\ \vdots \\ D_k X_{k,\text{loc}} \end{bmatrix} \\
&= (H_k \otimes I_m) \begin{bmatrix} H_m D_1 X_{1,\text{loc}} \\ H_m D_2 X_{2,\text{loc}} \\ \vdots \\ H_m D_k X_{k,\text{loc}} \end{bmatrix}. \qquad (1)
\end{aligned}
$$

Additionally, inserting a common sampling matrix $S \in \mathbb{R}^{m\times m}$ before each $H_m D_k X_{k,\text{loc}}$ yields

$$(H_k \otimes I_m)\begin{bmatrix} SH_mD_1X_{1,\text{loc}} \\ SH_mD_2X_{2,\text{loc}} \\ \vdots \\ SH_mD_kX_{k,\text{loc}} \end{bmatrix} = (H_k \otimes I_m)(I_k \otimes SH_m)\begin{bmatrix} D_1X_{1,\text{loc}} \\ D_2X_{2,\text{loc}} \\ \vdots \\ D_kX_{k,\text{loc}} \end{bmatrix}$$

$$= (H_k \otimes I_m)\,(I_k \otimes S)\,(I_k \otimes H_m)\begin{bmatrix} D_1X_{1,\text{loc}} \\ \vdots \\ D_kX_{k,\text{loc}} \end{bmatrix}$$

$$= (I_k \otimes S)\,(H_k \otimes I_m)\,(I_k \otimes H_m)\begin{bmatrix} D_1X_{1,\text{loc}} \\ \vdots \\ D_kX_{k,\text{loc}} \end{bmatrix} \qquad \text{(since } (H_k \otimes I_m)(I_k \otimes S) = (I_k \otimes S)(H_k \otimes I_m))$$

$$= (I_k \otimes S)\,(H_k \otimes H_m)\begin{bmatrix} D_1X_{1,\text{loc}} \\ \vdots \\ D_kX_{k,\text{loc}} \end{bmatrix}$$

$$=:\; B\,H_{km}\begin{bmatrix} D_1X_{1,\text{loc}} \\ \vdots \\ D_kX_{k,\text{loc}} \end{bmatrix}, \quad \text{where } B := I_k \otimes S, \quad H_{km} := H_k \otimes H_m.$$

If each client uses its own sampling matrix $S_i \in \mathbb{R}^{m\times m},\ i = 1, \ldots, k,$ then

$$(H_k \otimes I_m)\begin{bmatrix} S_1H_mD_1X_{1,\text{loc}} \\ S_2H_mD_2X_{2,\text{loc}} \\ \vdots \\ S_kH_mD_kX_{k,\text{loc}} \end{bmatrix} = (H_k \otimes I_m)\begin{bmatrix} S_1 & & \\ & \ddots & \\ & & S_k \end{bmatrix}(I_k \otimes H_m)\begin{bmatrix} D_1X_{1,\text{loc}} \\ \vdots \\ D_kX_{k,\text{loc}} \end{bmatrix}$$

$$= (H_k \otimes I_m)\,S_{\text{blk}}\,(I_k \otimes H_m)\begin{bmatrix} D_1X_{1,\text{loc}} \\ \vdots \\ D_kX_{k,\text{loc}} \end{bmatrix}, \quad \text{where } S_{\text{blk}} := \begin{bmatrix} S_1 & & \\ & \ddots & \\ & & S_k \end{bmatrix}.$$

A single global sampling matrix of the form $B = I_k \otimes S$ would require

$$S_{\text{blk}} = I_k \otimes S = \begin{bmatrix} S & & \\ & \ddots & \\ & & S \end{bmatrix}.$$

$$\text{Thus } S_{\text{blk}} = I_k \otimes S \iff S_1 = S_2 = \cdots = S_k = S.$$

If the S_i are not all identical (i.e., $\exists\, i \neq j :\ S_i \neq S_j$), then
$S_{\text{blk}} \neq I_k \otimes S$ for any S, so we cannot write

$$(H_k \otimes I_m)S_{\text{blk}}(I_k \otimes H_m) = B\,H_{km}\begin{bmatrix} D_1X_{1,\text{loc}} \\ \vdots \\ D_kX_{k,\text{loc}} \end{bmatrix} \qquad \text{with a single global } B \text{ of the form } I_k \otimes S.$$